Nonlinear Cournot and Bertrand-type dynamic triopoly with differentiated products and heterogeneous expectations

J. Andaluz [1,*], A.A. Elsadany [2], G. Jarne [1]

[1] Department of Economic Analysis, University of Zaragoza, Zaragoza, Spain

[2] Department of Basic Science, Faculty of Computers and Informatics, Ismailia 41522, Suez Canal University, Egypt

Abstract:

In a differentiated triopoly model with heterogeneous firms, the local stability of the Nash equilibrium under both quantity and price competition is analyzed. We find that the presence of a firm following a gradient rule based on marginal profits, and a player with adaptive expectations, determines the local stability of the Nash equilibrium, regardless the competition type, while the effects of the degree of product differentiation on the stability depend on the nature of products. Moreover, the Nash equilibrium is more stable under quantity competition than under price competition.

Keywords: Bertrand competition; Cournot competition; heterogeneous expectations; Stability; Bifurcation.

* Corresponding author. Department of Economic Analysis, University of Zaragoza, Gran Via, 2, 50005 Zaragoza, Spain
E-mail: jandaluz@unizar.es
Telephone: +34976762631, Fax: +34976761966

1. Introduction

Two classical models in the theory of oligopoly are those of Cournot [13] and Bertrand [7]. In Cournot's model the firms choose to compete on output quantity, and in Bertrand's model they choose to compete on price. Both models can be interpreted as static games where decisions are made simultaneously and where each firm maximizes its own profit, in a context of perfect and complete information[1].

However, there is a growing interest in analyzing Cournot and Bertrand competition in a dynamic setting. Assuming the discrete time scale, the properties of the resulting dynamic process are given by the way that the firms adjust their quantity or price levels that, in turn, depends on the formation of their expectations.

The naïve, or Cournot, expectations assume that firms use the latest available information. Thus, each firm expects its rivals to offer the same quantity or price in the current period as they did in the previous period and there is no retaliation. In this setting, it is concluded that for the duopoly case, the equilibrium is globally stable as it is deduced in the static context. However, under this expectation rule, in a seminal paper, Theocharis [24] showed that in a Cournot oligopoly with linear demand function and constant marginal costs, an increasing in the number of competitors plays a destabilizing role. Particularly, if the number of firms is higher than three, the Cournot-Nash equilibrium becomes unstable. This paradoxical result has been generalized by varying different assumptions, for example, by considering different shapes of the demand function (see [4], [21]), of the cost function [19], or both [28].

The burgeoning interest in nonlinear dynamic oligopolies has renewed the use of decisional mechanism. In this line, more realistic expectation rules than naïve expectations have been proposed. It is the case of gradient rule based on marginal profits (see [8], [23], [5]), and the adaptive expectations principle [9].

In the literature, most papers focus on games with homogeneous players, that is, firms that adopt the same expectation rule. However, it may be more realistic to assume that firms have heterogeneous expectations. Belonging to this group, we can cite [1], [2], and [26], who analyzed a duopoly model with homogeneous product, and assuming heterogeneous expectations. More recently, Fanti and Gori [16] analyze the dynamics of a horizontally[2] differentiated duopoly under Cournot competition, with heterogeneous players.

The analysis of oligopolies with more than two firms has been less addressed in the literature. In this line, [20], [3], and [11] are examples of homogeneous triopolies. The research on dynamics in games with more than two heterogeneous players is still poor. As exceptions, we can cite Elsadany et al [15], who consider an oligopoly game with four heterogeneous firms producing perfect substitute goods, and show that the stability of the Cournot-Nash equilibrium depends on the speed of adjustment of the gradient firm, and the player with adaptive expectations has a stabilizing effect on the game. On the other hand, in an oligopoly model with isoelastic inverse demand function and constant marginal costs, Tramontana et al [27] show that the stability region on the parametric space may enlarge by increasing the number of heterogeneous competitors.

The case of three heterogeneous firms producing differentiated goods is considered by Andaluz and Jarne [6]. These authors study a linear dynamic system of a differentiated triopoly under quantity and price competition, and assuming that two firms follow a gradient rule based on marginal profits and a firm revises its beliefs according to naïve expectations. It is showed that in the presence of both complement and substitute goods, the stability of the Nash equilibrium increases when goods tend to be independents. Moreover, the Nash equilibrium is more stable under quantity competition than under price competition.

The present article constitutes an extension of the model proposed by [6]. Firstly, we analyze the local stability of the Nash equilibrium, under both quantity and price competition, but assuming a nonlinear dynamic system of a differentiated triopoly. Secondly, we introduce more heterogeneity among players. Thus, we assume gradient rule, naïve expectations, and adaptive expectations hypothesis for the three firms competing in the market, respectively.

We conclude that there is a critical value of the adjustment speed of the gradient player, for which the Nash

[1] Otherwise, the Cournot and Bertrand models can be interpreted as conjectural variation models (see [10]). In Cournot's original model, each firm's conjecture is that the other firms are satisfied to continue selling their current quantity of output. However, from a purely game-theory perspective, the conjectural variations approach is theoretically unsatisfactory (see [25]).

[2] Horizontal product differentiation has been developed through the non-address approach, dating back to [10] and [12], and the address models, dating back to [18].

equilibrium loses its stability. This threshold is higher under Cournot competition than under Bertrand competition, and therefore, the Cournot-Nash equilibrium is more stable than the Bertrand-Nash equilibrium, regardless of the nature of products.

On the other hand, both under Bertrand and Cournot competition, it is deduced that, in the presence of complement goods, a lower degree of product differentiation (goods tend to be independent) stabilizes the Nash equilibrium. By contrast, assuming substitutes goods, there exists a level of product differentiation that ensures the maximal stability of the Nash equilibrium.

The remainder of the paper is organized as follows. Section 2 presents the model that is developed for this study. Section 3 develops the dynamics under Cournot competition. The dynamics under price competition is analyzed in Section 4. Section 5 closes the paper with the main conclusions.

2. The model

We extend the model formulated by [22] for the triopoly case. Specifically, we consider an economy with a monopolistic sector of three firms, each producing a differentiated good, and a competitive numeraire sector. There is a continuum of identical consumers with a utility function separable and linear in the numeraire good.

Denoting as q_i the quantity of good i, and p_i its price, the representative consumer maximizes the following utility function with respect to the quantities:

$$V(q_1,q_2,q_3)=u(q_1,q_2,q_3)-\sum_{i=1}^{3} p_i q_i \tag{1}$$

Function $u(q_1,q_2,q_3)$ is assumed to be quadratic and must be strictly concave. Particularly, we adopt the following specification:

$$u(q_1,q_2,q_3)=\alpha_1 q_1+\alpha_2 q_2+\alpha_3 q_3-\frac{1}{2}\left(\beta_1 q_1^2+\beta_2 q_2^2+\beta_3 q_3^2+2\gamma_{12}q_1q_2+2\gamma_{13}q_1q_3+2\gamma_{23}q_2q_3\right) \tag{2}$$

For the sake of simplicity, we assume that $\alpha_1=\alpha_2=\alpha_3=a>0$, $\beta_1=\beta_2=\beta_3=1$, and $\gamma_{12}=\gamma_{13}=\gamma_{23}=d$, with

$-\frac{1}{2} < d < 1$, in order to ensure the strict concavity of (2).

