Bass-Serre Theory

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Abstract

Groups are often first introduced as the set of automorphisms, or symmetries, of a given object. It is then natural to think that, in order to study the properties of a group, it can be more convenient to study the way it acts on an object, rather than the group itself. For instance, in Geometric Group Theory we study geometric and topological properties of a well-understood space upon which the group acts, and we then translate this geometric information into algebraic results.

Bass-Serre Theory deals with the case when this space is a graph, or more specifically a simply connected graph, that is, a tree. The quotient space under the group action should be a graph as well, so that this action is required to be *without inversion*. Our purpose is to develop the basic notions of this theory and prove the so-called Structure Theorem (Theorem 3.4.1), which gives a complete characterization of groups acting on trees. More precisely, we will aim to answer the following question:

What can be said of a group acting on a tree, provided that we understand the quotient graph and the stabilizers of the vertices and the edges?

It turns out, as we will see, that this is the only information we need to characterize the group.

To this end, in Chapter 1 we begin recalling some definitions and results from graph theory, and give the first example of a graph on which a group acts, which is the Cayley graph associated to a group presentation. Furthermore, this action is free, i.e. the vertex and edge stabilizers are all trivial. This graph is a tree if and only if the group is free.

In Chapter 2, we introduce some first examples of group acting on trees. The already mentioned free groups turn out to be characterized by the fact that they act freely on a tree, and we determine exactly on which trees they act in this way. This also provides a simple proof to Schreier's Theorem (Corollary 2.2.3) which states that any subgroup of a free group is free, and we deduce a simple relation between the index and the rank of these subgroups.

The basic constructions in Bass-Serre theory are free amalgamated products and HNN extensions, which correspond to groups acting on trees with a segment or a loop as quotient graph, respectively. We prove this characterization with the aid of the Normal Form Theorems which hold for these groups (Theorems 2.3.10 and 2.3.15). We also illustrate these constructions with several examples, such as $\mathrm{SL}_2(\mathbb{Z})$, Baumslag-Solitar groups or special cases of Artin groups.

Finally, in Chapter 3, we generalize the above results. We start with the definition of a graph of groups, which is a graph with groups associated to its vertices and edges in a certain way. This allows to define the fundamental group and the universal cover of a graph of groups, which are generalizations of the fundamental group and universal cover of a graph. The fundamental group acts on the universal cover, which is a tree, with quotient space isomorphic to the original graph and stabilizers isomorphic to the associated groups. In order to prove this, we use a generalized Normal Form Theorem (Corollary 3.2.3) which holds in these fundamental groups. Lastly, the Structure Theorem is a converse to this construction. It tells us that any group acting on a tree has this structure, so that we arrive at the desired characterization. This result can be used to understand better the structure of fundamental groups of graphs of groups; we illustrate this with a result on subgroups of free amalgamated products known as Kurosh's Theorem (Corollary 3.4.2).

Our main references are Chapter I of [11] and Chapter IV of [9].

Resumen

A menudo, los grupos son introducidos como el conjunto de automorfismos o simetrías de un objeto dado. Es por tanto natural pensar que, para estudiar las propiedades de un grupo, puede ser más práctico estudiar la manera en que actúa en un objeto, más que el grupo en sí. Por ejemplo, en la Teoría Geométrica de Grupos se estudian propiedades geométricas y topológicas de un espacio conocido en el que el grupo actúa, y luego se traduce esta información geométrica a resultados algebraicos.

La teoría de Bass-Serre se ocupa del caso en el que este espacio es un grafo, o más específicamente un grafo simplemente conexo, es decir, un árbol. El espacio cociente bajo la acción del grupo también debe ser un grafo, de modo que se requiere que esta acción sea *sin inversión*. Nuestro objetivo es desarrollar las nociones básicas de esta teoría y demostrar el llamado Teorema de estructura (teorema 3.4.1), que proporciona una caracterización completa de los grupos que actuán sobre árboles. Más concretamente, buscaremos dar respuesta a la siguiente pregunta:

¿Qué podemos decir de un grupo que actúa en un árbol, si conocemos el grafo cociente y los estabilizadores de los vértices y los ejes?

Resulta que, como veremos, esta es toda la información necesaria para caracterizar el grupo.

Con este fin, en el primer capítulo comenzamos recordando algunas definiciones y resultados de teoría de grafos, y damos el primer ejemplo de grafo en el que actúa un grupo, que es el grafo de Cayley asociado a la presentación de un grupo. Más aún, esta acción es libre, es decir, los estabilizadores de todos los vértices y los ejes son triviales. Este grafo es un árbol si y solo si el grupo es libre.

En el segundo capítulo, presentamos los primeros ejemplos de grupos actuando en árboles. Los ya mencionados grupos libres resultan estar caracterizados por el hecho de que actúan libremente en un árbol, y además determinamos exactamente en qué árboles actúan de esta manera. Esto proporciona también una demostración sencilla del Teorema de Schreier (corolario 2.2.3), según el que todo subgrupo de un grupo libre es libre, y deducimos una relación sencilla entre el índice y el rango de estos subgrupos.

Las construcciones básicas en la teoría de Bass-Serre son los productos amalgamados libres y las extensiones HNN, que corresponden a grupos actuando en árboles con un segmento o un lazo como grafo cociente, respectivamente. Demostramos esta caracterización con la ayuda de los teoremas de forma normal que se tienen para estos grupos (teoremas 2.3.10 y 2.3.15). Además, ilustramos estas construcciones con varios ejemplos, como $\mathrm{SL}_2(\mathbb{Z})$, grupos de Baumslag-Solitar o casos especiales de grupos de Artin.

Finalmente, en el tercer capítulo generalizamos los anteriores resultados. Comenzamos con la definición de un grafo de grupos, que es un grafo con grupos asociados a sus vértices y ejes de una determinada manera. Esto permite definir el grupo fundamental y la cubierta universal de un grafo de grupos, que son generalizaciones del grupo fundamental y la cubierta universal de un grafo. El grupo fundamental actúa en la cubierta universal, que es un árbol, con grafo cociente isomorfo al grafo original y estabilizadores isomorfos a los grupos asociados. Para demostrar esto, utilizamos un teorema de forma normal generalizado (corolario 3.2.3) que se tiene en estos grupos fundamentales. Para terminar, el teorema de estructura es un recíproco a esta construcción. Nos dice que todo grupo que actúa en un árbol

tiene esta estructura, con lo que llegamos a la caracterización que buscábamos. Este resultado puede usarse para entender mejor la estructura de los grupos fundamentales de grafos de grupos; ilustramos esto con un resultado sobre subgrupos de productos amalgamados libres conocido como Teorema de Kurosh (corolario 3.4.2).

Nuestras principales referencias son el capítulo I de [11] y el capítulo IV de [9].

Contents

Chapter 1

Graphs

1.1 Definitions and properties

Definition 1.1.1. A *graph* Γ consists of a set of *vertices* $X = \mathrm{V}(\Gamma)$ and a set of *edges* $Y = \mathrm{E}(\Gamma)$ together with maps $o, t : Y \to X$, $\bar{\cdot} : Y \to Y$ such that for every edge $y \in Y$, $\bar{\bar{y}} = y$, $\bar{y} \neq y$ and $o(y) = t(\bar{y})$. We call $o(y), t(y), \bar{y}$ the *origin*, *terminus* and *inverse* of y respectively. For any $y \in Y$ the set $\{y, \bar{y}\}$ is called a *geometric edge*. Intuitively, we only want these doubled edges so that we can walk in any sense along an oriented graph, but we can think of each geometric edge as a single edge.

A *morphism* between graphs $\Gamma \to \Delta$ is a pair of maps $\mathrm{V}(\Gamma) \to \mathrm{V}(\Delta)$, $\mathrm{E}(\Gamma) \to \mathrm{E}(\Delta)$ which are compatible in the obvious sense.

An *orientation* of the graph Γ is a subset $Y_+ \subseteq Y$ such that $Y = Y_+ \sqcup \overline{Y_+}$. We denote $Y_- := \overline{Y_+}$.

Definition 1.1.2. We may regard a graph as a topological space. With the same notation as above, let $T = X \sqcup (Y \times [0,1])$ where X, Y are given the discrete topology. The *topological realization* of the graph Γ is the quotient space T/R, where R is the equivalence relation generated by

$$\begin{array}{ll} (y,s) \sim (\bar{y}, 1-s) & \\ (y,0) \sim o(y) & \forall y \in Y, s \in [0,1]. \\ (y,1) \sim t(y) & \end{array}$$

Note that the edges $y, \bar{y}$ give rise to the same segment. We will identify a graph and its realization, and denote them in the same way.

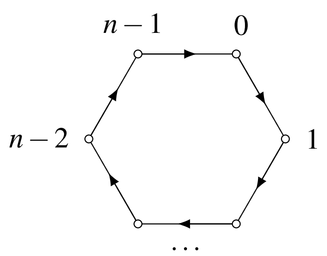
Definition 1.1.3. A *path* of length $n \geq 1$ in Γ is a morphism from the oriented graph

$P_n =$ 

into Γ. In particular, a graph isomorphic to P_1 is what we call a *segment*. A path can also be determined by an ordered sequence of edges $(y_1, \dots, y_n)$ with $t(y_i) = o(y_{i+1})$. We may consider infinite paths, corresponding to infinite sequences $(y_1, y_2, \dots)$.

A pair of the form $(y_i, y_{i+1}) = (y_i, \bar{y}_i)$ in the path is called a *backtracking*. Given an arbitrary path between two points of a graph, we can always construct inductively another path without backtracking, which we shall call a *reduced path*.

Similarly, a *circuit* or *cycle* of length n in Γ is a subgraph isomorphic to a cycle graph, i.e. a graph of the form

$C_n =$ 

Such a subgraph is defined by a reduced path $(y_1,\dots,y_n)$ such that the $t(y_i)$ are all distinct and $t(y_n)=o(y_1)$.

A cycle of length 1 is called a *loop*.

Definition 1.1.4. A graph is *combinatorial* if it has no cycle of length ≤ 2. In particular, we may describe any edge y in a combinatorial graph as the ordered pair $(o(y),t(y))$. These are precisely the graphs that can be defined from simplicial complexes.

Definition 1.1.5. A *tree* is a connected non-empty graph without cycles. In particular, a tree is combinatorial.

For any two vertices v,w in a tree, there exists a unique reduced path between them, and it is an injective path (this allows one to define a distance function on the set of vertices of a tree, by setting $d(v,w)$ to be equal to the length of this reduced path).

Now we turn our attention to subtrees of a graph Γ. The set of subgraphs of Γ that are trees, ordered by inclusion, is clearly directed. Thus by Zorn's lemma it has maximal elements, which we call *maximal subtrees* of Γ. If Γ is connected, then every maximal tree contains all the vertices of Γ, because in other case, we could adjoin a new vertex to the tree, contradicting maximality.

Lemma 1.1.6. *Let Γ be a connected graph with $|\mathrm{V}(\Gamma)|<\infty$. Then $|\mathrm{V}(\Gamma)|-|\mathrm{E}_+(\Gamma)|\leq 1$, and equality holds if and only if Γ is a tree.*

Proof. We first prove the equality for a tree. In the case of a single vertex, it is trivial. Also, it clearly remains true whenever we adjoin a terminal vertex together with a pair of edges $y,\bar{y}$. Since we can construct any given tree by this procedure, the equality is true for all finite trees.

Next, we prove the inequality in the general case. Let Γ' be a maximal subtree of Γ. Since it contains all vertices of Γ, we have $|\mathrm{V}(\Gamma')|=|\mathrm{V}(\Gamma)|$, $|\mathrm{E}(\Gamma')|\leq|\mathrm{E}(\Gamma)|$, and equality holds iff $\Gamma=\Gamma'$ i.e. Γ is a tree. Also, by the former case, we know $|\mathrm{E}_+(\Gamma')|=|\mathrm{V}(\Gamma')|-1$. Thus,

$$|\mathrm{E}_+(\Gamma)|=|\mathrm{V}(\Gamma)|-1+\left(|\mathrm{E}_+(\Gamma)|-\left|\mathrm{E}_+(\Gamma')\right|\right)$$

and the proposition follows. □

Definition 1.1.7. The *Euler characteristic* of a connected graph Γ with $|\mathrm{V}(\Gamma)|<\infty$ is defined as

$$\chi(\Gamma)=|\mathrm{V}(\Gamma)|-|\mathrm{E}_+(\Gamma)|. \tag{1.1}$$

The above lemma says then that $\chi(\Gamma)\leq 1$ and equality holds if and only if Γ is a tree.

As we said, any maximal subtree of a connected graph contains all of the vertices of the graph, and it is not difficult to show that a tree is a contractible space. Therefore, the homotopy type of a graph depends only on the edges that are left outside a given maximal subtree (the number of such edges does not depend on the choice of the maximal subtree). In particular we have

Theorem 1.1.8. *Any connected graph has the homotopy type of a bouquet of circles. Furthermore, a graph is a tree if and only if it is contractible.*

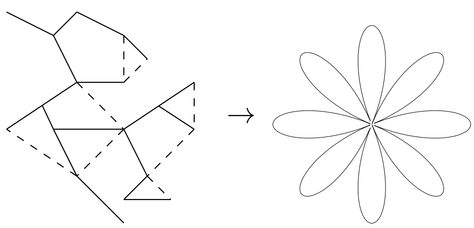


Figure 1.1: Contraction of a maximal subtree.

