# ABC of ladder operators for rationally extended quantum harmonic oscillator systems

**José F. Cariñena[a] and Mikhail S. Plyushchay[b]**

[a] *Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain*
[b] *Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile*

*E-mails: jfc@unizar.es, mikhail.plyushchay@usach.cl*

**Abstract**

The problem of construction of ladder operators for rationally extended quantum harmonic oscillator (REQHO) systems of a general form is investigated in the light of existence of different schemes of the Darboux-Crum-Krein-Adler transformations by which such systems can be generated from the quantum harmonic oscillator. Any REQHO system is characterized by the number of separated states in its spectrum, the number of 'valence bands' in which the separated states are organized, and by the total number of the missing energy levels and their position. All these peculiarities of a REQHO system are shown to be detected and reflected by a trinity ($\mathcal{A}^{\pm}$, $\mathcal{B}^{\pm}$, $\mathcal{C}^{\pm}$) of the basic (primary) lowering and raising ladder operators related between themselves by certain algebraic identities with coefficients polynomially-dependent on the Hamiltonian. We show that all the secondary, higher-order ladder operators are obtainable by a composition of the basic ladder operators of the trinity which form the set of the spectrum-generating operators. Each trinity, in turn, can be constructed from the intertwining operators of the two complementary minimal schemes of the Darboux-Crum-Krein-Adler transformations.

## 1 Introduction

There are two basic exactly solvable quantum mechanical systems which reveal themselves directly or indirectly in association with other systems and play a fundamental role in many physical theories and applications. One of them is a free particle characterized by a continuous spectrum. The other one is a harmonic oscillator with its infinite equidistant discrete spectrum of the bound states.

Free particle is essential, in particular, for understanding the properties of the soliton solutions in integrable systems. Quantum reflectionless potentials represent a snapshot of soliton solutions to the classical KdV equation. Initially, reflectionless potentials were obtained with the help of the method of the inverse scattering transform in solving the problem of theoretical construction of a solid dielectric medium that is perfectly transparent to electromagnetic radiation [1]. This important class of the systems can also be generated from the quantum free particle by means of Darboux transformations (DTs) and their generalization in the form of Darboux-Crum transformations (DCTs) [2, 3, 4, 5, 6, 7, 8, 9, 10]. Notice here that reflectionless systems appear for instance in the Gross-Neveu model [11] in the context of the hadron physics [12] and the physics of conducting polymers [13]. Any quantum reflectionless system with $n$ bound states in its spectrum is characterized by the presence of a nontrivial Lax-Novikov integral that is a differential operator of order $2n+1$ having a structure of a Darboux-dressed momentum operator of the free particle

system. It is this integral of motion that distinguishes the states in the doubly degenerate continuous part of the spectrum of a reflectionless system and detects all the non-degenerate bound states as well as a state at the very edge of the continuous part of the spectrum by annihilating them [14, 15]. It is the same operator that plays a fundamental role in the theory of nonlinear integrable systems [16, 17]. The peculiarity of reflectionless systems also reveals itself in the nature of the quantum mechanical supersymmetry associated with them. Instead of a usual linear or non-linear $\mathcal{N} = 2$ supersymmetric structure which appears in an extended system composed from a pair of quantum mechanical systems related by a DT or DCT, the extended system composed from a pair of reflectionless systems is described by the exotic nonlinear $\mathcal{N} = 4$ supersymmetry generated by two pairs of supercharges alongside with the two bosonic integrals of motion constructed from the Lax-Novikov integrals of the subsystems [15, 18]. The appearance of the exotic nonlinear supersymmetric structure associated with reflectionless systems is traced to the fact that any reflectionless Schrödinger system can be related to a free particle system not only by one but by two different DCTs due to the presence of the momentum operator in the quantum free particle system in the form of its integral of motion [1].

Some time ago, there has been discovered a new very broad class of exactly solvable systems which represent certain rational extensions of the quantum harmonic oscillator [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. The eigenstates of rationally extended quantum harmonic oscillator (REQHO) systems are given in terms of exceptional Hermite polynomials, and can be obtained from the quantum harmonic oscillator (QHO) system by an appropriate DCT, or its further generalization in the form of the Darboux-Crum-Krein-Adler transformation [21, 31, 32] [2]. In what follows we shall refer to any generalized Darboux transformation of the QHO with intertwining operators to be higher order differential operators as the Darboux-Crum-Krein-Adler transformation (DCKAT). Instead of a continuous spectrum of the free particle, the QHO is characterized by an infinite discrete spectrum of bound states. In spite of such a radical difference, the QHO is also a very peculiar system because its discrete spectrum is equidistant. As a consequence, instead of the Hermitian momentum operator integral that encodes reflectionless nature of the free particle, the QHO possesses a pair of Hermitian conjugate ladder operators which are spectrum-generating operators. Similarly to the relation between reflectionless and free particle systems, a given REQHO system can be obtained from the QHO by different DCKATs. Then one can expect by analogy with the pairs of reflectionless systems related by DCTs and the exotic supersymmetric structure associated with them that the REQHO systems should be characterized by some special properties. Particularly, it seems to be natural to expect the appearance of peculiarities related to the ladder operators for such a family of quantum systems.

It is worth to mention here that finite-gap systems, a limit case of which corresponds to reflectionless systems, and the QHO are generated by the periodic Darboux chains [42]. The last construction also produces Painlevé equations [42, 43, 44], that are intimately related with isomonodromic deformations of linear systems and integrability properties of nonlinear systems in partial derivatives [43, 44]. The REQHO systems are isomonodromic deformations of the QHO [45].

Some investigations on ladder operators in REQHO systems have already been realized in [24, 25, 46, 47, 48, 49, 50, 51]. This has been done, however, for some particular examples of the

[1] A general picture is more complicated, however. In the case of a coincidence of some or all discrete energy levels of the two subsystems, the supersymmetric reflectionless partners can be related directly by a DCT of a lower order, without a necessity to construct a chain of Darboux transformations via a free particle system. This happens due to the opening of a kind of a 'direct tunnelling channel' that can appropriately be understood from the standpoint of the picture of soliton scattering, see [15, 19].

[2] Exceptional Jacobi and Laguerre polynomials [33, 34, 35, 36, 37, 38, 39, 40, 41] can be associated with Darboux-Crum transformed free particle on finite interval (particle in infinite potential well) and isotonic oscillator, respectively; the isotonic oscillator, in turn, can be related to the QHO by a singular Darboux transformation.

REQHO systems or for some particular families of such systems, while the problem of construction of ladder operators and investigation of their properties for REQHO systems of a general form remains open. Note also that in the indicated works only some special aspects related to the ladder operators of the REQHO systems were studied. In particular, in a recent paper [51] we have considered the problem of construction of the ladder operators for the simplest case of the REQHO by exploiting the simplest DT which relates the system to the QHO. We also investigated there the discrete chains related with the obtained ladder operators given by a pair of Hermitian conjugate third order differential operators to be fermionic generators of the polynomially deformed bosonized $\mathfrak{osp}(1|2)$ superalgebra.

In the present article we investigate the problem of construction of ladder operators for REQHO systems of a general form in the light of existence of different DCKATs by which any such a system can be related to and generated from the QHO. In this point there shows up a similarity of the REQHO systems with reflectionless systems related with a free particle. We show that for any REQHO, there exists a trinity of the basic (primary) pairs of the lowering and raising ladder operators. This trinity is proved to form the set of the spectrum-generating operators which detects and reflects all the peculiar properties of a given REQHO system.

The paper is organized as follows. In the next Section we briefly review the properties of the DTs and their generalization in the form of the DCKATs. In Section 3 we discuss general schemes of the DCKATs for generation of the REQHO systems from the QHO by using different sets of physical and non-physical eigenstates of the latter. Any REQHO system can be characterized by the total number of separated states in the low part of its spectrum, by the number of the gapless 'valence bands' in which the eigenvalues of separated states are organized, and by the total number of missing energy levels and their position in the low part of the spectrum. In Section 4 we consider in detail the problem of construction of different ladder operators for the simplest REQHO system with one separated state and two missing energy levels in a unique gap that separates it from the equidistant infinite part of the spectrum. We investigate there the general properties and relations between the basic (primary) and secondary, higher-order ladder operators. In Section 5 we consider two more particular examples of the REQHO systems, one of which corresponds to a generalization of the system from Section 4. Another example corresponds to a REQHO system with one valence band composed from the two energy levels separated from the equidistant infinite part of the spectrum by the gap of two missing energy levels. The results and observations obtained in Sections 4 and 5 are generalized then in Section 6 for the case of the REQHO systems of a general form. Section 7 is devoted to a summary of the obtained results, where we also indicate some problems that could be interesting for further investigation.

## 2 Darboux-Crum-Krein-Adler transformations

Let $\psi_*(x)$ be a *nodeless* physical or non-physical *real* eigenfunction of a Hamiltonian operator $H = -\frac{d^2}{dx^2} + V(x)$ with eigenvalue $E_*$, $H\psi_* = E_*\psi_*$. We assume that potential $V(x)$ is a non-singular real function on all the real line $\mathbb{R}$. Define the first order differential operators

$$A \equiv \psi_* \frac{d}{dx}\frac{1}{\psi_*} = \frac{d}{dx} - \mathcal{W}\,, \qquad A^\dagger = -\frac{1}{\psi_*}\frac{d}{dx}\psi_*\,, \tag{2.1}$$

where $\mathcal{W} = \frac{\psi_*'}{\psi_*}$, $\psi_*' = \frac{d\psi_*}{dx}$. They factorize the shifted Hamiltonian, $H - E_* = A^\dagger A$, whose potential is given by $V = \mathcal{W}^2 + \mathcal{W}' + E_*$ in terms of the superpotential $\mathcal{W}$ and factorization energy $E_*$. The product with permuted operators $A$ and $A^\dagger$ defines a supersymmetric partner Hamiltonian, $AA^\dagger \equiv \breve{H} - E_*$, $\breve{H} = -\frac{d^2}{dx^2} + V_*$, for which $\breve{V} = \mathcal{W}^2 - \mathcal{W}' + E_*$. The relation of the potential $\breve{V}(x)$

of the superpartner $\breve{H}$ to the potential $V(x)$ of the system $H$ can be rewritten in a more convenient form for a further generalization,

$$\breve{V} = V - 2\,(\ln\psi_*)'' \,. \tag{2.2}$$

From factorization relations it follows that the operators $A$ and $A^\dagger$ intertwine the partner Hamiltonians,

$$AH = \breve{H}A\,, \qquad A^\dagger\breve{H} = HA^\dagger\,. \tag{2.3}$$

As a consequence, $A$ and $A^\dagger$ mutually map the eigenstates of the superpartners. Namely, if $\psi(x;E)$ is a physical or non-physical eigenstate of $H$ of eigenvalue $E \neq E_*$, $H\psi(x;E) = E\psi(x;E)$, then $\Psi(x;E) = A\psi(x;E)$ is an eigenstate of $\breve{H}$ of the same eigenvalue and of the same physical or non-physical nature. Vice versa, if $\Psi(x;E)$ is an eigenstate of $\breve{H}$ of eigenvalue $E \neq E_*$, then $A^\dagger\Psi(x;E)$ is an eigenstate of $H$ of the same eigenvalue and of the same nature.

If $\psi(x)$ is a solution of the second order differential equation $H\psi(x) = E\psi(x)$ for arbitrary value of $E$, a second, linearly independent solution of this equation is

$$\widetilde{\psi(x)} \equiv \psi(x)\int^x \frac{d\xi}{(\psi(\xi))^2}\,. \tag{2.4}$$

If $\psi(x)$ is a normalizable on $\mathbb{R}$ function, then $\widetilde{\psi(x)}$ is not normalizable, and vice versa. The properties of $A$ and $A^\dagger$ as the operators that mutually map the corresponding eigenstates $\psi(x)$ of $H$ and $\Psi(x)$ of $\breve{H}$ of eigenvalue $E \neq E_*$ are valid also for the associated eigenstates $\widetilde{\psi(x)}$ and $\widetilde{\Psi(x)}$. The case $E = E_*$ in this context is different. The eigenstate $\psi_*$ of $H$ of eigenvalue $E = E_*$ constitutes the kernel of the operator $A$, $A\psi_* = 0$. The same is valid for the state $\Psi_*(x) \equiv 1/\psi_*(x)$, which constitutes the kernel of $A^\dagger$, $A^\dagger\Psi_* = 0$, and is the eigenstate of $\breve{H}$ of the same eigenvalue $E = E_*$. In this special case of $E = E_*$, the operator $A$ transforms the state $\widetilde{\psi_*}$ into the state $\Psi_* = 1/\psi_* \in \ker(A^\dagger)$:

$$\widetilde{A\psi_*(x)} = \left(\psi_*\frac{d}{dx}\frac{1}{\psi_*}\right)\widetilde{\psi_*(x)} = \psi_*(x)\frac{d}{dx}\int^x \frac{d\xi}{(\psi_*(\xi))^2} = \frac{1}{\psi_*(x)}\,. \tag{2.5}$$

Analogously, the state $\widetilde{\Psi_*} = \widetilde{(1/\psi_*)}$ is transformed by $A^\dagger$ into $\psi_*$ being a kernel of $A$. More details can be found in [51] where the same notation is used.

Let us stress that for our constructions below it is important that if $\psi_*$ is a normalizable state, then the state $\widetilde{\psi_*}$ is not normalizable, and vice versa, and that the same property is valid for the pair of the states $\Psi_* = 1/\psi_*$ and $\widetilde{\Psi_*}$. Before the discussion of the concrete quantum systems we, however, completely neglect the questions of normalizability of the corresponding wave functions. Furthermore, here and in what follows we do not preoccupy about normalization of the states, and specify wave functions modulo a constant multiplication factor.

The DT construction can be generalized for the case of the DCKAT. The latter is generated on the basis of several seed eigenstates $\psi_{i_1}$, $\psi_{i_2}$,..., $\psi_{i_n}$ of $H$ of different eigenvalues $E_{i_k}$, $H\psi_{i_k} = E_{i_k}\psi_{i_k}$, $k = 1,\dots,n$, with $E_{i_k} \neq E_{i_{k'}}$ for $i_k \neq i_{k'}$. To get a nonsingular partner system $H_n$, these states should be such that their Wronskian $\mathbb{W}_n(x) \equiv \mathbb{W}(\psi_{i_1}(x), \psi_{i_2}(x), \dots, \psi_{i_n}(x)) = \det||\mathcal{F}(x)||$, $\mathcal{F}_{ij} = \frac{d^{i-1}}{dx^{i-1}}\psi_j$, $i = 1,\dots,n$, $j = i_1,\dots,i_n$, is a nodeless function. The partner system $H_n$ is then given by the potential

$$V_n(x) = V(x) - 2\,(\ln\mathbb{W}_n(x))''\,. \tag{2.6}$$

The eigenstates $\psi(x;E)$ of $H$ are mapped into the eigenstates $\Psi(x;E)$ of $H_n$ of the same eigenvalue $E$ via the relation

$$\Psi(x;E) = \frac{\mathbb{W}(\psi_{i_1}(x),\dots,\psi_{i_n}(x),\psi(x;E))}{\mathbb{W}_n(x)}\,. \tag{2.7}$$

In the case of $n = 1$, Eq. (2.6) reduces to Eq. (2.2) while (2.7) reduces to the relation corresponding to the case of the DT presented in the form $\Psi(x; E) = A_1\psi(x; E)$, with $\psi_{i_1} = \psi_*$, $A_1 = A$ and $H_1 = \breve{H}$. Furthermore, for $n > 1$ Eq. (2.7) can be presented in a form that generalizes the indicated DT's formula. For this we iteratively define two sequences of related differential operators $A_m$, $m = 1, \ldots$, and $\mathbb{A}_m$, $m = 0, 1, \ldots$, as follows: $\mathbb{A}_0 \equiv 1$,

$$A_m = (\mathbb{A}_{m-1}\psi_{i_m})\,\frac{d}{dx}\,\frac{1}{(\mathbb{A}_{m-1}\psi_{i_m})}\,, \qquad m = 1, 2, \ldots, \tag{2.8}$$

$$\mathbb{A}_n = A_n A_{n-1} \ldots A_1\,. \tag{2.9}$$

Let us denote $H_0 \equiv H$ and define $H_{m-1} = A_m^\dagger A_m + E_{i_m}$. We have $A_m A_m^\dagger = H_m - E_{i_m}$, and obtain a generalization of the intertwining relations (2.3), $A_m H_{m-1} = H_m A_m$, $A_m^\dagger H_m = H_{m-1} A_m^\dagger$. Then we can present (2.7) in the equivalent form

$$\Psi(x; E) = \mathbb{A}_n\psi(x; E)\,. \tag{2.10}$$

The $n$-th order differential operators $\mathbb{A}_n$ and $\mathbb{A}_n^\dagger$ intertwine the partner systems $H_0$ and $H_n$,

$$\mathbb{A}_n H_0 = H_n\mathbb{A}_n\,, \qquad \mathbb{A}_n^\dagger H_n = H_0\mathbb{A}_n^\dagger\,. \tag{2.11}$$

The products of operators $\mathbb{A}_n$ and $\mathbb{A}_n^\dagger$ turn out to be polynomials in the corresponding Hamiltonian operators with roots equal to the energies of factorization:

$$\mathbb{A}_n^\dagger\mathbb{A}_n = \prod_{k=1}^{n}(H_0 - E_{i_k})\,, \qquad \mathbb{A}_n\mathbb{A}_n^\dagger = \prod_{k=1}^{n}(H_n - E_{i_k})\,. \tag{2.12}$$

For non-singular partners $H_0$ and $H_n$, some or all of the intermediate second-order differential operators $H_m$ with $m = 1, \ldots, n-1$ can be singular. If we change the order of the seed states, the Wronskian is left invariant modulo a possible multiplication by $(-1)$, that does not change Eqs. (2.6) and (2.7). Differential operator $\mathbb{A}_n$ is not changed either, and from (2.7) one can conclude that the kernel of $\mathbb{A}_n$ is spanned by the complete set of the seed states, $\ker(\mathbb{A}_n) = \mathrm{span}\{\psi_{i_1}, \ldots, \psi_{i_n}\}$. However, under permutation in the order of the seed eigenstates the first order differential operators entering into the factorized form (2.9) of $\mathbb{A}_n$ are changed. As a result, the nature of some or of all of the intermediate systems $H_m$ can be changed from a non-singular (singular) to a singular (non-singular).