Maximizing (1), given (2), we deduce the inverse demand in the region of quantities where prices are positive:

$$\left.\begin{array}{l} p_1 = a - q_1 - dq_2 - dq_3 \\ p_2 = a - q_2 - dq_1 - dq_3 \\ p_3 = a - q_3 - dq_1 - dq_2 \end{array}\right\} \qquad (3)$$

From (3), we can obtain the corresponding direct demands as:

$$\left.\begin{array}{l} q_1 = \frac{a}{1+2d} - \frac{(1+d)p_1}{(1-d)(1+2d)} + \frac{dp_2}{(1-d)(1+2d)} + \frac{dp_3}{(1-d)(1+2d)} \\ q_2 = \frac{a}{1+2d} - \frac{(1+d)p_2}{(1-d)(1+2d)} + \frac{dp_1}{(1-d)(1+2d)} + \frac{dp_3}{(1-d)(1+2d)} \\ q_3 = \frac{a}{1+2d} - \frac{(1+d)p_3}{(1-d)(1+2d)} + \frac{dp_1}{(1-d)(1+2d)} + \frac{dp_2}{(1-d)(1+2d)} \end{array}\right\} \qquad (4)$$

From (4), it is deduced that, when $d = 0$, goods are independent and each firm is monopolist in the market for its variety. If $-\frac{1}{2} < d < 0$, goods are complements, and if instead $0 < d < 1$, goods are substitutes, being perfect substitutes for the limit case $d = 1$.

Firms have identical and constant marginal costs, $c > 0$. Therefore, the profits of firm i are given by $\Pi_i = (p_i - c)q_i$, $i = 1, 2, 3$.

3. Dynamics under Cournot competition

We consider a dynamic triopoly where firms compete on quantity. In this setting, the quantity demanded to each firm is given by (3) and the profit function of firm i is:

$$\Pi_i = p_i q_i - cq_i = \left(a - q_i - d \sum_{j=1, j\neq i}^{3} q_j - c \right) q_i, \quad i = 1,2,3 \qquad (5)$$

The marginal profit of the ith firm is:

$$\Phi_i = \frac{\partial \Pi_i}{\partial q_i} = a - c - 2q_i - d \sum_{j=1,\, j\neq i}^{3} q_i, \quad i = 1,2,3 \tag{6}$$

We consider a heterogeneous triopoly game in the sense that triopolists adopt different mechanisms to decide the output of each time period. We assume firm 1 follows a gradient rule based on marginal profits, firm 2 follows the best reply with naïve expectations, while firm 3 follows the best reply under an adaptive expectations scheme (henceforth, we call them gradient firm, naïve firm, and adaptive firm, respectively). Thus, the first firm increases or decreases the quantity of output that it produces at time t+1 depending on whether current marginal profits are either positive or negative[3]. That is, the output adjustment mechanism over time of the first firm is given by:

$$q_1^{t+1} = q_1^t + \alpha(q_1^t)\Phi_1^t = q_1^t + \alpha q_1^t (a - c - 2q_1^t - d(q_2^t + q_3^t)) \tag{7}$$

where $\alpha(q_1^t) = \alpha q_1^t$, being $\alpha > 0$ a parameter that captures firm 1's adjustment speed whit respect to a marginal change in profits.

The second firm is a naïve player which solves the following optimization problem:

$$q_2^{t+1} = \arg\max_{q_2} \Pi_2(q_1^{e,t+1}, q_2^t, q_3^{e,t+1}) \tag{8}$$

Using the profit expression (5) to solve (8), we obtain the best reply with naïve expectations:

$$q_2^{t+1} = \frac{a - c - d(q_1^t + q_3^t)}{2} \tag{9}$$

Finally, we suppose that firm 3 is an adaptive player. Hence, the third firm changes its output proportionally to the difference between the best reply with naïve expectations and the last period's output quantity. Then, the firm 3's decision mechanism is a partial adjustment towards the best response with naïve expectations, given by:

$$q_3^{t+1} = (1-\beta)q_3^t + \beta \frac{a - c - d(q_1^t + q_3^t)}{2}, \qquad 0 < \beta < 1 \tag{10}$$

[3] In the literature, this adjustment rule is usually called bounded rationality (see, among others, [26]) or "myopic" (see [14])

The parameter β measures how reluctant the adaptive player is to change its previous period choice following the suggestion given by the reaction function. If $\beta = 0$, the output level would never change, and if $\beta = 1$, it would be the best response with naïve expectations.

Using (7), (9) and (10), we obtain the following dynamic system:

$$\begin{cases} q_1^{t+1} = q_1^t + \alpha q_1^t \left[a - c - 2q_1^t - d(q_2^t + q_3^t) \right] \\ q_2^{t+1} = \dfrac{a - c - d(q_1^t + q_3^t)}{2} \\ q_3^{t+1} = (1-\beta) q_3^t + \beta \dfrac{a - c - d(q_1^t + q_2^t)}{2} \end{cases} \tag{11}$$

The nonlinear and discrete system (11) describes a Cournot triopoly game based on differentiated products and heterogeneous players. The dynamics is due to a three-dimensional nonlinear map that depends on five parameters. In the following section, the dynamic stability properties of this model will be discussed.

3.1. The equilibrium points and local stability

To study the qualitative behavior of the solutions of the nonlinear model (11), we study the existence of equilibrium points and their stability properties. We find the equilibrium points of system (11) imposing $q_i^{t+1} = q_i^t = q_i, i = 1, 2, 3.$ Then, they are the non-negative solutions of the following nonlinear system:

$$\left. \begin{array}{l} 0 = \alpha q_1 \left(a - c - 2q_1 - d(q_2 + q_3) \right) \\ q_2 = \dfrac{1}{2}(a - c - d(q_1 + q_3)) \\ 0 = -\beta q_3 + \dfrac{\beta}{2}\left(a - c - d(q_1 + q_2) \right) \end{array} \right\} \tag{12}$$

It is concluded that the dynamic system (11) has two equilibrium points: $E_1 = \left(0, \dfrac{a-c}{d+2}, \dfrac{a-c}{d+2} \right)$ and $E_2 = \left(\dfrac{a-c}{2(d+1)}, \dfrac{a-c}{2(d+1)}, \dfrac{a-c}{2(d+1)} \right)$, where $a > c$ should hold to ensure nonnegative equilibrium points. The equilibrium point E_1 is a boundary equilibrium and E_2 is the unique Nash equilibrium point.