As a consequence, the fundamental group of any connected graph is free of rank $|\mathrm{E}_+(\Gamma)-\mathrm{E}_+(\Gamma')|$ (which equals $1-\chi(\Gamma)$ in the finite case), and Γ is simply connected if and only if it is a tree.

In particular, if we choose any subtree of a graph Γ and contract it to a single vertex to get a new graph Γ', then Γ is a tree if and only if Γ' is a tree.

1.2 The Cayley graph

Definition 1.2.1. A *presentation* of a group G is an expression of the form

$$G = \langle S \mid R \rangle$$

where S is a set of letters called *generators* and R is a set of words of $S \cup S^{-1}$ which are the only valid *relations*, i.e. the only non-trivial equalities that hold in the group G. We will assume that $S \cap S^{-1} = \varnothing$, although note that this is not possible when there are elements of order 2 among the generators.

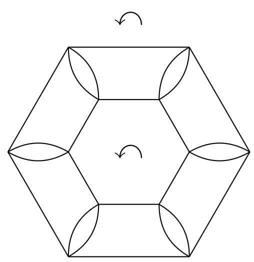
Example 1.2.2. The *free group* on n generators $F_n = \langle a_1, \ldots, a_n \mid \rangle$ has no relations.

A group is an algebraic structure, but we can enrich our understanding of it by regarding it as a geometric object. One very good way to do this is via the following construction.

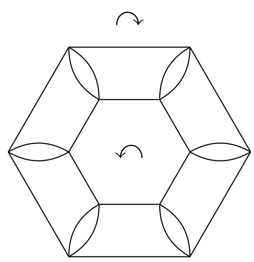
Definition 1.2.3. Let G be a group with a presentation $\langle S \mid R \rangle$. We define the *Cayley graph* $\Gamma(G,S)$ as the oriented graph having the elements of G as vertices, and for each $g \in G$ and each $s \in S$, a geometric edge with extremes g and gs. The positively oriented edges have origin g and terminus gs.

The group G acts on $\Gamma(G,S)$ by left multiplication, and as long as $S \cap S^{-1} = \varnothing$, the action preserves orientation, that is, edges in Y_+ are mapped to Y_+ and edges in Y_- are mapped to Y_-. Furthermore, G acts freely on the vertices and on the edges. One may analogously write in the above definition sg in place of gs, in which case G acts on the graph by right multiplication instead.

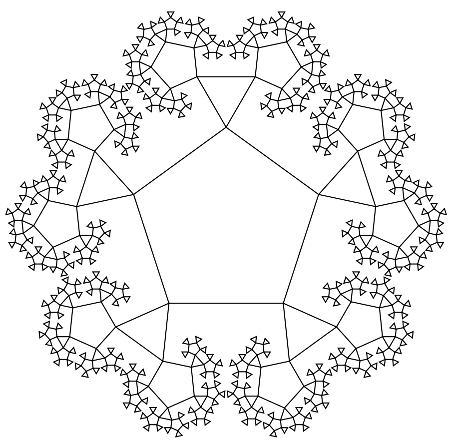
Examples 1.2.4.



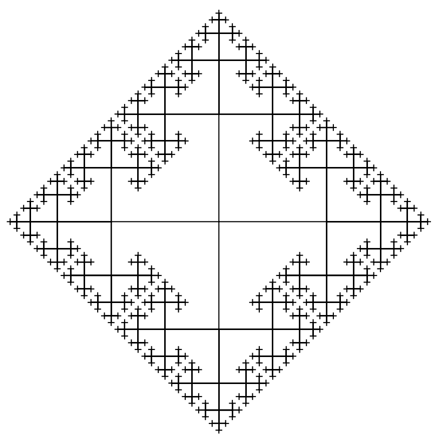
$\mathbb{Z}_2 \times \mathbb{Z}_6 = \langle a, b \mid a^2 = b^6 = 1, ab = ba \rangle$



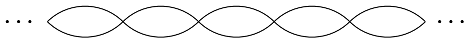
$D_{12} = \langle r, s \mid s^2 = r^6 = 1, r^s = r^{-1} \rangle$



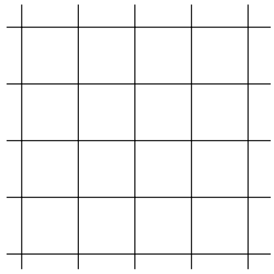
$\mathbb{Z}_3 * \mathbb{Z}_5 = \langle a, b \mid a^3 = b^5 = 1 \rangle$



$F_2 = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$



$\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1 \rangle$



$\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle$

Subgroup cosets, abelian groups, normal and central subgroups, splittings, direct and free products, and a number of group theoretic notions can be seen to have their geometric analogue; it can be very instructive to understand these concepts both from the algebraic and the geometric point of view. In particular, we see that cycles in the Cayley graph correspond precisely to relations between the elements of S. Therefore, we obtain

Theorem 1.2.5. $\Gamma(G,S)$ *is a tree if and only if G is a free group with basis S.*

Remark 1.2.6. A graph can be regarded as a complex, with the vertices as 0-dimensional cells and the edges as 1-dimensional cells. We next give an idea on how to construct higher dimensional complexes on which G acts by permuting the cells. Starting off with the Cayley graph as the 1-skeleton, one can construct a 2-complex by gluing 2-dimensional disks along the words in the relations; this gives rise to a simply connected space X, called *Cayley complex*, on which G acts freely. It is then the universal covering of the orbit space $G \backslash X$, which is again a 2-complex (known as *presentation complex*). In general, this complex is not aspherical (i.e. its higher homotopy groups are not trivial, or, equivalently, its universal covering, the Cayley complex, is not contractible), but by repeatedly gluing higher dimensional cells, one can kill these groups while leaving the fundamental group G untouched. The resulting space is then an *Eilenberg-Maclane* space, or $K(G,1)$, which is, by definition, a connected, aspherical complex with G as its fundamental group. This is a great source of interaction between algebra and topology; for instance, homology and cohomology groups of a group G can be defined to be those of a $K(G,1)$ (this can be shown not to depend on the specific constructed space). We refer to [2] for more details.

Example 1.2.7. The Cayley graph of $\mathbb{Z}_2 = \langle a \mid a^2 \rangle$ is isomorphic to $\mathbb{S}^1$. Then gluing two disks (we can think of each being based at one of the two vertices), we obtain the Cayley complex $\mathbb{S}^2$, which is the universal covering of the real projective plane $\mathbb{RP}^2$. Of course, one has $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$. In this case we see that $\mathbb{S}^2$ is not contractible; hence $\mathbb{RP}^2$ is not a $K(\mathbb{Z}_2,1)$. In fact, it can be shown that a group with torsion cannot have a finite-dimensional Eilenberg-Maclane space.

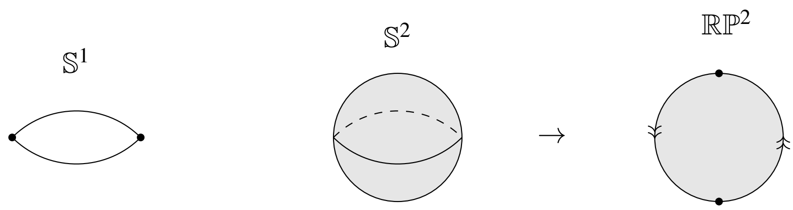


Figure 1.2: Cayley graph, Cayley complex and 2-presentation complex of $\mathbb{Z}_2$.

Chapter 2

Groups acting on trees

2.1 General setting

As a first step towards the structure theorem, in this chapter we will study some basic constructions of groups which act on trees. In fact, we shall see that these actions completely characterize these groups. We will always consider left actions, but of course one can reformulate the theory in terms of right actions.

Definition 2.1.1. Let X be a graph on which G acts. An *inversion* is a pair $(g,y) \in G \times \mathrm{E}(X)$ such that $gy = \bar{y}$. If there is no such pair, we say that G acts *without inversion*. This is the same as saying that there is an orientation of X preserved by G.

Example 2.1.2. As we have noted in the previous chapter, G acts without inversion on the Cayley graph $\Gamma(G,S)$ when $S \cap S^{-1} = \varnothing$.

If G acts without inversion on X, we can define the quotient graph $G\backslash X$ in the obvious way. Notice that if it did not act without inversion, the quotient of $\mathrm{E}(X)$ would not meet the requirements to be the set of edges of a graph. From now on, we shall assume, unless otherwise stated, that all group actions on graphs are without inversion.

Proposition 2.1.3. *Let X be a connected graph on which G acts. Then every subtree of $G\backslash X$ lifts to a subtree of X.*

Proof. Let T' be a subtree of $G\backslash X$, and let Ω be the set of subtrees of X that project injectively into T'. Ω is a directed set under the relation of inclusion. Let T_0 be a maximal element of Ω, and let T_0' be its image in T'. We claim that $T' = T_0'$, from which the proposition follows.

Assume that $T' \neq T_0'$, so there is an edge $y' \in T' - T_0'$. Since T' is connected, we can assume that $o(y') \in T_0'$, and then necessarily $t(y') \notin T_0'$, since otherwise we would get a cycle in T'. Let now y be a lift of y'. Since we can replace y by any gy, we can assume that $o(y) \in T_0$. Let now T_1 be the graph obtained by adjoining the vertex $t(y)$ and the edges $y, \bar{y}$ to T_0. Since this makes $t(y)$ a terminal vertex in T_1 and T_0 is a tree, T_1 is as well a tree, which properly contains T_0 and which projects injectively into T', contradicting the maximality of T_0. Therefore, $T' = T_0'$ as we claimed. ☐

Definition 2.1.4. Let X be a connected graph on which G acts. A *tree of representatives* of X modulo G is any subtree T of X that is the lift of a maximal tree in $G\backslash X$. Since X (hence $G\backslash X$) is connected, such a maximal tree contains all the vertices in $G\backslash X$, which is equivalent to saying that every orbit of G in $\mathrm{V}(X)$ contains exactly one element of $\mathrm{V}(T)$.

A *fundamental domain* of X modulo G is a subgraph of X that maps isomorphically onto $G\backslash X$.

If $G\backslash X$ is a tree, it is clear that any tree of representatives is a fundamental domain. In the case when X itself is a tree, which is the case that we will devote ourselves to, the converse is obviously true:

Proposition 2.1.5. *Let G act on a tree X. Then a fundamental domain of X modulo G exists if and only if $G\backslash X$ is a tree.*

Proof. Let T be a fundamental domain. Since X is connected, $G\backslash X$ is connected and so is T. Then, T is a connected subgraph of the tree X, so it is a tree. □

2.2 Free groups

We say that a group acts *freely* on a graph if it acts without inversion and no nontrivial element leaves a vertex fixed. For example, we have noted before that G acts freely on the Cayley graph $\Gamma(G,S)$. In particular, theorem 1.2.5 shows that if G is free, then there is a tree upon which it acts freely. In the next theorem, we strengthen this result by proving the converse. More precisely:

Theorem 2.2.1. *Let G act freely on a tree X. Choose a tree T of representatives of X modulo G and an orientation $Y_+ \subseteq \mathrm{E}(X)$ preserved by G.*

- (i) *Let $S \subseteq G$ be the set of nontrivial elements $g \in G$ for which there is an edge in Y_+ with origin in T and terminus in gT. Then G is free with basis S.*
- (ii) *If $|\mathrm{V}(G\backslash X)| < \infty$, then* $\operatorname{rank} G = 1 - \chi(G\backslash X)$.

Proof. (i) Since G acts freely on X and $T \to G\backslash X$ is injective, the map $g \mapsto gT$ is a bijection of G onto the orbit GT. In particular, we can form the quotient graph $X' = X/GT$ by contracting each tree gT to a single vertex which we denote (gT). Then X' is still a tree. The inverse of the bijection $g \mapsto (gT)$ can be regarded as a bijection $\alpha : \mathrm{V}(X') \to \mathrm{V}(\Gamma(G,S)) = G$. We claim that α can be extended to an isomorphism $\alpha : X' \to \Gamma(G,S)$, so that $\Gamma(G,S)$ will be a tree and by theorem 1.2.5, assertion (i) will follow.

Notice that $\mathrm{E}(X') = \mathrm{E}(X) - \mathrm{E}(GT)$. We give X' the orientation induced by that of X, i.e. $Y'_+ = Y_+ \cap \mathrm{E}(X')$. We want the map α to be a morphism of oriented graphs. To do that it suffices to define $\alpha : Y'_+ \to \mathrm{E}_+(\Gamma(G,S))$. Let $y \in Y'_+$, and let $(gT) = o(y)$ and $(g'T) = t(y)$. Then, the edge y connects gT to $g'T$ in X, so $s = g^{-1}g' \in S$. We define $\alpha(y) = (g, gs)$. Injectivity of α then follows from injectivity of $\alpha : \mathrm{V}(X') \to \mathrm{V}(\Gamma(G,S))$ and the fact that X', being a tree, is combinatorial. Surjectivity is also immediate by construction, and we are done!

