Accurate Bidiagonal Decomposition of Totally Positive Cauchy-Vandermonde Matrices and Applications

Ana Marco[a], José-Javier Martínez[a,*], Juan Manuel Peña[b]

[a]*Departamento de Física y Matemáticas, Universidad de Alcalá, Alcalá de Henares, Madrid 28871, Spain*
[b]*Departamento de Matemática Aplicada, Universidad de Zaragoza, Zaragoza 50009, Spain*

Abstract

Cauchy-Vandermonde matrices play a fundamental role in rational interpolation theory and in other fields. When all their corresponding nodes are different and positive and all poles are different and negative and follow adequate orderings, these matrices are totally positive. In this paper we provide fast algorithms for computing bidiagonal factorizations of these matrices and their inverses with high relative accuracy. These algorithms can be used to solve with high relative accuracy other algebraic problems, such as the computation of all singular values, all eigenvalues or the solution of certain linear systems. The error analysis of the algorithm for computing the bidiagonal factorization and the corresponding perturbation theory are also performed.

Keywords: Cauchy-Vandermonde matrix, Totally positive matrix, Neville elimination, Bidiagonal decomposition, High relative accuracy
2000 MSC: 65F05, 65F15, 65D05, 15A23, 15B05, 15B48

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*Corresponding author.
E-mail addresses: ana.marco@uah.es (A. Marco), jjavier.martinez@uah.es (J. J. Martínez), jmpena@unizar.es (J. M. Peña)
This research has been partially supported by Spanish Research Grant MTM2015-65433-P(MINECO/FEDER) from the Spanish Ministerio de Economía y Competitividad. A. Marco and J. J. Martínez are members of the Reseach Group ASYNACS (Ref. CCEE2011/R34) of Universidad de Alcalá. J. M. Peña is also supported by the Gobierno de Aragón.

1. Introduction

Matrices with a special structure arise in many fields and frequently present computational advantages, although they are usually ill-conditioned. Cauchy-Vandermonde matrices are $n \times n$ matrices in which the columns $1, \ldots, l$ form a rectangular Cauchy matrix and the columns $l+1, \ldots, n$ form a rectangular Vandermonde matrix. They naturally arise when computing rational interpolates with prescribed poles [28]. This type of interpolation has applications in control systems [30]. Cauchy-Vandermonde matrices also appear in connection with the numerical solution of singular integral equations [9, 18], as well as in numerical quadrature [31] and rational models of regression and E-optimal design [14, 17].

Recall that a matrix is *totally positive* (resp., *strictly totally positive*) if all its minors are nonnegative (resp., positive), and they are also called in the literature as totally nonnegative (resp., totally positive) [8, 29]. In [25], starting from a natural extension to the rational case of the well-known Newton basis of the polynomial case (see [2] and Section 4.6 of [13]), it was proved that a Cauchy-Vandermonde matrix is strictly totally positive if all its corresponding nodes are different and positive and all poles are different and negative (with precise orderings that we detail at the beginning of Section 3). When multiple poles are permitted, the total positivity property can fail, as shown in [27]. Interestingly, in [17] it is shown that some of its theoretical results cannot be extended to the case of several multiple poles. On the other hand, the theoretical interest of total positivity in regression is shown in [14].

A fast algorithm for solving Vandermonde linear systems was presented by Björck and Pereyra in [2]. For Cauchy linear systems, a fast Björck-Pereyra type algorithm was given in [3], and for the case of Cauchy-Vandermonde linear systems, an algorithm based on the use a Newton-type basis appears in [25]. Other fast algorithms for the inverses of Cauchy-Vandermonde matrices and confluent Cauchy-Vandermonde matrices are considered in [9] and [32], respectively.

A nonsingular totally positive matrix A also possesses a unique bidiagonal decomposition denoted by $\mathcal{BD}(A)$. If we have $\mathcal{BD}(A)$ to high relative accuracy, then we can apply the algorithms of [20, 21] to solve many algebraic problems with A to high relative accuracy. In [26] a bidiagonal factorization of the inverse of a Cauchy-Vandermonde matrix A was given, and this factorization is closely related to its bidiagonal decomposition $\mathcal{BD}(A)$ as it will be seen in Section 3. In the present paper, starting from the work done in[26], a fast and accurate algorithm for computing the $\mathcal{BD}(A)$ of a strictly totally positive Cauchy-Vandermonde matrix is given. Analogously to the approach of [7, 20, 23, 24], the error analysis of the algorithm and the study of the corresponding perturbation theory are performed. This study leads to finding an appropriate structured condition number.

The rest of the paper is organized as follows. Section 2 introduces Neville elimination, a key theoretical tool for our approach. Section 3 presents the bidiagonal factorization of totally positive Cauchy-Vandermonde matrices and their inverses. In Section 4 we provide a fast and accurate algorithm for computing the bidiagonal decomposition $\mathcal{BD}(A)$ of a totally positive Cauchy-Vandermonde matrix A. In fact, with the terminology of [5], our algorithm is a NIC (no inaccurate cancellation) algorithm. A NIC algorithm permits "true subtractions" (i.e., subtraction of numbers with different sign) only for initial data, and it can be performed to high relative accuracy.

In Section 5, the algorithms for computing to high relative accuracy the eigenvalues and singular values of totally positive Cauchy-Vandermonde matrices are presented. The accurate solution of linear systems is also considered, and if the right hand side of the linear system has alternating signs then the high relative accuracy is also guaranteed. In particular, the computation of the inverses of totally positive Cauchy-Vandermonde matrices to high relative accuracy is ensured. This computation is very important, for instance, in the problem of rational regression and optimal design [17]. Section 6 introduces the error analysis of the bidiagonal factorization algorithm of Section 4, and Section 7 presents the perturbation analysis. Structured condition numbers can be much smaller than the usual condition numbers (see [5]). This explains that we can perform accurate computations even with very ill-conditioned matrices. Finally, numerical examples are presented in Section 8.

2. Neville elimination and total positivity

To make this paper as self-contained as possible, we will briefly recall in this section some basic results on Neville elimination and total positivity which will be essential for obtaining the results presented in Section 3. Our notation follows the notation used in [10] and [11]. Given $k, n \in \mathbb{N}$ $(1 \leq k \leq n)$, $Q_{k,n}$ will denote the set of all increasing sequences of k positive integers less than or equal to n.

Let A be a square real matrix of order n. For $k \leq n$, $m \leq n$, and for any $\alpha \in Q_{k,n}$ and $\beta \in Q_{m,n}$, we will denote by $A[\alpha|\beta]$ the $k \times m$ submatrix of A containing the rows numbered by α and the columns numbered by β.

The fundamental theoretical tool for obtaining the results presented in this paper is the *Neville elimination* [10, 11, 12], a procedure that makes zeros in a matrix adding to a given row an appropriate multiple of the previous one.

Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be a square matrix of order n. The Neville elimination of A consists of $n-1$ steps resulting in a sequence of matrices $A_1 := A \rightarrow A_2 \rightarrow \ldots \rightarrow A_n$, where $A_t = (a_{i,j}^{(t)})_{1 \leq i,j \leq n}$ has zeros below its main diagonal in the $t-1$ first columns. The matrix A_{t+1} is obtained from A_t $(t = 1, \ldots, n-1)$ by using the following formula:

$$a_{i,j}^{(t+1)} := \begin{cases} a_{i,j}^{(t)} , & \text{if} \quad i \leq t \\ a_{i,j}^{(t)} - (a_{i,t}^{(t)}/a_{i-1,t}^{(t)})a_{i-1,j}^{(t)} , & \text{if} \quad i \geq t+1 \text{ and } j \geq t+1 \\ 0 , & \text{otherwise.} \end{cases} \tag{2.1}$$

In this process the element

$$p_{i,j} := a_{i,j}^{(j)} \qquad 1 \leq j \leq n, \quad j \leq i \leq n$$

is called (i,j) *pivot* of the Neville elimination of A. The process would break down if any of the pivots $p_{i,j}$ $(j \leq i < n)$ is zero. In that case we can move the corresponding rows to the bottom and proceed with the new matrix, as described in [10]. The Neville elimination can be done without row exchanges if all the pivots are nonzero, as it will happen in our situation. The pivots $p_{i,i}$ are called *diagonal pivots*. If all the pivots $p_{i,j}$ are nonzero, then $p_{i,1} = a_{i,1}\,\forall i$ and, by Lemma 2.6 of [10]

$$p_{i,j} = \frac{\det A[i-j+1, \ldots, i|1, \ldots, j]}{\det A[i-j+1, \ldots, i-1|1, \ldots, j-1]} \qquad 1 < j \leq i \leq n. \tag{2.2}$$

The element

$$m_{i,j} = \frac{p_{i,j}}{p_{i-1,j}} \qquad 1 \leq j \leq n-1, \quad j < i \leq n, \tag{2.3}$$

is called *multiplier* of the Neville elimination of A. The matrix $U := A_n$ is upper triangular and has the diagonal pivots on its main diagonal.