## 3 Generation of the REQHO systems from the QHO

Let us turn now to the specific example of the QHO system given by the Hamiltonian operator $H_{\rm osc} = -\frac{d^2}{dx^2} + x^2$. Its bound eigenstates are described by (not normalized here) wave functions

$$\psi_n(x) = H_n(x)e^{-x^2/2} \tag{3.1}$$

which correspond to eigenvalues $E_n = 2n + 1$, $n = 0, 1, \ldots$, where $H_n(x)$ are Hermite polynomials. The change of variable $x \to i\,x$ generates a change of the sign of the Hamiltonian, $H_{\rm osc} \to -H_{\rm osc}$, and so, transforms physical eigenstates $\psi_n(x)$ into non-physical eigenstates $\psi_n^-(x) = \mathcal{H}_n(x)e^{x^2/2}$ of $H_{\rm osc}$ of eigenvalues $E_n^- = -(2n + 1)$, $n = 0, 1, \ldots$, where $\mathcal{H}_n(x) = H_n(i\,x)$. Unlike $H_n(x)$, the polynomials $\mathcal{H}_n(x)$ with even index have no real zeros while the unique real zero of these polynomials with odd index is at $x = 0$. More details can be found in [51].

The choice of the ground state $\psi_0 = e^{-x^2/2}$ as a seed eigenfunction $\psi_*$ for the DT in (2.1) generates the first order differential operators

$$A_1 = \frac{d}{dx} + x \equiv a^- \,, \qquad A_1^\dagger = -\frac{d}{dx} + x \equiv a^+ \,. \tag{3.2}$$

They factorize the shifted Hamiltonian, $a^+a^- = H_{\rm osc} - 1$, and satisfy the commutation relation

$$[a^-, a^+] = 2 \,. \tag{3.3}$$

As a consequence we have

$$[H_{\rm osc}, a^\pm] = \pm 2a^\pm \,. \tag{3.4}$$

The two relations in (3.4) can be rewritten equivalently in the form

$$a^- H_{\rm osc} = (H_{\rm osc} + 2)a^- \,, \qquad a^+(H_{\rm osc} + 2) = H_{\rm osc}a^+ \,, \tag{3.5}$$

and mean that $a^-$ and $a^+$ are the ladder lowering and raising operators of the QHO. Relations in (3.5) can also be interpreted as that $a^-$ and $a^+$ intertwine the QHO system $H_{\rm osc} = H_0$ with the SUSY-partner system $H_1 = H_{\rm osc} + 2$ which is just the shifted QHO. Since $a^-\psi_0 = 0$, from the point of view of the DT one can consider the shifted system $H_1$ as the QHO with the removed ground state. But since $a^-\psi_n = \psi_{n-1}$, the partner system $H_1$ is the same QHO with the Hamiltonian shifted for $+2$. In this DT picture, the ground state $\psi_0$ of $H_{\rm osc}$ is created by application of $A_1^\dagger = a^+$ to the function $\widetilde{\psi_0^-} = \widetilde{(1/\psi_0)}$ that is a non-physical eigenstate of $H_1$ of eigenvalue $E = 1$.

The DCKAT generated on the basis of the set of the seed eigenstates $\psi_0, \ldots, \psi_{n-1}, \psi_n$, $n = 1, \ldots,$ produces the intertwining operators $\mathbb{A}^-_{n+1} = (a^-)^{n+1}$ and $\mathbb{A}^+_{n+1} = \mathbb{A}^\dagger_{n+1} = (a^+)^{n+1}$, which intertwine the QHO with the SUSY-partner system $H_{n+1} = H_{\rm osc} + 2(n+1)$,

$$\mathbb{A}^-_{n+1} H_{\rm osc} = H_{n+1}\mathbb{A}^-_{n+1}, \qquad \mathbb{A}^+_{n+1} H_{n+1} = H_{\rm osc}\mathbb{A}^+_{n+1} \,. \tag{3.6}$$

The case $n = 0$ here reproduces the relations of the DT generated by the choice $\psi_* = \psi_0$. Note that in the case of $n = 1$ the permutation of the seed states in the DCKAT construction, $(\psi_0, \psi_1) \to (\psi_1, \psi_0)$, gives rise to a singular operator

$$A_1 = \psi_1 \frac{d}{dx}\frac{1}{\psi_1} = \frac{d}{dx} + x - \frac{1}{x} \equiv a^-_{\rm iso} \,. \tag{3.7}$$

This operator acting on the second chosen seed eigenstate $\psi_0$ gives a function $-\frac{1}{x}e^{-x^2/2} = -1/\psi_1^-$, which according to (2.8) generates the second factorization operator $A_2 = \frac{d}{dx} + x + \frac{1}{x}$ that also is singular. The product of these two singular operators gives the same second-order non-singular intertwining operator [3] as the scheme with the $(\psi_0, \psi_1)$ pair: $A_2A_1 = (a^-)^2 = \mathbb{A}^-_2$. The intermediate Hamiltonian $H_1$ in this case is a singular at $x = 0$ operator corresponding to the (shifted) quantum isotonic oscillator, $A_1^\dagger A_1 = H_{\rm osc} - 3$, $A_1A_1^\dagger = H_1 - 3$, which is given by the potential $V_1(x) = x^2 + \frac{2}{x^2} + 2$. We also have here the relations $A_2^\dagger A_2 = H_1 - 1$, $A_2A_2^\dagger = H_{\rm osc} + 3$.

The choice of a nodeless non-physical state $\psi_* = \psi_0^-$ corresponding to factorization energy $E_* = -1$ in the DT construction gives

$$A_1 = \frac{d}{dx} - x = -a^+ \,, \qquad A_1^\dagger = -a^- \,. \tag{3.8}$$

[3] For the discussion of related phenomena in a context of the quantum second-order supersymmetry anomaly and coupling-constant metamorphosis see ref. [52]

We obtain the same ladder operators of the QHO, but now they will intertwine the QHO with the SUSY-partner system $H_1 = H_{\rm osc} - 2$,

$$a^+ H_{\rm osc} = (H_{\rm osc} - 2)a^+ , \qquad a^-(H_{\rm osc} - 2) = H_{\rm osc}a^- . \tag{3.9}$$

Here the action of $A_1$ on the non-physical eigenstate $\widetilde{\psi_0^-} = \widetilde{(1/\psi_0)}$ of $H_0 = H_{\rm osc}$ of eigenvalue $E = -1$ produces a physical ground state $\psi_0$ for the partner system $H_1$.

In the case of the DCKAT generated on the basis of the seed eigenstates $\psi_0^-, \ldots, \psi_{n-1}^-, \psi_n^-$, $n = 1, \ldots$, we obtain a partner system with the added $n+1$ bound states in the lower part of the spectrum of the QHO which is described by the shifted QHO Hamiltonian $H_{n+1} = H_{\rm osc} - 2(n+1)$.

The described DCKATs based on the choice of the seed eigenstates $\psi_0, \ldots, \psi_n$ or $\psi_0^-, \ldots, \psi_n^-$ reflect the property of the special shape invariance of the QHO system.

Before we proceed further, let us summarize briefly the main features of the non-singular DT and DCKAT schemes based on other choices of the sets of physical, $\psi_n$, and non-physical, $\psi_n^-$, eigenstates of the QHO as the seed eigenstates [29, 47, 50]. This will generalize the preceding discussion and will allow us to generate the REQHO systems.

Consider the DCKAT generated on the basis of the physical eigenstates $\psi_{i_1}$, $\psi_{i_2}$, ..., $\psi_{i_n}$, and non-physical eigenstates $\psi_{j_1}^-$, $\psi_{j_2}^-$, ..., $\psi_{j_l}^-$, of the QHO. The peculiarity of the physical and non-physical eigenstates in both families is that they have a form of polynomials multiplied by exponential functions $e^{-x^2/2}$ and $e^{x^2/2}$, respectively. As a result we obtain a quantum system described by a potential to be a rational function. In order a new quantum system generated by the DCKAT be non-singular, the complete set of the seed states has to be composed from the blocks of the states $(\psi_0^-, \ldots, \psi_{j_1-1}^-, \psi_{j_1}^-)$, $(\psi_{j_2}^-, \ldots, \psi_{j_2+l_2}^-), \ldots, (\psi_{j_r}^-, \ldots, \psi_{j_r+l_r}^-)$, $(\psi_0, \ldots, \psi_{i_1-1}, \psi_{i_1})$, $(\psi_{i_2}, \ldots, \psi_{i_2+2m_2^+ +1}), \ldots, (\psi_{i_s}, \ldots, \psi_{i_s+2m_s^+ +1})$, where $j_2 = j_1 + 2m_1^- + 3$, $j_{k+1} = j_k + 2m_k^- + 3$, $i_1 < i_2$, $i_k + 2m_k^+ + 1 < i_{k+1}$, and $m_k^+$, $m_k^-$ and $l_k$ can take values $0, 1, \ldots$. The total number of such blocks can be arbitrary and blocks that include the states $\psi_0$ and $\psi_0^-$ can be absent. Up to a possible constant shift, the generated system will have the gapped spectrum of the QHO with the deleted levels appearing at the positions of energies corresponding to physical states $\psi_i$ in these blocks, and with new, added energy levels appearing at the positions of energies of non-physical eigenstates $\psi_j^-$. In other words, the inclusion of physical states $\psi_i$ into the generating set of the seed states eliminates the energy levels, while the inclusion of non-physical states $\psi_j^-$ introduces corresponding additional energy levels into the spectrum. Each gap in the spectrum of the resulting system contains an even number of the missing energy levels. As in the simplest examples we considered above with the partner system to be the same but the shifted QHO, the same REQHO system can be produced by DCKATs based on different choices of the sets of the seed eigenstates. Different choices of the seed states generate different intertwining operators which relate REQHO system with the QHO. As a consequence, as we shall see, there exist different ladder operators for the same REQHO, which possess different properties. Below we first consider some simple concrete examples of the REQHO systems. This will allow us to investigate in detail the families of the DCKATs associated with a given REQHO system, to identify different ladder operators, and to study their properties as well as to establish the relations between them. Then the results will be developed for the case of the REQHO systems of a general form.

## 4 Simplest REQHO system and its ladder operators

A simplest REQHO system can be produced by taking a nodeless non-physical eigenstate of energy $E = -5$ of $H_{\rm osc}$,

$$\psi_2^- = (1 + 2x^2)e^{x^2/2} . \tag{4.1}$$

The first order differential operators

$$A^- \equiv \psi_2^- \frac{d}{dx} \frac{1}{\psi_2^-} = \frac{d}{dx} - x - \frac{4x}{2x^2+1}\,, \qquad A^+ = (A^-)^\dagger \tag{4.2}$$

factorize the shifted QHO Hamiltonian

$$A^+A^- = -\frac{d^2}{dx^2} + x^2 + 5 = H_{\rm osc} + 5 \equiv H\,. \tag{4.3}$$

Their permuted product generates a simplest REQHO system,

$$A^-A^+ = -\frac{d^2}{dx^2} + x^2 + 3 + 8\frac{2x^2-1}{(2x^2+1)^2} \equiv \breve{H}\,. \tag{4.4}$$

We have the intertwining relations

$$A^-H = \breve{H}A^-\,, \qquad A^+\breve{H} = HA^+\,, \tag{4.5}$$

from which it follows that the systems $H$ and $\breve{H}$ are almost isospectral, and operators (4.2) provide a map between the eigenstates of the QHO and the REQHO systems. The excited eigenstates of $\breve{H}$ are the bound states

$$\Psi_n(x) = A^-\psi_n(x)\,, \qquad E_n = 6 + 2(n-1)\,, \qquad n = 1, 2, \dots, \tag{4.6}$$

where $\psi_n(x)$ are the QHO eigenstates (3.1). In correspondence with the general relation (2.5), the ground state and its energy are

$$\Psi_0 = A^-\widetilde{\psi_2^-} = \frac{1}{\psi_2^-}\,, \qquad E_0 = 0\,. \tag{4.7}$$

In correspondence with a general picture described above, this state constitutes the kernel of the operator $A^+$. The ladder operators for $\breve{H}$ can be constructed by the Darboux-dressing of the ladder operators $a^\pm$ of the QHO,

$$\mathcal{A}^\pm = A^-a^\pm A^+\,. \tag{4.8}$$

We have

$$[\breve{H}, \mathcal{A}^\pm] = \pm 2\mathcal{A}^\pm\,, \tag{4.9}$$

and

$$\mathcal{A}^+\mathcal{A}^- = \breve{H}(\breve{H}-2)(\breve{H}-6)\,, \qquad \mathcal{A}^-\mathcal{A}^+ = \breve{H}(\breve{H}+2)(\breve{H}-4). \tag{4.10}$$

Due to the last factor in (4.8) and in correspondence with relations (4.10), the ground-state of the REQHO of zero energy, $\Psi_0 = 1/\psi_2^-$, is annihilated by both ladder operators $\mathcal{A}^-$ and $\mathcal{A}^+$. Another peculiarity is that the kernel of the lowering operator $\mathcal{A}^-$ also contains the first excited physical state $\Psi_1 = A^-\psi_0$ of energy $E = 6$, and the non-physical state $A^-\psi_1^-$ which is the eigenstate of $\breve{H}$ of the eigenvalue $E = 2$. The three-dimensional kernel of the lowering ladder operator $\mathcal{A}^-$ is therefore

$$\ker(\mathcal{A}^-) = \text{span}\,\{\Psi_0, A^-\psi_1^-, \Psi_1\}\,. \tag{4.11}$$

Besides the ground state $\Psi_0$, the kernel of $\mathcal{A}^+$ includes the two states $A^-\psi_3^-$ and $A^-\psi_0^-$, which are non-physical eigenstates of $\breve{H}$ of the eigenvalues $-2$ and 4,

$$\ker(\mathcal{A}^+) = \text{span}\,\{\Psi_0, A^-\psi_3^-, A^-\psi_0^-\}\,. \tag{4.12}$$

We denote $(\alpha_1)$ this scheme based on the DT with generating function $\psi_2^-$,

$$(\alpha_1) = \{\psi_2^-\}. \tag{4.13}$$

Up to a global shift, the same REQHO system can also be produced by means of any of the DCKAT schemes

$$(\alpha_2) = \{\psi_0^-, \psi_3^-\}, \dots, (\alpha_{n+1}) = \{\psi_0^-, \psi_1^-, \dots, \psi_{n-1}^-, \psi_{n+2}^-\}. \tag{4.14}$$

The presence of the first $n$ states $\psi_0^-, \psi_1^-, \dots, \psi_{n-1}^-$ in the scheme $(\alpha_{n+1})$ gives rise to the addition of the corresponding energy levels into the spectrum of the QHO while the inclusion of the state $\psi_{n+2}^-$ results finally in the generation of the shifted gapped REQHO system $H_{n+1} = \breve{H} - 2n$. The intertwining operators between $H$ defined by (4.3) and $H_{n+1}$ in this case are

$$\mathbb{A}_{n+1} = A^-(a^+)^n, \qquad \mathbb{A}_{n+1}^\dagger = (a^-)^n A^+, \tag{4.15}$$

where $A^-$ and $A^+$ are given by Eq. (4.2). Combining the intertwining operators of the schemes $(\alpha_{n+1})$ and $(\alpha_1)$, we can construct the higher-order ladder operators

$$A^-\mathbb{A}_{n+1}^\dagger = A^-(a^-)^n A^+ \equiv \mathcal{A}_n^-, \qquad \mathbb{A}_{n+1}A^+ = A^-(a^+)^n A^+ = (\mathcal{A}_n^-)^\dagger \equiv \mathcal{A}_n^+, \tag{4.16}$$