(ii) Let $Z = \{y \in Y_+ \mid o(y) \in T, t(y) \notin T\}$. By the above proof, we have a bijection $Z \to S$, whence $|Z| = |S|$. The image $G\backslash T$ of T in $G\backslash X$ is a maximal tree. We provide $G\backslash X$ with the induced orientation $G\backslash Y_+$. We then have that $G\backslash Y_+ = \mathrm{E}_+(G\backslash T) \sqcup G\backslash Z$, and also that $Z \to G\backslash Z$ is bijective. Thus, if $|\mathrm{V}(G\backslash T)| = |\mathrm{V}(G\backslash X)| < \infty$,

$$|G\backslash Y_+| - |\mathrm{V}(G\backslash X)| = |G\backslash Z| + |\mathrm{E}_+(G\backslash T)| - |\mathrm{V}(G\backslash T)| = |G\backslash Z| - 1 = |S| - 1,$$

where we have used lemma 1.1.6. □

Topological interpretation 1. Because G acts freely on X, X is the universal covering of the quotient space $G\backslash X$, and G is isomorphic to the fundamental group $\pi_1(G\backslash X)$. Since $G\backslash T$ is a maximal subtree of $G\backslash X$, the quotient $(G\backslash X)/(G\backslash T)$ is a bouquet of circles, and it has the same homotopy type as $G\backslash X$. The fundamental group G is then the free group with generators corresponding to each circle in the bouquet. These circles, suitably oriented, correspond to the elements of $G\backslash Z$, and hence to the elements of S, as we have seen above.

The structure theorem we are aiming for will be nothing but a generalization of this. It will allow us to describe a group acting on a tree as the *fundamental group of a graph of groups* which we will later define.

Remark 2.2.2. In fact, with the same idea one can generalize theorem 1.2.5. Let $G = \langle S \mid R\rangle$ be a presentation of G, so that $G = F(S)/\langle\langle R\rangle\rangle$. Then, the free action of $F(S)$ induces a free action of $\langle\langle R\rangle\rangle$ on the tree $\Gamma(F(S),S)$, and the quotient graph is isomorphic to $\Gamma(G,S)$. As a consequence, $\langle\langle R\rangle\rangle \cong \pi_1(\Gamma(G,S))$.

As a beautiful application of theorem 2.2.1, we get a very simple proof of an otherwise tricky result in group theory:

Corollary 2.2.3 (Schreier's theorem)**.** *Every subgroup H of a free group G is free. Moreover, if* $|G:H| = n < \infty$*, then* $\operatorname{rank} H - 1 = n(\operatorname{rank} G - 1)$.

Proof. If G is free, we can make it act freely on a tree X (for example, on the Cayley graph). Then it is clear that H also acts freely on X, so it is free by theorem 2.2.1. For the second assertion, notice that we can choose X so that $|\mathrm{V}(G\backslash X)| < \infty$ (for example, for the Cayley graph associated to a basis of G, this number is 1). The formula then just follows from theorem 2.2.1(ii) and the fact that $|\mathrm{V}(H\backslash X)| = n|\mathrm{V}(G\backslash X)|$ and $|\mathrm{E}(H\backslash X)| = n|\mathrm{E}(G\backslash X)|$. □

Topological interpretation 2. One can as well give a nice topological proof of this theorem. Let Y be a bouquet of circles with $\pi_1(Y) = G$. This space satisfies good connectedness properties, and thus there is a one-to-one correspondence between coverings of Y and subgroups of G. Hence there exists a covering $p: E \to Y$ such that via the induced monomorphism, $\pi_1(E) \cong H$. Since E is a covering space of a graph, it is a graph itslef, and so its fundamental group is free. The rank formula then follows from the fact that E is an n-fold covering of Y, so $\chi(E) = n\chi(Y)$. We refer to [10, §85] for details.

2.3 Free amalgamated products and HNN extensions

In this section we introduce two constructions which are basic to combinatorial group theory. Given two groups G_1, G_2, one constructs their *free product* $G_1 * G_2$ as the group generated by both keeping their original relations without adding any new ones. In other words, one has the presentation

$$G_1 * G_2 = \langle G_1, G_2 \mid \rangle$$

where we understand the G_i to carry each both their elements and their relations, i.e. if $G_i = \langle S_i \mid R_i \rangle$, then $G_1 * G_2 = \langle S_1, S_2 \mid R_1, R_2 \rangle$. Now, perhaps we would like instead to add some relations between some elements of each group. How can we do that? This is precisely what free amalgamated products will do for us.

Definition 2.3.1. Suppose we are given groups G_1, G_2 and A, and for $i = 1,2$, a monomorphism $\alpha_i : A \to G_i$. We define the *free amalgamated product* of the G_i along A by means of the α_i as

$$G_1 *_A G_2 = \langle G_1, G_2 \mid \alpha_1(a) = \alpha_2(a)\, \forall a \in A \rangle.$$

Remark 2.3.2. One can define in the same way the free amalgamated product of an arbitrary family of groups $\{G_i\}_{i \in I}$ along a common subgroup A, denoted $*_A G_i$. We will restrict ourselves to the case of two factors for simplicity.

Definition 2.3.3. Let H be a group, and let $A, B \leq H$ be two subgroups of H together with an isomorphism $\varphi : A \to B$. Let t be an element of infinite order not in H. We let

$$H*_{A,B,t} = \langle H, t \mid t^{-1}at = \varphi(a)\, \forall a \in A \rangle$$

denote the *HNN extension* of H with base group A and stable letter t (HNN stands for G. Higman, B.H. Neumann and H. Neumann).

If $A = H$, the extension is said to be *ascending*, while if $B = H$, it is called *descending*. Notice that $H*_{A,B,t} = H*_{B,A,t^{-1}}$ and thus we may write any descending extension as ascending. If $A \neq H$ and $B \neq H$, we call the extension *strict*. For an extension both ascending and descending, one has $H*_{H,H,t} = H \rtimes_\varphi \langle t \rangle$.

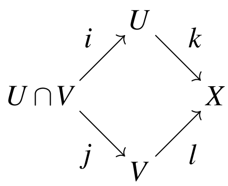
Remark 2.3.4. Let L be a group with two monomorphisms $\alpha_1, \alpha_2 : L \to H$, and let $A = \alpha_1(L), B = \alpha_2(L)$ and $\varphi = \alpha_2 \alpha_1^{-1} : A \to B$. Then, we can rewrite

$$H*_{A,B,t} = \langle H, t \mid t^{-1}\alpha_1(s)t = \alpha_2(s)\, \forall s \in L \rangle =: H*_{L,t}\,.$$

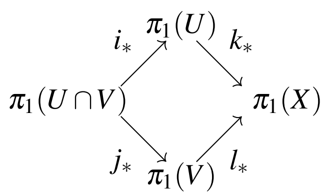
Topological interpretation 3. Both free amalgamated products and HNN extensions involve two subgroups and an isomorphism between them; they might be called the ”disconnected case” and the ”connected case” of one basic idea, based on an analogy between these two group constructions and the operation of gluing of topological spaces. It is in fact in this context where their motivation is most naturally found.

On the one hand, amalgamation is the group-theoretic analogue of pasting two different topological spaces together along a common connected subspace. The Seifert-Van Kampen theorem makes this analogy precise via fundamental groups:

Theorem 2.3.5 (Seifert-Van Kampen)**.** *Let U,V be two open subspaces of X such that $X = U \cup V$, so we have the following diagram of inclusions:*

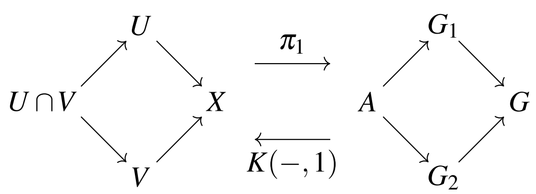


Assume that all the involved spaces are arcwise connected, and that the inclusions i, j are π_1-injective. Then, the induced group diagram



is an amalgamation diagram, i.e. $\pi_1(X) \cong \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$.

In fact, considering a less restrictive definition of amalgamation, the hypothesis that the induced maps i_*, j_* be injective is not necessary. We can then express the Seifert-Van Kampen theorem by saying that π_1 ”preserves amalgams”. In fact, it is a theorem of Whitehead (cf. [2, II.7.3]) that we can go in the other direction, provided that we restrict to π_1-injective inclusions: any group amalgamation diagram with injective maps can be realized by a space diagram. The spaces involved can furthermore be realized by Eilenberg-Maclane complexes; this can be used to derive in an easy way Mayer-Vietoris sequences in group homology and cohomology.



On the other hand, there is an analogous correspondence between HNN extensions and fundamental groups of topological spaces in which a subspace has been glued back on another homeomorphic subspace (so we ”attach a handle” to the space). These two group constructions allow, then, the description of the fundamental group of any reasonable geometric gluing.

We will now take a look at some examples of group amalgamations and HNN extensions. As one can imagine from the comments above, many of these examples arise in algebraic topology.

Examples 2.3.6. 1. The *Baumslag-Solitar groups* are defined as

$$\mathrm{BS}(n,m) = \langle a,t \mid t^{-1}a^n t = a^m \rangle, \quad n,m \in \mathbb{Z}.$$

Putting $H = \langle a \rangle$, $A = \langle a^n \rangle$, $B = \langle a^m \rangle$ and $\varphi : A \to B$ given by $\varphi(a^n) = a^m$, we see that $\mathrm{BS}(n,m) = H*_{A,B,t}$ is an HNN extension with an infinite cyclic group as base. The extension is ascending (resp. descending) if and only if $|n| = 1$ (resp. $|m| = 1$).

These groups are the source of many interesting examples. For the ascending case, one can see that $\mathrm{BS}(1,m) \cong \langle t \rangle \ltimes \mathbb{Z}\left[\frac{1}{m}\right]$ where the action of t^k is given by multiplication by m^k, for all $k \in \mathbb{Z}$. Indeed, using additive notation, $\langle\langle a \rangle\rangle \cong \mathbb{Z}\left[\frac{1}{m}\right]$ via $t^{-k}a^l t^k \mapsto lm^k$, $k,l \in \mathbb{Z}$ with $\gcd(m,l) = 1$. On the other hand, we have an obvious epimorphism $\mathrm{BS}(1,m) \to \langle t \rangle$ which splits and has kernel $\langle\langle a \rangle\rangle$, whence $\mathrm{BS}(1,m) \cong \langle t \rangle \ltimes \langle\langle a \rangle\rangle$ and the assertion follows. As a consequence, $\mathrm{BS}(1,m)$, being the semi-direct product of two abelian groups, is metabelian (that is, the derived subgroup G' is abelian). In particular, these are solvable groups.

2. Consider the special linear group with integer coefficients

$$\mathrm{SL}_2(\mathbb{Z}) = \{A \in M_{2\times 2}(\mathbb{Z}) \mid \det A = 1\}.$$

We claim that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elementary matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, which are of order 4 and 6 respectively. Indeed, let $T = S^{-1}R = -SR = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which is of infinite order with $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ $\forall n \in \mathbb{Z}$, let $G = \langle S,T \rangle$, and let us see that $\mathrm{SL}_2(\mathbb{Z}) = G$. For every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $n \in \mathbb{Z}$, we have

$$SA = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}, \quad T^n A = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}.$$

If $c = 0$, since $\det A = 1$ and A has integer entries, then A must be of the form

$$\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} = \pm T^{\pm m}$$

which, since $S^2 = -I_2$, is an element of G. Suppose now that $c \neq 0$. If $|a| \geq |c|$, divide a by c, so that $a = cq + r$ with $0 \leq r \leq |c|$. Then, $T^{-q}A$ has upper left entry $a - qc = r$, which has smaller modulus than the lower left entry c. Applying S switches these entries (with a sign change). If $r \neq 0$, we can then divide again the upper left entry $-c$ by the lower left entry r. Iterating this procedure of Euclidean division+multiplication by powers of T+multiplication by S, we eventually obtain a matrix with lower left entry 0, which reduces to the first case and completes the proof.

Note that the intersection $\langle S \rangle \cap \langle R \rangle$ is the cyclic group generated by $-I_2 = S^2 = R^3$ (this is in fact the center of $\mathrm{SL}_2(\mathbb{Z})$). We shall show later with the aid of Bass-Serre theory that this is the only relation that holds between S and R, i.e. $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, where each cyclic group is generated by the mentioned matrices.