The local stability of the equilibrium points of the three-dimensional system (11) depends on the eigenvalues of the Jacobian matrix of (11). This matrix is given by:

$$J(q_1,q_2,q_3)=\begin{pmatrix} 1+\alpha\left[a-c-4q_1-d(q_2+q_3)\right] & -\alpha d q_1 & -\alpha d q_1 \\ -\frac{d}{2} & 0 & -\frac{d}{2} \\ -\frac{\beta d}{2} & -\frac{\beta d}{2} & 1-\beta \end{pmatrix} \tag{13}$$

Proposition 1. *The boundary equilibrium point E_1 is unstable (a saddle point).*

Proof. In order to prove this result, we calculate the eigenvalues of the Jacobian matrix (13) evaluated in E_1:

$$J(E_1)=\begin{pmatrix} 1+\alpha\frac{(a-c)(2-d)}{d+2} & 0 & 0 \\ -\frac{d}{2} & 0 & -\frac{d}{2} \\ -\frac{\beta d}{2} & -\frac{\beta d}{2} & 1-\beta \end{pmatrix}$$

The three eigenvalues are, $\lambda_1=1+\alpha\frac{(a-c)(2-d)}{d+2}$ and $\lambda_{2,3}=\frac{1-\beta\pm\sqrt{(1-\beta)^2+\beta d^2}}{2}$. We have $\lambda_1>1$ given that $\alpha>0,\ a-c>0$ and $-\frac{1}{2}<d<1$. Moreover as $0<\beta<1,$ it is verified that $\left|\lambda_{2,3}\right|<1$. □

Next, we discuss the local stability of the Nash equilibrium. The Jacobian matrix (13) evaluated in this equilibrium is given by:

$$J(E_2)=\begin{pmatrix} 1-\alpha\frac{a-c}{d+1} & -\alpha\frac{d(a-c)}{2(d+1)} & -\alpha\frac{d(a-c)}{2(d+1)} \\ -\frac{d}{2} & 0 & -\frac{d}{2} \\ -\frac{\beta d}{2} & -\frac{\beta d}{2} & 1-\beta \end{pmatrix} \tag{14}$$

The eigenvalues of (14) are the solution of its characteristic equation which can be reduced to the cubic equation $\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0$, where the coefficients are given by:

$$A_1 = \beta - 2 + \alpha\frac{a-c}{d+1}$$
$$A_2 = 1 - \beta - \frac{\beta d^2}{4} - \alpha\frac{a-c}{d+1}\left[1 - \beta + \frac{d^2}{4}(1+\beta)\right]$$
$$A_3 = \frac{\beta d^2}{4}\left[1 - \alpha\frac{a-c}{d+1}\right] + \frac{\alpha d^2(a-c)}{4(d+1)}[1 - \beta + \beta d]$$

According to the Jury's criterion (see [17]), the equilibrium point E_2 is locally asymptotically stable if:

$$\left.\begin{array}{l} 1 + A_1 + A_2 + A_3 > 0 \\ 1 - A_1 + A_2 - A_3 > 0 \\ 1 - A_2 + A_1A_3 - A_3^{\,2} > 0 \\ 1 + A_2 - A_1A_3 - A_3^{\,2} > 0 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} 1 + A_1 + A_2 + A_3 > 0 \\ 1 - A_1 + A_2 - A_3 > 0 \\ 1 - A_2 + A_1A_3 - A_3^{\,2} > 0 \\ 3 - A_2 > 0 \end{array}\right\} \tag{15}$$

Proposition 2

The Cournot-Nash equilibrium E_2 is locally asymptotically stable provided that:

$$\alpha < \alpha^C = \frac{2(1+d)\left[8 - \beta(4+d^2)\right]}{(a-c)\left[2(4+d^2) + \beta(d-2)(2+d+d^2)\right]} \tag{16}$$

Proof.

For all $\alpha > 0, a > c > 0, \frac{-1}{2} < d < 1, 0 < \beta < 1$, it is verified:

i) $1 + A_1 + A_2 + A_3 = \frac{\alpha\beta(a-c)(d-2)^2}{4} > 0.$

ii) $3 - A_2 > 0$, because $A_2 < 1$.

It is deduced:

$$1 - A_1 + A_2 - A_3 > 0 \Leftrightarrow \alpha < \alpha_1 = \frac{2(1+d)\left[8 - \beta(4+d^2)\right]}{(a-c)\left[2(4+d^2) + \beta(d-2)(2+d+d^2)\right]}$$

To solve the inequality $1 - A_2 + A_1A_3 - A_3^{\,2} > 0$, we consider the equation $1 - A_2 + A_1A_3 - A_3^{\,2} = 0$, that is a quadratic equation in α with coefficients depending on the rest of parameters. The discriminant of this equation is always nonnegative, then the equation $1-A_2+A_1A_3-A_3^2=0$ has two real solutions, α_2 and α_3, such that for all $a>c>0, \frac{-1}{2}<d<1, 0<\beta<1$:

$$\alpha_3<0 \text{ and } \begin{cases} \alpha_2>0 \text{ if } \beta>\frac{1}{2-d} \\ \alpha_2<0 \text{ if } \beta<\frac{1}{2-d} \end{cases}$$

Moreover, it is deduced that:

$$1-A_2+A_1A_3-A_3^2>0 \Leftrightarrow \begin{cases} \alpha<\alpha_2 \text{ if } \beta>\frac{1}{2-d} \\ \forall\alpha>0 \text{ if } \beta<\frac{1}{2-d} \end{cases}$$

For $\beta>\frac{1}{2-d}$, comparing the thresholds α_1 and α_2, we obtain that $\alpha_1<\alpha_2$.

From these results, we conclude that the threshold of adjustment speed ensuring the local asymptotic stability of the Nash equilibrium is given by:

$$\alpha^C=\alpha_1=\frac{2(1+d)\left[8-\beta(4+d^2)\right]}{(a-c)\left[2(4+d^2)+\beta(d-2)(2+d+d^2)\right]}$$ □

From this proposition, it is deduced that the local stability of the Nash equilibrium strongly depends on the adjustment speed of gradient firm. Particularly, if this firm is excessively reactive (the value of α is high), the Nash equilibrium loses the stability and complex dynamics for the whole system may emerge. Therefore, it is interesting to analyze the effects of the other parameters on the value of the threshold α^C. We will consider the influence of β and *d*.

Taking the derivative of (16) with respect to β we obtain:

$$\frac{\partial\alpha^C}{\partial\beta}=-\frac{4d^2(1+d)(2+d)^2}{(a-c)\left[2(4+d^2)+\beta(d-2)(2+d+d^2)\right]^2}<0 \tag{17}$$

As we have noted, a value of β equal to 1 would indicate that the adaptive firm applies naïve expectations, and then, a high value of this parameter makes the third player more inclined to change quantity with respect to the previous period, favoring the presence of instability.

To evaluate the impact of the degree of product differentiation, we calculate:

$$\frac{\partial \alpha^C}{\partial d} = 4\frac{(8+10d^2+4d^3+d^4)\beta^2-(32-8d+16d^2+8d^3+d^4)\beta+8(4-2d-d^2)}{(a-c)\left[2(4+d^2)+\beta(d-2)(2+d+d^2)\right]^2} \tag{18}$$

In the parametric plane[4] (d,β), the sign of (18) changes in the curve:

$$\beta^C(d) = \frac{32+d\left[-8+d(4+d)^2\right]-d(2+d)\sqrt{d^4+12d^3+76d^2+80d+16}}{2\left[8+d^2(10+4d+d^2)\right]} \tag{19}$$

As is showed in Figure 1, the curve (19) divides the parametric plane in two regions. In light region, it is deduced that $\frac{\partial \alpha^C}{\partial d} > 0$, while in the dark region $\frac{\partial \alpha^C}{\partial d} < 0$.