The *complete Neville elimination* of a matrix A consists of performing the Neville elimination of A for obtaining U and then continue with the Neville elimination of U^T. The (i,j) pivot (respectively, multiplier) of the complete Neville elimination of A is the (j,i) pivot (respectively, multiplier) of the Neville elimination of U^T, if $j \geq i$. When no row exchanges are needed in the Neville elimination of A and U^T, we say that the complete Neville elimination of A can be done without row and column exchanges, and in this case the multipliers of the complete Neville elimination of A are the multipliers of the Neville elimination of A if $i \geq j$ and the multipliers of the Neville elimination of A^T if $j \geq i$ (see p. 116 of [12]).

A detailled error analysis of Neville elimination has been carried out in [1]. However, our approach uses results related to Neville elimination as a theoretical tool but it does not apply the Neville elimination algorithm for obtaining the bidiagonal factorization of Cauchy-Vandermonde matrices.

The Neville elimination characterizes the strictly totally positive matrices as follows [10]:

Theorem 2.1. A matrix is strictly totally positive if and only if its complete Neville elimination can be performed without row and column exchanges, the multipliers of the Neville elimination of A and A^T are positive, and the diagonal pivots of the Neville elimination of A are positive.

3. Bidiagonal factorization of Cauchy-Vandermonde matrices

A matrix

$$A = \begin{pmatrix} \frac{1}{x_1-d_1} & \cdots & \frac{1}{x_1-d_l} & 1 & x_1 & \cdots & x_1^{n-l-1} \\ \frac{1}{x_2-d_1} & \cdots & \frac{1}{x_2-d_l} & 1 & x_2 & \cdots & x_2^{n-l-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n-d_1} & \cdots & \frac{1}{x_n-d_l} & 1 & x_n & \cdots & x_n^{n-l-1} \end{pmatrix}$$

is called a *Cauchy-Vandermonde matrix* for the nodes $\{x_i\}_{1\leq i\leq n}$ and the poles $\{d_j\}_{1\leq j\leq l}$ because if $l=0$ it is a classical Vandermonde matrix and if $l=n$ it is a classical Cauchy matrix.

Let us observe that the Cauchy-Vandermonde matrix A is the coefficient matrix of the linear system associated with the following interpolation problem in the basis

$$\mathcal{B} = \{v_i(x)\}_{1\leq i\leq n} = \left\{\frac{1}{x-d_1}, \frac{1}{x-d_2}, \ldots, \frac{1}{x-d_l}, 1, x, x^2, \ldots, x^{n-l-1}\right\}.$$

Given the interpolation nodes $\{x_i : \ i = 1, \ldots, n\}$ and the interpolation data $\{b_i : \ i = 1, \ldots, n\}$, find the function

$$f(x) = \sum_{k=1}^{n} c_k v_k(x)$$

(a rational function with prescribed poles) such that $f(x_i) = b_i$ for $i = 1, \ldots, n$.

As it can be found in [25], the Cauchy-Vandermonde matrices are strictly totally positive when the nodes $\{x_i\}_{1 \le i \le n}$ and the poles $\{d_j\}_{1 \le j \le l}$ satisfy $0 < x_1 < x_2 < \ldots < x_n$ and $0 < -d_1 < -d_2 < \ldots < -d_l$. This result can also be seen as a consequence of our Theorem 3.2.

In the situation we are considering the following two theorems hold:

Theorem 3.1. Let $A = (a_{i,j})_{1 \le i,j \le n}$ be a Cauchy-Vandermonde matrix for the basis $\mathcal{B}$ whose nodes satisfy $0 < x_1 < x_2 < \ldots < x_n$ and whose poles satisfy $0 < -d_1 < -d_2 < \ldots < -d_l$. Then A^{-1} admits a factorization in the form

$$A^{-1} = G_1 G_2 \cdots G_{n-1} D^{-1} F_{n-1} F_{n-2} \cdots F_1,$$

where F_i, G_i $(i = 1, \ldots, n-1)$ are $n \times n$ bidiagonal matrices of the form

$$F_i = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & -m_{i+1,i} & 1 & & \\ & & & & -m_{i+2,i} & 1 & \\ & & & & & \ddots & \ddots \\ & & & & & & -m_{n,i} \ 1 \end{bmatrix}, G_i^T = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & -\widetilde{m}_{i+1,i} & 1 & & \\ & & & & -\widetilde{m}_{i+2,i} & 1 & \\ & & & & & \ddots & \ddots \\ & & & & & & -\widetilde{m}_{n,i} \ 1 \end{bmatrix}$$

and D is a diagonal matrix of order n

$$D = \mathrm{diag}\{p_{1,1}, p_{2,2}, \ldots, p_{n,n}\}.$$

The quantities $m_{i,j}$ are the multipliers of the Neville elimination of the Cauchy-Vandermonde matrix A, and have the expression

$$m_{ij} = \frac{p_{i,j}}{p_{i-1,j}} = \frac{a_{i,j}^{(j)}}{a_{i-1,j}^{(j)}}, \tag{3.1}$$

where $j = 1, \ldots, n-1$, $i = j+1, \ldots, n$ and

$$a_{ij}^{(j)} = \begin{cases} \frac{\prod_{k=i-j+1}^{i-1}(x_i-x_k)\prod_{r=1}^{j-1}(-d_j+d_r)}{\prod_{r=1}^{j-1}(x_i-d_r)\prod_{k=i-j+1}^{i}(x_k-d_j)}, & j \leq l, \\ \\ \frac{\prod_{k=i-j+1}^{i-1}(x_i-x_k)}{\prod_{r=1}^{l}(x_i-d_r)}, & j > l. \end{cases} \tag{3.2}$$

The quantities $\widetilde{m}_{i,j}$ are the multipliers of the Neville elimination of A^T and their expression is

$$\widetilde{m}_{i,j} = \frac{(A^T)_{i,j}^{(j)}}{(A^T)_{i-1,j}^{(j)}}, \tag{3.3}$$

where $j = 1, \ldots, n-1$,$i = j+1, \ldots, n$ and

$$(A^T)_{ij}^{(j)} = \begin{cases} \frac{\prod_{k=1}^{j-1}(x_j-x_k)\prod_{r=i-j+1}^{i-1}(-d_i+d_r)}{\prod_{r=i-j+1}^{i-1}(x_j-d_r)\prod_{k=1}^{j}(x_k-d_i)}, & i \leq l, \\ \\ \frac{\prod_{k=1}^{j-1}(x_j-x_k)}{\prod_{r=i-j+1}^{l}(x_j-d_r)}, & i-j+1 \leq l < i, \\ \\ \prod_{k=1}^{j-1}(x_j - x_k), & i-j = l, \\ \\ x_j^{i-j-l}\prod_{k=1}^{j-1}(x_j - x_k), & l < i-j. \end{cases} \tag{3.4}$$

Finally, the ith diagonal element of D $(i = 1, \ldots, n)$ is the diagonal pivot $p_{ii} = a_{ii}^{(i)}$ of the Neville elimination of A.