$[\breve{H}, \mathcal{A}_n^\pm] = \pm 2n\mathcal{A}_n^\pm$, where $\mathcal{A}_1^\pm = \mathcal{A}^\pm$. This in particular means that the third-order differential operators (4.8), which have the nature of the Darboux-dressed QHO operators $a^\pm$, can also be considered as the ladder operators generated via a composition of the intertwining operators corresponding to the schemes $(\alpha_1)$ and $(\alpha_2)$, $\mathcal{A}^- = A^-(a^-A^+)$, $\mathcal{A}^+ = (A^-a^+)A^+$. The use of the scheme $(\alpha_n)$ with $n > 2$ instead of $(\alpha_2)$ in such a composition provides us therefore with the $(\alpha_1)$-Darboux-dressed form (4.16) of the higher-order ladder operators $(a^\pm)^n$ of the QHO. The following relations between $\mathcal{A}^-$ and $\mathcal{A}_n^-$ are valid,

$$(\mathcal{A}^-)^n = \prod_{j=1}^{n-1}(\breve{H} + 2j)\cdot\mathcal{A}_n^- = \mathcal{A}_n^-\cdot\prod_{j=1}^{n-1}(\breve{H} - 2j), \tag{4.17}$$

and analogous relations are obtained from them for $\mathcal{A}^+$ and $\mathcal{A}_n^+$ by the Hermitian conjugation. In a more general case the composition of the intertwining operators of the schemes $(\alpha_n)$ and $(\alpha_m)$ with $n > m$ generates the higher-order ladder operators $\mathcal{A}_{n-m}^\pm$,

$$\mathbb{A}_{m+1}\mathbb{A}_{n+1}^\dagger = \prod_{j=1}^{m}(\breve{H} - 4 - 2j)\cdot\mathcal{A}_{n-m}^-, \qquad \mathbb{A}_{n+1}\mathbb{A}_{m+1}^\dagger = \mathcal{A}_{n-m}^+\cdot\prod_{j=1}^{m}(\breve{H} - 4 - 2j). \tag{4.18}$$

The REQHO system (4.4 ) can also be generated via the DCKAT based on the physical eigenstates $\psi_1$ and $\psi_2$. We denote this scheme, which eliminates two neighbour energy levels $E = 8$ and $E = 10$ in the spectrum of the shifted QHO (4.3), as $(\beta_2)$:

$$(\beta_2) = \{\psi_1, \psi_2\}. \tag{4.19}$$

We denote $\mathbb{B}_2^\pm$ the second-order intertwining operators $\mathbb{A}_2$ and $\mathbb{A}_2^\dagger$ constructed on the basis of these two states according to (2.9),

$$\mathbb{B}_2^- \equiv A_{\rm iso}^- a_{\rm iso}^-, \qquad \mathbb{B}_2^+ \equiv (\mathbb{B}_2^-)^\dagger = a_{\rm iso}^+ A_{\rm iso}^+. \tag{4.20}$$

The operator $a_{\rm iso}^-$ is defined in Eq. (3.7), and $a_{\rm iso}^+ = (a_{\rm iso}^-)^\dagger$. The result of the action of the operator $a_{\rm iso}^-$ on the QHO's eigenstate $\psi_2$ can be presented in terms of its physical and non-physical

eigenstates in the form $a^-_{\rm iso}\psi_2 = -\psi_0\psi_2^-/\psi_1^- \equiv \phi$. The first-order differential operators $A^\pm_{\rm iso}$ are generated by the function $\phi(x)$, according to (2.8),

$$A^-_{\rm iso} = \phi(x)\frac{d}{dx}\frac{1}{\phi(x)} = \frac{d}{dx} + x + \frac{1}{x} - \frac{4x}{1+2x^2}\,, \qquad A^+_{\rm iso} = (A^-_{\rm iso})^\dagger\,. \tag{4.21}$$

We have

$$a^+_{\rm iso}a^-_{\rm iso} = H - 8\,, \qquad a^-_{\rm iso}a^+_{\rm iso} = H_{\rm iso}\,, \tag{4.22}$$

where

$$H_{\rm iso} = -\frac{d^2}{dx^2} + x^2 + \frac{2}{x^2} - 1 \tag{4.23}$$

is the shifted isotonic oscillator to be singular at $x = 0$, and $H$ corresponds to the shifted Hamiltonian of the QHO defined in (4.3). We also have the relations

$$A^+_{\rm iso}A^-_{\rm iso} = H_{\rm iso} - 2\,, \qquad A^-_{\rm iso}A^+_{\rm iso} = \breve{H} - 4\,, \tag{4.24}$$

where $\breve{H}$ is the Hamiltonian of the REQHO system defined in Eq. (4.4). From (4.22) and (4.24) we find

$$\mathbb{B}_2^- H = (\breve{H} + 6)\mathbb{B}_2^-\,, \qquad \mathbb{B}_2^+(\breve{H} + 6) = H\mathbb{B}_2^+\,, \tag{4.25}$$

and

$$\mathbb{B}_2^+\mathbb{B}_2^- = (H - 8)(H - 10)\,, \qquad \mathbb{B}_2^-\mathbb{B}_2^+ = (\breve{H} - 2)(\breve{H} - 4)\,. \tag{4.26}$$

By the construction, $\ker(\mathbb{B}_2^-) = {\rm span}\,\{\psi_1, \psi_2\}$. One can also see that $\ker(\mathbb{B}_2^+) = {\rm span}\,\{A^-\psi_0^-, A^-\psi_1^-\} = {\rm span}\,\{\mathbb{B}_2^-\widetilde{\psi_1}, \mathbb{B}_2^-\widetilde{\psi_2}\} = {\rm span}\,\{\frac{\psi_1}{\mathbb{W}_2}, \frac{\psi_2}{\mathbb{W}_2}\}$, where $\mathbb{W}_2(x) = \mathbb{W}(\psi_1, \psi_2)(x) = -\psi_2^-(x)\,e^{-\frac{3}{2}x^2}$.

The DCKATs corresponding to the $(\alpha_1)$- and $(\beta_2)$-schemes are in some sense complementary. The $(\alpha_1)$-scheme introduces effectively a new energy level into the spectrum of the QHO below its ground-state energy at the distance equal to the tripled distance between equidistant energy levels. The $(\beta_2)$-scheme makes a similar job but by deleting the first two excited energy levels in the spectrum of the QHO. Since the Wronskian in the DCKAT in the latter scheme includes the additional exponential factor $e^{-\frac{3}{2}x^2}$ in comparison with the structure of the non-physical eigenstate $\psi_2^-$, this produces the additional constant shift $+6$ in the potential generated by means of relation (2.6) that is reflected in Eq. (4.25), cf. Eq. (2.11). From Eq. (4.25) it follows that if $\psi(x; E)$ is an eigenstate of $H$ of energy $E$, then $\mathbb{B}_2^-\psi(x; E)$ is an eigenstate of $\breve{H}$ of energy $(E - 6)$. As a consequence, all the spectrum of the system generated by the $(\beta_2)$-scheme will be shifted for $-6$ in comparison with the spectrum of the REHQO system (4.4) produced via the $(\alpha_1)$-scheme. In correspondence with this picture, the ground-state $\Psi_0$ of $\breve{H}$ can alternatively be constructed from the QHO ground-state $\psi_0$, $\Psi_0 = \mathbb{B}_2^-\psi_0$, cf. (4.7). The excited states $\Psi_{n+1}$ of energy $E_{n+1} = 6 + 6n$ with $n = 0, \ldots$ can be presented in the alternative to (4.6) form $\Psi_{n+1} = \mathbb{B}_2^-\psi_{n+3}$. All this gives a possibility for the construction of another pair of ladder operators for the REQHO system (4.4) with the properties rather different to those of the ladder operators we obtained by using only the $(\alpha)$-schemes. For this we take now the composition of the intertwining operators of the $(\alpha_1)$- and $(\beta_2)$- schemes to construct the operators

$$\mathcal{C}^- = \mathbb{B}_2^- A^+\,, \qquad \mathcal{C}^+ = A^-\mathbb{B}_2^+\,. \tag{4.27}$$

Instead of (4.9) and (4.10), they satisfy the relations

$$[\breve{H}, \mathcal{C}^\pm] = \pm 6\,\mathcal{C}^\pm\,, \tag{4.28}$$

and

$$\mathcal{C}^+\mathcal{C}^- = \breve{H}(\breve{H}-8)(\breve{H}-10)\,, \qquad \mathcal{C}^-\mathcal{C}^+ = (\breve{H}+6)(\breve{H}-2)(\breve{H}-4)\,. \tag{4.29}$$

Like $\mathcal{A}^\pm$, these are third-order differential operators of the nature of ladder operators. However, acting on eigenstates of the REQHO system $\breve{H}$, they change the energies not in 2 but in 6. In this aspect they are somewhat similar to the higher-order ladder operators $\mathcal{A}_3^\pm$, which are fifth-order differential operators discussed above. The essential difference of $\mathcal{C}^-$ from $\mathcal{A}_n^-$ and in particular $\mathcal{A}^-$ is that in correspondence with the first relation from (4.29), the kernel of $\mathcal{C}^-$ is composed only from physical eigenstates of $\breve{H}$,

$$\ker(\mathcal{C}^-) = \mathrm{span}\,\{\Psi_0, \Psi_2, \Psi_3\}\,. \tag{4.30}$$

The energies 0, 8 and 10 of these states are the roots of the third-degree polynomial in the first identity in (4.29). Also, unlike $\mathcal{A}^+$, the kernel of the raising operator $\mathcal{C}^+$ is composed only from the non-physical eigenstates of $\breve{H}$,

$$\ker(\mathcal{C}^+) = \mathrm{span}\,\{A^-\psi_5^-, A^-\psi_1^-, A^-\psi_0^-\} = \mathrm{span}\,\{\mathbb{B}_2^-\psi_3^-, \mathbb{B}_2^-\widetilde{\psi_1}, \mathbb{B}_2^-\widetilde{\psi_2}\}. \tag{4.31}$$

In correspondence with the second relation in (4.29), the eigenvalues of the states in (4.31) are $-6$, 2 and 4. From the point of view of the structure of the kernels and commutation relations (4.28), the ladder operators $\mathcal{C}^\pm$ are similar to the third-order differential operators $(a^\pm)^3$ in the QHO system. However, unlike $\mathcal{C}^-$, the operator $(a^-)^3$ annihilates the three lowest physical eigenstates of the QHO of the three subsequent values of energy. The first exited state $\Psi_1$ of the REQHO system of energy $E = 6$ does not belong to the kernel of $\mathcal{C}^-$ and is annihilated by $(\mathcal{C}^-)^2$: $\mathcal{C}^-\Psi_1 = \Psi_0$, $(\mathcal{C}^-)^2\Psi_1 = 0$.

The following relations can be established by comparing the kernels of the operators on both sides of the equalities,

$$A^+\mathbb{B}_2^- = -(a^-)^3\,, \qquad \mathbb{B}_2^+A^- = -(a^+)^3\,. \tag{4.32}$$

These and their analogous relations for other REQHO systems will play important role in what follows. From them one can find in particular the operator identities $a^-\mathbb{B}_2^+ = -(a^+)^2A^+$, $(a^-)^2\mathbb{B}_2^+ = -a^+A^+(\breve{H}-2)$, as well as the Hermitian conjugate ones.

One can introduce additionally the operators $a^\pm$ inside the factorized structure of the operators $\mathcal{C}^\pm$. In this way one can construct the operators

$$\mathcal{C}_{n+1}^- = \mathbb{B}_2^-(a^-)^nA^+\,, \qquad \mathcal{C}_{n+1}^+ = A^-(a^+)^n\mathbb{B}_2^+\,, \quad n = 0, \dots\,, \tag{4.33}$$

with the implied identification $\mathcal{C}_1^\pm = \mathcal{C}^\pm$ for $n = 0$. They satisfy the relation $[\breve{H}, \mathcal{C}_{n+1}^\pm] = \pm(6+2n)\mathcal{C}_{n+1}^\pm$. These operators can be treated either as the QHO operators $(a^\pm)^n$ dressed by the intertwining generators of the $(\alpha_1)$ and $(\beta_2)$ schemes, or as the operators produced by intertwining operators (4.15) from the $(\alpha_{n+1})$ scheme and those from the same $(\beta_2)$ scheme. The kernel of $\mathcal{C}_{n+1}^-$ is composed only by the physical eigenstates of $\breve{H}$, while the kernel of $\mathcal{C}_{n+1}^+$ is spanned only by its non-physical eigenstates. For instance, $\ker(\mathcal{C}_2^-) = \mathrm{span}\,\{\Psi_0, \Psi_1, \Psi_3, \Psi_4\}$. With the help of identities (4.32) we also find that $(\mathcal{C}^\pm)^n = (-1)^{n+1}\mathcal{C}_{3(n-1)+1}^\pm$, $n = 1, \dots$ .

Analogously to $\mathcal{A}^\pm$, we also introduce the operators

$$\mathcal{B}^\pm = \mathbb{B}_2^-a^\pm\mathbb{B}_2^+\,. \tag{4.34}$$

Unlike the third-order ladder operators $\mathcal{A}^\pm$ and $\mathcal{C}^\pm$, the $\mathcal{B}^\pm$ are fifth-order differential operators. They satisfy the relations

$$[\breve{H}, \mathcal{B}^\pm] = \pm 2\mathcal{B}^\pm \tag{4.35}$$

and

$$\mathcal{B}^+\mathcal{B}^- = \breve{H}(\breve{H}-2)(\breve{H}-6)(\breve{H}-4)^2\,, \qquad \mathcal{B}^-\mathcal{B}^+ = \breve{H}(\breve{H}+2)(\breve{H}-4)(\breve{H}-2)^2\,. \tag{4.36}$$

The kernel of $\mathcal{B}^-$ involves two physical and three non-physical eigenstates of $\breve{H}$,

$$\ker(\mathcal{B}^-) = \mathrm{span}\,\{\Psi_0, \Psi_1, A^-\psi_0^-, A^-\psi_1^-, A^-\widetilde{\psi_0^-}\}\,. \tag{4.37}$$

The eigenvalue $E=4$ of the non-physical eigenstates $A^-\psi_0^-$ and $A^-\widetilde{\psi_0^-}$ in the kernel of $\mathcal{B}^-$ corresponds to the double root of the last factor in the first relation in (4.36). The kernel of the increasing ladder operator $\mathcal{B}^+$ includes only one physical eigenstate,

$$\ker(\mathcal{B}^+) = \mathrm{span}\,\{\Psi_0, A^-\psi_0^-, A^-\psi_1^-, A^-\psi_3^-, A^-\widetilde{\psi_1^-}\}\,. \tag{4.38}$$

The eigenvalue $E=2$ of the non-physical eigenstates $A^-\psi_1^-$ and $A^-\widetilde{\psi_1^-}$ in the kernel of the increasing ladder operator $\mathcal{B}^+$ corresponds to the double root of the last factor in the second relation in (4.36). By analogy with (4.16), one can consider the higher-order ladder operators

$$\mathcal{B}_n^\pm = \mathbb{B}_2^-(a^\pm)^n\mathbb{B}_2^-\,, \tag{4.39}$$

$[\breve{H}, \mathcal{B}_n^\pm] = \pm 2n\mathcal{B}_n^\pm$, with the identification $\mathcal{B}_1^\pm = \mathcal{B}^\pm$. The lowering operator $\mathcal{B}^-$ can be related to the ladder operators $\mathcal{A}^-$ and $\mathcal{C}^-$ via the identities

$$\mathcal{B}^- = \mathcal{A}^-(\breve{H}-4)\,, \qquad \mathcal{B}_2^- = (\mathcal{A}^-)^2\,, \qquad \mathcal{B}_3^- = -\mathcal{C}^-(\breve{H}-2)(\breve{H}-4)\,, \tag{4.40}$$

$\mathcal{B}_4^- = -\mathcal{C}_2^-(\breve{H}-2)(\breve{H}-4)$, $\mathcal{B}_5^- = -\mathcal{C}^-(\mathcal{A}^-)^2$, $\mathcal{B}_6^- = (\mathcal{C}^-)^2(\breve{H}-2)(\breve{H}-4)$, etc. The increasing operator $\mathcal{C}^+$ is related to $\mathcal{A}^+$ and $\mathcal{C}^+$ via the conjugate identities. Similarly to (4.17), for degrees $n>1$ of $\mathcal{B}^-$ we have

$$(\mathcal{B}^-)^n = \prod_{j=1}^{n-1}(\breve{H}-2+2j)(\breve{H}-4+2j)\cdot\mathcal{B}_n^-\,, \tag{4.41}$$

and an analogous relation for $(\mathcal{B}^+)^n$.

