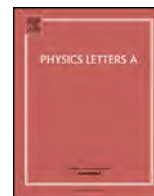




Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla



Superintegrable systems with a position dependent mass: Kepler-related and oscillator-related systems

Manuel F. Rañada

Dep. de Física Teórica and IUMA, Universidad de Zaragoza, 50009 Zaragoza, Spain

ARTICLE INFO

Article history:

Received 25 March 2016

Accepted 4 May 2016

Available online xxxx

Communicated by C.R. Doering

Keywords:

Integrability

Superintegrability

Killing vectors

Position dependent mass

Quadratic constants of motion

Complex functions

ABSTRACT

The superintegrability of two-dimensional Hamiltonians with a position dependent mass (pdm) is studied (the kinetic term contains a factor m that depends of the radial coordinate). First, the properties of Killing vectors are studied and the associated Noether momenta are obtained. Then the existence of several families of superintegrable Hamiltonians is proved and the quadratic integrals of motion are explicitly obtained. These families include, as particular cases, some systems previously obtained making use of different approaches. We also relate the superintegrability of some of these pdm systems with the existence of complex functions endowed with interesting Poisson bracket properties. Finally the relation of these pdm Hamiltonians with the Euclidean Kepler problem and with the Euclidean harmonic oscillator is analyzed.

© 2016 Published by Elsevier B.V.

1. Introduction

It is known that some Liouville integrable systems, as the harmonic oscillator or the Kepler problem, admit more constants of motion than degrees of freedom; they are called superintegrable. Therefore, a Hamiltonian H with two degrees of freedom is said to be integrable if it admits an integral of motion J_2 in addition to the Hamiltonian, and superintegrable if it admits two integrals of motion, J_1 and J_2 , that Poisson commute and a third independent integral J_3 . The integral J_3 has vanishing Poisson bracket with H but not necessarily with J_1 and J_2 .

The mass m has been traditionally considered as a constant in the theory of physical systems admitting a Hamiltonian description. A consequence of this is that the study of superintegrable systems has been mainly focused on two and three degrees of freedom natural Hamiltonian systems (that is, kinetic term plus a potential) with a constant mass; in geometric terms this means that the configuration space Q is an Euclidean space or a constant curved space (spherical or hyperbolic). Nevertheless, in these last years the interest for the study of systems with a position dependent mass has become a matter of great interest and has attracted a lot of attention to many authors. It seems therefore natural to enlarge the study of superintegrability to include systems with a position dependent mass.

It is known that the Liouville formalism characterize the Hamiltonians that are integrable but it does not provide a method for obtaining the constants of motion; therefore it has been necessary to carry out several different methods for searching integrals of motion (Noether symmetries, Hidden symmetries, Lax pairs formalism, bi-Hamiltonian structures, etc.). In a recent paper Szumin-ski et al. studied [1] families of Hamiltonians of the form

$$H_{nk} = \frac{1}{2} r^{n-k} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + r^n U(\phi),$$

(n and k are integers) and then, making use of some previous results of Morales-Ruiz and Ramis related with the differential Galois group of variational equations [2–4], they derive necessary conditions for the integrability of such systems. Then using some rather involved mathematics (related with the hypergeometric differential equation) they arrive to a certain number of Hamiltonians and prove that four of them, given by

$$\mathcal{H}_1 = \frac{1}{2} r^6 \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) - r \cos \phi, \quad (n = 1, k = -5)$$

$$\mathcal{H}_2 = \frac{1}{2} \frac{1}{r^2} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{1}{r} \cos \phi, \quad (n = -1, k = 1)$$

$$\mathcal{H}_3 = \frac{1}{2} r^4 \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{1}{r} \cos \phi, \quad (n = -1, k = -5)$$

$$\mathcal{H}_4 = \frac{1}{2} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) - r \cos \phi, \quad (n = 1, k = 1)$$

E-mail address: mfran@unizar.es.

<http://dx.doi.org/10.1016/j.physleta.2016.05.007>

0375-9601/© 2016 Published by Elsevier B.V.

are superintegrable (two independent constants in addition to the Hamiltonian). The fourth Hamiltonian is in fact a rather simple Euclidean system but the other three are really interesting and deserve to be studied with detail.

In this paper we will study the existence of superintegrability and we will construct the constants of motion using as starting point the properties of the Killing vectors.