Proof. It can be found in [26]. □

Theorem 3.2. Let $A = (a_{i,j})_{1\leq i,j\leq n}$ be a Cauchy-Vandermonde matrix for the basis $\mathcal{B}$ whose nodes satisfy $0 < x_1 < x_2 < \ldots < x_n$ and whose poles satisfy $0 < -d_1 < -d_2 < \ldots < -d_l$. Then A admits a factorization in the form

$$A = F_{n-1}F_{n-2}\cdots F_1 D G_1 \cdots G_{n-2}G_{n-1},$$

where F_i, G_i^T $(i = 1, \ldots, n-1)$ are $n \times n$ bidiagonal matrices of the form

$$F_i = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & m_{i+1,1} & 1 & & \\ & & & & m_{i+2,2} & 1 & \\ & & & & & \ddots & \ddots \\ & & & & & m_{n,n-i} & 1 \end{bmatrix}, G_i^T = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \widetilde{m}_{i+1,1} & 1 & & \\ & & & & \widetilde{m}_{i+2,2} & 1 & \\ & & & & & \ddots & \ddots \\ & & & & & \widetilde{m}_{n,n-i} & 1 \end{bmatrix}$$

and D is the $n \times n$ diagonal matrix

$$D = (d_{i,j})_{1 \le i,j \le n} = \operatorname{diag}\{p_{1,1}, p_{2,2}, \dots, p_{n,n}\}.$$

The expressions of the multipliers $m_{i,j}$ ($j = 1, \dots, n-1; \quad i = j+1, \dots, n$) of the Neville elimination of A, the multipliers $\widetilde{m}_{i,j}$ ($j = 1, \dots, n-1; \quad i = j+1, \dots, n$) of the Neville elimination of A^T, and the diagonal pivots $p_{i,i}$ ($i = 1, \dots, n$) of the Neville elimination of A are also in this case given by Eq.(3.1) and Eq.(3.2), Eq.(3.3) and Eq.(3.4), and Eq.(3.2), respectively.

Proof. Since the matrix A is strictly totally positive, by Theorem 2.1, the complete Neville elimination of A can be performed without row and column exchanges providing the bidiagonal factorization of A in the statement of this theorem (see [12]). $\square$

It must be observed that the matrices F_i ($i = 1, \dots, n-1$) and the matrices G_j ($j = 1, \dots, n-1$) that appear in the bidiagonal factorization of A are not the same bidiagonal matrices that appear in the bidiagonal factorization of A^{-1} , nor their inverses (see Theorem 3.1 and Theorem 3.2). The multipliers of the Neville elimination of A and A^T give us the bidiagonal factorization of A and A^{-1}, but obtaining the bidiagonal factorization of A from the bidiagonal factorization of A^{-1} (or vice versa) is not straightforward. See [12] for a more detailed explanation.

4. The algorithm

In this section we present a fast and accurate algorithm for computing $\mathcal{BD}(A)$ for a totally positive Cauchy-Vandermonde matrix A. Let us point out here that, given A, the matrix $\mathcal{BD}(A)$ represents *both* the bidiagonal decomposition of A, and that of its inverse A^{-1} (see Theorem 3.1 and Theorem 3.2).

Given the nodes $\{x_i\}_{1 \le i \le n}$ and the poles $\{d_j\}_{1 \le j \le l}$, the algorithm returns a matrix $M = \mathcal{BD}(A)$ such that

$$\begin{array}{ll} M_{i,i} = p_{i,i} & i = 1, \dots, n, \\ M_{i,j} = m_{i,j} & j = 1, \dots, n-1; \ i = j+1, \dots, n, \\ M_{i,j} = \widetilde{m}_{j,i} & i = 1, \dots, n-1; \ j = i+1, \dots, n, \end{array}$$

where $m_{i,j}$ are the multipliers of the Neville elimination of A, $\widetilde{m}_{i,j}$ are the multipliers of the Neville elimination of A^T and $p_{i,i}$ are the diagonal pivots of the Neville elimination of A.

The computations of the $m_{i,j}$ and the $p_{i,i}$ are developed by using the algorithm for constructing the $a_{i,j}^{(j)}$ presented in [26].

The computation of the $\widetilde{m}_{ij}$ are developed by using the following proposition and an appropriate recursion process.

Proposition 4.1. Let $A = (a_{i,j})_{1\le i,j\le n}$ be a Cauchy-Vandermonde matrix for the basis $\mathcal{B}$ whose nodes satisfy $0 < x_1 < x_2 < \ldots < x_n$ and whose poles satisfy $0 < -d_1 < -d_2 < \ldots < -d_l$. The multipliers $\widetilde{m}_{i,j}$ of the Neville elimination of A^T are:

$$\widetilde{m}_{i,j} = \begin{cases} \frac{\prod_{r=i-j+1}^{i-1}(-d_i+d_r)\prod_{k=1}^{j}(x_k-d_{i-1})}{\prod_{r=i-j}^{i-2}(-d_{i-1}+d_r)\prod_{k=1}^{j}(x_k-d_i)} \cdot \frac{x_j-d_{i-j}}{x_j-d_{i-1}}, & 1 < i \le l,\ 1 \le j \le i-1, \\ \\ \frac{\prod_{k=1}^{j}(x_k-d_l)}{\prod_{r=l-j+1}^{l-1}(-d_l+d_r)} \cdot \frac{x_j-d_{l-j+1}}{x_j-d_l}, & i = l+1,\ 2 \le j \le l, \\ \\ x_j, & l+2 \le i \le n,\ 1 \le j \le i-l-1, \\ \\ x_j - d_{l+k-j}, & l+1 \le i \le n,\ i-l \le j \le i-1. \end{cases}$$

Proof. Using Eq.(3.3) and Eq.(3.4) the expresions for the $\widetilde{m}_{ij}$ are easily obtained. $\square$

In order to facilitate the understanding of the error analysis presented in Section 6 we include here the pseudocode of the algorithm.

Computation of the $m_{i,j}$ and $p_{i,i}$ given by Eq.(3.1) and Eq.(3.2) (See [26]):

- G_j computation:

```
G_1 = 1
for j = 2 : l
  G_j = d_1 - d_j
  for k = 2 : j - 1
    G_j = G_j (d_k - d_j)
  end
end
```

- $B_{i,j}$ computation:

$B_{1,1} = x_1 - d_1$

```
for j = 2 : l
    B_{j,j} = x_1 - d_j
        for k = 2 : j
            B_{j,j} = B_{j,j}(x_k - d_j)
        end
end
for j = 1 : l
    for i = j + 1 : n
        B_{i,j} = B_{i-1,j} (x_i - d_j)/(x_{i-j} - d_j)
    end
end
```

$$\texttt{for } j = 2 : l \quad B_{j,j} = x_1 - d_j, \quad \texttt{for } k = 2 : j \quad B_{j,j} = B_{j,j}(x_k - d_j), \quad B_{i,j} = B_{i-1,j}\frac{x_i - d_j}{x_{i-j} - d_j}$$

- $a_{ij}^{(j)}$ computation:

```
for i = 1 : n
    a_{i,1}^{(1)} = 1/(x_i - d_1)
end
for j = 1 : l - 1
    for i = j + 1 : n
        a_{i,j+1}^{(j+1)} = a_{i,j}^{(j)} · G_{j+1}/G_j · B_{i,j}/B_{i,j+1} · (x_i - x_{i-j})/(x_i - d_j)
    end
end
for i = l + 1 : n
    a_{i,l+1}^{(l+1)} = a_{i,l}^{(l)} · B_{i,l}/G_l · (x_i - x_{i-l})/(x_i - d_l)
end
for j = l + 1 : n - 1
    for i = j + 1 : n
        a_{i,j+1}^{(j+1)} = a_{i,j}^{(j)} · (x_i - x_{i-j})
    end
end
```

$$a_{i,1}^{(1)} = \frac{1}{x_i - d_1}, \quad a_{i,j+1}^{(j+1)} = a_{i,j}^{(j)} \cdot \frac{G_{j+1}}{G_j} \cdot \frac{B_{i,j}}{B_{i,j+1}} \cdot \frac{x_i - x_{i-j}}{x_i - d_j}, \quad a_{i,l+1}^{(l+1)} = a_{i,l}^{(l)} \cdot \frac{B_{i,l}}{G_l} \cdot \frac{x_i - x_{i-l}}{x_i - d_l}, \quad a_{i,j+1}^{(j+1)} = a_{i,j}^{(j)} \cdot (x_i - x_{i-j})$$