A generalization of the $(\beta_2)$-scheme corresponds to the family of the DCKAT schemes

$$(\beta_3) = \{\psi_0, \psi_2, \psi_3\}\,, \ldots, (\beta_{n+2}) = \{\psi_0, \ldots, \psi_{n-1}, \psi_{n+1}, \psi_{n+2}\}\,. \tag{4.42}$$

In the case of the scheme $(\beta_{n+2})$, the intertwining operators constructed according to the prescription (2.9) are $\mathbb{B}_{n+2}^- \equiv \mathbb{B}_2^-(a^-)^n$ and $\mathbb{B}_{n+2}^+ = (a^+)^n\mathbb{B}_2^+$. The schemes $(\beta_{n+2})$ do not give anything new but allow us to re-interprete the already discussed higher-order ladder operators $\mathcal{C}_{n+1}^\pm$ as those produced via the composition of the $(\beta_{n+2})$ and $(\alpha_1)$ schemes, $\mathcal{C}_{n+1}^- = \mathbb{B}_{n+2}^-A^+$, $\mathcal{C}_{n+1}^+ = A^-\mathbb{B}_{n+2}^+$. Analogously, $\mathcal{B}_{n+1}^- = \mathbb{B}_{n+2}^-\mathbb{B}_2^+$, $\mathcal{B}_{n+1}^+ = \mathbb{B}_2^-\mathbb{B}_{n+2}^+$.

Besides the two infinite families $(\alpha_n)$, $n=1,\ldots$, and $(\beta_n)$, $n=2,\ldots$, which involve as the seed eigenfunctions either only non-physical or only physical eigenstates of the QHO, there are two additional, 'intermediate' schemes which simultaneously include eigenstates of both types. These are the schemes

$$(\gamma_2) = \{\psi_0, \psi_1^-\}\,, \qquad (\gamma_3) = \{\psi_0, \psi_1, \psi_0^-\}\,. \tag{4.43}$$

The intertwining operators in the case of the scheme $(\gamma_2)$ are the second-order differential operators $\mathbb{A}_2 = A^-a^-$ and $\mathbb{A}_2^\dagger = a^+A^+$, while in the scheme $(\gamma_3)$, the intertwining operators are the third-order differential operators $\mathbb{A}_3 = A^-(a^-)^2$ and $\mathbb{A}_3^\dagger = (a^+)^2A^+$. These operators have a structure similar to that of the intertwining operators in the family of the schemes $(\alpha_n)$. With their help

we do not obtain anything essentially new for the construction of the ladder operators for the REQHO system $\breve{H}$ in comparison with the already discussed structures. Indeed, employing the superposition of the intertwining operators from the $(\beta_2)$-scheme and either $(\gamma_2)$- or $(\gamma_3)$- schemes, one can construct the ladder operators

$$\mathcal{C}^-_{-n} \equiv \mathbb{B}^-_2 (a^+)^n A^+ , \qquad \mathcal{C}^+_{-n} \equiv A^- (a^-)^n \mathbb{B}^+_2 , \qquad n = 1, 2 . \tag{4.44}$$

Here $n = 1$ and $n = 2$ correspond, respectively, to the $(\gamma_2)$- and $(\gamma_3)$- schemes. Operators $\mathcal{C}^\pm_{-1}$ are fourth-order differential operators, while $\mathcal{C}^\pm_{-2}$ are fifth-order differential operators. They, however, are not independent but can be expressed in terms of the already constructed intertwining operators. Namely, we have, in particular,

$$\mathcal{C}^\pm_{-1} = -\mathcal{A}^\pm_2 , \qquad \mathcal{C}^-_{-2} = -(\breve{H} - 2)\mathcal{A}^- , \qquad \mathcal{C}^+_{-2} = -(\breve{H} - 4)\mathcal{A}^+ . \tag{4.45}$$

These relations can be established by comparing the kernels of $\mathcal{C}^-_{-1}$ and $\mathcal{C}^-_{-2}$, $\ker(\mathcal{C}^-_{-1}) = \mathrm{span}\{\Psi_0, A^-\psi^-_0, \Psi_1, \Psi_2\}$, $\ker(\mathcal{C}^-_{-2}) = \mathrm{span}\{\Psi_0, A^-\psi^-_1, A^-\psi^-_0, A^-\widetilde{\psi^-_0}, \Psi_1\}$, with the kernels of $\mathcal{A}^-_2$ and of the operator $(\breve{H} - 2)\mathcal{A}^- = \mathcal{A}^-(\breve{H} - 4)$, respectively, and by comparison of the signs before the leading derivative terms in the corresponding pairs of operators. Due to the identities (4.32), a generalization of the operators (4.44) for $n > 2$ does not give us anything new since $\mathcal{C}^\pm_{-3} = -\breve{H}(\breve{H} - 2)(\breve{H} - 4)$. This last relation as well as relations (4.45) can also be obtained by employing the identities (4.32). We also have the operator identities

$$\mathcal{A}^+\mathcal{C}^+_n = (\breve{H} - 2)\mathcal{C}^+_{n+1} , \qquad \mathcal{A}^-\mathcal{C}^-_n = -\mathcal{A}^-_{n+3} , \qquad n = 1, \ldots , \tag{4.46}$$

$$\mathcal{A}^+\mathcal{C}^- = -(\breve{H} - 6)\mathcal{A}^-_2 , \qquad \mathcal{A}^-\mathcal{C}^+ = -(\breve{H} + 2)\mathcal{A}^+_2 , \tag{4.47}$$

as well as the Hermitian conjugate relations.

Let us look in more detail at the already mentioned similarity between the operators $\mathcal{C}^\pm$ and $\mathcal{A}^\pm_3$. Using the first identity from (4.32) and factorization relation (4.4), we obtain $\mathcal{A}^-_3 = A^-(a^-)^3A^+ = -\breve{H}\mathcal{C}^-$ and $\mathcal{A}^+_3 = -(\breve{H} - 6)\mathcal{C}^+$. These relations are similar to those in (4.45). Employing Eq. (4.17), we find that

$$(\mathcal{A}^-)^3 = -\breve{H}(\breve{H} + 2)(\breve{H} + 4)\mathcal{C}^- = -\mathcal{C}^-(\breve{H} - 6)(\breve{H} - 4)(\breve{H} - 2) . \tag{4.48}$$

This relation from the point of view of the kernels of the involved operators corresponds to the following picture. The kernel of the operator $\mathcal{A}^-$ is formed by the two physical eigenstates $\Psi_0$ and $\Psi_1$ and by one non-physical eigenstate $A^-\psi^-_1$. We have also the relations [51] $\mathcal{A}^-(A^-\widetilde{\psi^-_1}) = \Psi_0$, $\mathcal{A}^-(A^-\psi^-_0) = A^-\psi^-_1$, $\mathcal{A}^-\Psi_2 = \Psi_1$, $\mathcal{A}^-(A^-\widetilde{\psi^-_0}) = A^-\widetilde{\psi^-_1}$, $\mathcal{A}^-(A^-\widetilde{\psi_0}) = A^-\psi^-_0$, $\mathcal{A}^-\Psi_3 = \Psi_2$. As a result we obtain

$$\ker(\mathcal{A}^-)^3 = \mathrm{span}\{\Psi_0, A^-\psi^-_1, \Psi_1, A^-\widetilde{\psi^-_1}, A^-\psi^-_0, \Psi_2, A^-\widetilde{\psi^-_0}, A^-\widetilde{\psi_0}, \Psi_3\} . \tag{4.49}$$

The pairs of states $(A^-\psi^-_1, A^-\widetilde{\psi^-_1})$, $(A^-\psi^-_0, \widetilde{\psi^-_0})$ and $(\Psi_1 = A^-\psi_0, A^-\widetilde{\psi_0})$ constitute, respectively, the kernels of the factors $(H - 2)$, $(H - 4)$ and $(H - 6)$ in (4.48). The remaining three physical eigenstates $\Psi_0$, $\Psi_2$ and $\Psi_3$ in (4.49) correspond to the kernel of the operator $\mathcal{C}^-$.

According to Eq. (4.48) and its conjugate version, the ladder operators $\mathcal{C}^\pm$ can be generated by $\mathcal{A}^\pm$. Then with taking into account Eq. (4.40) and all the described relations, we conclude that in the case of the simplest REQHO system given by the Hamiltonian $\breve{H}$ defined in (4.4) all the set of the ladder operators can be obtained, in principle, from the compositions of the ladder operators $\mathcal{B}^+$ and $\mathcal{B}^-$.

In conclusion of this section let us note, however, that in comparison with the QHO picture, the peculiarity of the system (4.4) in particular is that its ground-state $\Psi_0$ cannot be achieved from physical states by action of the lowering operators $\mathcal{A}^-$ and $\mathcal{B}^-$ which are differential operators of orders 3 and 5. Like the first-order differential operator $a^-$ in the QHO, the ladder operators $\mathcal{A}^-$ and $\mathcal{B}^-$ decrease the energy values of $\breve{H}$ in 2, but they produce the ground-state by acting on the non-physical eigenstate $A^-\widetilde{\psi_1^-}$ of the eigenvalue $E = 2$. The ground-state $\Psi_0$ of zero energy can be achieved, however, by application of the lowering operator $\mathcal{C}^-$, which is a third-order differential operator, to the physical eigenstate $\Psi_1$ with eigenvalue $E = 6$.

We also notice here that the ladder operators (4.8) for the REQHO system of the simplest form (4.4) were constructed (without employing the Darboux-dressing procedure) in [20] where this model was introduced and investigated for the first time. Later these ladder operators were constructed, particularly, in [25], [46, 47] and recently in [51] via the Darboux-dressing prescription based on the non-physical seed state we used here. The ladder operators (4.27) we constructed on the basis of the $(\alpha_1)$- and $(\beta_2)$- schemes by employing the analogy with reflectionless quantum systems [14, 15] where the corresponding Lax-Novikov integrals of motion can be generated either by Darboux-dressing of the free particle momentum operator or by 'gluing' two different intertwining operators that act in the opposite directions. The same last mentioned method also allows ones to generate the Lax-Novikov integrals for periodic finite-gap systems where the Darboux-dressing mechanism can not be applied, see [18]. In reflectionless and finite-gap systems, however, the two glued intertwiners always are differential operators of the 'opposite', even and odd, differential orders, but both intertwine the two corresponding partner systems without additional relative displacement. It is because of the relative displacement in intertwining relations (4.5) and (4.25) that here we obtain the ladder operators $\mathcal{C}^\pm$ while in reflectionless and finite-gap systems analogous procedure generates the integrals of motion. Within the same framework we used here and based on employing the two schemes with physical and non-physical seed states of the quantum harmonic oscillator, the ladder operators (4.27) were introduced earlier in [47] (but without exploiting the indicated analogy with generation of the Lax-Novikov integrals) and derived later in [48] for more general families related to multi indexed exceptionnal orthogonal polynomials. By another method such ladder operators were introduced for the system (4.4) even earlier in [24]. The ladder operators (4.34) constructed here by the Darboux-dressing procedure based on physical seed states seems were not discussed earlier in the literature.

With subsequent analysis we shall see that the trinity of the basic ladder operators ($\mathcal{A}^\pm$, $\mathcal{B}^\pm$, $\mathcal{C}^\pm$) admits a natural generalization for the case of REQHO systems of a general form, and that each pair of the conjugate lowering and raising ladder operators detects and reflects some specific properties of a corresponding quantum system.

## 5 Two further examples of the REQHO systems

A REQHO system generated by the DT based on the non-physical state $\psi_{2n}^-$ with $n > 1$ is similar to the considered REQHO system generated by the DT based on $\psi_2^-$. In this case the gap in the spectrum of the REQHO corresponds, up to a global shift, to the missing $2n$ energy levels with $E = 3, \dots, 4n + 1$ in the spectrum of the QHO. We also have here the two infinite families of the DCKAT schemes of the structures which generalize those of the case $n = 1$. For instance, in the case of $n = 2$, we have $\psi_4^- = (4x^4 + 12x^2 + 3)e^{x^2/2}$, and the two infinite families of the schemes are

$$(\alpha_1) = \{\psi_4^-\}, \qquad (\alpha_2) = \{\psi_0^-, \psi_5^-\}, \qquad (\alpha_{n+1}) = \{\psi_0^-, \dots, \psi_{n-1}^-, \psi_{n+4}^-\}, \tag{5.1}$$

and

$$(\beta_4) = \{\psi_1, \psi_2, \psi_3, \psi_4\}, \qquad (\beta_{n+4}) = \{\psi_0, \dots, \psi_{n-1}, \psi_{n+1}, \psi_{n+2}, \psi_{n+3}, \psi_{n+4}\}. \tag{5.2}$$

In addition, we have the 'intermediate' schemes whose sets of seed states include both physical and non-physical eigenstates of the QHO. These are

$$(\gamma_2)=\{\psi_0,\psi_3^-\}\,,\quad (\gamma_3)=\{\psi_0,\psi_1,\psi_2^-\}\,,\quad (\gamma_4)=\{\psi_0,\psi_1,\psi_2,\psi_1^-\}\,,\quad (\gamma_5)=\{\psi_0,\psi_1,\psi_2,\psi_3,\psi_0^-\}\,. \tag{5.3}$$

The scheme $(\alpha_1)$ generates the intertwining operators $A^-=\psi_4^-\frac{d}{dx}\frac{1}{\psi_4^-}$, and $A^+=(A^-)^\dagger$. They allow us to construct ladder operators that are third-order differential operators, the Darboux-dressed ladder operators of the QHO, $\mathcal{A}^\pm=A^-a^\pm A^+$. They satisfy the relations of the form (4.9), $[\breve{H},\mathcal{A}^\pm]=\pm 2\mathcal{A}^\pm$, with

$$\breve{H}\equiv A^-A^+=-\frac{d^2}{dx^2}+x^2+7+32\frac{4x^6+4x^4+3x^2-6}{(4x^4+12x^2+3)^2}\,, \tag{5.4}$$

and similarly to the already considered case, here both ladder operators $\mathcal{A}^\pm$ annihilate the ground state of $\breve{H}$ which is $\Psi_0=1/\psi_4^-=A^-\widetilde{\psi_4^-}$. Besides the ground state $\Psi_0$ of energy $E=0$, the kernel of $\mathcal{A}^-$ contains, the first excited state $\Psi_1=A^-\psi_0$ of energy $E=10$ and one non-physical eigenstate $A^-\psi_3^-$ of energy $E=2$. The kernel of $\mathcal{A}^+$ contains besides the ground state $\Psi_0$ also two non-physical eigenstates $A^-\psi_0^-$ and $A^-\psi_5^-$ of $\breve{H}$ of eigenvalues $E=8$ and $E=-2$. The intertwining operators corresponding to the DCKAT scheme $(\beta_4)=\{\psi_1,\psi_2,\psi_3,\psi_4\}$ are the fourth order differential operators constructed in accordance with Eq. (2.9), which by analogy with the already considered case we denote here as $\mathbb{B}_4^-$ and $\mathbb{B}_4^+=(\mathbb{B}_4^-)^\dagger$. Then we define another pair of ladder operators via a composition of the intertwining operators of this $(\beta_4)$-scheme and of the $(\alpha_1)$-scheme, $\mathcal{C}^-=\mathbb{B}_4^-A^+$, $\mathcal{C}^+=A^-\mathbb{B}_4^+$. Unlike the previously discussed case of the REQHO system (4.4), these are fifth-order differential operators, which satisfy the relations $[\breve{H},\mathcal{C}\pm]=\pm 10\mathcal{C}^\pm$. The ladder operator $\mathcal{C}^-$ annihilates five physical eigenstates of $\breve{H}$, which are the ground state $\Psi_0$ and the states $\Psi_{j+1}=A^-\psi_j$, $j=1,2,3,4$, with the energy values $E_0=0$ and $E_{j+1}=10+2j$. Like in the REQHO system we considered before, the first excited state $\Psi_1=A^-\psi_0$ of energy $E_1=10$ here does not belong to the kernel of the decreasing ladder operator $\mathcal{C}^-$, and we have $\mathcal{C}^-\Psi_1=\psi_0$, $(\mathcal{C}^-)^2\Psi_1=0$. The kernel of $\mathcal{C}^+$ is composed only from non-physical eigenstates. Yet another pair of the ladder operators corresponds to $\mathcal{B}^\pm=\mathbb{B}_4^-a^\pm\mathbb{B}_4^+$, which are differential operators of order 9. Like $\mathcal{A}^\pm$, they satisfy the relations $[\breve{H},\mathcal{B}^\pm]=\pm 2\mathcal{B}^\pm$. The kernel of the lowering ladder operator $\mathcal{B}^-$ is spanned by two physical eigenstates $\Psi_0$ and $\Psi_1$ of energies 0 and 10, and seven non-physical eigenstates of $\breve{H}$ of eigenvalues 8 (twice), 6 (twice), 4 (twice) and 2, $\ker(\mathcal{B}^-)=\text{span}\{\Psi_0,\Psi_1,A^-\psi_0^-,A^-\widetilde{\psi_0^-},\Psi_1,A^-\psi_1^-,A^-\widetilde{\psi_1^-},\Psi_1,A^-\psi_2^-,A^-\widetilde{\psi_2^-},A^-\psi_3^-\}$. The kernel of $\mathcal{B}^+$ is spanned by the ground-state $\Psi_0$ and by eight non-physical eigenstates of $\breve{H}$.