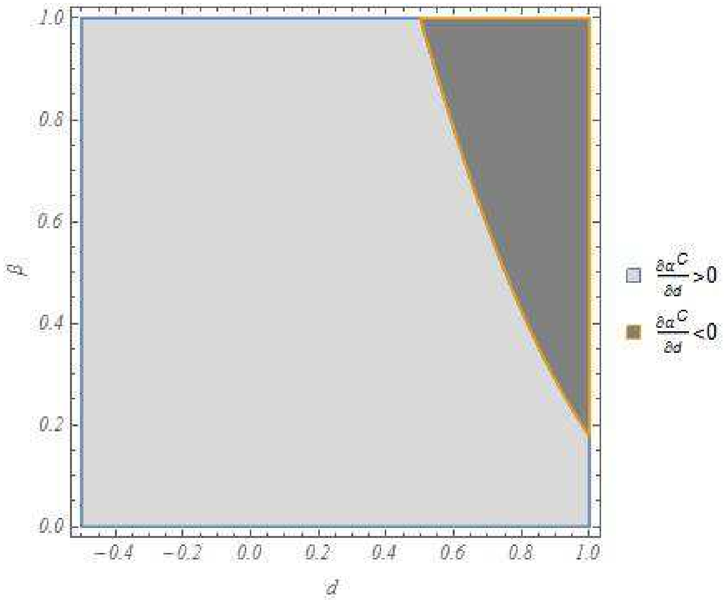


Fig. 1. Sign of $\frac{\partial \alpha^C}{\partial d}$.

From this result it is deduced that, independent of the behavior of the adaptive player, in the presence of complementary goods ($d < 0$), the Nash equilibrium becomes more stable when goods tend to be more

[4] For simplicity, we assume $a = 2$ and $c = 1$.

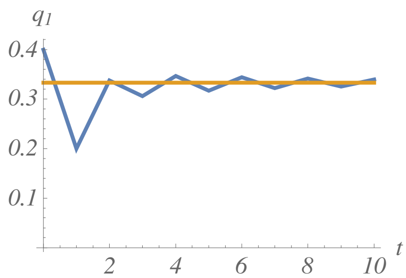
independent, being this result in line with the conclusions obtained in the literature (see [16] for the duopoly case, and [6] in a context of a linear dynamic system of a differentiated triopoly). By contrast, this result is not reproduced in the presence of substitute goods. Assuming $d > 0$, the curve (19) represents the locus of points (d, β) such that the expression (18) is equal to zero and α^C reaches a maximum value. Thus, for each value of β, there is a critical value of substitutability for which, the threshold of adjustment speed of the firm 1 ensuring the stability of the Nash equilibrium is maximum. This surprising result shows that a higher independence between goods does not necessarily imply a higher stability of the Nash equilibrium.

From an economic point of view, this last result is related with the inter-temporal strategic substitutability of quantities in the presence of substitute goods (see [22]) which, in turn, it is influenced by the relationship between β and d. Specifically, as β is closer to one, the value of d that maximizes the level of strategic substitutability between gradient firm's quantity and the competitors' output goes to zero.

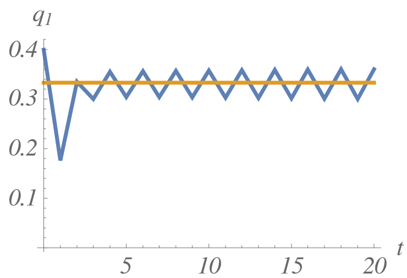
3.2. Numerical simulations

This section illustrates the obtained analytical results through different simulations of the model (11). Moreover, we show different non-equilibrium dynamics generated by the model.

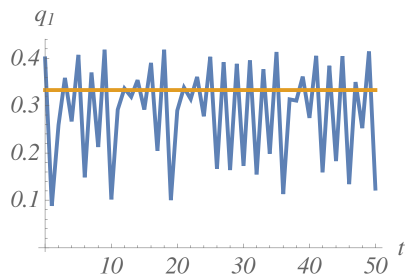
Being $a = 2$, $c = 1$, $\beta = 0.5$, $d = 0.5$, and assuming the parameter α free, the next figures show that an increase in the firm 1's adjustment speed destabilizes the Nash equilibrium leading to increasingly complex attractors. The attractor in the Figure 2a is the Nash equilibrium, in Figure 2b a 2-cycle and in Figure 2c we can observe complex behavior. Figure 3 shows the strange attractor which attracts the trajectory shown in Figure 2c.



a . $\alpha = 2.5$



b . $\alpha = 2.8$



c . $\alpha = 3.9$

Fig. 2 . Trajectories of q_1 for the values of parameters $a = 2$, $c = 1$, $d = 0.5$, $\beta = 0.5$, α given in each figure and initial conditions $q_1^0 = 0.4, q_2^0 = 0.4, q_3^0 = 0.4$.

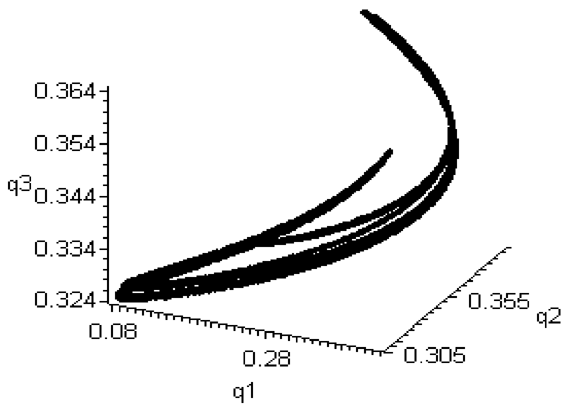


Fig. 3 . The strange attractor of (11) for the values of parameters $a = 2$, $c = 1$, $d = 0.5$, $\beta = 0.5$, $\alpha = 3.9$.

The loss of stability of Nash equilibrium, and the appearance of increasingly complex attractors, when the adjustment speed increase, is corroborated in Figure 4. In this figure, it is shown the bifurcation diagram of q_1 with respect to the adjustment of speed, with $a = 2$, $c = 1$, $d = 0.5$ and $\beta = 0.5$. We see that the Nash equilibrium loses its stability via flip bifurcations, leading to cyclic attractors and strange attractors.

This result is in line with the conclusions obtained in [1], [26], [16], and [6], among others.

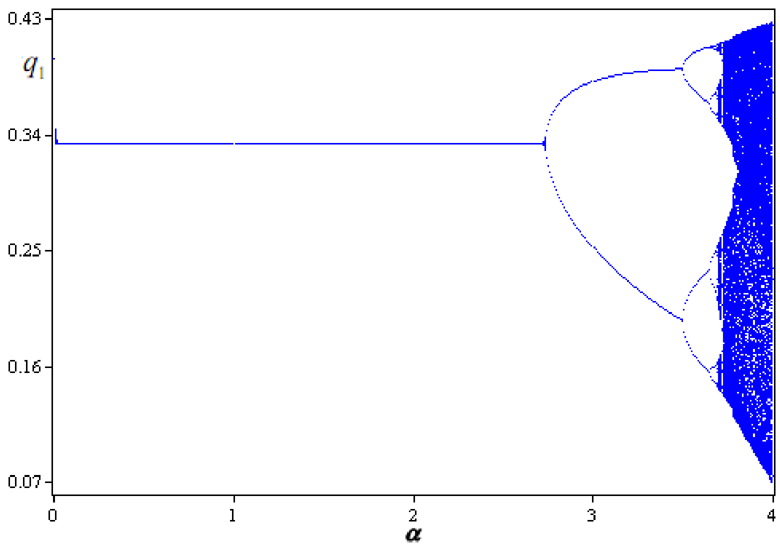
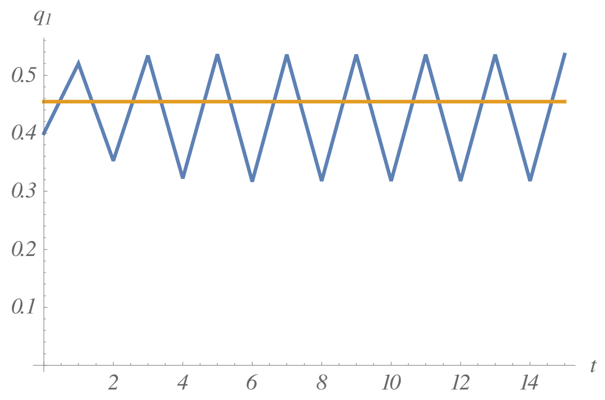