- $m_{i,j}$ computation:

```
for j = 1 : n
  for i = j + 1 : n
    m_{i,j} = a^{(j)}_{i,j} / a^{(j)}_{i-1,j}
  end
end
```

- $p_{i,i}$ computation:

```
for i = 1 : n
  p_{i,i} = a^{(i)}_{i,i}
end
```

Computation of the $\widetilde{m}_{i,j}$ given by the expression in Proposition 4.1:

for $j = 2 : l$
  $\widetilde{m}_{j,1} = \frac{x_1 - d_{j-1}}{x_1 - d_j}$
end

$\widetilde{m}_{l+1,1} = x_1 - d_1$

for $j = l + 2 : n$
  $\widetilde{m}_{j,1} = x_1$
end

for $j = 2 : l$
  for $i = 1 : j - 2$
    $\widetilde{m}_{j,i+1} = \widetilde{m}_{j,i} \cdot \frac{(d_{j-i} - d_j)(x_i - d_{j-1})(x_{i+1} - d_{j-i-1})}{(d_{j-i-1} - d_{j-1})(x_i - d_{j-i})(x_{i+1} - d_j)}$
  end
end

if $l > 1$
  $\widetilde{m}_{l+1,2} = \widetilde{m}_{l+1,1} \cdot \frac{x_2 - d_{l-1}}{d_{l-1} - d_l}$
end

for $i = 2 : l - 1$
  $\widetilde{m}_{l+1,i+1} = \widetilde{m}_{l+1,i} \cdot \frac{(x_i - d_l)(x_{i+1} - d_{l-i})}{(d_{l-i} - d_l)(x_i - d_{l-i+1})}$
end

for $j = l + 2 : n$
  for $i = 2 : j - l - 1$
    $\widetilde{m}_{j,i} = x_i$
  end

```
  for $i = j-l : j-1$
    $\widetilde{m}_{j,i} = x_i - d_{j-i}$
  end
end
```

5. Accurate computations with Cauchy-Vandermonde matrices

Three important problems in numerical linear algebra (linear system solving, eigenvalue computation and singular value computation) are considered in this section for the case of a strictly totally positive Cauchy-Vandermonde matrix. The bidiagonal factorization of the Cauchy-Vandermonde matrix (or its inverse) allows us to develop accurate and efficient algorithms for solving each one of these problems.

Let us observe here that one could try to solve these problems by using standard algorithms. However the solution provided by them will generally be less accurate since Cauchy-Vandermonde matrices are ill conditioned [5] and these algorithms can suffer from inaccurate cancellation, since they do not take into account the structure of the matrix, which is crucial in our approach.

5.1. Linear system solving

The fast and accurate solution of structured linear systems is a problem that has been studied in the field of numerical linear algebra for different types of structured matrices (see, for example, [2, 3, 4, 6, 22, 23, 25]). Now we will consider this problem for the case of totally positive Cauchy-Vandermonde matrices.

Let $Ax = b$ be a linear system whose coefficient matrix A is the square Cauchy-Vandermonde matrix of order n for the nodes $\{x_i\}_{1\leq i\leq n}$ and the poles $\{d_j\}_{1\leq j\leq l}$, where $0 < x_1 < x_2 < \ldots < x_n$ and $0 < -d_1 < -d_2 < \ldots < -d_l$. The following algorithm solves $Ax = b$ with a computational cost of $O(n^2)$ arithmetic operations [26].

INPUT: The nodes $\{x_i\}_{1\leq i\leq n}$, the poles $\{d_j\}_{1\leq j\leq l}$ and the data vector $b \in \mathbb{R}^n$.

OUTPUT: The solution vector $x \in \mathbb{R}^n$.

- *Step 1:* Computation of the bidiagonal decomposition of A^{-1}.

- *Step 2:* Computation of

$$x = A^{-1}b = G_1G_2\cdots G_{n-1}D^{-1}F_{n-1}F_{n-2}\cdots F_1b$$

Step 1 is performed by using the implementation in MATLAB of the algorithm presented in Section 4. Step 2 can be carried out by using the algorithm `TNSolve` of P. Koev [19]. Given the bidiagonal factorization of the matrix A, `TNSolve` solves $Ax = b$ by using backward substitution.

The accuracy of the whole algorithm is guaranteed provided that b has alternating sign pattern (see Subsection 3.2 of [21]).

5.2. Eigenvalue computation

Let A be a square Cauchy-Vandermonde matrix of order n for the nodes $\{x_i\}_{1\leq i\leq n}$ and the poles $\{d_j\}_{1\leq j\leq l}$, where $0 < x_1 < x_2 < \ldots < x_n$ and $0 < -d_1 < -d_2 < \ldots < -d_l$. The following algorithm computes accurately the eigenvalues of A.

INPUT: The nodes $\{x_i\}_{1\leq i\leq n}$ and the poles $\{d_j\}_{1\leq j\leq l}$.
OUTPUT: A vector $x \in \mathbb{R}^n$ containing the eigenvalues of A.

- *Step 1:* Computation of the bidiagonal decomposition of A by using the algorithm included in Section 4.

- *Step 2:* Given the result of Step 1, computation of the eigenvalues of A by using the algorithm `TNEigenvalues`.

`TNEigenvalues` is an algorithm of P. Koev [20] which computes accurate eigenvalues of a totally positive matrix starting from its bidiagonal factorization. The computational cost of `TNEigenvalues` is of $O(n^3)$ arithmetic operations (see [20]) and its implementation in MATLAB can be taken from [19]. In this way, as the computational cost of Step 1 is of $O(n^2)$ arithmetic operations, the cost of the whole algorithm is of $O(n^3)$ arithmetic operations.

5.3. Singular values and condition number

Let A be a square Cauchy-Vandermonde matrix of order n for the nodes $\{x_i\}_{1\leq i\leq n}$ and the poles $\{d_j\}_{1\leq j\leq l}$, where $0 < x_1 < x_2 < \ldots < x_n$ and $0 < -d_1 < -d_2 < \ldots < -d_l$. The following algorithm computes accurately the singular values of A.

INPUT: The nodes $\{x_i\}_{1\leq i\leq n}$ and the poles $\{d_j\}_{1\leq j\leq l}$.
OUTPUT: A vector $x \in \mathbb{R}^n$ containing the singular values of A.

- *Step 1:* Computation of the bidiagonal decomposition of A by using the implementation in MATLAB of the algorithm included in Section 4.

- *Step 2:* Given the result of Step 1, computation of the singular values by using `TNSingularValues`.

`TNSingularValues` is an algorithm of P. Koev that computes accurate singular values of a totally positive matrix starting from its bidiagonal factorization [20]. Its computational cost is of $O(n^3)$ and its implementation in MATLAB can be found in [19]. Taking this complexity into account, the computational cost of our algorithm for computing the singular values of a totally positive Cauchy-Vandermonde matrix is of $O(n^3)$ arithmetic operations.

Let us observe that the accurate computation of the singular values of A allows us to compute the *2-norm* condition number of A accurately, just dividing the largest by the smallest singular value of A. An example illustrating this fact is included in Section 8.

6. Error Analysis

In this section the error analysis of the algorithm included in Section 4 for computing the bidiagonal factorization of a square strictly totally positive Cauchy-Vandermonde matrix is carried out.