We recall that a Killing vector field X in a Riemannian manifold (M, g) , is the (infinitesimal) generator of a symmetry of the metric g (that is, X is a generator of isometries); in geometric terms X must be solution of the equation $\mathcal{L}_X g = 0$ where \mathcal{L}_X denotes the Lie derivative. If M is of dimension n then the metric admits at most $d = \frac{1}{2}n(n+1)$ linearly independent Killing vector fields (constant curvature spaces admit the maximum number; for example if M is the Euclidean plane $M = \mathbb{E}^2$ then $d = 3$).

If the configuration space of a system is a Riemannian manifold (Q, g) then g determines a kinetic Lagrangian $L_g = T_g = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j$ such that the associated motion is just the geodesic motion, and the Killing vectors of (Q, g) determine the constants of motion for the geodesic trajectories (the so-called Noether momenta). In most of cases the addition of a potential $V(x)$ to the kinetic Lagrangian L_g destroys these first integrals but, in some cases, the new system admits first integrals of second order in the momenta whose quadratic terms are determined by Killing tensors of valence $p = 2$ built from Killing vectors. We recall that Killing tensor \mathbf{K} of valence p defined in a Riemannian manifold (M, g) is a symmetric $(p, 0)$ tensor satisfying the Killing tensor equation

$$[\mathbf{K}, g]_S = 0$$

where $[\cdot, \cdot]_S$ denotes the Schouten bracket (bilinear operator representing the natural generalization of the Lie bracket of vector fields) [5,6]. In the case $p = 2$ the Killing tensor \mathbf{K} determines a homogeneous quadratic function $F_K = K^{ij}p_i p_j$ and then the Killing equation can be rewritten as the vanishing of the Poisson bracket of two functions

$$\{K^{ij}p_i p_j, g^{ij}p_i p_j\} = 0.$$

This means that the function F_K is a first integral of the geodesic flow determined by the Hamiltonian $\mathcal{H} = (1/2)g^{ij}p_i p_j$. From a practical viewpoint this means that quadratic term of the integrals of the Hamiltonian $H = T_g + V$ can be expressed as a sum of products of the Noether momenta.

The three Hamiltonians \mathcal{H}_j , $j = 1, 2, 3$, studied in [1] can be considered as Hamiltonians with position dependent masses (pdm) $m = 1/r^6$, $m = r^2$, and $m = 1/r^4$, respectively. In geometric terms this means that they are defined in non-Euclidean spaces.

The following three points summarize the contents of this paper.

- We will study the existence of superintegrable systems with a position dependent mass (pdm) of the form $m_n = r^{2n}$ using the geometric formalism as an approach. We first obtain the Killing vectors for the corresponding metrics (that are conformal metrics) and then we obtain the expressions of the Noether momenta. The following step is the obtainment of the quadratic integrals.
- In fact, as a result of our approach we obtain that the three particular cases above mentioned are not exceptional values (with distinguishing properties) but just particular values in a more general situation. Moreover the above three Hamiltonians \mathcal{H}_j , $j = 1, 2, 3$, obtained in [1] are the particular cases ($k_0 = 0, k_1 = -1, k_2 = 0$) of the following more general functions

$$H_1 = \frac{1}{2}r^6 \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + k_0 r^2 + r(k_1 \cos \phi + k_2 \sin \phi),$$

$$H_2 = \frac{1}{2} \frac{1}{r^2} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + \frac{k_0}{r^2} + \frac{1}{r} (k_1 \cos \phi + k_2 \sin \phi),$$

$$H_3 = \frac{1}{2} r^4 \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + \frac{k_0}{r^2} + \frac{1}{r} (k_1 \cos \phi + k_2 \sin \phi).$$

- We obtain several families of superintegrable Hamiltonians with a position dependent mass (pdm) of the form $m_n = r^{2n}$ but with different potentials $U(r, \phi)$. An important property is that these new potentials, that have also the form of a linear combination with coefficients k_0, k_1 , and k_2 , can be considered as the m_n -deformed versions of the Euclidean superintegrable potentials V_a and V_b (related with the harmonic oscillator), V_c (related with the Kepler problem), and V_d (also related with the Kepler problem), first obtained in [7] and then studied by many authors (see [8] and references therein).

We close this Introduction with the following comments.

First, the study of systems with a position dependent mass is a matter highly studied in these last years but, in most of cases, these studies are related with the problem of the quantization (because the problem of order in the quantization of the kinetic term); the study presented in this paper is concerned with only the classical case and, although different, it has a close relation with the study presented in [9].