Other, secondary ladder operators can be constructed by introducing the QHO ladder operators $(a^\pm)^n$ inside the factorized structures of the basic ladder operators $\mathcal{A}^\pm$, $\mathcal{B}^\pm$ and $\mathcal{C}^\pm$, or by considering compositions of the intertwining operators corresponding to (5.1), (5.2) and (5.3) schemes analogously to how it was done for the simplest REQHO system. The secondary, higher-order ladder operators can also be generated by taking the products of the basic ladder operators. Analogously to (4.32), we also have here the relations $A^+\mathbb{B}_4^-=-(a^-)^5$, $\mathbb{B}_4^+A^-=-(a^+)^5$. As an analog of relation (4.48) we have

$$(\mathcal{A}^-)^5=-\mathcal{C}^-(\breve{H}-10)(\breve{H}-8)(\breve{H}-6)(\breve{H}-4)\,, \tag{5.5}$$

and the relation

$$\mathcal{B}^-=(\breve{H}-2)(\breve{H}-4)(\breve{H}-6)\mathcal{A}^- \tag{5.6}$$

is analogous here to the first relation in (4.40). As in the case of the simplest REQHO system considered in the previous section, for the REQHO system described by the Hamiltonian(5.4) all

the ladder operators can be generated and extracted from the powers of the basic ladder operators $\mathcal{B}^{\pm}$.

Let us consider yet another example of the REQHO system in which two states are separated by a gap from the infinite equidistant part of the spectrum. A simplest system of such a nature can be generated by employing the minimal $(\alpha)$-scheme

$$(\alpha_2) = \{\psi_2^-, \psi_3^-\}\,. \tag{5.7}$$

Let us shift the Hamiltonian of the QHO for $+7$ and denote $H = H_{\rm osc} + 7$, for which the potential is $V(x) = x^2 + 7$ and the spectrum is $E_n = 8 + 2n$, $n = 0, 1, \ldots$. The Wronskian here is $\mathbb{W}_2(x) = e^{x^2}(3 + 4x^4)$. The second order DCKAT based on (5.7) produces the partner system which is the REQHO system $\breve{H} = -\frac{d^2}{dx^2} + \breve{V}(x)$, where in accordance with (2.6),

$$\breve{V}(x) = 3 + x^2 + 32x^2\frac{4x^4 - 9}{(3 + 4x^4)^2}\,. \tag{5.8}$$

Its gapped spectrum is

$$E_0 = 0\,, \quad E_1 = 2\,, \quad E_{2+n} = 8 + 2n\,, \quad n = 0, 1, \ldots\,. \tag{5.9}$$

The intertwining second order differential operators constructed via (2.9) on the basis of the seed QHO eigenstates (5.7) we denote as $\mathbb{A}_2^-$ and $\mathbb{A}_2^+ = (\mathbb{A}_2^-)^\dagger$. The kernel of $\mathbb{A}_2^-$ is spanned by the states $\psi_2^-$ and $\psi_3^-$, while the kernel of $\mathbb{A}_2^+$ is spanned by the lowest physical eigenstates of $\breve{H}$ of the energies 0 and 2, which can be obtained from the QHO non-physical eigenstates $\widetilde{\psi_2^-}$ and $\widetilde{\psi_3^-}$, $\Psi_0 = \mathbb{A}_2^-\widetilde{\psi_2^-}$, $\Psi_1 = \mathbb{A}_2^-\widetilde{\psi_3^-}$. The operators $\mathbb{A}_2^+$ and $\mathbb{A}_2^-$ satisfy the relations

$$\mathbb{A}_2^+\mathbb{A}_2^- = H(H-2)\,, \qquad \mathbb{A}_2^-\mathbb{A}_2^+ = \breve{H}(\breve{H}-2)\,, \tag{5.10}$$

and $\mathbb{A}_2^- H = \breve{H}\mathbb{A}_2^-$, $\mathbb{A}_2^+\breve{H} = H\mathbb{A}_2^+$. We construct the ladder operators $\mathcal{A}^{\pm}$ for $\breve{H}$ by the Darboux-dressing of the QHO operators $a^{\pm}$,

$$\mathcal{A}^{\pm} = \mathbb{A}_2^- a^{\pm}\mathbb{A}_2^+\,. \tag{5.11}$$

These fifth-order differential operators satisfy the relations $[\breve{H}, \mathcal{A}^{\pm}] = \pm 2\mathcal{A}^{\pm}$, and

$$\mathcal{A}^+\mathcal{A}^- = \breve{H}(\breve{H}-2)^2(\breve{H}-4)(\breve{H}-8)\,, \qquad \mathcal{A}^-\mathcal{A}^+ = (\breve{H}+2)(\breve{H})^2(\breve{H}-2)(\breve{H}-6)\,. \tag{5.12}$$

The kernel of the lowering operaror is $\ker(\mathcal{A}^-) = \mathrm{span}\{\Psi_0, \Psi_1, \Psi_2, \widetilde{\Psi_1}, \mathbb{A}_2^-\psi_1^-\}$. Here the first three states are the lowest three physical eigenstates of $\breve{H}$ of energies $E = 0$, 2 and 8, respectively, and the two last states are non-physical eigenstates of energies $E = 2$ and 4. The indicated energies correspond to the roots of the polynomial in the first equality in (5.12). The kernel of the increasing operator $\mathcal{A}^+$ is spanned by the two lowest eigenstates $\Psi_0$ and $\Psi_1$ and by the three non-physical eigenstates: $\ker(\mathcal{A}^+) = \mathrm{span}\{\Psi_0, \Psi_1, \mathbb{A}_2^-\psi_4^-, \widetilde{\Psi_0}, \mathbb{A}_2^-\psi_0^-\}$. The energies of these five eigenstates correspond to zeros of the polynomial in the second equality in (5.12).

The same system, up to a global shift, can also be generated via the complementary minimal $(\beta_2)$-scheme

$$(\beta_2) = \{\psi_2, \psi_3\}\,. \tag{5.13}$$

The Wronskian of the seed states in this case is $\mathbb{W}_2(x) = \mathbb{W}(\psi_2, \psi_3) = e^{-x^2}(3+4x^4)$. The potential calculated according to (2.6) is the potential (5.8) shifted for $+8$. Let us denote the corresponding intertwining second order differential operators constructed on the basis of these seed states as $\mathbb{B}_2^-$ and $\mathbb{B}_2^+ = (\mathbb{B}_2^-)^\dagger$. They satisfy the relations

$$\mathbb{B}_2^- H = (\breve{H}+8)\mathbb{B}_2^-\,, \qquad \mathbb{B}_2^+(\breve{H}+8) = H\mathbb{B}_2^+\,, \tag{5.14}$$

and

$$\mathbb{B}_2^+\mathbb{B}_2^- = (H-12)(H-14)\,, \qquad \mathbb{B}_2^-\mathbb{B}_2^+ = (\breve{H}-4)(\breve{H}-6)\,. \tag{5.15}$$

Note that here $\mathbb{B}_2^-(x) = -\mathbb{A}_2^-(ix)$. The kernel of $\mathbb{B}_2^-$ is spanned by the seed states (5.13), whereas the kernel of $\mathbb{B}_2^+$ is spanned by non-physical eigenstates of $\breve{H}$ of energies $E=4$ and $E=6$: $\ker(\mathbb{B}_2^+) = \mathrm{span}\,\{\mathbb{B}_2^-\widetilde{\psi_2}, \mathbb{B}_2^-\widetilde{\psi_3}\} = \mathrm{span}\,\{\mathbb{A}_2^-\psi_0^-, \mathbb{A}_2^-\psi_1^-\}$. The ladder operators

$$\mathcal{C}^- = \mathbb{B}_2^-\mathbb{A}_2^+\,, \qquad \mathcal{C}^+ = \mathbb{A}_2^-\mathbb{B}_2^+ \tag{5.16}$$

are differential operators of the order four. They obey the relations

$$[\breve{H}, \mathcal{C}^\pm] = \pm 8\,\mathcal{C}^\pm\,, \tag{5.17}$$

and

$$\mathcal{C}^+\mathcal{C}^- = \breve{H}(\breve{H}-2)(\breve{H}-12)(\breve{H}-14)\,, \qquad \mathcal{C}^-\mathcal{C}^+ = (\breve{H}+8)(\breve{H}+6)(\breve{H}-4)(\breve{H}-6)\,. \tag{5.18}$$

The lowering operator $\mathcal{C}^-$ annihilates the four physical states $\Psi_0, \Psi_1, \Psi_4, \Psi_5$ of the energies $E = 0, 2, 12, 14$. The kernel of $\mathcal{C}^+$ is spanned only by non-physical eigenstates of energies $E = -8, -6, 4, 6$, $\ker(\mathcal{C}^+) = \mathrm{span}\,\{\mathbb{B}_2^-\psi_3^-, \mathbb{B}_2^-\psi_2^-, \mathbb{B}_2^-\widetilde{\psi_2}, \mathbb{B}_2^-\widetilde{\psi_3}\} = \mathrm{span}\,\{\mathbb{A}_2^-\psi_7^-, \mathbb{A}_2^-\psi_6^-, \mathbb{A}_2^-\psi_1^-, \mathbb{A}_2^-\psi_0^-\}$. Changing the order of the second order operators in the factorized form of the ladder operators in (5.16), we obtain the operator identities

$$\mathbb{B}_2^+\mathbb{A}_2^- = (a^+)^4\,, \qquad \mathbb{A}_2^+\mathbb{B}_2^- = (a^-)^4\,, \tag{5.19}$$

cf. (4.32). Again, these relations can be verified by comparing the kernels of the corresponding operators.

Yet another pair of the ladder operators corresponds to differential operators of order 5,

$$\mathcal{B}^\pm = \mathbb{B}_2^- a^\pm \mathbb{B}_2^+\,. \tag{5.20}$$

They satisfy relations $[\breve{H}, \mathcal{B}^\pm] = \pm 2\mathcal{B}^\pm$, and

$$\mathcal{B}^+\mathcal{B}^- = \breve{H}(\breve{H}-4)(\breve{H}-6)^2(\breve{H}-8)\,, \qquad \mathcal{B}^-\mathcal{B}^+ = (\breve{H}+2)(\breve{H}-2)(\breve{H}-4)^2(\breve{H}-6)\,. \tag{5.21}$$

In correspondence with the first relation in (5.21), the kernel of the lowering operator is spanned by the ground-state $\Psi_0$ and the eigenstate $\Psi_2$ at the bottom of the equidistant infinite part of the spectrum as well as by the three non-physical eigenstates of $\breve{H}$, $\ker(\mathcal{B}^-) = \mathrm{span}\,\{\Psi_0, \Psi_2, \mathbb{A}_2^-\psi_0^-, \mathbb{A}_2^-\psi_1^-, \mathbb{A}_2^-\widetilde{\psi_0^-}\}$. The kernel of the increasing operator $\mathcal{B}^+$ is spanned by the first excited physical eigenstate $\Psi_1$ of energy 2, and by four non-physical eigenstates of $\breve{H}$ of eigenvalues $-2$, 4 (twice) and 6, $\ker(\mathcal{B}^+) = \mathrm{span}\,\{\Psi_1, \mathbb{A}_2^-\psi_3^-, \mathbb{A}_2^-\psi_1^-, \mathbb{A}_2^-\widetilde{\psi_1^-}, \mathbb{A}_2^-\psi_0^-\}$. Here $\mathcal{B}^-$ and $\mathcal{A}^-$ are related by the operator identity

$$\breve{H}\mathcal{B}^- = (\breve{H}-4)\mathcal{A}^-\,, \tag{5.22}$$

and the relation between the increasing ladder operators $\mathcal{B}^+$ and $\mathcal{A}^+$ is given by Hermitian conjugation of (5.22). Note that in comparison with the first relation in (4.40) and relation (5.6) in the examples of the REQHO systems with one separated energy level here the relation (5.22) contains a Hamiltonian-dependent factor before the operator $\mathcal{B}^-$. Coherently with this, in the REQHO system under consideration the lowering operator $\mathcal{B}^-$ in comparison with $\mathcal{A}^-$ annihilates only the lowest state $\Psi_0$ in the separated part of the spectrum. The raising operator $\mathcal{B}^+$ annihilates another separated state $\Psi_1$ in comparison with both separated states $\Psi_0$ and $\Psi_1$ annihilated by $\mathcal{A}^+$. This

difference can be understood if we note that the polynomial in the first identity in (5.21) does not have the root 2. Rewriting relation (5.22) in the equivalent form $\mathcal{B}^-(\breve{H}-2)=\mathcal{A}^-(\breve{H}-6)$, we see then that the annihilation of the state $\Psi_1$ from the kernel of $\mathcal{A}^-$ is provided by the factor $(\breve{H}-2)$ on the left hand side of the identity. In the same way one can understand the difference in the kernels of the raising operators $\mathcal{B}^+$ and $\mathcal{A}^+$ by looking at the roots of the polynomial in the second identity in (5.21) and by taking into account the identity relation $\mathcal{B}^+\breve{H}=\mathcal{A}^+(\breve{H}-4)$ to be conjugate to (5.22).

The same (up to a global shift) system $\breve{H}$ given by the potential (5.8) can be produced by using the higher-order $(\alpha)$- and $(\beta)$-schemes,

$$(\alpha_{n+2})=\{\psi_0^-,\dots,\psi_{n-1}^-,\psi_{n+2}^-,\psi_{n+3}^-\}\,,\qquad (\beta_{n+2})=\{\psi_0,\dots,\psi_{n-1},\psi_{n+2},\psi_{n+3}\}\,,\tag{5.23}$$

$n=1,\dots$, and by the two intermediate $(\gamma)$-schemes,

$$(\gamma_3)=\{\psi_0,\psi_1^-,\psi_2^-\}\,,\qquad (\gamma_4)=\{\psi_0,\psi_1,\psi_0^-,\psi_1^-\}\,.\tag{5.24}$$

The secondary, higher-order ladder operators can be generated here in the same way as for the REQHO systems with one added gapped bound state. Here the relation

$$(\mathcal{A}^-)^4=\mathcal{C}^-(\breve{H}-8)^2(\breve{H}-6)^2(\breve{H}-4)^2(\breve{H}-2)(\breve{H}-10)\,.\tag{5.25}$$

is analogous to the relations (4.48) and (5.5), and shows that the ladder operators $\mathcal{C}^\pm$ can be generated by the ladder operators $\mathcal{A}^\pm$.

## 6 REQHO systems of a general form and their ladder operators

We generalize now our analysis of the particular examples for the case of REQHO systems of a general form, for which we construct the ladder operators and investigate their properties.

Each REQHO system can be generated by employing the appropriate DCKAT based on any of the schemes from the two infinite families. The $(\alpha)$-type schemes include only non-physical eigenstates of the QHO chosen as the seed states. The $(\beta)$-type schemes involve only the corresponding physical eigenstates of the QHO. Besides, there exists also a finite number of intermediate $(\gamma)$-type schemes which simultaneously use the eigenstates of both types. Let us denote $\mathbb{A}^-_{n_+}$ and $\mathbb{A}^+_{n_+}$ the mutually conjugate intertwining operators constructed on the basis of the $(\alpha)$-type scheme with a minimal number $n_+$ of seed non-physical eigenstates. They are differential operators of order $n_+$. Analogously, let us denote the intertwining operators constructed on the basis of the $(\beta)$-scheme with a minimal number $2n_-$ of seed physical eigenstates as $\mathbb{B}^-_{2n_-}$ and $\mathbb{B}^+_{2n_-}$. By the construction we have $\mathbb{A}^-_{n_+}\psi^-_{j_s}=0$ and $\mathbb{B}^-_{2n_-}\psi_{i_s}=0$, where $\psi^-_{j_s}$ are $n_+$ non-physical eigenstates of the QHO which are the seed states in the $(\alpha_{n_+})$-scheme, $\psi^-_{j_s}\in(\alpha_{n_+})$, while $\psi_{i_s}$ are $2n_-$ physical eigenstates which are used in the $(\beta_{2n_-})$-scheme, $\psi_{i_s}\in(\beta_{2n_-})$. Such two minimal $(\alpha_{n_+})$- and $(\beta_{2n_-})$-schemes are complementary similarly to the schemes $(\alpha_1)$ and $(\beta_2)$ in the case of the simplest REQHO system we considered in detail above. The corresponding Wronskians in these two schemes have the form $\mathbb{W}_{n_+}=\exp\left(\frac{1}{2}x^2n_+\right)\phi(x)$ and $\mathbb{W}_{2n_-}=c\exp\left(-x^2n_-\right)\phi(x)$, where $\phi(x)$ is some nodeless polynomial function, and $c$ is some constant. We fix the additive constant shifts in the Hamiltonians $H$ of the QHO and $\breve{H}$ of the REQHO in such a way that $\mathbb{A}^-_{n_+}H=\breve{H}\mathbb{A}^-_{n_+}$ and that the ground-state $\Psi_0$ of $\breve{H}$ has zero energy, $E_0=0$. Then the lowest state $\Psi_{n_+}=\mathbb{A}^-_{n_+}\psi_0$ in the equidistant infinite part of the spectrum of $\breve{H}$ will be characterized by the energy value

$$E_{n_+}=2(n_++2n_-)\equiv 2\Delta\tag{6.1}$$

that also will be the energy of the ground-state $\psi_0$ of the shifted QHO Hamiltonian $H$, $H\psi_0 = E_{n_+}\psi_0$. Then we have $H = H_{\rm osc} - 1 + 2\Delta$. For the other basic lowering intertwining operator $\mathbb{B}^-_{2n_-}$ we have the relation

$$\mathbb{B}^-_{2n_-} H = (\breve{H} + 2\Delta)\mathbb{B}^-_{2n_-} \,. \tag{6.2}$$

This means in particular that if $\psi(x;E)$ is an eigenstate of $H$ of energy $E$ and if $\mathbb{B}^-_{2n_-}\psi(x;E) \neq 0$, then this latter state will be eigenstate of $\breve{H}$ of the eigenvalue $(E - 2\Delta)$, $\breve{H}\left(\mathbb{B}^-_{2n_-}\psi(x;E)\right) = (E-2\Delta)\mathbb{B}^-_{2n_-}\psi(x;E)$.