For our error analysis we use the standard model of floating point arithmetic (see section 2.2 of [16]):

Let x, y be floating point numbers and ϵ be the *machine precision*,

$$fl(x \odot y) = (x \odot y)(1+\delta)^{\pm 1}, \quad \text{where } |\delta| \leq \epsilon, \quad \odot \in \{+, -, \times, /\}.$$

The following theorem shows that our algorithm computes the bidiagonal decomposition of a Cauchy-Vandermonde matrix accurately in floating point arithmetic. The result given in the theorem does not include the case in which $l = 0$, that is, the case in which the given matrix is a Vandermonde matrix.

Theorem 6.1. Let A be a square Cauchy-Vandermonde matrix for the nodes $\{x_i\}_{1\leq i\leq n}$ and the poles $\{d_j\}_{1\leq j\leq l}$, where $0 < x_1 < x_2 < \ldots < x_n$ and $0 < -d_1 < -d_2 < \ldots < -d_l$, $l > 0$. Let $\mathcal{BD}(A) = (b_{i,j})_{1\leq i,j\leq n}$ be the matrix representing the exact bidiagonal decomposition of A and $(\widehat{b}_{i,j})_{1\leq i\leq n}$ be the matrix representing the computed bidiagonal decomposition of A by means of the algorithm presented in Section 4 in floating point arithmetic with machine precision ϵ. Then

$$|\widehat{b}_{i,j} - b_{i,j}| \leq \frac{(16ln - 4n - 12l + 7)\epsilon}{1 - (16ln - 4n - 12l + 7)\epsilon} b_{i,j}, \quad i, j = 1, \ldots, n.$$

Proof. Accumulating the relative errors in the style of Higham (see Chapter 3 of [16], [7], [20] and [23]) in the computation of the $m_{i,j}$ by means of the algorithm included in Section 4 we obtain

$$|\widehat{m}_{i,j} - m_{i,j}| \leq \frac{(16ln - 4n - 12l + 7)\epsilon}{1 - (16ln - 4n - 12l + 7)\epsilon} m_{i,j}, \tag{6.1}$$

for $j = 1, \ldots, n-1$ and $i = j+1, \ldots, n$, where $\widehat{m}_{i,j}$ are the multipliers $m_{i,j}$ computed in floating point arithmetic. Proceeding in the same way for the computation of the $\widetilde{m}_{i,j}$ we derive

$$|\widehat{\widetilde{m}}_{i,j} - \widetilde{m}_{i,j}| \leq \begin{cases} \frac{(12l-21)\epsilon}{1-(12l-21)\epsilon} \widetilde{m}_{i,j}, & l \geq 3, \\ \\ \frac{5\epsilon}{1-5\epsilon} \widetilde{m}_{i,j}, & l = 2, \\ \\ \frac{\epsilon}{1-\epsilon} \widetilde{m}_{i,j}, & l = 1, \end{cases} \tag{6.2}$$

for $j = 1, \ldots, n-1$ and $i = j+1, \ldots, n$, where $\widehat{\widetilde{m}}_{i,j}$ are the multipliers $\widetilde{m}_{i,j}$ computed in floating point arithmetic. Analogously

$$|\widehat{p}_{i,i} - p_{i,i}| \leq \frac{(8ln - 2n - 2l + 3)\epsilon}{1 - (8ln - 2n - 2l + 3)\epsilon} p_{i,i}, \quad i = 1, \ldots, n, \tag{6.3}$$

where $\widehat{p}_{i,i}$ are the diagonal pivots $p_{i,i}$ computed in floating point arithmetic. Therefore, looking at the inequalities given by Eq.(6.1), Eq.(6.2) and Eq.(6.3) and taking into account that $\widehat{m}_{i,j}$, $\widehat{\widetilde{m}}_{i,j}$ and $\widehat{p}_{i,i}$ are the entries of $(\widehat{b}_{i,j})_{1 \leq i,j \leq n}$, we conclude that

$$|\widehat{b}_{i,j} - b_{i,j}| \leq \frac{(16ln - 4n - 12l + 7)\epsilon}{1 - (16ln - 4n - 12l + 7)\epsilon} b_{i,j}, \quad i, j = 1, \ldots, n. \quad \Box$$

7. Perturbation theory

In Section 7 of [20] it is proved that if a totally positive matrix A is represented as a product of nonnegative bidiagonal matrices, then small relative perturbations in the entries of the bidiagonal factors cause only small relative perturbations in the eigenvalues and singular values of A. More precisely (see Corollary 7.3 in [20]), $\mathcal{BD}(A)$ determines the eigenvalues and the singular values of A accurately, and the appropriate structured condition number of each eigenvalue and/or singular value with respect to perturbations in $\mathcal{BD}(A)$ is at most $2n^2$.

These results make clear the importance, in the context of our work, of the study of the sensitivity of the $\mathcal{BD}(A)$ of a Cauchy-Vandermonde matrix with respect to perturbations in the nodes x_i. In this section we prove that small relative perturbations in the nodes of a Cauchy-Vandermonde matrix A produce only small relative perturbations in its bidiagonal factorization $\mathcal{BD}(A)$ provided that the relative gaps are not too small.

We begin by defining the quantities which lead to the finding of an appropriate condition number, in a similar way to the work carried out in [7, 20, 23, 24].

Definition 7.1. Let A be a square strictly totally positive Cauchy-Vandermonde matrix for the nodes $\{x_i\}_{1\le i\le n}$ and the poles $\{d_j\}_{1\le j\le l}$, and let $x_i' = x_i(1+\delta_i)$ be the perturbed nodes for $1 \le i \le n$, where $|\delta_i| \ll 1$. We define:

$$rel\_gap_x \equiv \min_{i\neq j} \frac{|x_i - x_j|}{|x_i| + |x_j|},$$

$$rel\_gap_{xd} \equiv \min_{i,j} \frac{|x_i - d_j|}{|x_i|},$$

$$\theta \equiv \max_i \frac{|x_i - x_i'|}{|x_i|} = \max_i |\delta_i|,$$

$$\alpha \equiv \min\{rel\_gap_x, rel\_gap_{xd}\},$$

$$\kappa_{CV} \equiv \frac{1}{\alpha},$$

where $\theta \ll rel\_gap_x, rel\_gap_{xd}$.

The following proposition will be useful in proving Theorem 7.3.

Proposition 7.2. Let $A = (a_{i,j})_{1\le i,j\le n}$ be a Cauchy-Vandermonde matrix for the basis $\mathcal{B}$ whose nodes satisfy $0 < x_1 < x_2 < \ldots < x_n$ and whose poles satisfy $0 < -d_1 < -d_2 < \ldots < -d_l$. The multipliers $m_{i,j}$ of the Neville elimination of A are

$$m_{i,j} = \begin{cases} \frac{\prod_{k=i-j+1}^{i-1}(x_i-x_k)\prod_{r=1}^{j-1}(x_{i-1}-d_r)}{\prod_{k=i-j}^{i-2}(x_{i-1}-x_k)\prod_{r=1}^{j-1}(x_i-d_r)} \cdot \frac{x_{i-j}-d_j}{x_i-d_j}, & j=1,\ldots,l;\ i=j+1,\ldots,n, \\ \\ \frac{\prod_{k=i-l}^{i-1}(x_i-x_k)\prod_{r=1}^{l}(x_{i-1}-d_r)}{\prod_{k=i-l-1}^{i-2}(x_{i-1}-x_k)\prod_{r=1}^{l}(x_i-d_r)}, & j=l+1;\ i=l+2,\ldots,n, \\ \\ \frac{\prod_{k=i-j+1}^{i-1}(x_i-x_k)\prod_{r=1}^{l}(x_{i-1}-d_r)}{\prod_{k=i-j}^{i-2}(x_{i-1}-x_k)\prod_{r=1}^{l}(x_i-d_r)}, & j=l+2,\ldots,n;\ i=j+1,\ldots,n, \end{cases}$$

and the pivots $p_{i,i}$ of the Neville elimination of A are:

$$p_{i,i} = \begin{cases} \frac{\prod_{k=1}^{i-1}(x_i-x_k)\prod_{r=1}^{i-1}(-d_i+d_r)}{\prod_{r=1}^{i-1}(x_i-d_r)\prod_{k=1}^{i}(x_k-d_i)}, & i=1,\ldots,l, \\ \\ \frac{\prod_{k=1}^{i-1}(x_i-x_k)}{\prod_{r=1}^{l}(x_i-d_r)}, & i=l+1,\ldots,n. \end{cases}$$

Proof. The expressions for the m_{ij} are easily obtained by using Eq.(3.1) and Eq.(3.2). The expressions for the $p_{i,i}$ follow from Eq.(3.2). $\square$

The next theorem is the main result of this section. It shows that, when $1 \le l < n$, small relative perturbations in the nodes of a Cauchy-Vandermonde matrix A produce only small relative perturbations in its bidiagonal factorization $\mathcal{BD}(A)$ provided that the relative gaps are not too small. The case $l = 0$ is the Vandermonde case (see [2, 15]). The case $l = n$ is the Cauchy case (see [3]).