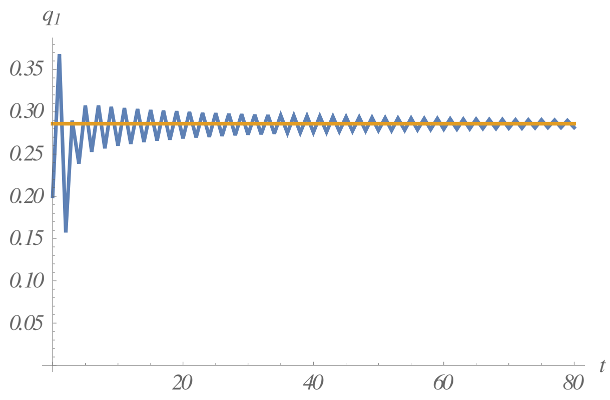
Fig. 4. Bifurcation diagram of q_1 with respect to α for the values of parameters $a = 2$, $c = 1$, $d = 0.5$ and $\beta = 0.5$.

In Figures 5 and 6, we illustrate the effect of the parameter d on the threshold α^C in the presence of substitute goods through simulations (keeping $a = 2$, $c = 1$, $\beta = 0.5$).

A higher independence between goods (d tends to zero) may destabilize the Nash equilibrium (see Figure 5a versus Figure 2a). On the other hand, there are cases where the Nash equilibrium is more stable when the goods are more substitutes (d tends to one) (see Figure 5b versus Figure 2b). This result contrasts with the previous conclusions offered by [16] for the duopoly case, and by [6] for the triopoly case in a linear dynamic system.



a. $d = 0.1, \alpha = 2.5, q_1^0 = 0.4, q_2^0 = 0.4, q_3^0 = 0.4$



b. $d = 0.75, \alpha = 2.8, q_1^0 = 0.2, q_2^0 = 0.2, q_3^0 = 0.2$

Fig. 5 . Trajectories of q_1 for the values of parameters $a = 2$, $c = 1$, $\beta = 0.5$ and d, α and initial conditions given in each figure.

Figure 6 shows the bifurcation diagram of q_1 with respect to d, with $a = 2$, $c = 1$, $\alpha = 2.5$ and $\beta = 0.5$.

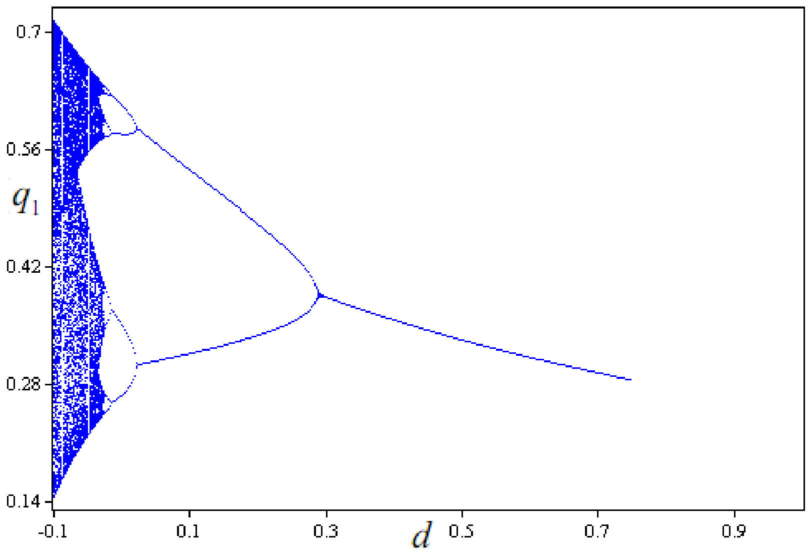


Fig. 6. Bifurcation diagram of q_1 with respect to d for the values of parameters $a = 2$, $c = 1$, $\alpha = 2.5$ and $\beta = 0.5$.

4. Dynamics under Bertrand competition

In this section, we consider a dynamic triopoly where firms compete on price rather than quantity. In this setting, the quantity demanded to each firm is given by expressions (4).

At each period t, every firm must form an expectation of the rival's price in the next time period in order to determine the corresponding profit-maximizing prices for period $t + 1$. As in the Cournot case, it is assumed that firm 1 is a gradient player, such that, this firm decides to increase or decrease the price in next period, as it experiences positive or negative marginal profit. While firm 2 is a naïve player, and firm 3 thinks with adaptive expectations. By using (4) and (5), the nonlinear three-dimensional system that describes the game dynamics is given by:

$$\begin{cases} p_1^{t+1} = p_1^t + \alpha p_1^t \dfrac{(1-d)a+(d+1)c-2(d+1)p_1^t+dp_2^t+dp_3^t}{(1-d)(1+2d)} \\ p_2^{t+1} = \dfrac{(1-d)a+(d+1)c+dp_1^t+dp_3^t}{2(d+1)} \\ p_3^{t+1} = (1-\beta)p_3^t + \beta\dfrac{(1-d)a+(d+1)c+dp_1^t+dp_2^t}{2(d+1)} \end{cases} \tag{20}$$

4.1 Equilibrium points and local stability

The steady states of (20) are defined by the non-negative solutions of the following system:

$$\left.\begin{array}{l} \alpha p_1 \dfrac{(1-d)a+(d+1)c-2(d+1)p_1+dp_2+dp_3}{(1-d)(1+2d)} = 0 \\ p_2 = \dfrac{(1-d)a+(d+1)c+dp_1+dp_3}{2(d+1)} \\ -\beta p_3 + \beta\dfrac{(1-d)a+(d+1)c+dp_1+dp_2}{2(d+1)} = 0 \end{array}\right\} \tag{21}$$

From (21) we obtain two equilibrium points: $S_1 = \left(0, \dfrac{(1-d)a+(1+d)c}{d+2}, \dfrac{(1-d)a+(1+d)c}{d+2}\right)$ and $S_2 = \left(\dfrac{(1-d)a+(1+d)c}{2}, \dfrac{(1-d)a+(1+d)c}{2}, \dfrac{(1-d)a+(1+d)c}{2}\right)$, being S_1 a boundary equilibrium and S_2 the unique Nash equilibrium point.