In terms of the operators $\mathbb{A}^\pm_{n_+}$ and $\mathbb{B}^\pm_{2n_-}$ we construct the three pairs of the basic ladder operators

$$\mathcal{A}^\pm = \mathbb{A}^-_{n_+} a^\pm \mathbb{A}^+_{n+} \,, \tag{6.3}$$

$$\mathcal{B}^\pm = \mathbb{B}^-_{2n_-} a^\pm \mathbb{B}^+_{2n-} \,, \tag{6.4}$$

and

$$\mathcal{C}^- = \mathbb{B}^-_{2n_-}\mathbb{A}^+_{n_+} \,, \qquad \mathcal{C}^+ = \mathbb{A}^-_{n_+}\mathbb{B}^+_{2n_-} \,. \tag{6.5}$$

The operators $\mathcal{A}^\pm$ and $\mathcal{B}^\pm$ are differential operators of orders $2n_+ + 1$ and $4n_- + 1$, respectively, while the ladder operators $\mathcal{C}^\pm$ are differential operators of order $n_+ + 2n_-$. These basic ladder operators satisfy the relations

$$[\breve{H}, \mathcal{A}^\pm] = \pm 2\mathcal{A}^\pm \,, \qquad [\breve{H}, \mathcal{B}^\pm] = \pm 2\mathcal{B}^\pm \,, \qquad [\breve{H}, \mathcal{C}^\pm] = \pm 2\Delta\, \mathcal{C}^\pm \,. \tag{6.6}$$

We also have the operator identities

$$\mathcal{A}^+\mathcal{A}^- = \mathcal{P}_{\mathcal{A}}(\breve{H}) \,, \qquad \mathcal{A}^-\mathcal{A}^+ = \mathcal{P}_{\mathcal{A}}(\breve{H} + 2) \,, \tag{6.7}$$

$$\mathcal{B}^+\mathcal{B}^- = \mathcal{P}_{\mathcal{B}}(\breve{H}) \,, \qquad \mathcal{B}^-\mathcal{B}^+ = \mathcal{P}_{\mathcal{B}}(\breve{H} + 2) \,, \tag{6.8}$$

and

$$\mathcal{C}^+\mathcal{C}^- = \mathcal{P}_{\mathcal{C}}(\breve{H}) \,, \qquad \mathcal{C}^-\mathcal{C}^+ = \mathcal{P}_{\mathcal{C}}(\breve{H} + 2\Delta) \,, \tag{6.9}$$

where

$$\mathcal{P}_{\mathcal{A}}(\breve{H}) = (\breve{H} - 2\Delta)\mathcal{P}_{\mathbb{A}}(\breve{H} - 2)\mathcal{P}_{\mathbb{A}}(\breve{H}) \,, \quad \mathcal{P}_{\mathcal{B}}(\breve{H}) = \breve{H}\mathcal{P}_{\mathbb{B}}(\breve{H} + 2\Delta - 2)\mathcal{P}_{\mathbb{B}}(\breve{H} + 2\Delta) \,, \tag{6.10}$$

$$\mathcal{P}_{\mathcal{C}}(\breve{H}) = \mathcal{P}_{\mathbb{B}}(\breve{H})\mathcal{P}_{\mathbb{A}}(\breve{H}) \,. \tag{6.11}$$

The polynomial $\mathcal{P}_{\mathbb{A}}$ of order $n_+$ is defined here by $\mathcal{P}_{\mathbb{A}}(\breve{H}) \equiv \mathbb{A}^-_{n_+}\mathbb{A}^+_{n_+} = \breve{H}\prod_{i=1}^{n_+-1}(\breve{H} - E_i)$, where $E_i$, $i = 1, \ldots, n_+ - 1$ , are nonzero eigenvalues of the corresponding excited separated (gapped) physical eigenstates $\Psi_i$ of $\breve{H}$. Together with zero energy $E_0 = 0$ of the ground state $\Psi_0$, the energy values of the $n_+$ gapped physical eigenstates of $\breve{H}$ are the shifted by the constant $2\Delta = 2(2n_- + n_+)$ energies of the corresponding non-physical eigenstates $\psi^-_{j_s}$ of $H$ which appear as the seed states in the minimal $(\alpha_{n_+})$-scheme. The permuted product of the intertwining operators gives here $\mathbb{A}^+_{n_+}\mathbb{A}^-_{n_+} = \mathcal{P}_{\mathbb{A}}(H)$. The polynomial $\mathcal{P}_{\mathbb{B}}$ of order $2n_-$ is defined via the relation $\mathcal{P}_{\mathbb{B}}(H) \equiv \mathbb{B}^+_{2n_-}\mathbb{B}^-_{2n_-} = \prod_{j=1}^{2n_-}(H - E^-_j)$, where by $E^-_j$ we denote the shifted for the same constant $2\Delta$ energies of the physical eigenstates $\psi_{i_s}$ of $H$ which are present as the seed states in the minimal $(\beta_{2n_-})$-scheme. For the permuted

product of the intertwining operators we have $\mathbb{B}^-_{2n_-}\mathbb{B}^+_{2n_-} = \mathcal{P}_{\mathbb{B}}(\breve{H}+2\Delta)$. Let us also note here a useful relation

$$\mathcal{P}_{\mathbb{A}}(\breve{H})\mathcal{P}_{\mathbb{B}}(\breve{H}+2\Delta) = \prod_{j=0}^{2n_-+n_+-1}(\breve{H}-2j)\,, \tag{6.12}$$

that reflects the complementarity of the minimal $(\alpha_{n_+})$- and $(\beta_{2n_-})$-schemes.

The operator $\mathcal{A}^-$ annihilates all the $n_+$ physical eigenstates $\Psi_0,\ldots,\Psi_{n_+-1}$ of the system $\breve{H}$ whose energies lie below the infinite equidistant part of the spectrum and are separated from it by some gap of $2n_0$ missing energy levels, $n_0 \geq 1$. Between these $n_+$ separated energy levels there can appear $g$, $0 \leq g < n_+$, 'internal' gaps each one containing an even number of missing energy levels. We name the $g+1$ sets of energy levels in the lower separated part of the spectrum which do not contain internal gaps as valence bands. If $g > 0$, we denote by $2n_1,\ldots,2n_g$ the number of missing energy levels in the corresponding internal energy gaps assuming that the highest value $g$ of index $i$ in $n_i$ corresponds here to the lowest energy gap in the spectrum. The total number of the missing energy levels $2(n_0+\ldots+n_g)$ is equal to the number $2n_-$ of the physical eigenstates $\psi_{i_s}$ which participate as the seed states in the minimal $(\beta_{2n_-})$-scheme. In addition to the gapped physical eigenstates $\Psi_0,\ldots,\Psi_{n_+-1}$, the operator $\mathcal{A}^-$ also annihilates the lowest state $\Psi_{n_+}$ of energy $E_{n_+} = 2\Delta$ in the infinite equidistant part of the spectrum due to the presence of the operator $a^-$ in its structure. Besides, $\mathcal{A}^-$ annihilates some $n_+$ non-physical eigenstates of $\breve{H}$. Note in particular that

$$\Psi_0 = \widetilde{\mathbb{A}^-_{n_+}\psi^-_{j_+}} = \mathbb{B}^-_{2n_-}\psi_0\,, \qquad \Psi_{n_+-1} = \widetilde{\mathbb{A}^-\psi^-_{2n_0}} = \mathbb{B}^-_{2n_-}\psi_{j_+-2n_0}\,, \tag{6.13}$$

$$\Psi_{n_+} = \mathbb{A}^-_{n_+}\psi_0 = \mathbb{B}^-_{2n_-}\psi_{2n_-+n_+}\,. \tag{6.14}$$

Here $j_+ = 2n_- + n_+ - 1$ is the maximal value of the index $j_s$ of the non-physical eigenstates $\psi^-_{j_s} \in \ker(\mathbb{A}^-_{n_+})$ from the minimal $(\alpha_{n_+})$-scheme; it coincides with the maximal value of the index $i_s$ of the physical eigenstates $\psi_{i_s} \in \ker(\mathbb{B}^-_{2n_-})$ from the minimal scheme $(\beta_{2n_-})$. The energy values of the indicated $2n_+ + 1$ physical and non-physical eigenstates from the kernel of $\mathcal{A}^-$ are the roots of the polynomial $\mathcal{P}_{\mathcal{A}}(\breve{H})$ which appears in the first identity in (6.7). The kernel of $\mathcal{A}^+$ is spanned by the $n_+$ lowest separated physical eigenstates $\Psi_0,\ldots,\Psi_{n_+-1}$, and by some $n_+ + 1$ non-physical eigenstates of $\breve{H}$. The energy values of these eigenstates from $\ker(\mathcal{A}^+)$ correspond to the roots of the polynomial $\mathcal{P}_{\mathcal{A}}(\breve{H}+2)$ which appears in the second relation in (6.7).

The kernel of the ladder operator $\mathcal{C}^-$ is spanned by $n_+ + 2n_-$ physical states, $n_+$ of which, $\Psi_0,\ldots,\Psi_{n_+-1}$, correspond to the lowest separated (gapped) energy values. The other $2n_-$ eigenstates of $\breve{H}$ in the kernel of $\mathcal{C}^-$ are the physical states $\mathbb{A}^-_{n_+}\psi_{i_s}$ in a lower part of the infinite equidistant part of the spectrum, where $\psi_{i_s} \in \ker(\mathbb{B}^-_{2n_-})$. The number of those 'supplementary' states in the lower part of the equidistant spectrum which are not annihilated by $\mathcal{C}^-$ and whose eigenvalues lie below the highest energy value of a physical eigenstate from $\ker(\mathcal{C}^-)$ is equal to $n_+$. The number of the sequential lowest states at the very bottom of the equidistant infinite part of the spectrum of $\breve{H}$ which are not annihilated by $\mathcal{C}^-$ is equal to the number of the physical states in the lowest valence band of the separated part of the spectrum. The kernel of $\mathcal{C}^+$ will be spanned by some $n_+ + 2n_-$ non-physical eigenstates. The energies of the corresponding eigenstates from the kernels of the ladder operators $\mathcal{C}^-$ and $\mathcal{C}^+$ correspond to the roots of the polynomials in $\breve{H}$ which appear in the first and the second identities in (6.9).

The kernel of the ladder operator $\mathcal{B}^-$ contains $g+1$ physical eigenstates, each one lying at the very bottom of each valence band. Besides, it also contains the physical eigenstate $\Psi_{n_+}$ of energy $E_{n_+} = 2\Delta$ which is the lowest state of the equidistant infinite part of the spectrum. In addition, $\ker(\mathcal{B}^-)$ contains $2n_-$ non-physical eigenstates of the form $\Psi^{non}_{i_s} \equiv \widetilde{\mathbb{B}^-\psi_{i_s}}$, which correspond to the

missing energy values in the gaps. Finally, it also involves $2n_- - (g+1)$ non-physical eigenstates of the form $\widetilde{\Psi^{non}_{i_s}}$ for all values of the index $i_s$ except those $g+1$ values, each one of which corresponds to a lowest eigenstate in each gap. The kernel of the increasing ladder operator $\mathcal{B}^+$ contains $g+1$ physical eigenstates whose eigenvalues lie at the top of each valence band. It also involves $4n_- - g$ non-physical eigenstates from the gaps. Eigenvalues of the eigenstates from the kernels of the ladder operators $\mathcal{B}^-$ and $\mathcal{B}^+$ correspond to the roots of the polynomials which appear, respectively, in the first and the second identities in (6.8).

The relations of the form (4.32) and (5.19) are valid for the basic intertwining operators of the arbitrary REQHO system we consider here. Indeed, the operator $\mathbb{B}^-_{2n_-}$ annihilates all the $2n_-$ physical eigenstates $\psi_{i_s}$ of the QHO which participate as the seed states in the $(\beta_{2n_-})$-scheme. On the other hand, when $\mathbb{B}^-_{2n_-}$ acts on the $n_+$ 'supplementary' eigenstates in the lower part of the spectrum of the QHO, it transforms these states into the separated lowest $n_+$ physical eigenstates of the REQHO system $\breve{H}$ which constitute the kernel of the intertwining operator $\mathbb{A}^+_{n_+}$. In particular, as we saw $\mathbb{B}^-_{2n_-}$ maps the ground-state $\psi_0$ of the shifted QHO, $H\psi_0 = 2\Delta\psi_0$, into the zero energy ground-state $\Psi_0$ of $\breve{H}$, $\mathbb{B}^-_{2n_-}\psi_0 = \Psi_0$, $\breve{H}\Psi_0 = 0$. We conclude then that the composite operator $\mathbb{A}^+_{n_+}\mathbb{B}^-_{2n_-}$ annihilates all the $2n_- + n_+$ eigenstates $\psi_n$ of the QHO with $n = 0, \ldots, 2n_- + n_+ - 1$. But the same job is made by the lowering ladder operator $(a^-)^{2n_-+n_+}$ of the QHO. From here we obtain the operator equalities

$$\mathbb{A}^+_{n_+}\mathbb{B}^-_{2n_-} = (-1)^{n_+}(a^-)^{2n_-+n_+}\,, \qquad \mathbb{B}^+_{2n_-}\mathbb{A}^-_{n_+} = (-1)^{n_+}(a^+)^{2n_-+n_+}\,. \tag{6.15}$$

The relations (6.15) reflect the complementary nature of the involved minimal $(\alpha_{n_+})$- and $(\beta_{2n_-})$-schemes. The identities (6.15) are employed to establish the operator identities in (6.9). They are also essential for the analysis of the kernels of the basic ladder operators.

Similarly to the simplest case of the REQHO system, one can construct other pairs of secondary ladder operators different from the described basic ladder operators $\mathcal{A}^\pm$, $\mathcal{B}^\pm$ and $\mathcal{C}^\pm$. This can be done effectively by introducing additional factors $(a^\pm)^n$ inside the structure of these operators: $\mathcal{A}^\pm_n \equiv \mathbb{A}^-_{n_+}(a^\pm)^n\mathbb{A}^+_{n_+}$, $\mathcal{B}^\pm_n \equiv \mathbb{B}^-_{2n_-}(a^\pm)^n\mathbb{B}^+_{2n_-}$, $n = 1, \ldots$, $\mathcal{A}^\pm_1 = \mathcal{A}^\pm$, $\mathcal{B}^\pm_1 = \mathcal{B}^\pm$, and $\mathcal{C}^+_{n+1} \equiv \mathbb{A}^-_{n_+}(a^+)^n\mathbb{B}^+_{2n_-}$, $\mathcal{C}^-_{n+1} \equiv \mathbb{B}^-_{2n_-}(a^-)^n\mathbb{A}^+_{n_+}$, where $n = 0, \ldots$, $\mathcal{C}^\pm_1 = \mathcal{C}^\pm$. One can also consider the operators $\mathcal{C}^-_{-n} = \mathbb{B}^-_{2n_-}(a^+)^n\mathbb{A}^+_{n_+}$, $\mathcal{C}^+_{-n} = \mathbb{A}^-_{n_+}(a^-)^n\mathbb{B}^+_{2n_-}$ with $n = 1, \ldots, 2n_- + n_+ - 1$. In $\mathcal{C}^\pm_{-n}$ we restrict the values of the index $n$ from above having in mind the identity

$$\mathcal{C}^-_{-(2n_-+n_+)} = (-1)^{n_+}\,\mathcal{P}_{\mathbb{A}}(\breve{H})\,\mathcal{P}_{\mathbb{B}}(\breve{H} + 2\Delta)\,, \tag{6.16}$$

see Eq. (6.12), and so, for $n > 2n_- + n_+ - 1$ these operators do not provide essentially new structures. The secondary, higher-order ladder operators can also be obtained by taking the compositions of the intertwining operators of the corresponding $(\alpha)$-, $(\beta)$- and $(\gamma)$-schemes. They also are generated via the composition of the basic ladder operators $\mathcal{A}^\pm$, $\mathcal{B}^\pm$ and $\mathcal{C}^\pm$. In particular, the quadratic compositions of $\mathcal{A}^\pm$ and $\mathcal{C}^\pm$ are given by (6.7), (6.9), and by the relations

$$(\mathcal{A}^+)^2 = \mathcal{P}_{\mathbb{A}}(\breve{H} - 2)\mathcal{A}^+_2\,, \quad (\mathcal{C}^+)^2 = (-1)^{n_+}\mathcal{C}^+_{2n_-+n_++1}\,, \quad \mathcal{A}^+\mathcal{C}^- = (-1)^{n_+}(\breve{H} - 2\Delta)\mathcal{A}^-_{2n_-+n_+-1}\,, \tag{6.17}$$

$$\mathcal{A}^+\mathcal{C}^+ = \mathcal{P}_{\mathbb{A}}(\breve{H} - 2)\mathcal{C}^+_1\,, \quad \mathcal{A}^-\mathcal{C}^- = (-1)^{n_+}\mathcal{A}^-_{2n_-+n_++1}\,, \quad \mathcal{A}^-\mathcal{C}^+ = \mathcal{P}_{\mathbb{A}}(\breve{H} + 2)\cdot\mathcal{C}^+_{-1}\,, \tag{6.18}$$

and by the relations conjugate to (6.17) and (6.18). The relation

$$(\mathcal{A}^-)^{2n_-+n_+} = (-1)^{n_+}\prod_{l=0}^{2n_-+n_+-1}\mathcal{P}_{\mathbb{A}}(\breve{H} + 2l)\cdot\mathcal{C}^- \tag{6.19}$$

shows that as in the considered particular cases of the REQHO systems, the ladder operators $\mathcal{C}^{\pm}$ can be generated by the operators $\mathcal{A}^{\pm}$. Also, the following operator identity is valid:

$$(\mathcal{A}^{-})^{2n_{-}+n_{+}-1} = (-1)^{n_{+}} \frac{1}{\breve{H}} \prod_{j=0}^{2n_{-}+n_{+}-2} \mathcal{P}_{\mathbb{A}}(\breve{H}+2j)\, \mathcal{C}_{-1}^{-}\,. \tag{6.20}$$