3. Let Γ be a combinatorial graph. The *right angled Artin group* (RAAG for short) associated to Γ is the group generated by the vertices of Γ, such that two vertices commute if and only if there is an edge between them. That is:

$$A_\Gamma = \langle \mathrm{V}(\Gamma) \mid [v,w] = 1 \quad \forall \overset{v}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{w}{\circ} \in \mathrm{E}(\Gamma) \rangle.$$

Let now v,w be two vertices such that there is no edge between them in Γ, and take the subgraphs $\Gamma_1 = \Gamma - v$, $\Gamma_2 = \Gamma - w$ and $\Gamma_1 \cap \Gamma_2 = \Gamma - \{v,w\}$. For each $u \in \mathrm{V}(\Gamma_1 \cap \Gamma_2)$, let u_i denote the corresponding element in A_{Γ_i}. Because there is no edge joining v and w in Γ, one has $\mathrm{E}(\Gamma) = \mathrm{E}(\Gamma_1 \cup \Gamma_2)$, hence no new relations apart from those in the A_{Γ_i} are added to A_Γ, except those of identifying the elements u_i in the intersection. In other words,

$$A_\Gamma = \langle A_{\Gamma_1}, A_{\Gamma_2} \mid u_1 = u_2 \ \forall u \in \mathrm{V}(\Gamma_1 \cap \Gamma_2)\rangle = A_{\Gamma_1} *_{A_{\Gamma_1 \cap \Gamma_2}} A_{\Gamma_2}.$$

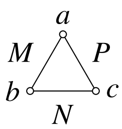
Repeating this process on each subgraph, one can write any RAAG as an iterated free amalgamated product. Notice that the procedure ends when we arrive to a clique (a full subgraph), whose RAAG is a free abelian group of rank equal to the number of vertices.

We can write RAAGs as iterated HNN extensions as well. Let $v \in \mathrm{V}(\Gamma)$, put $\Gamma_1 = \Gamma - v$, and let $\mathrm{lk}_\Gamma(v)$ be the full subgraph induced by the vertices that are joined to v in Γ, i.e. $\{w \in \mathrm{V}(\Gamma) \mid \overset{v}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{w}{\circ} \in \mathrm{E}(\Gamma)\}$. Adjoining the vertex v to Γ_1 to recover Γ translates in the RAAG to adding all the relations encoded by $\mathrm{lk}_\Gamma(v)$. Hence we see that

$$A_\Gamma = \langle A_{\Gamma_1}, v \mid [v,w] \ \forall w \in \mathrm{lk}_\Gamma(v)\rangle = A_{\Gamma_1} *_{A_{\mathrm{lk}_\Gamma(v)}, v},$$

since $[v,w] = 1 \Leftrightarrow v^{-1}wv = w$.

4. The above are particular examples of *Artin groups*, which are defined by means of a combinatorial graph Γ with each geometric edge y labeled by an integer N_y. A particularly simple case is that of a *triangle Artin group*, given by the presentation



$$A_{MNP} = \langle a,b,c \mid (a,b)_M = (b,a)_M, (b,c)_N = (c,b)_N, (c,a)_P = (a,c)_P\rangle$$

where $(a,b)_M$ denotes the alternating word $abab\ldots$ of length M. Suppose that $P = 2$, let $x = ab$ and $y = cb$, and consider a new presentation of A_{MN2} with generators b,x,y. The relation $(a,b)_M = (b,a)_M$ is replaced by $bx^mb^{-1} = x^m$ when $M = 2m$, and by $bx^mb = x^{m+1}$ when $M = 2m+1$. We denote this relation by $r_M(b,x)$. Note that $yx^{-1} = ca^{-1}$, so relation $ac = ca$ can be replaced by $yx^{-1} = bx^{-1}yb^{-1}$. This gives the following presentation

$$A_{MN2} = \langle b,x,y \mid r_M(b,x), r_N(b,y), bx^{-1}yb^{-1} = yx^{-1}\rangle.$$

In the case when both M,N are even, we see that this is the presentation of an HNN extension with base group freely generated by x,y and stable letter b. More generally, one can show that if $M \geq N \geq P$ and either $P > 2$ or $N > 3$, then A_{MNP} splits as either a free amalgamated product or an HNN extension of finitely generated free groups. The idea to do this is to consider a special class of graphs which admit a "good" orientation, which includes the cases where $P > 2$. Both these and the remaining cases are dealt with through a nice geometric argument using the 2-presentation complex; see [5] and [6].

5. Let $G = \langle S \mid r\rangle$ be a finitely generated one-relator group. This group is torsion-free if and only if r is not a proper power (cf. [2, p. 37]). In such case, when the exponent sum in r of some generator t equals zero, it can be shown that G is an HNN extension of a one-relator group whose defining relator is strictly shorter than r, and with f.g. free associated subgroups. If there is no zero exponent sum generator, we can still embed G in such an HNN extension. This result is very useful, since it can be used to prove certain statements by induction on the length of the relator, and it is usually proved as part of the proof of a theorem of Magnus known as *Freiheitssatz* about a certain free subgroup of G; see [9, IV.5.1]. The idea is to replace the generators by their conjugates by powers of t, which allows to rewrite r with fewer letters. The only "bad" point of this is that one is generally left with infinitely many generators.

Two classical examples of one-relator groups are

$$\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1,b_1]\ldots[a_g,b_g]\rangle,\ g \geq 1,$$

which is the fundamental group of the genus g closed orientable surface F_g (i.e. a connected sum of g toruses), and

$$\langle c_1,\dots,c_k \mid c_1^2\dots c_k^2\rangle, k\geq 2,$$

which is the fundamental group of the non-orientable closed surface with k crosscaps N_k (i.e. a connected sum of k real projective planes). This is easily proved by induction, showing that F_g (resp. N_k) can be obtained by gluing the sides of a $4g$-gon (resp. $2k$-gon) labeled by the word $[a_1,b_1]\dots[a_g,b_g]$ (resp. $c_1^2\dots c_k^2$).

2.3.1 Normal forms

In a free group, each element has a unique reduced expression in terms of a basis. This is no longer true when there are relations in the group that may allow to write very differently the same element, but in some cases we still get lucky and have what we call a *normal form*, which is (essentially) unique for each element. Even so, for many problems one is usually only interested in being able to deduce that a given element is *not* trivial.

Definition 2.3.7. Let $G = H*_{A,B,t}$. The word $g_0t^{\varepsilon_1}g_1\dots t^{\varepsilon_n}g_n$, where $g_0\in H$ and $\varepsilon_i=\pm1$, is said to be *reduced* if there is no consecutive subword $t^{-1}g_it$ with $g_i\in A$ or tg_it^{-1} with $g_i\in B$.

Two distinct reduced words may be equal in G. To actually get normal forms we will need a further refinement.

Definition 2.3.8. Let S_A (resp. S_B) be a set of representatives of the right cosets of A (resp. of B) in H, both containing the identity. A *normal form* is a word $g_0t^{\varepsilon_1}g_1\dots t^{\varepsilon_n}g_n$ where

(a) $g_0\in H$,

(b) $\varepsilon_i=-1\Rightarrow g_i\in S_A$,

(c) $\varepsilon_i=1\Rightarrow g_i\in S_B$, and

(d) there is no consecutive subword $t^{\varepsilon}1t^{-\varepsilon}$.

To get an intuition on this definition, notice that the defining relations of G can be rewritten as

$$t^{-1}a=\varphi(a)t^{-1}\,\forall a\in A \quad \text{or} \quad tb=\varphi^{-1}(b)t\,\forall b\in B.$$

We can view these as *quasi-commuting* relations. Indeed, they allow us to move elements in A or B to either side of t, via φ. By working from right to left, we can then show that every element of G is equal to a normal form $g_0t^{\varepsilon_1}g_1\dots t^{\varepsilon_n}g_n$.

Example 2.3.9. Let $F=\langle x,y\rangle$ and let $G=F*_{x,y^2,t}=\langle x,y,t\mid t^{-1}xt=y^2\rangle$. As representatives of $\langle x\rangle$-cosets, we choose all freely reduced words on x and y which do not begin with x, while as representatives of $\langle y^2\rangle$-cosets, we choose all freely reduced words on x and y which do not begin with y^k, $k\geq 2$. We can calculate the normal form of the element $xyt^{-1}x^3ty^5xyt^{-1}x^3y^3$ using the above relations:

$$xyt^{-1}x^3ty^5xyt^{-1}x^3y^3=xyt^{-1}x^3ty^5xy^7t^{-1}y^3=xyt^{-1}x^5tyxy^7t^{-1}y^3=xy^{12}xy^7t^{-1}y^3.$$

Theorem 2.3.10 (Normal form theorem for HNN extensions)**.**

(i) (Britton's lemma). *If $g_0t^{\varepsilon_1}g_1\dots t^{\varepsilon_n}g_n=1$ in G, then either $n=0$ and $g_0=1$, or $n\geq 1$ and the word is not reduced.*

(ii) *Every element of G has a unique representation as a normal form.*

Sketch of proof. One first shows that both statements are equivalent. Then, to prove (ii) the idea is to make G act on the set of normal forms by multiplication on the left and reduction to normal form. The proof can be found at [9, IV.2.1]. □

Remark 2.3.11. 1. As a consequence of this theorem, the natural homomorphism $H \to G$ is injective.

2. For most purposes, there is no need to choose coset representatives. What is important is that H is embedded in G and that we have a criterion to tell when words of G do not represent the identity.

Corollary 2.3.12. *Every element of finite order in $G = H*_{A,B,t}$ is conjugate to an element of H.*

Proof. A word $g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n}$ is called *cyclically reduced* if all its cyclic permutations are reduced. Clearly, every element of G is conjugate to an alement that admits a cyclically reduced expression.

Let $g \in G$ such that $g^m = 1$, and let $h = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n}$ be a cyclically reduced word with h conjugate of g. We claim that $h \in H$. Indeed, if $h \notin H$ i.e. $n \geq 1$, then $h^m = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} \dots g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} \neq 1$ by Britton's lemma. □

Corollary 2.3.13. *$H*_{A,B,t}$ is torsion-free if and only if H is torsion-free.*

We turn now to free amalgamated products.

Definition 2.3.14. Let $G = G_1 *_A G_2$ be an amalgam of two groups. The word $c_1 \dots c_n$ is called *reduced* if

(a) $c_i \in G_1$ or G_2 alternately (so consecutive c_i's belong to different factors),

(b) if $n \geq 2$, $c_i \notin A \, \forall i \geq 1$, and

(c) if $n = 1$, $c_1 \neq 1$.

Theorem 2.3.15 (Normal form theorem for free amalgamated products)**.** *If $c_0 \dots c_n$ is a reduced word, then $c_0 \dots c_n \neq 1$ in G. In particular, the inclusions $G_1, G_2 \to G$ are embeddings.*

Sketch of proof. The idea is to construct an embedding of G into the HNN extension $(G_1 * G_2)*_{A,t}$ which sends reduced words to reduced words, and then use theorem 2.3.10. See [9, IV.2.6]. □

As with HNN extensions, there is an equivalent statement of the normal form theorem involving a choice of coset representatives of the common subgroup A in each factor:

Theorem 2.3.16. *Let $G = G_1 *_A G_2$, and let $S_i \subseteq G_i$ be sets of right coset representatives of G_i/A, both containing the identity. Then any $g \in G$ can be written uniquely in the form*

$$g = a s_1 \dots s_n$$

where $a \in A$, $s_k \in S_1 - 1$ or $S_2 - 1$ and no two consecutive s_k, s_{k+1} belong to the same factor G_i. We call this the normal form *of g.*

Corollary 2.3.17. *Let $G = G_1 *_A G_2$, and assume that A is a normal subgroup of both G_i. Then, A is normal in G and $G/A \cong (G_1/A) * (G_2/A)$.*

Proof. Let $g \in G$, and let us see that $Ag = gA$. Keeping the notation of theorem 2.3.16, we write $g = a_0 s_1 \dots s_n$ where $a_0 \in A$ and, to fix ideas, $s_1 \in G_1$. Since A is normal in G_1, there exists $a_1 \in A$ such that $a_0 s_1 = s_1 a_1$. We repeat this operation writing $a_{k-1} s_k = s_k a_k$ for each k, so that

$$g = a_0 s_1 \dots s_n = s_1 a_1 s_2 \dots s_n = \dots = s_1 \dots s_n a_n.$$

Hence by the same argument $Ag = As_1 \dots s_n = s_1 \dots s_n A = gA$. The second statement is immediate using again normal forms, just noting that a set of coset representatives of G_i modulo A is the same as a set of coset representatives of G_i/A modulo 1. □

Example 2.3.18. Notice that the hypothesis of the corollary is trivially satisfied when the G_i are abelian, which is the case of many of our examples. For instance, the projective version of $\mathrm{SL}_2(\mathbb{Z})$ is

$$\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\langle -I_2 \rangle \cong (\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6)/\mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

Finally, with the same proof as in corollary 2.3.12, we have

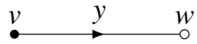
Corollary 2.3.19. *Every element of finite order in $G_1 *_A G_2$ is conjugate to an element of G_1 or G_2.*

Corollary 2.3.20. *$G_1 *_A G_2$ is torsion-free if and only if G_1 and G_2 are torsion-free.*

2.3.2 Action on trees

Given a group acting on a graph, we gave the name of fundamental domain to a subgraph that projects isomorphically onto the quotient graph. One might wish to characterize a group action through its fundamental domain (given that it exists). We also know that if the graph is a tree, the fundamental domain needs to be a tree as well. Perhaps then the simplest candidate we could think of is a segment:

(we colour its vertices differently to stress out the fact that they represent distinct orbits). It turns out that amalgams of two groups correspond precisely to this type of action!