Theorem 7.3. Let A and A' be strictly totally positive Cauchy-Vandermonde matrices for the poles $\{d_j\}_{1\le j\le l}$ $(1 \le l < n)$ and the nodes $\{x_i\}_{1\le i\le n}$ and $x'_i = x_i(1+\delta_i)$ for $i = 1,\ldots,n$, where $|\delta_i| \le \theta \ll 1$. Let $\mathcal{BD}(A)$ and $\mathcal{BD}(A')$ be the matrices representing the bidiagonal decomposition of A and the bidiagonal decomposition of A', respectively. Then

$$|(\mathcal{BD}(A'))_{i,j} - (\mathcal{BD}(A))_{i,j}| \le \begin{cases} \frac{(2l+2n-2)\kappa_{CV}\theta}{1-(2l+2n-2)\kappa_{CV}\theta}(\mathcal{BD}(A))_{i,j}, & l \le \frac{2n+1}{3}, \\ \\ \frac{(5l-3)\kappa_{CV}\theta}{1-(5l-3)\kappa_{CV}\theta}(\mathcal{BD}(A))_{i,j}, & l > \frac{2n+1}{3}. \end{cases}$$

Proof. Taking into account that $|\delta_i| \le \theta$, it can easily be shown that

$$x'_i - d_j = (x_i - d_j)(1+\delta'_{i,j}), \quad |\delta'_{i,j}| \le \frac{\theta}{rel\_gap_{xd}} \tag{7.1}$$

and

$$x'_i - x'_j = (x_i - x_j)(1 + \delta_{i,j}), \quad |\delta_{i,j}| \leq \frac{\theta}{rel\_gap_x}. \tag{7.2}$$

Accumulating the perturbations in the style of Higham (see Chapter 3 of [16], [7], [20] and [23]) using the formula for the $m_{i,j}$ in Proposition 7.2, and Eq.(7.1) and Eq.(7.2) we obtain

$$m'_{i,j} = m_{i,j}(1 + \bar{\delta}), \quad |\bar{\delta}| \leq \frac{(2l + 2n - 2)\kappa_{CV}\theta}{1 - (2l + 2n - 2)\kappa_{CV}\theta},$$

where $m'_{i,j}$ are the entries of $\mathcal{BD}(A')$ below the main diagonal. Proceeding in the same way by using the formula in Proposition 4.1 for the $\widetilde{m}_{i,j}$ we get

$$\widetilde{m}'_{i,j} = \widetilde{m}_{i,j}(1 + \bar{\delta}), \quad |\bar{\delta}| \leq \frac{2l\frac{\theta}{rel\_gap_{xd}}}{1 - 2l\frac{\theta}{rel\_gap_{xd}}},$$

where $\widetilde{m}'_{i,j}$ are the entries of $\mathcal{BD}(A')$ above the main diagonal. Analogously, and using in this case the formula for the $p_{i,i}$ in Proposition 7.2, we get

$$p'_{i,i} = p_{i,i}(1 + \bar{\delta}), \quad |\bar{\delta}| \leq \begin{cases} \frac{(5l-3)\kappa_{CV}\theta}{1-(5l-3)\kappa_{CV}\theta}, & i = 1, \ldots, l, \\ \\ \frac{(n+l-1)\kappa_{CV}\theta}{1-(n+l-1)\kappa_{CV}\theta}, & i = l+1, \ldots, n. \end{cases},$$

where $p'_{i,i}$ are the diagonal elements of $\mathcal{BD}(A')$. Finally, considering the last three inequalities we conclude that

$$|(\mathcal{BD}(A'))_{i,j} - (\mathcal{BD}(A))_{i,j}| \leq \begin{cases} \frac{(2l+2n-2)\kappa_{CV}\theta}{1-(2l+2n-2)\kappa_{CV}\theta}(\mathcal{BD}(A))_{i,j}, & l \leq \frac{2n+1}{3}, \\ \\ \frac{(5l-3)\kappa_{CV}\theta}{1-(5l-3)\kappa_{CV}\theta}(\mathcal{BD}(A))_{i,j}, & l > \frac{2n+1}{3}. \end{cases}$$

□

So, we see that the quantity $(2l+2n-2)\kappa_{CV}$ in the case in which $l \leq \frac{2n+1}{3}$, and the quantity $(5l-3)\kappa_{CV}$ in the case in which $l > \frac{2n+1}{3}$, is an appropriate structured condition number of A with respect to the perturbations in the data x_i. This result is analogous to the results of [7, 20] in the sense that the relevant quantities for the determination of a structured condition number are the *relative* separations between the nodes. In this case important quantities for the determination of a structured condition number are also the relative distances between the nodes and the poles.

Combining this theorem with Corollary 7.3 in [20], which states that small componentwise relative perturbations of $\mathcal{BD}(A)$ cause only small relative perturbation in the eigenvalues λ_i and singular values σ_i of A, we obtain that

$$|\lambda_i' - \lambda_i| \leq O(n^3 \kappa_{CV} \theta)\lambda_i \quad \text{and} \quad |\sigma_i' - \sigma_i| \leq O(n^3 \kappa_{CV} \theta)\sigma_i,$$

where λ_i' and σ_i' are the eigenvalues and the singular values of A'. That is to say, small relative perturbation in the nodes of a Cauchy-Vandermonde matrix A produce only small relative perturbations in its eigenvalues and in its singular values.

A formula for the perturbation of the solution of linear systems $Ax = b$ associated to generalized Vandermonde matrices starting from its $\mathcal{BD}(A)$, can be seen in Proposition 7.2 of [6].

8. Numerical experiments

Several numerical experiments showing the good performance of our algorithms are included in this section. The first example illustrates how the problems of linear system solving (in the case in which the data vector has alternating sign pattern), eigenvalue computation and condition number estimation of a given strictly totally positive Cauchy-Vandermonde matrix are all solved with high relative accuracy. In the second example we apply our algorithm for solving Cauchy-Vandermonde linear systems to the computation of a definite integral of a rational function by means of the method of partial fraction expansion. Finally, the third example is devoted to the computation of a definite integral by using a interpolatory quadrature formula. In this numerical experiment the solution of a linear system whose coefficient matrix is the transpose of a Cauchy-Vandermonde matrix is involved.

Example 8.1. Let A be the square Cauchy-Vandermonde matrix of order 12 for the nodes

$$1 < 2 < \frac{5}{2} < 4 < \frac{17}{4} < 5 < \frac{11}{2} < 6 < \frac{27}{4} < 7 < 8 < 9$$

and the poles

$$d_1 = -1, d_2 = -2, d_3 = -3, d_4 = -4, d_5 = -5.$$

We start by estimating the condition number with respect to the 2-norm of A, which is

$$\kappa_2(A) = \frac{\sigma_{max}}{\sigma_{min}},$$

where σ_{max} and σ_{min} are respectively the largest and the smallest singular value of A.