Second, quadratic superintegrability is a property very related with Hamilton–Jacobi (H–J) multiple separability (Schrödinger separability in the quantum case) and this property is also true for systems with a position dependent mass. This question (H–J separability approach to systems with a pdm) was studied in [9] (in this case the pdm depends on a parameter κ) and more recently in [10] (in this last case the pdm Hamiltonians studied were also related with those recently obtained through a differential Galois group analysis in [1]).

Third, the study of systems admitting generalizations of the Laplace–Runge–Lenz vector [11–19] and the study of generalizations of the Kepler problem (Kepler-related problems with closed trajectories) are two (related) questions highly studied (see [20] and references therein). We will see in the next sections that some of the pdm Hamiltonians studied in this paper are endowed with integrals of motion rather similar to the Laplace–Runge–Lenz vector (this is also true for the above mentioned functions H_1 and H_2) and therefore they belong to the family of generalized the Kepler problems.

2. Superintegrability with quadratic constants of motion in the Euclidean plane

We recall, in this Section, the existence in the Euclidean plane of four two-dimensional potentials V_j , $j = a, b, c, d$, that are superintegrable with quadratic integrals of motion.

- (a) The following potential, related with the harmonic oscillator,

$$V_a = \frac{1}{2} \omega_0^2 (x^2 + y^2) + \frac{k_1}{x^2} + \frac{k_2}{y^2} \tag{1}$$

is separable in Cartesian coordinates and polar coordinates. The constants of motion are the two one-dimensional energies and a third function related with the square of the angular momentum.

- (b) The following potential, related with the harmonic oscillator,

$$V_b = \frac{1}{2} \omega_0^2 (x^2 + 4y^2) + \frac{k_1}{x^2} + k_2 y \tag{2}$$

is separable in Cartesian coordinates and parabolic coordinates. The constants of motion are the two one-dimensional energies and a third function related with the Runge–Lenz vector.

(c) The following potential, related with the Kepler problem,

$$V_c = \frac{k_0}{\sqrt{x^2 + y^2}} + \frac{k_1}{y^2} + \frac{k_2 x}{y^2 \sqrt{x^2 + y^2}} \quad (3)$$

is separable in polar coordinates and parabolic coordinates. The first constant of motion is the Hamiltonian itself and the other two, I_{c2} and I_{c3} , are related with the square of the angular momentum and the Runge–Lenz vector.

(d) The following potential, related with the Kepler problem,

$$V_d = \frac{k_0}{\sqrt{x^2 + y^2}} + k_1 \frac{[\sqrt{x^2 + y^2} + x]^{1/2}}{\sqrt{x^2 + y^2}} + k_2 \frac{[\sqrt{x^2 + y^2} - x]^{1/2}}{\sqrt{x^2 + y^2}} \quad (4)$$

is separable in two different systems of parabolic coordinates and the two constants of motion, I_{d2} and I_{d3} , are related with the Runge–Lenz vector.

In the following sections we will study Hamiltonians with a pdm. We will prove that the systems obtained in [1] are just particular cases of a much more general situation and we will present all the results making use of a notation that stress the relation of the new Hamiltonians (to be denoted as H_{nj} , $j = a, b, c, d$) with the above mentioned Euclidean systems.

3. Position dependent mass, Killing vectors and Noether momenta

A position dependent mass $m_n = 1/r^{2n}$ determines a kinetic Lagrangian $L_n = T_n$ and an associated metric ds_n^2 given by

$$T_n = \frac{1}{2} \frac{1}{r^{2n}} (v_r^2 + r^2 v_\phi^2), \quad ds_n^2 = \frac{1}{r^{2n}} (dr^2 + r^2 d\phi^2). \quad (5)$$

This metric admits three symmetries; the invariance under rotations (generated by $X_J = \partial/\partial\phi$) and two other symmetries generated by the Killing vectors X_1 and X_2 given by

$$X_1 = r^n \left(\cos(k_n \phi) \frac{\partial}{\partial r} + \frac{1}{r} \sin(k_n \phi) \frac{\partial}{\partial \phi} \right),$$

$$X_2 = r^n \left(\sin(k_n \phi) \frac{\partial}{\partial r} - \frac{1}{r} \cos(k_n \phi) \frac{\partial}{\partial \phi} \right),$$

where, for ease of the notation, we introduce k_n for $k_n = n - 1$. Every Killing vector X determines an associated Noether momenta P (so many Noether momenta as Killing vectors) that represents a constant of motion for the geodesic motion; so, in this case, we have the angular momentum $p_\phi = v_\phi / r^{2(n-1)}$ and the other two given by

$$i(X_1)\theta_L = \frac{1}{r^n} \left(\cos(k_n \phi) v_r + r \sin(k_n \phi) v_\phi \right),$$

$$i(X_2)\theta_L = \frac{1}{r^n} \left(\sin(k_n \phi) v_r - r \cos(k_n \phi) v_\phi \right),$$

where θ_L is the Cartan 1-form

$$\theta_L = \left(\frac{\partial L}{\partial v_r} \right) dr + \left(\frac{\partial L}{\partial v_\phi} \right) d\phi.$$