In this case, the Jacobian matrix of the three-dimensional dynamic system (20) adopts the expression:

$$J(p_1,p_2,p_3)=\begin{pmatrix} 1+\alpha\dfrac{a(1-d)+(d+1)c-4(1+d)p_1+dp_2+dp_3}{(1-d)(2d+1)} & \dfrac{\alpha dp_1}{(1-d)(2d+1)} & \dfrac{\alpha dp_1}{(1-d)(2d+1)} \\ \dfrac{d}{2(1+d)} & 0 & \dfrac{d}{2(1+d)} \\ \dfrac{\beta d}{2(d+1)} & \dfrac{\beta d}{2(d+1)} & 1-\beta \end{pmatrix} \qquad (22)$$

Proposition 3. *The boundary equilibrium point S_1 is unstable (a saddle point).*

Proof. The Jacobian matrix (22) evaluated in S_1 is:

$$J(S_1)=\begin{pmatrix} 1+\alpha\dfrac{\left[a(1-d)+(d+1)c\right](3d+2)}{(1-d)(2d+1)(d+2)} & 0 & 0 \\ \dfrac{d}{2(1+d)} & 0 & \dfrac{d}{2(1+d)} \\ \dfrac{\beta d}{2(d+1)} & \dfrac{\beta d}{2(d+1)} & 1-\beta \end{pmatrix}$$

Its three eigenvalues are, $\lambda_1=1+\alpha\dfrac{\left[a(1-d)+(d+1)c\right](3d+2)}{(1-d)(2d+1)(d+2)}$ and $\lambda_{2,3}=\dfrac{1-\beta\pm\sqrt{(1-\beta)^2+\dfrac{\beta d^2}{(1+d)^2}}}{2}$. We have $\lambda_1>1$ due to $\alpha>0$, $a>c$ and $-\dfrac{1}{2}<d<1$. Moreover as $0<\beta<1$, it is verified that $\left|\lambda_{2,3}\right|<1$. □

The Jacobian matrix (22) evaluated in the Nash equilibrium S_2 is given by:

$$J(S_2)=\begin{pmatrix} 1-\dfrac{2\alpha(1+d)p^*}{(1-d)(1+2d)} & \dfrac{\alpha dp^*}{(1-d)(1+2d)} & \dfrac{\alpha dp^*}{(1-d)(1+2d)} \\ \dfrac{d}{2(1+d)} & 0 & \dfrac{d}{2(1+d)} \\ \dfrac{\beta d}{2(d+1)} & \dfrac{\beta d}{2(d+1)} & 1-\beta \end{pmatrix}, \text{ with } p^*=\dfrac{a(1-d)+c(1+d)}{2} \qquad (23)$$

The eigenvalues of (23) are the solution of its characteristic equation which can be reduced to the cubic equation $\lambda^3+B_1\lambda^2+B_2\lambda+B_3=0$ where the coefficients are given by:

$$B_1 = \beta - 2 + \frac{2\alpha(1+d)}{(1-d)(1+2d)} p^*$$

$$B_2 = 1 - \beta - \frac{\beta d^2}{4(1+d)^2} - \frac{\alpha\left[4(1+d)^2(1-\beta) + d^2(1+\beta)\right]}{2(1+d)(1+2d)(1-d)} p^*$$

$$B_3 = \frac{\beta d^2}{4(1+d)^2} + \frac{\alpha d^2}{2(1+d)(1+2d)(1-d)}\left[1 - \frac{\beta(2+3d)}{1+d}\right] p^*$$

Proposition 4

The Bertrand-Nash equilibrium S_2 is locally asymptotically stable provided that:

$$\alpha < \alpha^B = \frac{2(1-d)(1+2d)\left[-8(1+d)^2 + \beta(4+8d+5d^2)\right]}{\left[a(1-d)+c(1+d)\right]\left[\beta(2+3d)(2+3d+2d^2) - 2(1+d)(4+8d+5d^2)\right]} \tag{24}$$

Proof.

For all $\alpha > 0, a > c > 0, \frac{-1}{2} < d < 1, 0 < \beta < 1$, the following holds:

i) $1 + B_1 + B_2 + B_3 = \dfrac{(3d+2)^2 \alpha\beta p^*}{2(1+d)^2(1-d)(1+2d)} > 0.$

ii) $3 - B_2 > 0$, because $B_2 < 1$.

The other two conditions define the threshold of the firm 1's adjustment speed for the local asymptotic stability of the Nash equilibrium.

$$1 - B_1 + B_2 - B_3 > 0 \Leftrightarrow \alpha < \alpha_1 = \frac{2(1-d)(1+2d)\left[-8(1+d)^2 + \beta(4+8d+5d^2)\right]}{\left[a(1-d)+c(1+d)\right]\left[\beta(2+3d)(2+3d+2d^2) - 2(1+d)(4+8d+5d^2)\right]}$$

The equation $1 - B_2 + B_1B_3 - B_3^2 = 0$ has two real solutions, α_2 and α_3, such that $\alpha_3 < 0$ and the sign of α_2 in parametric plane[5] (d, β) is shown in Figure 7.

[5] For simplicity, we assume $a = 2$ and $c = 1$.

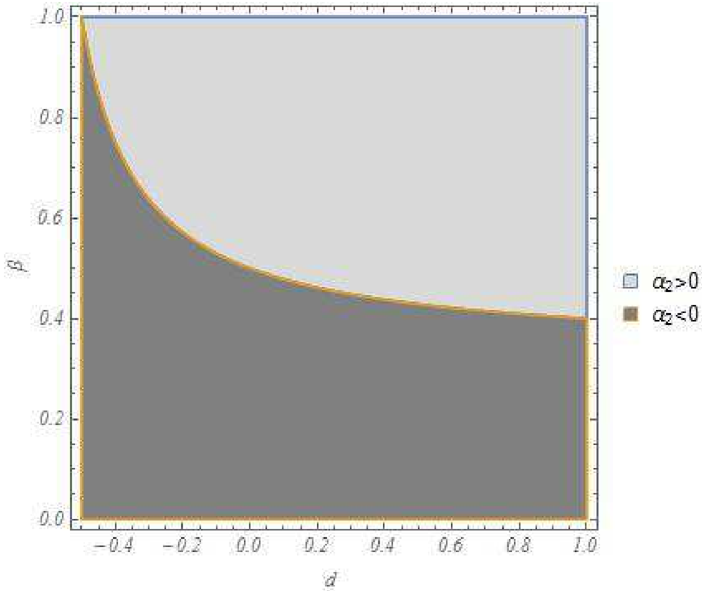


Fig. 7. Sign of α_2.

For all $a > c > 0, \frac{-1}{2} < d < 1, 0 < \beta < 1$, it is deduced that:

$$1 - B_2 + B_1 B_3 - B_3^{\ 2} > 0 \Leftrightarrow \begin{cases} \alpha < \alpha_2 & \text{if } (d,\beta) \text{ is in light region in Figure 7} \\ \forall \alpha > 0 & \text{if } (d,\beta) \text{ is in dark region in Figure 7} \end{cases}$$

Comparing the thresholds α_1 and α_2 in light region in Figure 7, we obtain that $\alpha_1 < \alpha_2$.

From these results, we conclude that the threshold of adjustment speed ensuring the local asymptotic stability of the Nash equilibrium is given by:

$$\alpha^B = \alpha_1 = \frac{2(1-d)(1+2d)\left[-8(1+d)^2 + \beta(4+8d+5d^2)\right]}{\left[a(1-d)+c(1+d)\right]\left[\beta(2+3d)(2+3d+2d^2) - 2(1+d)(4+8d+5d^2)\right]} \qquad \square$$

As in the quantity competition case, it is deduced that the local stability of the Nash equilibrium is determined by the adjustment speed of gradient player, given that an increase in the parameter α constitutes a source of complexity.