In Table 1 we present the condition number $\kappa_2(A)$ obtained in *Maple* by dividing the largest by the smallest singular value of A (computed by using 50-digit arithmetic) and the relative errors obtained when computing it by means of:

1. The ratio of the largest by the smallest singular value of A obtained by means of the algorithm presented in Section 5.3 (column labeled by `MMP`).
2. The command `cond(A,2)` from MATLAB.

$\kappa_2(A)$	`MMP`	`cond(A,2)`
$5.8038e+17$	$2.0e-15$	$7.1e-02$

Table 1: Condition number of the Cauchy-Vandermonde matrix of Example 8.1.

The relative error is obtained by using the value $\kappa_2(A)$ in the first column of Table 8.1.

As the results of the experiment shows, our algorithm is adequate for estimating the condition number of a strictly totally positive Cauchy-Vandermonde matrix with respect to the 2-norm .

We have also checked a similar example with a Cauchy-Vandermonde matrix of order 25. We must observe that even for this moderate value of n the condition number of the matrix increases to $\kappa_2(A) = 4.3e + 39$, and nevertheless the relative errors are similar to those obtained for the matrix of order 12.

Let us consider now the vector with alternating sign pattern

$$b^T = (10, -4, 2, -1, 3, -5, 4, -7, 5, -2, 6, -3).$$

Our aim is solving the Cauchy-Vandermonde linear system $Ax = b$.

In Table 2 we show the results obtained when solving this linear system by using

1. The algorithm presented in Section 5.1 (column labeled by `MMP`).
2. The command $A\backslash b$ from MATLAB.

We compute the relative error of a solution x of the linear system $Ax = b$ by means of the formula

$$err = \frac{\| x - x_e \|_2}{\| x_e \|_2},$$

where x_e is the exact solution of the linear system computed in *Maple*.

MMP	$A\backslash b$
$5.2e-16$	$3.1e-06$

Table 2: Solution of $A\backslash b$ in Example 8.1.

Finally, we include in Table 3 the eigenvalues λ_i of A and the relative errors obtained when computing them by means of

- The algorithm presented in Section 5.2 (column labeled by `MMP`).
- The command `eig` from MATLAB.

The relative error of each eigenvalue is computed by using the eigenvalues calculated in *Maple* with 50-digit arithmetic.

The results appearing in Table 3 show that, while the command `eig` from MATLAB only computes the largest eigenvalues with high relative accuracy, our algorithm computes all the eigenvalues with high relative accuracy. In particular, the smallest eigenvalue is computed by `eig` from MATLAB with a relative error of $9.4e-02$, while using our approach it is computed with a relative error of $2.6e-16$.

We have also checked a similar example with the same Cauchy-Vandermonde matrix of order 25 and condition number $\kappa_2(A) = 4.3e+39$. The relative errors in the computation of eigenvalues are similar to those obtained for the matrix of order 12.

Example 8.2. In this example we will compute a definite integral of a rational function $f(x)$ by the standard method of computing the partial fraction expansion of $f(x)$. The coefficients of the partial fraction expansion will be calculated by solving a rational interpolation problem whose corresponding linear system is a Cauchy-Vandermonde linear system.

λ_i	MMP	eig
$5.6e+05$	$2.1e-16$	$2.1e-16$
$4.5e+03$	$1.4e-15$	$2.6e-15$
$8.8e+01$	$3.2e-16$	$5.8e-14$
$2.5e+00$	$3.5e-16$	$2.4e-13$
$6.5e-01$	$1.7e-16$	$5.0e-12$
$5.5e-02$	$7.6e-16$	$2.0e-10$
$6.7e-03$	$3.9e-16$	$3.2e-09$
$2.8e-04$	$1.9e-16$	$4.1e-08$
$1.2e-05$	0	$1.2e-06$
$2.6e-07$	$1.0e-15$	$7.2e-05$
$2.9e-09$	$5.7e-16$	$2.5e-03$
$6.1e-12$	$2.6e-16$	$9.4e-02$

Table 3: Eigenvalues of the Cauchy-Vandermonde matrix of Example 8.1.

Let us consider

$$f(x) = \frac{p(x)}{(x+\frac{1}{2})(x+\frac{3}{4})(x+1)(x+2)},$$

where $p(x)$ is the monic Legendre orthogonal polynomial of degree 12 on $[0,4]$. This rational function $f(x)$ has the simple poles $d_1 = -\frac{1}{2}, d_2 = -\frac{3}{4}, d_3 = -1$ and $d_4 = -2$. We are interested in computing the integral

$$\int_0^1 f(x)dx.$$

Taking into account the degree of $p(x)$, the basis of the interpolation space will be

$$\left\{\frac{1}{x+\frac{1}{2}}, \frac{1}{x+\frac{3}{4}}, \frac{1}{x+1}, \frac{1}{x+2}, 1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\right\}.$$

If $\{c_i\}_{1\le i\le 13}$ are the coefficients of the interpolating function (i.e. the partial fraction expansion of $f(x)$), then the integral has the value

$$c_1 \log 3 + c_2 \log \frac{7}{3} + c_3 \log 2 + c_4 \log \frac{3}{2} + \sum_{i=5}^{13} c_i \frac{1}{i-4}.$$

Since we can freely choose the interpolation nodes $\{x_i\}$ for obtaining the interpolation data $f(x_i)$, we choose those nodes positive, and therefore (taking into account that the poles are $d_1 = -\frac{1}{2}, d_2 = -\frac{3}{4}, d_3 = -1$ and $d_4 = -2$) the corresponding Cauchy-Vandermonde matrix will be strictly totally positive.

In addition, if the vector containing the interpolation data has an alternating sign pattern, we will have high relative accuracy in the solution vector. We get this sign pattern by using the twelve roots of the orthogonal polynomial $p(x)$ and selecting $\{x_i\}$ adequately. For instance, we have chosen the nodes $\{x_i\}$:

$$\frac{1}{50}, \frac{4}{25}, \frac{6}{25}, \frac{16}{25}, 1, \frac{8}{5}, 2, \frac{12}{5}, 3, \frac{16}{5}, \frac{18}{5}, \frac{98}{25}, \frac{199}{50}.$$

In this situation the condition number of the Cauchy-Vandermode matrix A of the linear system corresponding to the interpolation problem is $\kappa_2(A) = 1.2e + 13$.

The solution of this linear system by using our algorithm is the vector c containing the computed coefficients of the partial fraction expansion of $f(x)$. If c_e is the vector containing the exact rational coefficients of the partial fraction expansion of $f(x)$, the relative error using the 2-norm is

$$\frac{\| c - c_e \|_2}{\| c_e \|_2} = 2.4e - 16,$$

while solving the linear system by means of the command $A \backslash b$ of MATLAB (considering the exact matrix A computed by *Maple*) the relative error is $6.1e - 09$.

It must be observed that when the vector b has alternating sign pattern, the second stage of our algorithm for solving Cauchy-Vandermonde linear systems, that is, the computation of the product

$$G_2 \cdots G_{n-1} D^{-1} F_{n-1} F_{n-2} \cdots F_1 b,$$

can be done with high relative accuracy. This is a consequence of the checkerboard sign pattern of A^{-1}, which derives from the fact that A is a strictly totally positive matrix. This important property was already observed in an analogous situation in the paper [15], devoted to the error analysis of the Björck-Pereyra algorithm for Vandermonde systems.

By using these computed coefficients c we obtain the following value for the integral:

$$v = -3.121549073642774e - 03.$$

The relative error when comparing with the value v_e computed in *Maple* with 50-digit arithmetic is

$$\frac{|v - v_e|}{|v_e|} = 5.2e - 08,$$

while using the coefficients computed by solving the system with the command $A\backslash b$ of MATLAB the relative error is $1.9e - 07$.

Let us point out that even using the exact partial fraction expansion there is a problem of cancellation in the computation of this definite integral, a problem so important that even using the exact coefficients of the partial fraction expansion the relative error in this computation is $5.5e - 08$.