Theorem 2.3.21. *Let G be a group acting on a graph X, and let $T =$* 

be a segment of X. Suppose that T is a fundamental domain of X modulo G. Let G_v, G_w and $G_y = G_{\overline{y}}$ be the stabilizers of the vertices and edges of T. Let $\psi : G_v *_{G_y} G_w \to G$ be the homomorphism induced by the inclusions $G_v \to G$ and $G_w \to G$. Then one has

(i) *ψ is injective if and only if X contains no cycles.*

(ii) *ψ is surjective if and only if X is connected.*

(iii) *ψ is an isomorphism if and only if X is a tree.*

Proof. Notice first that the stabilizers indeed satisfy $G_y = G_{\overline{y}} = G_v \cap G_w$, since $y, \overline{y}$ are the only edges with vertices v and w, and no element of G can interchange v and w (they are inequivalent mod G).

The key observation is that we can relate reduced paths in X to reduced words in $G_v *_{G_y} G_w$, as follows.

Let $(g_1y_1, \dots, g_ny_n)$ be a reduced path in X, where $g_i \in G$, $y_i \in \{y, \overline{y}\}$. Passing to $G\backslash X \cong T$, we see that $\overline{y_i} = y_{i-1}$. Let then $v_i = o(y_i) = t(y_{i-1})$. We have that $g_i \equiv g_{i-1} \bmod G_{v_i}$, since

$$g_iv_i = g_io(y_i) = o(g_iy_i) = t(g_{i-1}y_{i-1}) = g_{i-1}t(y_{i-1}) = g_{i-1}v_i,$$

and $g_i \not\equiv g_{i-1} \bmod G_y = G_{\overline{y}}$, since

$$g_i\overline{y_i} = \overline{g_iy_i} \neq g_{i-1}y_{i-1} = g_{i-1}\overline{y_i},$$

as an equality would yield a backtracking. Therefore, any reduced path must be of the form

$$(c_0y_0, c_0c_1y_1, \dots, c_0 \dots c_ny_n) \tag{2.1}$$

where $\overline{y_i} = y_{i-1}$ and $c_i \in G_{o(y_i)} - G_y \, \forall i \geq 1$.

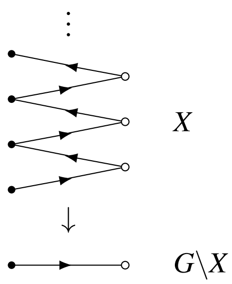


Figure 2.1: Reduced paths in X fold "like an accordion" when projected onto $G\backslash X$.

(i) The reduced path (2.1) is a cycle if and only if $c_0 \dots c_n t(y_n) = c_0 o(y_0)$ i.e. $\overline{y_0} = y_n$ (so any cycle must be of even length!) and $c_1 \dots c_n \in G_{o(y_0)}$. In conclusion, the existence of a cycle in X is equivalent to the existence of a sequence $c_1, \dots, c_{n+1} \in G_v - G_y$ or $G_w - G_y$ alternately such that $1 = c_1 \dots c_{n+1}$ in G. Since by theorem 2.3.15 an element of this form can never be trivial in $G_v *_{G_y} G_w$, this is equivalent to saying that ψ is not injective.

(ii) In view of (2.1), constructing a reduced path in X between any two given vertices amounts to constructing a normal form in $G_v *_{G_y} G_w$ that equals a specific element in G, which is in turn equivalent to surjectivity of ψ.

(iii) This is just the intersection of the first two assertions. □

Remark 2.3.22. As we have noted during the proof, if X contains any cycles, then they must be of even length. That is the same to say that X is a bipartite graph.

To complete this characterization, we now prove the converse.

Theorem 2.3.23. *Let $G = G_1 *_A G_2$ be an amalgam of two groups. Then there is a tree X unique up to isomorphism on which G acts, with fundamental domain a segment $T = \overset{v}{\bullet}\xrightarrow{\;y\;}\overset{w}{\circ}$, and whose vertices and edges have $G_v = G_1$, $G_w = G_2$ and $G_y = G_{\overline{y}} = A$ as their respective stabilizers.*

Proof. We really have no choice as to how to construct X. Namely, we must take

$$\begin{aligned} &\mathrm{V}(X) = G/G_1 \sqcup G/G_2, \\ &\mathrm{E}(X) = G/A \sqcup \overline{G/A}, \\ &o(gA) = gG_1, \\ &t(gA) = gG_2. \end{aligned}$$

If we set $v = G_1$, $w = G_2$ and $y = G_A$, then G acts on X by left multiplication with the segment $T = \overset{v}{\bullet}\xrightarrow{\;y\;}\overset{w}{\circ}$ as a fundamental domain and with G_1, G_2 and G_A as stabilizers. Theorem 2.3.21 then shows that X is a tree. □

Remark 2.3.24. 1. The existence of the tree X associated to G is little more than a reformulation of the normal form theorem for amalgamanted free products. Nevertheless, the tree is a very convenient tool for keeping track of the combinatorics of normal forms. It is often considerably easier to prove things about G by using X than it is to work directly with normal forms.

2. In the situation of corollary 2.3.17, the tree associated to $G = G_1 *_A G_2$ is the same as the one associated to $G/A = (G_1/A) * (G_2/A)$.

3. The cardinality of the set of geometric edges passing through the vertex gG_i is equal to the index $|G_i : A|$ (this is called the *valency* of the vertex).

Example 2.3.25. We return to $G = \mathrm{SL}_2(\mathbb{Z})$ with the same notation as in example 2 of 2.3.6. This group acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by Möbius transformations, via the so-called *modular action*:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Let $\omega = e^{\frac{2\pi i}{3}}$ (i.e. the only primitive 3rd root of unity which lies in $\mathbb{H}$). We claim that the stabilizers of i and ω under the modular action are

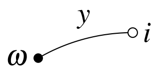
(a) $G_i = \langle S \rangle$ and

(b) $G_\omega = \langle R \rangle$.

Indeed, $\frac{ai+b}{ci+d} = i \Leftrightarrow ai+b = di-c \Leftrightarrow a = d$ and $b = -c$, hence $1 = ad-bc = a^2+b^2$ and, since $a,b \in \mathbb{Z}$, either $(a,b) = \pm(1,0)$ and the matrix is $\pm I_2$, or $(a,b) = \pm(0,1)$ and the matrix is $\pm S$. Since $\langle S \rangle = \{\pm I_2, \pm S\}$, and S does indeed fix i, this proves (a).

On the other hand, since $\omega^2+\omega+1 = 0$, we have $\frac{a\omega+b}{c\omega+d} = \omega \Leftrightarrow a\omega+b = c\omega^2+d\omega = (d-c)\omega - c \Leftrightarrow b = -c$ and $a = d-c = d+b$. Thus $1 = ad-bc = a(a-b)+b^2 = (a-\frac{b}{2})^2+\frac{3}{4}b^2$, and the only integer solutions of this equation are $(a,b) = \pm(1,0), \pm(0,1), \pm(1,1)$, which yield 6 possible values for the matrix that turn out to be the powers of R. Since R indeed fixes ω, this proves (b).

Notice that for every $A \in G, z \in \mathbb{C}$, we have $Az = (-A)z$. This means that $-I_2$ acts trivially, and this induces an action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$. In fact, it can be shown (see [3]) that the stabilizer of any point not belonging to the orbits Gi or $G\omega$ is precisely $\{\pm I_2\}$.

Now consider the circular arc $y = \left\{e^{i\theta} \mid \frac{\pi}{2} < \theta < \frac{2\pi}{3}\right\}$ joining i and ω, and let $T =$ 

. The stabilizer of the arc y is $G_y = \langle I_2 \rangle$. We have that G acts on the set $X = GT$ of translates of T with fundamental domain T; moreover, one can show that X is a tree, so by corollary 2.3.21 it follows that

$$\mathrm{SL}_2(\mathbb{Z}) = G_i *_{G_y} G_\omega \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6,$$

as promised.

To picture this tree in a fancy way, recall that the upper half plane is a model for the hyperbolic plane. Another model is the Poincaré disk, which we show in the next figure with ω at its center. There is a tiling of the disk by ideal hyperbolic triangles (they have their vertices at infinity) which is compatible with the action of $\mathrm{SL}_2(\mathbb{Z})$, i.e. the triangles are shuffled by this action. The points in the orbit $G\omega$ correspond then to the barycenters of the triangles, while the points in the orbit Gi correspond to the intersections of the edges of the triangles with the hyperbolic segments joining translates of ω in adjacent triangles.

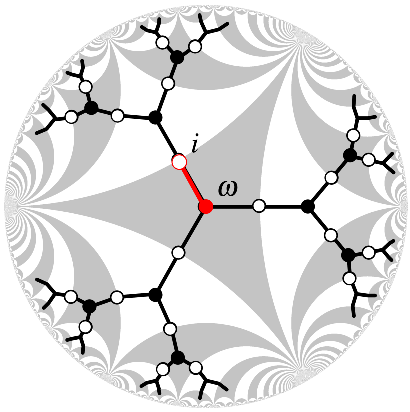


Figure 2.2: Tree associated with $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{PSL}_2(\mathbb{Z})$, as seen in the Poincaré disk.

Just like we did for free groups, we are now able to easily prove some group theoretical results with the aid of Bass-Serre theory.

Proposition 2.3.26. *Let $H \leq G = G_1 *_A G_2$, and suppose that H intersects trivially every conjugate of G_1 or G_2. Then H is free.*

Proof. Let X be the tree associated to G. The hypothesis on H is equivalent to that H acts freely on X; therefore by theorem 2.2.1, H is free. □

Example 2.3.27. Let $G = \mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 = \langle S, R \mid S^4 = R^6 = 1, S^2 = R^3 \rangle$, and consider its derived subgroup G', generated by all the commutators $[a,b], a,b \in G$. This is a normal subgroup, and the quotient G/G' is, by definition, the abelianization of G:

$$\begin{aligned} G_{\mathrm{ab}} &= \langle S, R \mid S^4 = R^6 = 1, S^2 = R^3, [S,R] = 1 \rangle = \langle S, R^2 \mid S^4 = (R^2)^3 = 1, [S,R^2] = 1 \rangle \\ &\cong \mathbb{Z}_4 \times \mathbb{Z}_3 = \mathbb{Z}_{12}. \end{aligned}$$

It is easy to check that G' intersects trivially the subgroups generated by S and R; for example, we see by looking at the presentation of G_{ab} that their images in the quotient have the same order as in G. Hence G' is a free subgroup of G of index $|G:G'| = |G_{\text{ab}}| = 12$, and G' acts freely on the tree X associated to G. Each vertex (resp. edge) G-orbit splits in $|G:G'G_1|$ or $|G:G'G_2|$ (resp. $|G:G'A|$) H-orbits, since G_1, G_2, A are the corresponding stabilizers. Therefore by theorem 2.2.1(ii) we can identify G' with the fundamental group of $G'\backslash X$ and

$$\operatorname{rank} G' = 1 - \left(\left|\mathrm{V}(G'\backslash X)\right| - \left|\mathrm{E}(G'\backslash X)\right|\right) = 1 - 12\left(\frac{1}{4} + \frac{1}{6} - \frac{1}{2}\right) = 2.$$

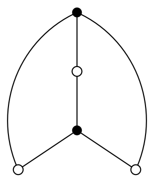


Figure 2.3: $G'\backslash X$ has $3+2=5$ vertices and 6 edges.

Next, we give a nice application to free products:

Proposition 2.3.28. *The kernel R of the canonical projection $p: A*B \to A\times B$ is a free group, with basis the set of nontrivial commutators $\{[a,b] \mid a \in A-1, b \in B-1\}$.*

Remark 2.3.29. In the case that A, B are abelian, then R is the derived subgroup of $A*B$.

Proof. Clearly, $R\cap A = R\cap B = 1$, so, since R is normal in $G = A*B$, $R-1$ does not meet any conjugate of A or B. Hence by proposition 2.3.26, R is free, or more precisely, it acts freely on the tree X associated to G, defined by

$$\begin{aligned} &\mathrm{V}(X) = G/A \sqcup G/B,\\ &\mathrm{E}(X) = G \sqcup \overline{G},\\ &o(g) = gA,\\ &t(g) = gB. \end{aligned}$$

Let us describe the quotient graph $R\backslash X$. The orbit set of the edges is

$$\mathrm{E}(R\backslash X) = R\backslash G \sqcup \overline{R\backslash G} \cong A\times B \sqcup \overline{A\times B},$$

since $R = \ker p$. For the vertices, let $\pi_A : A\times B \to A$ and $\pi_B : A\times B \to B$ denote the canonical projections. Since $\ker \pi_B p = AR$ and $\ker \pi_A p = BR$, we have

$$\mathrm{V}(R\backslash X) = R\backslash G/A \sqcup R\backslash G/B = G/AR \sqcup G/BR \cong B \sqcup A$$

since R is normal, where the double classes $R\backslash G/A$ are the classes of G/A modulo R. On the element level, we are identifying

$$\begin{array}{rcll} \text{edges: } Rg & \leftrightarrow & p(g) & \\ \text{vertices: } RgA = gAR & \leftrightarrow & \pi_B p(g) & \text{(this is the "B-part" of } g) \\ RgB = gBR & \leftrightarrow & \pi_A p(g) & \text{("A-part" of } g) \end{array}$$

Hence $\forall (a,b) \in A\times B, \forall g \in p^{-1}(a,b)$,

$$o(a,b) \equiv o(Rg) = Ro(g) = RgA \equiv \pi_B p(g) = \pi_B(a,b) = b$$

and similarly, $t(a,b) = a$.