We deduce the effect of d and β on the threshold of the adjustment speed.

$$\frac{\partial \alpha^B}{\partial \beta} = -\frac{4(1-d)d^2(1+d)(2+d)^2(1+2d)}{\left[a(1-d)+c(1+d)\right]\left[\beta(2+3d)(2+3d+2d^2) - 2(1+d)(4+8d+5d^2)\right]^2} < 0 \qquad (25)$$

Therefore, as in the Cournot case, the greater the value of β, the lower the stability of the Nash equilibrium is, regardless of the value of the parameters a, c and d.

The expression of the derivative $\frac{\partial \alpha^B}{\partial d}$ is more complicated. From computational calculus, we obtain that in the parametric plane[6] (d, β), the sign of the derivative $\frac{\partial \alpha^B}{\partial d}$ changes in the curve:

$$\beta^B(d) = \frac{\begin{bmatrix} -32+216d+1272d^2+2328d^3+ \\ 1983d^4+874d^5+187d^6 \end{bmatrix} + d(2+d)\sqrt{\begin{matrix} 144+240d-1028d^2-2548d^3-263d^4+ \\ 2676d^5+2122d^6+248d^7-295d^8 \end{matrix}}}{2(-8+48d+310d^2+604d^3+533d^4+240d^5+58d^6)} \quad (26)$$

In Figure 8 (in order to see better the curve $\beta(d)$, a zoom of Figure 8 is shown), $\frac{\partial \alpha^B}{\partial d} > 0$ in light region and $\frac{\partial \alpha^B}{\partial d} < 0$ in dark region.

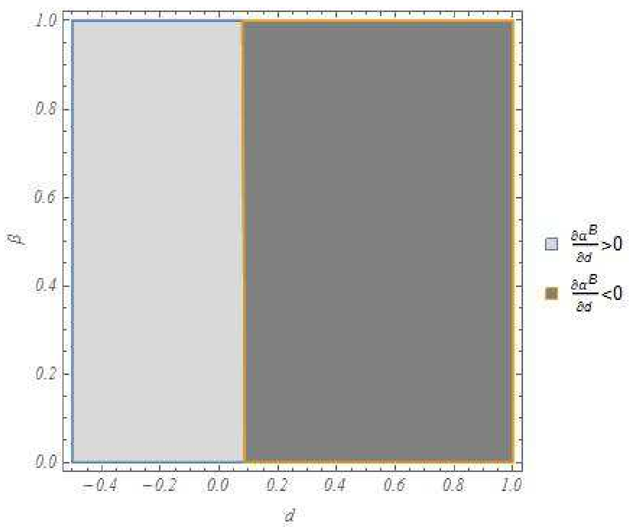
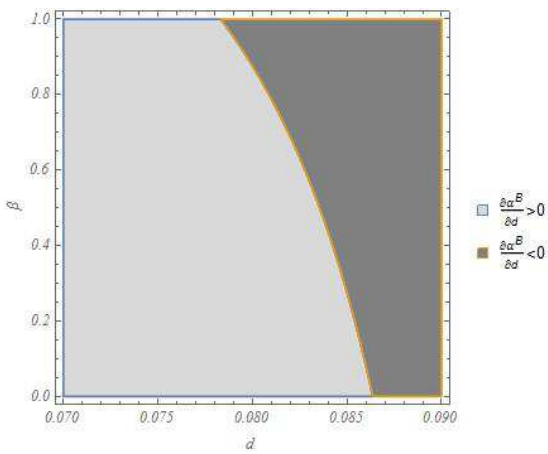


Fig. 8. Sign of $\frac{\partial \alpha^B}{\partial d}$.



Zoom of Fig. 8.

Therefore, the conclusions obtained under Cournot competition are reproduced when firms compete on price. In the presence of complement goods, a reduction in the degree of complementarity (d tends to zero), increases the stability of the Nash equilibrium. By contrast, this monotonous relationship is not reproduced assuming substitute goods. In this case, an increase in the degree of product differentiation may both stabilize

[6] Again, for simplicity, we assume $a = 2$ and $c = 1$.

and destabilize the Nash equilibrium, given that there is a degree of substitutability (being fixed β) for which the threshold of the adjustment speed is maximum (note that this result holds for positive values of d close to zero). In this case, this last result is related with the inter-temporal strategic complementarity of prices in the presence of substitute goods (see [22]).

4.2. Numerical simulations

As in the Cournot competition, we obtain that an increase in the adjustment speed of firm 1 destabilizes the Nash equilibrium leading to increasingly complex attractors. Figure 9 shows the bifurcation diagram of p_1 with respect to the adjustment speed, with $a = 2$, $c = 1$, $\beta = 0.5$, $d = 0.5$. We see that the Nash equilibrium loses its stability via flip bifurcations, leading to cyclic attractors and strange attractors.

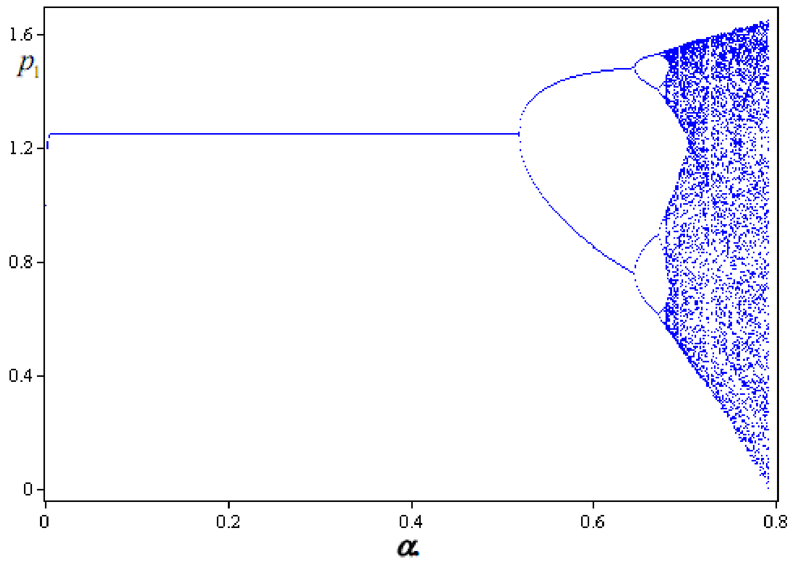


Fig. 9. Bifurcation diagram of p_1 with respect to α for the values of parameters $a = 2$, $c = 1$, $d = 0.5$ and $\beta = 0.5$.

From comparison of thresholds α^C and α^B, we conclude that the value of the adjustment speed where the Nash equilibrium loses its stability is lower under Bertrand competition than under Cournot competition for all parametric values $a > c > 0, \frac{-1}{2} < d < 1, 0 < \beta < 1$. This conclusion is shown in Figure 10, and it is in accordance with the results obtained in [6]. From the economic perspective, this result can be understood as the consequences on the stability of a fiercer competition under Bertrand competition than under Cournot competition regardless of whether the goods are substitutes or complements (see [22]).