Here the operator multiplier before $\mathcal{C}_{-1}^{-}$ is the polynomial of order $n_{+}(2n_{-}+n_{+}-1)-1$ in $\breve{H}$ since the $j=0$ term $\mathcal{P}_{\mathbb{A}}(\breve{H})$ in the product is equal to the factor $\breve{H}$ which cancels the multiplier $\frac{1}{\breve{H}}$ before the product symbol. We also have the identity which relates the operators $\mathcal{A}^{-}$ and $\mathcal{B}^{-}$,

$$(\breve{H}-2\Delta+2)\mathcal{P}_{\mathbb{A}}(\breve{H}+2)\,\mathcal{B}^{-} = (\breve{H}+2)\mathcal{P}_{\mathbb{B}}(\breve{H}+2\Delta)\,\mathcal{A}^{-}\,. \tag{6.21}$$

The analogous identity for $\mathcal{A}^{+}$ and $\mathcal{B}^{+}$ is obtained from (6.21) by Hermitian conjugation. In particular cases of the three REQHO systems considered in the previous two sections, relation (6.21) reduces to the first relation in (4.40) and to the identities (5.6) and (5.22).

In conclusion of this section, let us show that the trinity $(\mathcal{A}^{\pm},\mathcal{B}^{\pm},\mathcal{C}^{\pm})$ of the pairs of the lowering and raising ladder operators allows us to generate an arbitrary physical eigenstate from the ground state $\Psi_0$, and as a consequence, any two physical eigenstates can be related by the appropriate consecutive action of the basic ladder operators from the trinity. First, from the described properties of the operators and commutation relations (6.6) it follows that in the equidistant infinite part of the spectrum any two eigenstates can be related by the ladder operators $\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ in the same way as the ladder operators $a^{\pm}$ relate the states in the QHO system. The only difference will appear in the numerical coefficients which have to be included into the composition of the indicated basic operators when we work with the normalized eigenstates. If a valence band contains more than one eigenstate, different states in this band can be connected by application to them of the appropriate degrees of the lowering and raising operators $\mathcal{B}^{-}$ and $\mathcal{B}^{+}$. Note that within the valence band with $n_i$ states these operators satisfy the identity $(\mathcal{B}^{\pm})^{n_i}=0$. Recall also that the lowest state $\Psi_{n_+}$ in the equidistant infinite part of the spectrum is related with the ground state by the action of the ladder operators $\mathcal{C}^{\pm}$: $\Psi_0=\mathcal{C}^{-}\Psi_{n_+}$ and $\Psi_{n_+}=\mathcal{C}^{+}\Psi_0$. In the same way one can relate any state $\Psi_n$ of energy $0<E_n<2\Delta$ with $0<n\leq n_+-1$ from the separated part of the spectrum with the corresponding state $\Psi_{n_++n}$ of energy $E_n+2\Delta$ from the equidistant infinite part of the spectrum. Then, if a REQHO system contains more than one valence band, the ground state $\Psi_0$ from the lowest valence band can be related to some state $\Psi_l$ with eigenvalue $E_l$ from some higher valence band, for instance, by the following composition of the ladder operators: $\Psi_l=\mathcal{C}^{-}(\mathcal{A}^{+})^{r_l}\mathcal{C}^{+}\Psi_0$, $\Psi_0=\mathcal{C}^{-}(\mathcal{A}^{-})^{r_l}\mathcal{C}^{+}\Psi_l$, where $r_l=E_l/2$. This shows finally that the ladder operators from the trinity are the spectrum-generating operators for the REQHO system of a general form.

The described properties of the REQHO systems of a general form are illustrated by Figure 1.

## 7 Summary and outlook

In conclusion, we summarize the obtained results and indicate some interesting problems for further investigation.

The REQHO system of a general form is characterized by $n_+\geq 1$ low-lying energy levels which are separated from the higher equidistant infinite part of the spectrum by some gap of an even number $2n_0\geq 2$ of missing levels. Between separated energy levels there can be additional gaps of an even number of missing levels in each such a gap. As a result, in the lower part of the

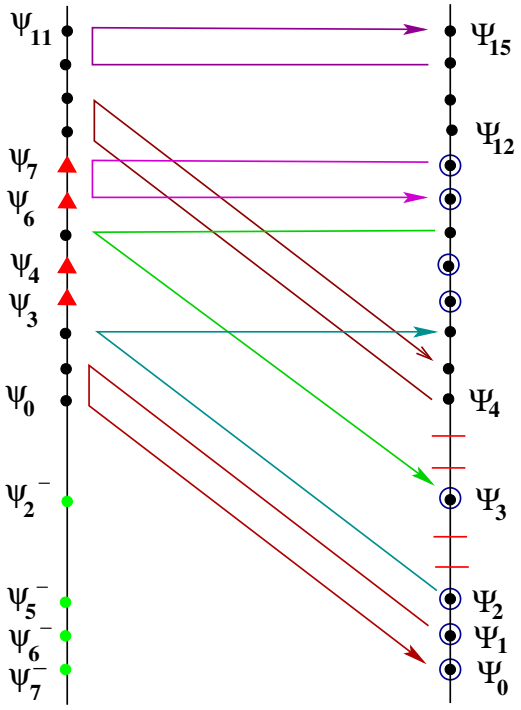


Figure 1: Example of the REQHO system with two gaps and four separated states organized into two valence bands of one and three states. The system is generated by the DCKATs based on the complementary schemes $(\alpha_4) = \{\psi_2^-, \psi_5^-, \psi_6^-, \psi_7^-\}$ and $(\beta_4) = \{\psi_3, \psi_4, \psi_6, \psi_7\}$. The following action of the ladder operators is shown: $\Psi_{15} = \mathcal{A}^+\Psi_{14}$, $\Psi_{10} = \mathcal{A}^-\Psi_{11}$, $\Psi_5 = \mathcal{B}^+\Psi_4$, $\Psi_0 = \mathcal{B}^-\Psi_1$, $\Psi_3 = \mathcal{C}^-\Psi_9$, $\Psi_6 = \mathcal{C}^+\Psi_2$, where $\mathcal{A}^\pm = \mathbb{A}_4^- a^\pm \mathbb{A}_4^+$, $\mathcal{B}^\pm = \mathbb{B}_4^- a^\pm \mathbb{B}_4^+$, $\mathcal{C}^- = \mathbb{B}_4^- \mathbb{A}_4^+$, $\mathcal{C}^+ = \mathbb{A}_4^- \mathbb{B}_4^+$. On the right, eight encircled states belong to the kernel of $\mathcal{C}^-$ and their energies correspond to the energies of the seed states from the complementary $(\alpha_4)$- and $(\beta_4)$- schemes. If the equidistant part of the spectrum of the REQHO system is moved down for the distance $2\Delta = 2(n_+ + 2n_-) = 16$, that corresponds to the difference between energies of the lowest state $\Psi_4$ in the equidistant part of the spectrum and the ground state $\Psi_0$, the encircled energy levels in the equidistant part of the spectrum take exactly a position of missing energy levels in the low, separated part of the spectrum. This refelects a complementarity of the minimal $(\alpha_4)$- and $(\beta_4)$- schemes, according to which we have $\Psi_0 = \mathbb{A}_4^- \widetilde{\psi_7^-} = \mathbb{B}_4^- \psi_0$, $\Psi_1 = \mathbb{A}_4^- \widetilde{\psi_6^-} = \mathbb{B}_4^- \psi_1$, $\Psi_2 = \mathbb{A}_4^- \widetilde{\psi_5^-} = \mathbb{B}_4^- \psi_2$, $\Psi_3 = \mathbb{A}_4^- \widetilde{\psi_2^-} = \mathbb{B}_4^- \psi_5$, $\Psi_{4+n} = \mathbb{A}_4^- \psi_n = \mathbb{B}_4^- \psi_{8+n}$, $n = 0, 1, \ldots$.

spectrum there appears a total number $2n_- \geq 2n_0$ of missing levels, and the separated part of the spectrum is organized into $(g+1) \geq 1$ 'valence bands'. Such a peculiar structure of the spectrum of the REQHO system characterized by the three integer numbers $(n_+, 2n_-, g+1)$ is detected and reflected by the trinity $(\mathcal{A}^\pm, \mathcal{B}^\pm, \mathcal{C}^\pm)$ of the pairs of the lowering and raising ladder operators. Any raising or lowering ladder operator from the trinity when acts on a physical eigenstate either transforms it into another physical eigenstate with the changed value of energy, or annihilates it.

The separated states are detected by $\mathcal{A}^-$ and $\mathcal{A}^+$ that are differential operators of order $2n_+ + 1$. Each one of these two operators annihilates all the $n_+$ states $\Psi_0, \ldots, \Psi_{n_+-1}$ in the valence bands. The operator $\mathcal{A}^-$ annihilates in addition the lowest state $\Psi_{n_+}$ in the equidistant infinite part of the spectrum. In this aspect the lowering ladder operator $\mathcal{A}^-$ has properties very similar to those of the Lax-Novikov integral in reflectionless systems we discussed in Section 1. Besides, the operators $\mathcal{A}^-$ and $\mathcal{A}^+$ annihilate some $n_+$ and $n_+ + 1$ non-physical eigenstates, respectively. Due to the relation $[\breve{H}, \mathcal{A}^\pm] = \pm 2\mathcal{A}^\pm$, the $\mathcal{A}^\pm$ act in the equidistant infinite part of the spectrum as the spectrum-generating operators like the ladder operators $a^\pm$ in the QHO system. Namely, they transform physical eigenstates $\Psi_j$ with $j \geq n_+$ into the states $\Psi_{j\pm 1}$ by shifting the energy values in $\pm 2$, and $\mathcal{A}^- \Psi_{n_+} = 0$. It is also worth to note here that the quadratic in the ladder operators $\mathcal{A}^\pm$ relations in (6.7) are analogous to the Burchnall-Chaundy polynomial identity [15, 18, 53] that relates the Lax-Novikov integral with the corresponding Hamiltonian of a reflectionless (or a finite-gap) system, and underlies the modern theory of integrable systems [54].

The ladder operators $\mathcal{B}^\pm$ are differential operators of order $4n_- + 1$, and each of them effectively counts the number $g+1$ of the valence bands and measures the size of each valence band. This is done as follows. The lowering operator $\mathcal{B}^-$ annihilates each physical eigenstate which lies at the very bottom of each valence band, whereas the raising operator $\mathcal{B}^+$ annihilates each state at the very top of each valence band. So, if a valence band contains only one state, this state is annihilated by both $\mathcal{B}^-$ and $\mathcal{B}^+$, and in such a one-dimensional valence band the action of $\mathcal{B}^\pm$ is similar to that of $\mathcal{A}^\pm$. However, if we have a valence band with more than one state, the operators $\mathcal{B}^+$ and $\mathcal{B}^-$, unlike $\mathcal{A}^\pm$, act in such a band as the raising and lowering operators, $[\breve{H}, \mathcal{B}^\pm] = \pm 2\mathcal{B}^\pm$, which satisfy there the relations $(\mathcal{B}^\pm)^{n_i} = 0$, where $n_i$ is the number of states in the band. Like $\mathcal{A}^-$, the operator $\mathcal{B}^-$ annihilates the lowest state $\Psi_{n_+}$ in the equidistant infinite part of the spectrum. The kernels of $\mathcal{B}^-$ and $\mathcal{B}^+$ also include some $4n_- - g - 1$ and $4n_- - g$ non-physical eigenstates, respectively. Similarly to $\mathcal{A}^\pm$, in the equidistant infinite part of the spectrum the operators $\mathcal{B}^\pm$ also act as the spectrum-generating operators.

Under the action of the ladder operators $\mathcal{A}^\pm$ and $\mathcal{B}^\pm$ the $n_+$ states from the valence bands turn out to be completely disconnected from the physical states in the equidistant infinite part of the spectrum. Both parts of the spectrum are connected by means of the third pair of the mutually conjugate ladder operators $\mathcal{C}^+$ and $\mathcal{C}^-$, that are differential operators of order $n_+ + 2n_-$. The kernel of the lowering operator $\mathcal{C}^-$ is spanned by physical eigenstates, $n_+$ of which correspond to all the $n_+$ eigenstates from the valence bands. The rest $2n_-$ states from $\ker(\mathcal{C}^-)$ are some eigenstates with energy levels lying in the low part of the equidistant infinite part of the spectrum. The positions of those $2n_-$ energy levels correspond to the missing energy levels in the gaps moved up for the distance $2\Delta = 2n_+ + 4n_- \geq 6$ which is exactly equal to the distance between the energy levels of the ground state $\Psi_0$ and the lowest state $\Psi_{n_+}$ in the equidistant infinite part of the spectrum. The kernel of $\mathcal{C}^+$ is spanned by some non-physical eigenstates only. Due to the relation $[\breve{H}, \mathcal{C}^\pm] = \pm 2\Delta\, \mathcal{C}^\pm$, the operators $\mathcal{C}^+$ and $\mathcal{C}^-$ act as the raising and lowering operators changing the energy for $\pm 2\Delta$. All the states from the valence bands are obtained by application of the lowering operator $\mathcal{C}^-$ to those low-lying states with energies $2\Delta \leq E < 4\Delta$ in the equidistant infinite part of the spectrum which are not annihilated by it. In particular, the lowest state $\Psi_{n_+}$ with energy $E_{n_+} = 2\Delta$ in the equidistant infinite part of the spectrum is transformed by $\mathcal{C}^-$ into

the ground-state $\Psi_0$ of zero energy. The state $\Psi_{n_+}$, in turn, can be obtained from $\Psi_0$ by action of the raising operator $\mathcal{C}^+$, and is also generated from the state $\Psi_{n_++\Delta}$ of energy $E_{n_++\Delta} = 4\Delta$ by applying to the latter the lowering operator $\mathcal{C}^-$. As a consequence of the described properties, any two states in the spectrum of a REQHO system can be related by an appropriate consecutive action of the basic ladder operators from the trinity. In particular, arbitrary excited state from any valence band or from the equidistant infinite part of the spectrum can be obtained from the ground state $\Psi_0$. This means that the basic ladder operators from the trinity are the spectrum-generating operators of the REQHO system.

The energies of all the physical and non-physical eigenstates of the kernels of the lowering operators $\mathcal{A}^-$, $\mathcal{B}^-$ and $\mathcal{C}^-$ are the roots of the corresponding polynomials in $\breve{H}$ which appear on the right hand side in the first relations from equations (6.7), (6.8) and (6.9), respectively. The eigenvalues of the physical and non-physical eigenstates from the kernels of the conjugate operators $\mathcal{A}^+$, $\mathcal{B}^+$ and $\mathcal{C}^+$ are the roots of the corresponding polynomials which appear in the second operator identity relations in the same equations. The basic ladder operators $\mathcal{A}^-$ and $\mathcal{B}^-$ satisfy the two-term identity relation (6.21) which is linear in both of these operators but involve the coefficients that are certain polynomials in the Hamiltonian $\breve{H}$. The presence of such polynomial coefficients reflects a difference in action of these operators on the states in a separated part of the spectrum. The operators $\mathcal{A}^-$ and $\mathcal{C}^-$ are related by the operator identity of the form (6.19). Proceeding from these relations, one can obtain the identity that relates the operators $\mathcal{B}^-$ and $\mathcal{C}^-$, and by conjugation one can find the identities that relate the raising operators of the trinity.