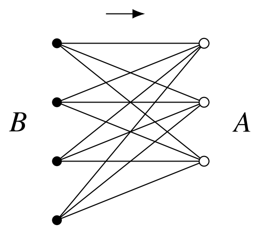


Figure 2.4: $R\backslash X$.

We construct a maximal subtree of $R\backslash X$ choosing the edges corresponding to pairs $(a,1)$ and $(1,b)$. Since $p_{|A}$ and $p_{|B}$ are injective, we can lift these edges to a and b in $\mathrm{E}(X)$, respectively. On the other hand, since $A\cap B = 1$ in G, we can lift the vertices $b\in B$ to bA and $a\in A$ to aB. This indeed defines a subgraph T of X, so, since X is a tree and T projects injectively onto a maximal subtree of $R\backslash X$, it is a tree of representatives. It has vertices $B/A\sqcup A/B$ and edges $A\cup B\sqcup\overline{A\cup B}$, where we denote $B/A = \{bA \mid b\in B\}$ and analogously for A/B.

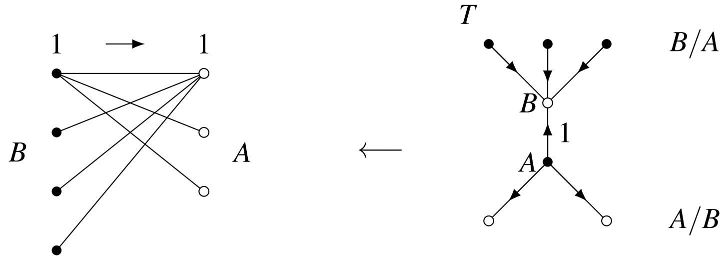
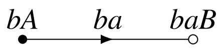


Figure 2.5: Maximal subtree and its lift T, the usual way one draws it.

Finally, to apply theorem 2.2.1, we are going to see that the elements $r\in R-1$ such that

- there exists $g\in G-(A\cup B)$ (a positive edge not in T) with
- $gA\in B/A$ (origin in T) and
- $gB\in rA/B$ (terminus in the translate $rT\neq T$)

are precisely the commutators $[b,a], a\in A-1, b\in B-1$. Since $g\notin A\cup B$, from the second condition we have $g = ba$ for some $b\in B-1, a\in A-1$, so the segment we are looking at is



(notice that ba is a lift of one of the edges (a,b) of $R\backslash X$ which we have *not* lifted to T; in fact, it is the only edge in this fibre to have its origin in T and its terminus in a different translate rT). Now, from the third condition $baB\in rA/B$ we deduce that $ba = ra_1b_1$ for some $a_1\in A$, $b_1\in B$. We assumed that $r\in R$, and R acts freely on X, so this equation must have a unique solution! We find it by passing to $A\times B$:

$$ba = ra_1b_1 \Rightarrow (a,b) = p(ba) = p(ra_1b_1) = (a_1,b_1),$$

since $r\in R = \ker p$. Hence, $a_1 = a$ and $b_1 = b$, and

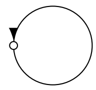
$$r = bab^{-1}a^{-1} = [b,a],$$

which completes the proof (note that $[a,b] = [b,a]^{-1}$). □

Example 2.3.30. Using the above result, we get yet another proof of the fact that the derived subgroup of $G = \mathrm{SL}_2(\mathbb{Z})$ is free of rank 2: we just need to pass to the quotient $G/\langle -I_2\rangle = \mathrm{PSL}_2(\mathbb{Z})$, which is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$, and apply the proposition. We conclude that G' is freely generated by the matrices

$$SRS^{-1}R^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad SR^2S^{-1}R^{-2} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

We have studied so far the case when a group acts on a tree and the quotient graph is a segment. Another very simple situation is when the quotient is a loop:



Theorem 2.3.31. *Let $G = H*_{A,B,t}$ be an HNN extension. Then, up to isomorphism there is a unique graph X on which G acts, such that there is a segment $T = \overset{v}{\circ}\xrightarrow{\;y\;}\overset{w}{\circ}$ in X whose vertices and edges have stabilizers $G_v = H$, $G_w = tHt^{-1}$ and $G_y = G_{\overline{y}} = A$, respectively, and the action is transitive on vertices and transitive on geometric edges (equivalently, the quotient graph is a loop). Moreover, X is a tree.*

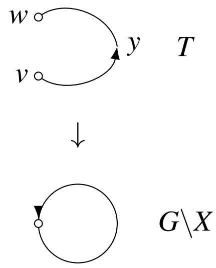


Figure 2.6: The segment projecting onto the loop.

Proof. As in theorem 2.3.23, we have no choice but to set

$$\begin{aligned}
&\mathrm{V}(X) = G/H,\\
&\mathrm{E}(X) = G/A \sqcup \overline{G/A},\\
&o(gA) = gH,\\
&t(gA) = gtH.
\end{aligned}$$

Notice that the map t is well defined, as $at = t\varphi(a)\,\forall a \in A$. Letting $v = H$, $w = tH$ and $y = A$, then G acts on X by left multiplication with the segment $T = \overset{v}{\circ}\xrightarrow{\;y\;}\overset{w}{\circ}$ having the required stabilizers. It is clear too that the action is transitive on vertices and geometric edges, so all that is left is to show that X is a tree. As before, this is done by relating reduced paths and reduced words.

Note that the vertices H and gH are adjacent if and only if there is $g_0 \in H$ with $g = g_0t$ or $g = g_0t^{-1}$. Let $h \in G$ with normal form $h = g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\dots t^{\varepsilon_n}g_n$. Hence one has a path with sequence of vertices

$$(H, g_0t^{\varepsilon_1}H, g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}H, \dots, g_0t^{\varepsilon_1}\dots g_{n-1}t^{\varepsilon_n}H = hH)$$

and since the action of G is transitive on the vertices, this shows that X is connected.

To see that X has no cycles, it suffices to see that it has no cycles passing through the vertex H. Assume that there exists such a cycle of length $n \geq 1$. Its sequence of vertices must be of the form

$$(H, g_0t^{\varepsilon_1}H, g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}H, \dots, g_0t^{\varepsilon_1}\dots g_{n-1}t^{\varepsilon_n}H = H)$$

(with $g_i \in H$), i.e. there is $g_n \in H$, $g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\dots t^{\varepsilon_n}g_n = 1$. As the cycle has no backtracking, we have that $\forall i = 1, \dots, n-1$,

$$g_{i-1}t^{\varepsilon_i}g_it^{\varepsilon_{i+1}}H \neq H,$$

i.e. $t^{\varepsilon_i}g_it^{\varepsilon_{i+1}} \notin H$, so in particular it is not in A or B, and then by Britton's lemma, $g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\dots t^{\varepsilon_n}g_n \neq 1$, a contradiction. □

For the converse to theorem 2.3.31, which tells us that if a group acts on a tree with a loop as quotient then it is an HNN extension, we shall wait until we have proved the general structure theorem, so that we will obtain this result as an easy particular case.

Chapter 3

Graphs of groups

3.1 Fundamental groups

We come to a key definition in Bass-Serre theory, which will allow us to generalize the constructions in the previous chapter.

Definition 3.1.1. A *graph of groups* (G,Y) consists of a graph Y, a group G_v for each $v \in \mathrm{V}(Y)$, and for each $y \in \mathrm{E}(Y)$ a group G_y together with a monomorphism $G_y \to G_{t(y)}$, which we denote $a \mapsto a^y$. One requires in addition that $G_{\overline{y}} = G_y$, so really, we have monomorphisms $G_y \to G_{t(y)}$ and $G_y \to G_{o(y)}$. We denote by G_y^y and $G_y^{\overline{y}}$ the images of G_y under these monomorphisms, respectively.

Let Y be a connected graph, and let (G,Y) be a graph of groups. The procedure to construct the *fundamental group* of (G,Y) can be summarized as follows. We choose a maximal subtree T of Y, and construct a group G_T by iterated free amalgamated products of the vertex groups along the edge groups. For every edge y of $Y - T$, G_T contains a pair of isomorphic subgroups G_y^y and $G_y^{\overline{y}}$, so that we can extend G_T by iterated HNN-extensions with stable letters y. The resulting group, as we will see, is independent of the choice of T.

In order to do this, we begin by defining first an "auxiliary" group $F(G,Y)$ generated by the vertex groups G_v and the edges y of Y, subject to the relations

$$\overline{y} = y^{-1} \quad \text{and} \quad y a^y y^{-1} = a^{\overline{y}}, \quad \forall y \in \mathrm{E}(Y), a \in G_y.$$

We will not distinguish an element of G_v from its image in $F(G,Y)$ via inclusion, since, as we will later prove, this inclusion is an embedding.

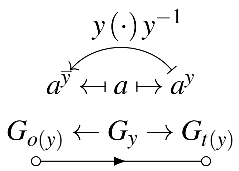


Figure 3.1: Situation in each segment of (G,Y). The notation $(\cdot)^y$ is not to be confused with conjugation by y, which we will always denote $y(\cdot)y^{-1}$.

Definition 3.1.2. Let c be a path in Y with origin $o(c) = v_0$. We let $y_1, \ldots, y_n$ denote the edges of c, and $v_i = o(y_{i+1}) = t(y_i)$. Let $\mu = r_0, \ldots, r_n$ be a sequence of elements $r_i \in G_{v_i}$. Then the word in $F(G,Y)$

$$(c, \mu) = r_0 y_1 r_1 y_2 \ldots y_n r_n$$

is called a *word of type c*.

The role of the edges y in the group $F(G,Y)$ should be understood just as that of the "glue" that allows one to put all the vertex groups G_v together. In the actual group we are interested in, which we next introduce, we make this glue be "almost" invisible (more precisely, we only care about it in the non-contractible part of the graph).

Definition 3.1.3. Fix a maximal subtree T of Y. We construct the *fundamental group of* (G,Y) *at* T by quotienting out of $F(G,Y)$ the elements $y \in \mathrm{E}(T)$, i.e.

$$\pi_1(G,Y,T) = F(G,Y)/\langle\langle \mathrm{E}(T)\rangle\rangle.$$

In particular, notice that in $\pi_1(G,Y,T)$ one has $a^{\overline{y}} = a^y \, \forall y \in \mathrm{V}(T), a \in G_y$.

Remark 3.1.4. Let R be the normal subgroup of $\pi_1(G,Y,T)$ generated by the images of the G_v. It follows from the definition that the quotient $\pi_1(G,Y,T)/R$ is defined by the generators $y \in \mathrm{E}(Y) - \mathrm{E}(T)$ and the relations $\overline{y} = y^{-1}$. This is nothing but $\pi_1(Y,T)$, the fundamental group in the ordinary sense of the graph Y relative to T. It is a free group, with a basis consisting of the geometric edges which do not belong to T.

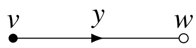
Definition 3.1.5. Fix a vertex v_0 of Y. We define the *fundamental group of* (G,Y) *at* v_0 as the set of words in $F(G,Y)$ whose type is a closed path (possibly with backtracking) based at v_0, i.e.

$$\pi_1(G,Y,v_0) = \{(c,\mu) \mid o(c) = t(c) = v_0\}.$$

It is immediate to check that $\pi_1(G,Y,v_0)$ is a subgroup of $F(G,Y)$. When G is the *trivial graph of groups* I, corresponding to $I_v = 1 \, \forall v \in \mathrm{V}(Y)$, then the group $\pi_1(I,Y,v_0)$ coincides with $\pi_1(Y,v_0)$, the fundamental group in the usual sense of the graph Y based at v_0 (more precisely, one obtains the combinatorial definition of this group). In the general case, the canonical projection $G \to I$ extends to an epimorphism $\pi_1(G,Y,v_0) \to \pi_1(Y,v_0)$, whose kernel is the normal subgroup of $\pi_1(G,Y,v_0)$ generated by the G_v.

Examples 3.1.6. 1. Suppose that $G_y = 1 \, \forall y \in \mathrm{E}(Y)$. Then $\pi_1(G,Y,T)$ is generated by the groups G_v and the elements $y \in \mathrm{E}(Y) - \mathrm{E}(T)$ subject only to the relations $\overline{y} = y^{-1}$. Thus taking into account the above remark, we have

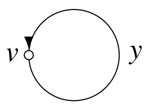
$$\pi_1(G,Y,T) = \pi_1(Y,T) * (*G_v).$$

2. If $T =$ 

is a segment, we have

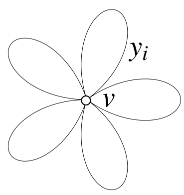
$$\pi_1(G,T,T) = \langle G_v, G_w \mid a^y = a^{\overline{y}} \, \forall a \in G_y \rangle = G_v *_{G_y} G_w.$$

More generally, if T is a tree, then all the $y \in \mathrm{E}(T)$ disappear in the quotient and $\pi_1(G,T,T)$ is an iterated free amalgamated product, which we shall call *tree product* and denote by G_T.