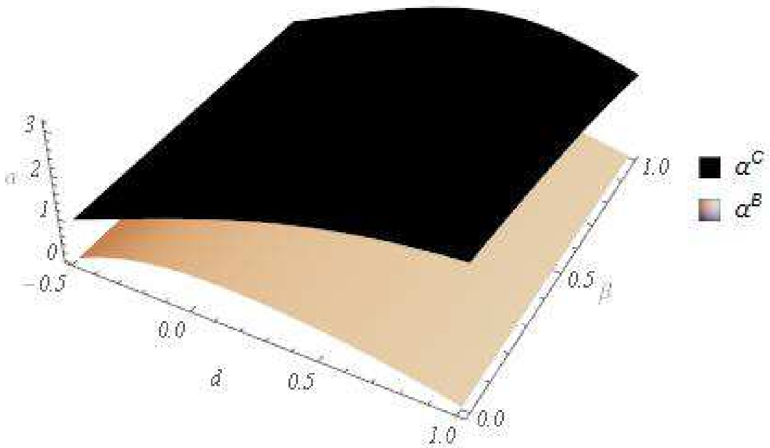


Fig. 10. Comparison between thresholds α^C and α^B for the values of parameters $a = 2$, $c = 1$.

5. Conclusion

In this paper, we study the dynamics in a triopoly game with product differentiation, and heterogeneous firms. By investigating the local stability of the equilibrium points, under both quantity and price competition, we find the following results.

First, under both quantity and price competition, the behavior of the gradient firm may constitute a source of instability. This happens whenever this player is sufficiently reactive, presenting a high value of the adjustment speed. This result corroborates the conclusions offered in the literature (see [26], [15], [16], and [6], among others).

Second, the firm of adaptive expectations has a stabilizing or destabilizing effect on the game, given that if this firm is more inclined to change quantity or price with respect to the previous period, the Nash equilibrium can lose the local stability. This result is in line with the conclusions offered by [26], and [15] in absence of product differentiation.

Third, we discuss the effect of the degree of product differentiation on the local stability of the Nash equilibrium. We find that under both Cournot and Bertrand competition, in the presence of complement goods, a greater independence among goods increases the stability of the Nash equilibrium, reproducing the previous results proposed in the literature (see [16] for the duopoly case, and [6], for a differentiated triopoly with linear dynamic system). However, assuming substitute goods, a monotonous relationship between degree of product differentiation and stability is not reproduced. Regardless of the competition type, there is a critical degree of substitutability for which the set of values of the adjustment speed for the gradient firm that ensures the local stability is largest. In consequence, a decrease of the degree of product differentiation may destabilize the Nash equilibrium.

Finally, the comparison between Cournot and Bertrand competition allows us to deduce that the Nash equilibrium is more stable in a quantity setting context, independently of the nature of the products (in line with the conclusions of [6]).

Acknowledgments

J. Andaluz and G. Jarne wish to thank the Spanish Ministry of Economics and Competitiveness (ECO2013-41353-P) and the Government of Aragon and FEDER (consolidated groups S10 and S13) for their financial support.

References

[1] H.N. Agiza, A.A. Elsadany, Nonlinear dynamics in the Cournot duopoly game with heterogeneous players, Physica A 320 (2003) 512-524.

[2] H.N. Agiza, A.A. Elsadany, Chaotic dynamics in nonlinear duopoly game with heterogeneous players, Appl. Math. Comput. 149 (2004) 843-860.

[3] A. Agliari, L. Gardini, T. Puu, Global bifurcations of basins in a triopoly game, Int. J. Bifurcat. Chaos 12 (2002) 2175-2207.

[4] E. Ahmed, H.N. Agiza, Dynamics of a Cournot game with n-competitors, Chaos Soliton. Fract. 9 (1998) 1513-1571.

[5] E. Ahmed, A.A. Elsadany, T. Puu, On Bertrand duopoly game with differentiated goods, Appl. Math. and Comput. 251 (2015) 169-179.

[6] J. Andaluz, G. Jarne, (2015). On the dynamics of economic games based on product differentiation, Math. Comput. Simulat. 113 (2015) 16-27.

[7] J. Bertrand, Révue de la Théorie Mathématique de la Richesse Sociale et des Recherches sur les Principles Mathématiques de la Théorie des Richesses, Journal des Savants (1883) 499-508.

[8] G.I. Bischi, A. Naimzada, Global analysis of a dynamic duopoly game with bounded rationality, In J.A. Filar, V. Gaitsgory, K. Mizukami (Eds), Advances in Dynamic Games and Applications, vol. 5, Birkhauser, 2000.

[9] G.I. Bischi, M. Kopel, Equilibrium selection in a nonlinear duopoly game with adaptive expectations, J. Econ. Behav. Organ. 46 (2001) 73-100.

[10] A.L. Bowley, The Mathematical Groundwork of Economics, Oxford University Press, Oxford, 1924.

[11] J.S. Cánovas, Reducing competitors in Cournot-Puu oligopoly, Nonlinear Anal.-Real World Appl. 13 (2012) 1772-1779.

[12] E. Chamberlin, The Theory of Monopolistic Competition, Harvard University Press, Cambridge MA, 1933.

[13] A. Cournot, Recherches sur les Principes Mathématiques de la Théorie des Richesses, Hachette, Paris, 1838.

[14] A. Dixit, Comparative Statics in Oligopoly, Int. Econ. Rev. 27 (1) (1986) 107-122.

[15] A.A. Elsadany, H.N. Agiza, E.M. Elabbasy, Complex dynamics and chaos control of heterogeneous quadropoly game, Appl. Math. Comput. 219 (2013) 11110-11118.

[16] L. Fanti, L. Gori, The dynamics of a differentiated duopoly with quantity competition, Econ. Model. 29 (2012) 421-427.

[17] G. Gandolfo, Economic Dynamics, Forth ed., Springer, Heidelberg, 2010.

[18] H. Hotelling, Stability in competition, Econ. J. 39 (1929) 41-57.

[19] K. Okuguchi, Adaptive expectations in an oligopoly model, Rev. Econ. Stud. 37 (1970) 233-237.

[20] T. Puu, The complex dynamics with three oligopolists, Chaos Soliton. Fract. 7 (1996) 2075-2081.

[21] T. Puu, On the stability of Cournot equilibrium when the number of competitors increases, J. Econ. Behav. Organ. 66 (2008) 445-456.

[22] N. Singh, X. Vives, Price and Quantity Competition in a Differentiated Duopoly, Rand J. Econ. 15 (1984) 546-554.

[23] R. Spiegler, Bounded Rationality in Industrial Organization, Oxford University Press, 2014.

[24] R.D. Theocharis, On the stability of the Cournot solution on the oligopoly problem, Rev. Econ. Stud. 27 (1960) 133-134.

[25] J. Tirole, The Theory of Industrial Organization, The MIT Press, Cambridge MA, 1988.

[26] F. Tramontana, Heterogeneous duopoly with isoelastic demand function, Econ. Model. 27 (2010) 350-357.

[27] F.Tramontana, A.A. Elsadany, X. Baogui, H.N. Agiza, Local stability of the Cournot solution with increasing heterogeneous competitors, Nonlinear Anal.-Real World Appl. 26 (2015) 150-160.

[28] W. Wu, Z. Chen, W.H. Ip, Complex nonlinear dynamics and controlling chaos in a Cournot duopoly economic model, Nonlinear Anal.-Real World Appl. 11 (2010) 4363-4377.