The operators $\mathcal{A}^\pm$ are constructed as the ladder operators $a^\pm$ of the QHO dressed by means of the Darboux-Crum-Krein-Adler intertwining operators $\mathbb{A}^-_{n_+}$ and $\mathbb{A}^+_{n_+}$ constructed on the basis of the minimal set of $n_+$ non-physical eigenstates of the QHO which are used as the seed states in the corresponding DCKAT based on the $(\alpha_{n_+})$-scheme, see Eq. (6.3). The operators $\mathcal{B}^\pm$ are constructed in the same way with the help of the intertwining operators $\mathbb{B}^-_{2n_-}$ and $\mathbb{B}^+_{2n_-}$ obtained on the basis of the minimal set of the $2n_-$ physical eigenstates which are employed as the seed states in the DCKAT in the corresponding $(\beta_{2n_-})$-scheme, see Eq. (6.4). The operators $\mathcal{C}^\pm$ can be obtained as the composition (6.5) of the corresponding intertwining operators from both indicated schemes of the DCKATs. The minimal schemes $(\alpha_{n_+})$ and $(\beta_{2n_-})$ are complementary, what is reflected in particular by the relations (6.12) and (6.15). The secondary, higher-order ladder operators can be constructed in analogous way by dressing the higher-order ladder operators $(a^\pm)^n$ of the QHO, or by the composition of the appropriate intertwining operators from the non-minimal $(\alpha_{n_++n})$ and $(\beta_{2n_-+n})$ schemes, or by employing the intertwining operators from the corresponding intermediate $(\gamma)$-type schemes which use both physical and non-physical eigenstates of the QHO as the seeds states of the corresponding DCKATs. The secondary ladder operators can also be generated via the appropriate composition of the basic (primary) ladder operators $\mathcal{A}^\pm$, $\mathcal{B}^\pm$ and $\mathcal{C}^\pm$.

It seems to be interesting to investigate the quantum mechanical systems related to the exceptional Laguerre and Jacobi orthogonal polynomials in the light of the results on the ladder operators obtained here. The results of such an investigation will be presented elsewhere.

The ladder operators $\mathcal{B}^\pm$ have a nature of the polynomially deformed bosonic creation and annihilation operators in the equidistant infinite part of the spectrum. On the other hand, these operators act trivially on the one-state valence bands whose singleton states are annihilated by both the lowering $\mathcal{B}^-$ and the raising $\mathcal{B}^+$ operators. The same operators reveal the properties of the deformed fermionic creation and annihilation operators in the valence bands consisting from two states. They have the properties of the deformed para-fermion creation and annihilation operators of order $n > 2$ in those valence bands which contain $n > 2$ eigenstates of $\breve{H}$. The interesting question is then if there exist some concrete physical systems which would reveal the spectrum of the REQHO systems. If so, it seems that the trinity of the ladder operators should play a

fundamental role in the physics associated with such systems. In the same direction the interesting question is whether the quantum mechanical REQHO systems and the structures associated with them can be generalized somehow for the case of the quantum fields.

In [46], the family of the REQHO systems with two separated states generated by non-physical seed states $\psi^-_{m_1}$ and $\psi^-_{m_2}$, $m_2 - m_1 \equiv \ell = 1 + 2r$, $r = 0, 1, \ldots$, $m_1 = 2k$, $k = 1, \ldots$, was considered. For such class of the systems, there the lowering, $c$, and increasing, $c^\dagger$, ladder operators of the differential order $2 + \ell$ were constructed by employing auxiliary systems some of which are singular and have a nature similar to that of the isotonic oscillator (4.23). In the simplest case $m_1 = 2$ and $m_2 = 3$ such a system corresponds to the REQHO system (5.8) we considered in Section 5. Like our fifth order ladder operator $\mathcal{B}^-$, the third order ladder operator $c$ from [46] annihilates the ground state $\Psi_0$ and the lowest state $\Psi_2$ in the infinite equidistant part of the spectrum. The increasing operator $c^\dagger$ like our $\mathcal{B}^+$ annihilates the excited state $\Psi_1$ in the separated two-state lower part of the spectrum. In the systems with $\ell > 1$, however, the kernel of the increasing operator $c^\dagger$ still includes only one physical state which is, again, the separated state $\Psi_1$, while our $\mathcal{B}^+$ operator annihilates both separated states $\Psi_0$ and $\Psi_1$. In addition to the separated ground state $\Psi_0$ and the lowest state $\Psi_2$ in the equidistant part of the spectrum, the kernel of the lowering operator $c$ in this case includes also the $\ell - 1$ excited states $\Psi_3, \ldots, \Psi_{2+\ell}$ in the equidistant part of the spectrum. In this aspect, the lowering operator $c$ from [46] has some similarity with our operator $\mathcal{C}^-$. But our ladder operator $\mathcal{C}^-$ is of differential order $2m_1 + 2r$ and its kernel includes some $2m_1 + 2r - 2$ excited states in the equidistant part of the spectrum together with both separated states $\Psi_0$ and $\Psi_1$. Thus, in the case of $\ell > 1$ the nature of the operators $c$ and $c^\dagger$ in the sense of the physical states which they annihilate is different from the nature of any of our lowering and increasing ladder operators $\mathcal{A}^-$, $\mathcal{A}^+$, $\mathcal{B}^-$, $\mathcal{B}^+$ and $\mathcal{C}^-$, $\mathcal{C}^+$. It would be interesting to investigate whether the analogs of the ladder operators $c$ and $c^\dagger$ from [46] can be constructed for REQHO systems containing more than two states in the lower separated part of the spectrum, and what is the exact relation of such ladder operators with our trinity ($\mathcal{A}^\pm$, $\mathcal{B}^\pm$, $\mathcal{C}^\pm$) of the ladder operators.


### Acknowledgements

JFC and MSP acknowledge support from research projects FONDECYT 1130017 (Chile), Proyecto Basal USA1555 (Chile), MTM2015-64166-C2-1 (MINECO, Madrid) and DGA E24/1 (DGA, Zaragoza). MSP is grateful for the warm hospitality at Zaragoza University. JFC thanks for the kind hospitality at Universidad de Santiago de Chile.


## References


[1] I. Kay and H. E. Moses, *"Reflectionless transmission through dielectrics and scattering potentials,"*, J. Appl. Phys. **27**, 1503 (1956).

[2] G. Darboux, *"Sur une proposition relative aux équations linéaires,"* C. R. Acad. Sci Paris **94**, 1456 (1882).

[3] E. Schrödinger, *"A method of determining quantum-mechanical eigenvalues and eigenfunctions,"* Proc. Roy. Irish Acad. (Sect. A) **46**, 9 (1940).

[4] L. Infeld and T. E. Hull, *"The factorization method,"* Rev. Mod. Phys. **23**, 21 (1951).

[5] M. M. Crum, *"Associated Sturm-Liouville systems,"* Quart. J. Math. Oxford **6**, 121 (1955).

[6] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, 1991).

[7] J. F. Cariñena and A. Ramos, *"Riccati equation, factorization method and shape invariance,"* Rev. Math. Phys. **12**, 1279 (2000) [arXiv:math-ph/9910020].

[8] J. F. Cariñena, D. J. Fernández and A. Ramos, *"Group theoretical approach to the intertwined Hamiltonians,"* Ann. Phys. **292**, 42 (2001) [arXiv:math-ph/0311029].

[9] B. Mielnik and O. Rosas-Ortiz, *"Factorization: little or great algorithm?,"* J. Phys. A **37**, 10007 (2004).

[10] J. F. Cariñena and A. Ramos, *"Generalized Bäcklund-Darboux transformations in one-dimensional quantum mechanics,"* Int. J. Geom. Methods Mod. Phys. **5**, 605 (2008).

[11] D. J. Gross and A. Neveu, *"Dynamical symmetry breaking in asymptotically free field theories,"* Phys. Rev. D **10**, 3235 (1974).

[12] R. F. Dashen, B. Hasslacher and A. Neveu, *"Semiclassical bound states in an asymptotically free theory,"* Phys. Rev. D **12**, 2443 (1975).

[13] D. K. Campbell and A. R. Bishop, *"Soliton excitations in polyacetylene and relativistic field theory models,"* Nucl. Phys. B **200**, 297 (1982).

[14] F. Correa, V. Jakubsky and M. S. Plyushchay, *"Aharonov-Bohm effect on AdS(2) and nonlinear supersymmetry of reflectionless Pöschl-Teller system,"* Annals Phys. **324**, 1078 (2009) [arXiv:0809.2854 [hep-th]].

[15] A. Arancibia, J. Mateos Guilarte and M. S. Plyushchay, *"Effect of scalings and translations on the supersymmetric quantum mechanical structure of soliton systems,"* Phys. Rev. D **87**, 045009 (2013) [arXiv:1210.3666 [math-ph]].

[16] E. D. Belokolos, A. I. Bobenko, V. Z. Enolskii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, (Springer, Berlin, 1994).

[17] F. Gesztesy and H. Holden, *Soliton Equations and their Algebro-Geometric Solutions*, (Cambridge Univ. Press, 2003).

[18] F. Correa, V. Jakubsky, L. M. Nieto and M. S. Plyushchay, *"Self-isospectrality, special supersymmetry, and their effect on the band structure,"* Phys. Rev. Lett. **101**, 030403 (2008) [arXiv:0801.1671 [hep-th]].

[19] A. Arancibia and M. S. Plyushchay, *"Transmutations of supersymmetry through soliton scattering, and self-consistent condensates,"* Phys. Rev. D **90**, no. 2, 025008 (2014) [arXiv:1401.6709 [hep-th]].

[20] S. Yu. Dubov, V. M. Eleonskii and N. E. Kulagin, *"Equidistant spectra of anharmonic oscillators,"* Zh. Eksp. Teor. Fiz. **102**, 814 (1992); Chaos **4**, 47 (1994).

[21] V. E. Adler, *"A modification of Crum's method,"* Theor. Math. Phys. **101**, 1381 (1994).

[22] B. F. Samsonov and I. N. Ovcharov, "Darboux transformation and exactly solvable potentials with quasi-equidistant spectrum," Russ. Phys. J. **38**, 765 (1995).

[23] V.G.Bagrov and B.F.Samsonov, *"Darboux transformation, factorization, and supersymmetry in one-dimensional quantum mechanics,"* Theor. Math. Phys. **104**, 1051 (1995).

[24] V. Spiridonov, *"Universal superpositions of coherent states and self-similar potentials,"* Phys. Rev. A **52**, 1909 (1995) [arXiv:quant-ph/9601030].

[25] G. Junker and P. Roy, *"Conditionally exactly solvable problems and non-linear algebras,"* *Phys. Lett. A* **232**, *155 (1997).*

[26] J. F. Cariñena, A. M. Perelomov, M. F. Rañada and M. Santander, *"A quantum exactly solvable nonlinear oscillator related to the isotonic oscillator,"* J. Phys. A: Math. Theor. **41**, 085301 (2008) [arXiv:0711.4899 [quant-ph]].

[27] J. M. Fellows and R. A. Smith, *"Factorization solution of a family of quantum nonlinear oscillators,"* J. Phys. A **42**, 335303 (2009) .

[28] J. Sesma, *"The generalized quantum isotonic oscillator,"* J. Phys. A **43**, 185303 (2010) [arXiv:1005.1227 [quant-ph]].

[29] D. Gómez-Ullate, Y. Grandati, and R. Milson, *"Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials,"* J. Phys. A **47**, 015203 (2014) [arXiv:1306.5143 [maph-ph]].

[30] A. M. Pupasov-Maksimov, *"Propagators of isochronous an-harmonic oscillators and Mehler formula for the exceptional Hermite polynomials,"* Annals of Physics **363**, 122 (2015) [arXiv:1502.01778 [maph-ph]].

[31] M. G. Krein, *"On a continuous analogue of a Christoffel formula from the theory of orthogonal polynomials,"* Dokl. Akad. Nauk SSSR **113**, 970 (1957).

[32] D. Gomez-Ullate, Y. Grandati, and R. Milson, *"Extended Krein-Adler theorem for the translationally shape invariant potentials,"* J. Math. Phys. **55**, 040201 (2014) [arXiv:1309.3756 [nlin.SI]].

[33] D. Gómez-Ullate, N. Kamran, and R. Milson, *"An extended class of orthogonal polynomials defined by a Sturm-Liouville problem,"* J. Math. Anal. Appl. **359** (2009) 352 [arXiv:0807.3939 [math-ph]].

[34] D. Gómez-Ullate, N. Kamran, and R. Milson, *"An extension of Bochners problem: exceptional invariant subspaces,"* J. Approx. Theory **162**, 897 (2010) [arXiv:0805.3376 [math-ph]].

[35] S. Odake and R. Sasaki, *"Infinitely many shape invariant potentials and new orthogonal polynomials,* Phys. Lett. B **679**, 414 (2009) [arXiv:0906.0142 [math-ph]].

[36] S. Odake and R. Sasaki, *"Another set of infinitely many exceptional (X) Laguerre polynomials,* Phys. Lett. B **684**, 173 (2010) [arXiv:0911.3442 [math-ph]].

[37] C. Quesne, *"Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry,"* J. Phys. A **41**, 392001 (2008) [arXiv:0807.4087 [quant-ph]].

[38] C. Quesne, *"Solvable rational potentials and exceptional orthogonal polynomials in supersymmetric quantum mechanics,"* SIGMA **5**, 084 (2009) [arXiv:0906.2331 [math-ph]].

[39] R. Sasaki, S. Tsujimoto and A. Zhedanov, *"Exceptional Laguerre and Jacobi polynomials and the corresponding potentials through Darboux-Crum transformations,"* J. Phys. A **43**, 315204 (2010) [arXiv:1004.4711 [math-ph]].

[40] Y. Grandati, *"Solvable rational extensions of the isotonic oscillator,"* Ann. Phys. **326**, 2074 (2011) [arXiv:1101.0055 [math-ph]].

[41] R. Sasaki, The Universe **2**, no. 2, 2 (2014) [arXiv:1411.2703 [math-ph]].

[42] A. P. Veselov and A. B. Shabat, *"Dressing chains and the spectral theory of the Schrödinger operator,"* Funct. Anal. Appl. **27**, 81 (1993).

[43] V. E. Adler, *"Nonlinear chains and Painlevé equations,"* Physica D **73**, 335 (1994).

[44] R. Willox and J. Hietarinta, *"Painlevé equations from Darboux chains: I. PIII - PV,"* J. Phys. A **36**, 10615 (2003).

[45] A. Oblomkov, *"Monodromy-free Schrödinger operators with quadratically increasing potentials,"* Theor. Math. Phys. **121**, 1574 (1999).

[46] I. Marquette and C. Quesne, *"Two-step rational extensions of the harmonic oscillator: exceptional orthogonal polynomials and ladder operators," J. Phys. A: Math. Theor.* **46**, *155201 (2013) [arXiv:1212.3474[math-ph]].*

[47] I. Marquette and C. Quesne, *"New ladder operators for a rational extension of the harmonic oscillator and superintegrability of some two-dimensional systems,"* J. Math. Phys. **54**, 102102 (2013) [arXiv:1303.7150 [math-ph]].

[48] I. Marquette and C. Quesne, *"Combined state-adding and state-deleting approaches to type III multi-step rationally-extended potentials: applications to ladder operators and superintegrability,"* J. Math. Phys. **55**, 112103 (2014) [arXiv:1402.6380 [math-ph]].

[49] I. Marquette, *"New families of superintegrable systems from k-step rational extensions, polynomial algebras and degeneracies,"* J. Phys.: Conf. Ser. **597**, 012057 (2015) [arXiv:1412.0312 [math-ph]].

[50] C. Quesne, *"Ladder operators for solvable potentials connected with exceptional orthogonal polynomials,"* J. Phys. Conf. Ser. **597**, 012064 (2015) [arXiv:1412.5874 [math-ph]].

[51] J. F. Cariñena and M. S. Plyushchay, *"Ground-state isolation and discrete flows in a rationally extended quantum harmonic oscillator,"* Phys. Rev. D **94**, no. 10, 105022 (2016) [arXiv:1611.08051 [hep-th]].

[52] M. S. Plyushchay, *"Schwarzian derivative treatment of the quantum second-order supersymmetry anomaly, and coupling-constant metamorphosis,"* Ann. Phys., in press [arXiv:1602.02179 [hep-th]].

[53] J.L. Burchnall, T.W. Chaundy, *Commutative ordinary differential operators*, Proc. London Math. Soc. Ser. 2, **21**, 420 (1923); *Commutative ordinary differential operators*, Proc. Royal Soc. London A **118**, 557 (1928).

[54] I. M. Krichever, *"Integration of nonlinear equations by the methods of algebraic geometry,* Func. Anal. Appl. **11**, 12 (1977); *"Baker-Akhiezer functions and integrable systems,"* in: *"Integrability. The Seiberg-Witten and Whitham Equations"*, Edited by H. W. Braden and I. M. Krichever, p. 1 (Gordon and Breach Science Publishers, Amsterdam, 2000).