Making use of the Legendre transformation we obtain the kinetic Hamiltonian

$$H_n = T_n = \frac{1}{2} r^{2n} \left(p_r^2 + \frac{1}{r^2} p_\phi^2 \right)$$

and the Hamiltonian expressions of the Noether momenta as linear functions of the canonical momenta

$$P_1 = r^n \left(p_r \cos(k_n \phi) + \frac{1}{r} p_\phi \sin(k_n \phi) \right),$$

$$P_2 = r^n \left(p_r \sin(k_n \phi) - \frac{1}{r} p_\phi \cos(k_n \phi) \right),$$

such that

$$\{P_1, T_n\} = 0, \quad \{P_2, T_n\} = 0, \quad \{p_\phi, T_n\} = 0.$$

4. Harmonic oscillator related Hamiltonians

In what follows we introduced potentials in the Lagrangian L_n (Hamiltonian H_n) in two steps. First central potentials ($V_{na} = 1/r^{2(n-1)}$ and $V_{nc} = r^{n-1}$) and then ϕ -dependent new terms.

4.1. Hamiltonian H_{na}

The first system to be studied with a position dependent mass $m_n = 1/r^{2n}$ is represented by a Hamiltonian with central potential $V_{na} = 1/r^{2k_n}$

$$H_{na} = T_n + \frac{k_0}{r^{2k_n}}, \quad k_n = n - 1, \quad n \neq 1. \quad (6)$$

It is superintegrable with the following three constants of motion

$$J_1 = p_\phi, \quad J_{11} = P_1^2 + 2 \frac{k_0}{r^{2k_n}} (\cos(k_n \phi))^2,$$

$$J_{22} = P_2^2 + 2 \frac{k_0}{r^{2k_n}} (\sin(k_n \phi))^2,$$

that satisfy the following properties

$$(i) dJ_1 \wedge dJ_{11} \wedge dJ_{22} \neq 0, \quad (ii) \{J_{11}, J_{22}\} = 0,$$

$$(iii) H_{na} = \frac{1}{2} (J_{11} + J_{22}).$$

A remarkable property is that the following function

$$J_{12} = J_{21} = P_1 P_2 + 2 \frac{k_0}{r^{2k_n}} \cos(k_n \phi) \sin(k_n \phi)$$

is also a constant of motion. These three integrals $\{J_{11}, J_{22}, J_{12}\}$ can be considered as the three components F_{ij} , $i, j = 1, 2$, of a Fradkin tensor [21]. Because of this the Hamiltonian H_{na} can be interpreted as representing an harmonic oscillator with a pdm $m_n = 1/r^{2n}$.

In a similar way to what happens in the Euclidean case [7,8], the above Hamiltonian, that it has a central potential V_{na} , admits the addition of two non-central new terms preserving the quadratic superintegrability. In this case we have

$$H_{na} = T_n + U_{na}(r, \phi),$$

$$U_{na} = \frac{k_0}{r^{2k_n}} + r^{2k_n} \left[\left(\frac{k_1}{\cos^2 k_n \phi} \right) + \left(\frac{k_2}{\sin^2 k_n \phi} \right) \right], \quad (7)$$

where k_0, k_1 , and k_2 are arbitrary constants. The three independent constants of motion are

$$J_{a1} = P_1^2 + 2 \frac{k_0}{r^{2k_n}} (\cos(k_n \phi))^2 + 2k_1 r^{2k_n} (\sec(k_n \phi))^2,$$

$$J_{a2} = P_2^2 + 2 \frac{k_0}{r^{2k_n}} (\sin(k_n \phi))^2 + 2k_2 r^{2k_n} (\csc(k_n \phi))^2$$

and

$$J_{a3} = p_\phi^2 + 2 \left[\left(\frac{k_1}{\cos^2 k_n \phi} \right) + \left(\frac{k_2}{\sin^2 k_n \phi} \right) \right].$$