3. If $Y =$ 

is a loop, then

$$F(G,Y) = \pi_1(G,Y,v) = \langle G_v, y \mid y a^y y^{-1} = a^{\overline{y}} \, \forall a \in G_y \rangle = G_v *_{G_y,y}.$$

4. If $Y =$ 

is a bouquet of n circles, then

$$F(G,Y) = \pi_1(G,Y,v) = \langle G_v, y_1, \dots, y_n \mid y_i a^{y_i} y_i^{-1} = a^{\overline{y_i}} \, \forall a \in G_{y_i}, i = 1, \dots, n \rangle.$$

We can regard this as a generalization of HNN extensions, with several stable letters instead of just one. Of course, when $G = I$ is the trivial graph of groups, we get the free group on n generators.

5. One can understand any fundamental group of a graph of groups as an iterated construction using at each step one of the above basic building blocks. Given a maximal subtree T, one exhibits the fundamental group as a generalized HNN extension in the above sense, with the tree product as base group and the edges which do not belong to T as stable letters:

$$\pi_1(G,Y,T) = \langle G_T, y \in \mathrm{E}_+(Y) - \mathrm{E}_+(T) \mid y a^y y^{-1} = a^{\bar{y}} \forall a \in G_y \rangle.$$

The two constructions we have given for a fundamental group of (G,Y) are in fact equivalent, so the situation is the same as for the usual fundamental group.

Theorem 3.1.7. *Let (G,Y) be a graph of groups, let $v_0 \in \mathrm{V}(Y)$ and let T be a maximal subtree of Y. The canonical projection $p: F(G,Y) \to \pi_1(G,Y,T)$ induces an isomorphism of $\pi_1(G,Y,v_0)$ onto $\pi_1(G,Y,T)$.*

Sketch of proof. One constructs the inverse $p^{-1}: \pi_1(G,Y,T) \to \pi_1(G,Y,v_0)$ by conjugation by the elements corresponding to reduced paths. See [11, prop. 20 of I.5]. $\square$

3.2 Reduced words

The following definition generalizes the notions of reduced words we gave in section 2.3.1.

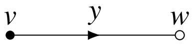
Definition 3.2.1. With the notation of def. 3.1.2, we say that the word $(c,\mu) = r_0 y_1 r_1 y_2 \dots y_n r_n$ of type c is *reduced* if it satisfies one of the following:

(a) $n = 0$ and $r_0 \neq 1$, or

(b) $n \geq 1$ and $r_i \notin G_{y_i}^{y_i}$ for every index i such that $y_{i+1} = \bar{y}_i$ (i.e. there is a backtracking).

Theorem 3.2.2. *If (c,μ) is a reduced word, then $(c,\mu) \neq 1$ in $F(G,Y)$. In particular, the inclusions $G_v \to F(G,Y)$ are embeddings.*

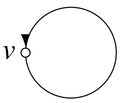
Sketch of proof. To prove the theorem, one shows first that the group $F(G,Y)$ can be constructed by smaller "building blocks", which allows to use an induction argument and reduce to the case of a group with a single geometric edge (see [11, lemma 8 of I.5]). But we have actually already done all the work in this situation:

(1) The case of a segment $T =$  .

In this case, a reduced word has the form $(c,\mu) = r_0 y^{\varepsilon_1} r_1 y^{\varepsilon_2} \dots y^{\varepsilon_n} r_n$, with $\varepsilon_i = \pm 1, \varepsilon_{i+1} = -\varepsilon_i, r_0 \in G_{v_0}$ and $r_i \in G_{v_i} - G_y^{y^{\varepsilon_i}}$, where $v_i = v$ or w alternately. If $n = 0$, then $r_0 \neq 1$. Now, taking the quotient, we have

$$\begin{array}{rcl} F(G,T) & \longrightarrow & \pi_1(G,T,T) = G_v *_{G_y} G_w \\ r_0 y^{\varepsilon_1} r_1 y^{\varepsilon_2} \dots y^{\varepsilon_n} r_n & \longmapsto & r_0 r_1 \dots r_n, \end{array}$$

and it follows from theorem 2.3.15 that $r_0 r_1 \dots r_n \neq 1$, whence $(c,\mu) \neq 1$.

(2) The case of a loop $Y = v$  y .

A reduced word takes the form $(c,\mu) = r_0 y^{\varepsilon_1} r_1 y^{\varepsilon_2} \dots y^{\varepsilon_n} r_n$, with $\varepsilon_i = \pm 1, r_i \in G_v$ and $r_i \notin G_y^{y^{\varepsilon_i}}$ if $\varepsilon_{i+1} = -\varepsilon_i$. But now $F(G,Y) = \pi_1(G,Y,v) = G_v *_{G_y,y}$, so the result reduces to Britton's lemma of theorem 2.3.10. $\square$

As an immediate consequence, we get the following generalization of theorems 2.3.10 and 2.3.15:

Corollary 3.2.3 (Normal form theorem, general case)**.** *Let T be a maximal subtree of Y, and let (c,μ) be a reduced word whose type c is a closed path based at the vertex v_0. Then the image of (c,μ) in $\pi_1(G,Y,T)$ is not* 1.

Proof. The hypothesis implies that $(c,\mu) \in \pi_1(G,Y,v_0)$, and by theorem 3.2.2, $(c,\mu) \neq 1$ in this group. The corollary then follows from theorem 3.1.7. $\square$

3.3 Universal coverings

Let (G,Y) be a connected graph of groups (with a fixed orientation) and T a maximal subtree of Y. In the same spirit as that of a covering of a topological space, we are going to construct the following objects:

- a graph $\widetilde{X} = \widetilde{X}(G,Y,T)$,
- an action of $\pi = \pi_1(G,Y,T)$ on $\widetilde{X}$,
- a morphism $\widetilde{X} \to Y$ which induces an isomorphism $\pi\backslash\widetilde{X} \cong Y$, and
- sections $\mathrm{V}(Y) \to \mathrm{V}(\widetilde{X})$ and $\mathrm{E}(Y) \to \mathrm{E}(\widetilde{X})$, denoted $v \mapsto \widetilde{v}$ and $y \mapsto \widetilde{y}$,

such that

- $\forall v \in \mathrm{V}(Y)$, the stabilizer $\pi_{\widetilde{v}}$ of $\widetilde{v}$ in π is G_v, and similarly,
- $\forall y \in \mathrm{E}(Y)$, $\pi_{\widetilde{y}} = \begin{cases} G_y^{\overline{y}} \subseteq G_{o(y)}, & \text{if } y \text{ is a positive edge,} \\ G_y^{y} \subseteq G_{t(y)}, & \text{if } y \text{ is a negative edge.} \end{cases}$

Just as we have been doing so far when constructing graphs, to achieve this we set

$$\begin{aligned} \mathrm{V}(\widetilde{X}) &= \bigsqcup_{v\in \mathrm{V}(Y)} \pi/G_v \\ \mathrm{E}_+(\widetilde{X}) &= \bigsqcup_{y\in \mathrm{E}_+(Y)} \pi/G_y^{\overline{y}} \quad \text{with} \quad \begin{array}{l} o(gG_y^{\overline{y}}) = gG_{o(y)} \\ t(gG_y^{\overline{y}}) = gyG_{t(y)} \end{array} \\ \mathrm{E}_-(\widetilde{X}) &= \bigsqcup_{y\in \mathrm{E}_-(Y)} \pi/G_y^{y} \quad \text{with} \quad \begin{array}{l} o(gG_y^{y}) = gy^{-1}G_{o(y)} \\ t(gG_y^{y}) = gG_{t(y)} \end{array} \end{aligned}$$

Notice that the definition of the extremes of negative edges is forced by that of the positive ones:

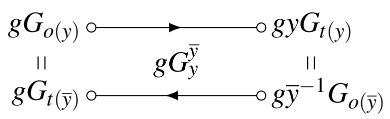


Figure 3.2: Edges in $\widetilde{X}$ (here $y \in \mathrm{E}_+(Y)$; recall that $G_{\overline{y}} = G_y$).

The sections then come off as $\widetilde{v} = 1\cdot\pi_{\widetilde{v}}$ and $\widetilde{y} = 1\cdot\pi_{\widetilde{y}}$, with $\pi_{\widetilde{v}}$ and $\pi_{\widetilde{y}}$ defined as above. Notice that the extremes definitions can be summarized by

$$\begin{array}{l} o(g\widetilde{y}) = gy^{-\varepsilon(y)}\widetilde{o(y)} \\ t(g\widetilde{y}) = gy^{1-\varepsilon(y)}\widetilde{t(y)} \end{array} \quad \text{where} \quad \varepsilon(y) = \begin{cases} 0 & \text{if } y \text{ is positive,} \\ 1 & \text{if } y \text{ is negative.} \end{cases}$$

To see that the action of π on $\widetilde{X}$ is a graph morphism, it remains to show that the stabilizer $\pi_{g\widetilde{y}} = g\pi_{\widetilde{y}}g^{-1}$ of $g\widetilde{y}$ is contained in the stabilizers of its extremes $o(g\widetilde{y}), t(g\widetilde{y})$. If y is positive, then

$$\begin{aligned} \pi_{o(g\widetilde{y})} &= \pi_{g\widetilde{o(y)}} = g\pi_{\widetilde{o(y)}}g^{-1} = gG_{o(y)}g^{-1} \supseteq gG_y^{\overline{y}}g^{-1} = g\pi_{\widetilde{y}}g^{-1} = \pi_{g\widetilde{y}}, \\ \pi_{t(g\widetilde{y})} &= \pi_{gy\widetilde{t(y)}} = gy\pi_{\widetilde{t(y)}}y^{-1}g^{-1} = gyG_{t(y)}y^{-1}g^{-1} \supseteq gyG_y^{y}y^{-1}g^{-1} = gG_y^{\overline{y}}g^{-1} = \pi_{g\widetilde{y}} \end{aligned}$$

(see figure 3.1); an analogous computation shows the same for negative edges (or you can use the fact that $\overline{g\widetilde{y}} = g\overline{\widetilde{y}} = g\widetilde{\overline{y}}$). Because we will use it later, we remark that for every $y \in \mathrm{E}(Y)$, we have the equality

$$\pi_{\widetilde{y}} = y^{1-\varepsilon(y)} G_y^y y^{\varepsilon(y)-1}.$$

We have now defined the graph $\widetilde{X}$ as well as the action of π on $\widetilde{X}$, and, by construction, $\pi\backslash\widetilde{X} = Y$. Notice that, if $y \in \mathrm{E}(T)$, then $G_{\widetilde{y}}^{y} = G_{\widetilde{y}}^{\overline{y}}$ and $o(g\widetilde{y}) = g\widetilde{o(y)}, t(g\widetilde{y}) = g\widetilde{t(y)}$ for all $g \in \pi$. In particular, $v \mapsto \widetilde{v}, y \mapsto \widetilde{y}$ define a lift $T \to \widetilde{T}$ of T into $\widetilde{X}$.

Examples 3.3.1. 1. When all the stabilizers are trivial, i.e. $G = I$ is the trivial graph of groups, we have $\pi = \pi_1(Y,T)$ and $\widetilde{X}$ is the universal covering (in the usual sense) of Y relative to T. In particular, π acts freely on the tree $\widetilde{X}$ and it is of course a free group. Notice that, by the proof of theorem 2.2.1, if Y has a single vertex, i.e. it is a bouquet of circles as in example 4 of 3.1.6, then $\widetilde{X}$ is the Cayley graph $\Gamma(\pi, y_1, \dots, y_n)$; otherwise, $\widetilde{X}$ contains infinitely many copies of T which we can shrink to recover the Cayley graph.

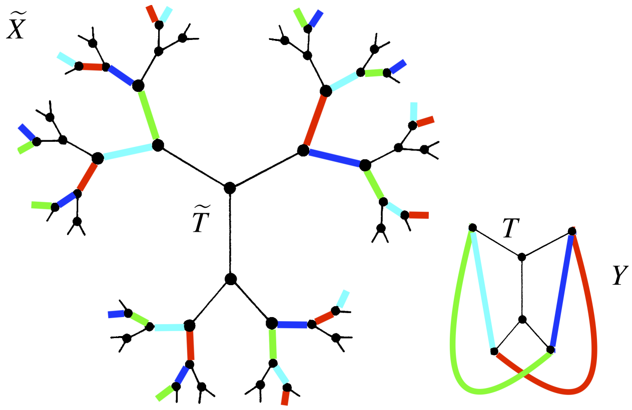
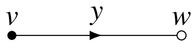
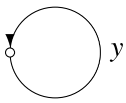


Figure 3.3: Universal covering relative to a maximal subtree.