Example 8.3. In this example we present an application of the solution of the dual linear system $A^T x = b$, where A is a Cauchy-Vandermonde matrix and b does not present the alternating sign pattern of the previous examples. This will imply less accuracy than in the previous examples in the computation of the solution of the linear system (see Eq.(8.1)). However, the relative error in the application of the formula for computing the integral happens to be smaller.

A key fact is the following result (see [20]):

$$\mathcal{BD}(A^T) = (\mathcal{BD}(A))^T.$$

Our aim is to approximate the definite integral

$$\int_0^1 \frac{e^t}{t+1} dx$$

by using an interpolatory quadrature formula that extends the Fejér's first rule to the rational case [31]. Fejér's first rule uses the zeros of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$ of the first kind in in $(-1, 1)$, which are

$$y_i = \cos\theta_k, \quad \theta_i = \frac{(2i-1)\pi}{2n}, \quad i = 1, \dots, n.$$

In this case we consider $n = 13$, and the nodes $\{x_i\}_{1 \le i \le 13}$ are $\{y_i\}_{1 \le i \le 13}$ shifted to the interval $(0, 1)$ by using

$$x_i = \frac{y_i + 1}{2} \in (0, 1).$$

Requiring, in the rational case, exactness for the functions

$$\frac{1}{x+1}, 1, x, x^2, \dots, x^{11},$$

the weights $\{w_i\}_{1\leq i\leq 13}$ of the quadrature formula are computed by solving a linear system whose coefficient matrix is the transpose of the Cauchy-Vandermonde matrix A for the nodes $\{x_i\}_{1\leq i\leq 13}$ and the pole $d_1 = -1$. The condition number of A is $\kappa_2 = 7.8e + 09$.

If w_e is the vector containing the exact weights $\{w_i\}$ computed in *Maple* and w is the vector containing the weights computed by solving the system $A^T x = b$ by using our approach, the relative error obtained in the computation of w is:

$$\frac{\| w - w_e \|_2}{\| w_e \|_2} = 8.5e - 08. \tag{8.1}$$

The relative error obtained in the evaluation of the integral when using the weights in w is $2.0e - 16$. In its computation we have used the exact value of the definite integral computed in *Maple*.

Let us observe that, if we proceed in the same way by using the Fejér's first rule in the polynomial case instead of the rational case, that is, requiring the exactness of the quadrature formula for the functions

$$1, x, x^2, \dots, x^{12},$$

the relative error obtained in the approximation of the definite integral is $1.1e - 13$.

References

[1] P. Alonso, M. Gasca, J. M. Peña, Backward error analysis of Neville elimination, Appl. Numer. Math. 23 (1997) 193–204.

[2] A. Björck, V. Pereyra, Solution of Vandermonde systems of equations, Math. Comput. 24 (1970) 893–906.

[3] T. Boros, T. Kailath, V. Olshevsky, A fast parallel Björck-Pereyra-type algorithm for solving Cauchy linear equatios, Linear Algebra Appl. 302/303 (1999) 265-293.

[4] J. Delgado, J. M. Peña, Accurate computations with collocation matrices of rational bases, Appl. Math. Comput. 219 (2013) 4354-4364.

[5] J. Demmel, I. Dumitriu, O. Holtz and P. Koev, Accurate and efficient expression evaluation and linear algebra, Acta Numer. 17 (2008) 87–145.

[6] J. Demmel, P. Koev, The accurate and efficient solution of a totally positive generalized Vandermonde linear system, SIAM J. Matrix Anal. Appl. 27(1) (2005) 142–152.

[7] J. Demmel, P. Koev, Accurate SVDs of polynomial Vandermonde matrices involving orthonormal polynomials, Linear Algebra Appl. 417 (2006) 382–396.

[8] S. M. Fallat, C. R. Johnson, Totally Nonnegative Matrices, Princeton University Press, Princeton, NJ, 2011.

[9] T. Finck, G. Heinig, K. Rost, An inversion formula and fast algorithms for Cauchy-Vandermonde matrices, Linear Algebra Appl. 183 (1993) 179-191.

[10] M. Gasca, J. M. Peña, Total positivity and Neville elimination, Linear Algebra Appl. 165 (1992) 25–44.

[11] M. Gasca, J. M. Peña, A matricial description of Neville elimination with applications to total positivity, Linear Algebra Appl. 202 (1994) 33–45.

[12] M. Gasca, J. M. Peña, *On Factorizations of Totally Positive Matrices*, in: M. Gasca and C. A. Michelli (Eds.), Total Positivity and Its Applications, Kluwer Academic Publishers, Dordrecht, 1996, pp. 109–130.

[13] G. H. Golub, C. F. Van Loan, Matrix Computations, fourth ed., Johns Hopkins University Press, Baltimore, 2013.

[14] B. Heiligers, Totally positive regression: E-optimal designs, Metrika 54 (2001), 191–213.

[15] N. J. Higham, Error analysis of the Björck-Pereyra algorithms for solving Vandermonde systems, Numer. Math. 50 (1987), 613–632.

[16] N. J. Higham, Accuracy and Stability of Numerical Algorithms, 2nd ed., SIAM, Philadelphia, 2002.

[17] L. A. Imhof, W. J. Studden, E-Optimal designs for rational models, The Annals of Statistics 29 (2001), 763–783.

[18] P. Junghanns and D. Oestreich, Numerische Lösung des Staudammproblems mit Drainage. Z. Angew. Math. Mech. 69 (1989) 83–92.

[19] P. Koev, `http://www.math.sjsu.edu/~koev`

[20] P. Koev, Accurate eigenvalues and SVDs of totally nonnegative matrices, SIAM J. Matrix Anal. Appl. 21 (2005), 1–23.

[21] P. Koev, Accurate computations with totally nonnegative matrices, SIAM J. Matrix Anal. Appl. 29 (2007) 731–751.

[22] A. Marco, J.-J. Martínez, Accurate computations with Said-Ball-Vandermonde matrices, Linear Algebra Appl. 432 (2010) 2894–2908.

[23] A. Marco, J.-J. Martínez, Accurate computations with totally positive Bernstein–Vandermonde matrices, Electron. J. Linear Algebra 26 (2013) 357–380.

[24] A. Marco, J.-J. Martínez, Bidiagonal decomposition of rectangular totally positive Said-Ball-Vandermonde matrices: Error analysis, perturbation theory and applications, Linear Algebra Appl. 495 (2016) 90–107.

[25] J. J. Martínez, J. M. Peña, Fast algorithms of Björck-Pereyra type for solving Cauchy-Vandermonde systems, Appl. Numer. Math. 26 (1998) 343-352.

[26] J. J. Martínez, J. M. Peña, Factorizations of Cauchy-Vandermonde matrices, Linear Algebra Appl. 284 (1998) 229-237.

[27] J. J. Martínez, J. M. Peña, Factorizations of Cauchy-Vandermonde matrices with one multiple pole, in: O. Pordavi (Ed.), Recent Research on Pure and Applied Algebra, Nova Sci., Hauppage, NY, 2003, pp. 85–95.

[28] G. Mühlbach, Interpolation by Cauchy-Vandermonde systems and applications, Numerical Analysis 2000, Vol. II: Interpolation and extrapolation. J. Comput. Appl. Math. 122 (2000), no. 1-2, 203–222.

[29] A. Pinkus, Totally Positive Matrices, Cambridge University Press, Cambridge, UK, 2010.

[30] A. Ribalta, State space realizations of rational interpolants with prescribed poles, Systems Control Lett. 43 (2001) 379–386.

[31] J. A. C. Weideman, D. P. Laurie, Quadrature rules based on partial fraction expansions, Numer. Algor. 24 (2000) 159–178.

[32] W. Xiang, L. Linzhang, Factorizations of confluent Cauchy-Vandermonde matrices, Appl. Math. Comput. 182 (2006), no. 2, 1667-1672.