2. If $Y =$ 

is a segment, then $\widetilde{X}$ is the tree associated with the amalgam $\pi = G_v *_{G_y} G_w$, which we constructed in theorem 2.3.23.

3. If $Y = v$ 

is a loop, then $\widetilde{X}$ is the tree associated with $\pi = G_v *_{G_y, y}$, which we constructed in theorem 2.3.31.

In the examples above, the graph $\widetilde{X}$ turned out to be a tree. We next prove that this holds in general.

Theorem 3.3.2. *With the definition above, $\widetilde{X}$ is a tree.*

Proof. We first show that $\widetilde{X}$ is connected. For every $y \in \mathrm{E}(Y)$, one of the extremes of $\widetilde{y}$ belongs to $\widetilde{T}$ ($o(\widetilde{y}) = \widetilde{o(y)}$ if y is positive; $t(\widetilde{y}) = \widetilde{t(y)}$ if y is negative). This shows that the smallest subgraph W of $\widetilde{X}$ which contains all the $\widetilde{y}$ is connected; moreover, the π-translates of W cover $\widetilde{X}$, i.e. $\pi W = \widetilde{X}$. It then suffices to show that there is a subset $S \subseteq \pi$ which generates π and such that $W \cup sW$ is connected for all $s \in S$. Indeed, this will imply, by induction on n, that $W \cup s_1 W \cup s_1 s_2 W \cup \cdots \cup s_1 \dots s_n W$ is connected for any $s_1, \dots, s_n \in S \cup S^{-1}$.

We take S to be the union of the G_v ($v \in \mathrm{V}(Y)$) and the $y \in \mathrm{E}(Y)$. If $s \in G_v$, the graphs W and sW have a common vertex $\widetilde{v}$, so $W \cup sW$ is connected. Likewise, W and yW have a common vertex ($o(\widetilde{y}) = o(y\widetilde{y})$ if y is positive; $t(\widetilde{y}) = t(y\widetilde{y})$ if y is negative), so we are done.

To show that $\widetilde{X}$ is a tree, it now suffices to prove that it does not contain any cycle of length $n \geq 1$. Assume $\widetilde{c}$ is such a cycle, let $s_1\widetilde{y}_1, \dots, s_n\widetilde{y}_n$ be the sequence of its edges, and let $v_0, \dots, v_n$ be the sequence of vertices of the projection c of $\widetilde{y}$ in Y (so $v_0 = v_n$). If we put $\varepsilon_i = \varepsilon(y_i)$, we have

$$s_i y_i^{1-\varepsilon_i} G_{v_i} = t(s_i\widetilde{y}_i) = o(s_{i+1}\widetilde{y}_{i+1}) = s_{i+1} y_{i+1}^{-\varepsilon_{i+1}} G_{v_i}$$

where indices are taken mod n. Now putting $q_i = s_i g_i^{-\varepsilon_i}$, this means that $q_i y_i r_i = q_{i+1}$ with $r_i \in G_{v_i}$. Hence, $y_i r_i = q_i^{-1} q_{i+1}$ and by multiplying we obtain

$$y_1 r_1 \dots y_n r_n = 1. \tag{3.1}$$

Let (c,μ) be the word of type c defined by $\mu = 1, r_1, \dots, r_n$. We are going to prove that (c,μ) is reduced. Indeed, suppose that $y_{i+1} = \overline{y}_i = y_i^{-1}$. Then

$$r_i = y_i^{\varepsilon_i - 1} s_i^{-1} s_{i+1} y_{i+1}^{-\varepsilon_{i+1}} = y_i^{\varepsilon_i - 1} s_i^{-1} s_{i+1} y_i^{1-\varepsilon_i}.$$

We have to show that $r_i \notin G_{y_i}^{y_i}$ i.e.

$$s_i^{-1} s_{i+1} \notin y_i^{1-\varepsilon_i} G_{y_i}^{y_i} y_i^{\varepsilon_i - 1} = \pi_{\widetilde{y}_i} \Leftrightarrow s_i \pi_{\widetilde{y}_i} \neq s_{i+1} \pi_{\widetilde{y}_i},$$

which is true, since $s_{i+1}\widetilde{y}_i = s_{i+1}\overline{\widetilde{y}}_{i+1} = \overline{s_{i+1}\widetilde{y}_{i+1}} \neq s_i \widetilde{y}_i$ because $\widetilde{c}$ has no backtracking. Therefore, (c,μ) is reduced, so, since c is closed, the equality (3.1) contradicts corollary 3.2.3, and the proof is complete. □

Definition 3.3.3. By the above result, $\widetilde{X}$ is simply connected, which justifies that we call it the *universal covering* of the graph of groups (G,Y) relative to T.

3.4 The structure theorem

We have so far exhibited a procedure to construct, out of a graph of groups (G,Y), a group π which acts on a tree $\widetilde{X}$ with stabilizers isomorphic to the groups G_v, G_y and orbit space isomorphic to Y. The theorem we are going to prove says that *any* group acting on a tree has this structure. More precisely, let G be a group which acts on a connected graph X. We shall see that, if X is a tree, then G can be identified with the fundamental group of a certain graph of groups (G,Y), where $Y = G\backslash X$ and the vertex and edge groups correspond to the stabilizers of the action of G. The construction is really very natural; we simply need to define things in the only way they can be defined to make everything work.

We begin with the construction of (G,Y). As above, set $Y = G\backslash X$ and let T be a maximal subtree of Y, $j : T \to X$ a lifting of T (so jT is a tree of representatives of X modulo G), and fix an orientation of Y. We extend j to a section $j : \mathrm{E}(Y) \to \mathrm{E}(X)$ such that $j\overline{y} = \overline{jy}$: it suffices to define it for positive edges, in which case we choose jy so that $o(jy) \in \mathrm{V}(jT)$ (we then have $o(jy) = jo(y)$). Since $t(jy)$ and $jt(y)$ project both to $t(y)$ in Y, we can choose $\gamma_y \in G$ such that $t(jy) = \gamma_y jt(y)$. We extend $y \mapsto \gamma_y$ to all of $\mathrm{E}(Y)$ by $\gamma_{\overline{y}} = \gamma_y^{-1}$ and $\gamma_y = 1$ if $y \in \mathrm{E}(T)$. Hence for each $y \in \mathrm{E}(Y)$, we have the following equalities, which already remind us of $\widetilde{X}$:

$$o(jy) = \gamma_y^{-\varepsilon(y)} jo(y),$$
$$t(jy) = \gamma_y^{1-\varepsilon(y)} jt(y).$$

The graph of groups (G,Y) is then defined by

$$G_v = G_{jv}, \quad v \in \mathrm{V}(Y),$$
$$G_y = G_{jy}, \quad y \in \mathrm{E}(Y),$$

where G_{jv}, G_{jy} are the stabilizers of jv, jy in G, and $G_y \to G_{t(y)}$ is given by $a \mapsto a^y = \gamma_y^{\varepsilon(y)-1} a \gamma_y^{1-\varepsilon(y)}$, which is well defined since $\gamma_y^{\varepsilon(y)-1} G_{jy} \gamma_y^{1-\varepsilon(y)} \subseteq G_{jt(y)} \, \forall y \in \mathrm{E}(Y)$.

Let now $\phi : \pi = \pi_1(G,Y,T) \to G$ be the homomorphism defined by the inclusions $G_v \to G$ and by $\phi(y) = \gamma_y$, and let

$$\begin{aligned} \psi : \widetilde{X} = \widetilde{X}(G,Y,T) &\longrightarrow X \\ g\widetilde{v} &\longmapsto \psi(g\widetilde{v}) = \phi(g) jv, \\ g\widetilde{y} &\longmapsto \psi(g\widetilde{y}) = \phi(g) jy. \end{aligned}$$

One can check that ψ is a graph morphism, and that it is ϕ-equivariant (i.e. it can be regarded as a π-map via ϕ).

Theorem 3.4.1 (Structure theorem)**.** *The following are equivalent:*

(i) *X is a tree.*

(ii) *$\psi : \widetilde{X} \to X$ is a graph isomorphism.*

(iii) *$\phi : \pi \to G$ is a group isomorphism.*

Proof. $(i) \Rightarrow (ii)$ Let W be the smallest subgraph of X which contains jy for all $y \in \mathrm{E}(Y)$. Each edge of W has an extreme in jT, and we have $GW = X$. Moreover, $W \subseteq \psi(\widetilde{X})$ by construction, and ϕ induces isomorphisms between the stabilizers of the corresponding vertices and edges in $\widetilde{X}$ and X, so ψ is locally injective. One can deduce from here, using some properties of graph automorphisms, that ψ is in fact an isomorphism (see [11, theorem 13 of I.5]).

$(ii) \Rightarrow (i)$ follows from theorem 3.3.2.

$(ii) \Rightarrow (iii)$: Let $v \in \mathrm{V}(Y)$. We have $\ker\phi \cap \pi_{\widetilde{v}} = 1$ because ϕ defines an isomorphism between $G_v = \pi_{\widetilde{v}}$ and G_{jv}. Hence, if $g \in \ker\phi - 1$, the two vertices $\widetilde{v}, g\widetilde{v}$ of $\widetilde{X}$ are distinct and have the same image jv in X; whence $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (ii)$ is clear by construction. □

We show now an example of application of this theorem to a group theoretical result, which helps to understand subgroups of free amalgamated products.

Corollary 3.4.2 (Kurosh's theorem)**.** *Let $G = G_1 *_A G_2$ and let H be a subgroup of G. For each $x \in G/G_i$, let $H_i^x = H \cap xG_ix^{-1}$, which is the stabilizer of x under the natural action of H on G/G_i. Suppose that H intersects trivially every conjugate of A. Then there exist a free subgroup $F \leq H$ and sets of right coset representatives X_i of G/G_i modulo H (i.e. in $H\backslash G/G_i$) such that*

$$H = F * (*H_1^x) * (*H_2^x),$$

where x ranges over the coset representatives.

Proof. Let X be the tree associated to G, so that G acts on X with a segment as fundamental domain. The stabilizers of the edges are the conjugates of A, and those of the vertices are the conjugates of the G_i. Since $H \leq G$, H acts on X as well and we can apply theorem 3.4.1. We see then that $H = \pi_1(H,Y,S)$ where $Y = H\backslash X$ and S is a maximal subtree of Y, a lifting of which is chosen in X. The hypothesis that $H - 1$ intersects trivially every conjugate of A is equivalent to saying that $H_y = 1$ for all $y \in \mathrm{E}(Y)$. By the first example in 3.1.6, we then have

$$H \cong F * (*H_v)$$

where $F \cong \pi_1(Y,S)$ is a free group and v ranges over $\mathrm{V}(S)$. On the other hand,

$$\mathrm{V}(X) \cong G/G_1 \sqcup G/G_2,$$
$$\mathrm{V}(S) \cong H\backslash G/G_1 \sqcup H\backslash G/G_2.$$

The lifting of S into X then defines systems of representatives $X_i \subseteq G/G_i$ of $H\backslash G/G_i$. If x belongs to X_i, the corresponding group H_v is $H \cap xG_ix^{-1}$, whence the theorem. □

Remark 3.4.3.

1. The condition $H \cap xAx^{-1} = 1$ is trivially satisfied when $A = 1$, i.e. G is a free product. Actually, the original formulation of Kurosh's theorem deals only with this case.

2. With the same proof, one obtains an analogous result for subgroups of HNN extensions: if $G = K*_{L,t}$ and $H \leq G$ such that H intersects trivially any conjugate of L, then

$$H = F * (*H \cap xKx^{-1})$$

where F is free and x ranges over G/K.

Bibliography

[1] R. Bieri, *Homological dimension of discrete groups*. 2nd ed. Queen Mary College, University of London, 1981.

[2] K.S. Brown, *Cohomology of groups*. Springer-Verlag, New York, 1982.

[3] K. Conrad, $\mathrm{SL}_2(\mathbb{Z})$. Available at `https://kconrad.math.uconn.edu/blurbs`.

[4] A. Hatcher, *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[5] K. Jankiewicz, *Residual finiteness of certain 2-dimensional Artin groups*. arXiv:2006.04717, 2020.

[6] K. Jankiewicz, *Splittings of triangle Artin groups*. arXiv:2103.01343, 2021.

[7] A. Karrass, W. Magnus and D. Solitar, *Combinatorial group theory. Presentations of groups in terms of generators and relations*. 2nd ed. Dover Publications, New York, 1976.

[8] A. Karrass, A. Pietrowski and D. Solitar, *Finite and infinite cyclic extensions of free groups*. J. Austral. Math. Soc., Vol. 16, pp. 458–466, 1973.

[9] R.C. Lyndon and P.E. Schupp, *Combinatorial group theory*. Springer-Verlag, Berlin, 1977.

[10] J.R. Munkres, *Topology*. 2nd ed. Prentice Hall, Upper Saddle River, 2000.

[11] J.-P. Serre, *Trees*. Springer-Verlag, Berlin, 1980.

[12] T. Weigel, *Groups acting on trees*. Notes of summer school SMI, Perugia, 2012.