Starting with the central potential $V_{na} = 1/r^{2k_n}$ we can also construct the following Hamiltonian

$$H'_{na} = T_n + U'_{na}(r, \phi),$$

$$U'_{na} = \frac{k_0}{r^{2k_n}} + \frac{1}{r^{k_n}} \left(k_1 \cos(k_n \phi) + k_2 \sin(k_n \phi) \right). \quad (8)$$

It has, in addition to the two quadratic constants J'_{a1} and J'_{a2} , similar to J_{a1} and J_{a2} , a linear in the momenta constant of motion

$$J'_{a3} = 2k_0 p_\phi + k_2 P_1 - k_1 P_2$$

determined by an exact Noether symmetry of the Lagrangian $L'_{na} = T_n - U'_{na}(r, \phi)$. Note that the Hamiltonian \mathcal{H}_3 [1] mentioned in the introduction appears as the particular case $n = 2$ of H'_{na} .

4.2. Hamiltonian H_{nb}

Now we consider the pdm Hamiltonian $H_{nb} = T_n + U_{nb}(r, \phi)$ where the potential U_{nb} takes the form

$$U_{nb} = \frac{k_0}{r^{2k_n}} \left(\cos^2(k_n \phi) + 4 \sin^2(k_n \phi) \right) + r^{2k_n} \left(\frac{k_1}{\cos^2 k_n \phi} \right) + \frac{k_2}{r^{k_n}} \sin(k_n \phi), \quad k_n = n - 1, \quad (9)$$

where $k_0, k_1,$ and k_2 are arbitrary constants. It is superintegrable with the following three independent integrals of motion

$$J_{b1} = P_1^2 + 2 \frac{k_0}{r^{2k_n}} (\cos(k_n \phi))^2 + 2k_1 r^{2k_n} (\sec(k_n \phi))^2,$$

$$J_{b2} = P_2^2 + 8 \frac{k_0}{r^{2k_n}} (\sin(k_n \phi))^2 + \frac{2k_2}{r^{k_n}} \sin(k_n \phi)$$

and

$$J_{b3} = P_1 p_\phi - \frac{k_0}{r^{3k_n}} \cos(k_n \phi) \sin(2k_n \phi) + k_1 r^{k_n} (\sec^3(k_n \phi) \sin(2k_n \phi)) - \frac{k_2}{2r^{2k_n}} \cos^2(k_n \phi).$$

5. Kepler related Hamiltonians

5.1. Hamiltonian H_{nc}

Now we consider a Hamiltonian with a position dependent mass $m_n = 1/r^{2n}$ and a central potential $V_{nc} = r^{n-1}$

$$H_{nc} = T_n + k_0 r^{n-1}, \quad n \neq 1. \quad (10)$$

It is superintegrable with the following three constants of motion

$$J_1 = p_\phi, \quad J_2 = P_2 p_\phi - k_0 \cos(k_n \phi),$$

$$J_3 = P_1 p_\phi + k_0 \sin(k_n \phi).$$

It is clear that J_2 and J_3 are quite similar to the two components of a two-dimensional Runge–Lenz vector. Because of this the Hamiltonian H_{nc} can be interpreted as representing a Kepler system with a pdm $m_n = 1/r^{2n}$.

There are three different ways of modifying the potential V_{nc} by introducing additional ϕ -dependent terms in such a way that the superintegrability is preserved. In the two first cases only one of the two Runge–Lenz-like constants is preserved; in the third case both Runge–Lenz-like constants are preserved (but then the integral J_1 disappears).

(c1) The following Hamiltonian

$$H_{nc1} = T_n + U_{nc1}(r, \phi),$$

$$U_{nc1} = k_0 r^{n-1} + r^{2k_n} \left[\left(\frac{k_1}{\sin^2 k_n \phi} \right) + k_2 \left(\frac{\cos k_n \phi}{\sin^2 k_n \phi} \right) \right], \quad (11)$$

has (in addition to the Hamiltonian itself) two functionally independent first integrals of the second order in the momenta

$$dJ_{c2} \wedge dJ_{c3} \wedge dH_{nc1} \neq 0,$$

$$\{J_{c2}, H_{nc1}\} = 0, \quad \{J_{c3}, H_{nc1}\} = 0,$$

given by

$$J_{c2} = p_\phi^2 + 2 \left[\left(\frac{k_1}{\sin^2 k_n \phi} \right) + k_2 \left(\frac{\cos k_n \phi}{\sin^2 k_n \phi} \right) \right],$$

$$J_{c3} = P_2 p_\phi - k_0 \cos(k_n \phi) - 2k_1 r^{k_n} (\csc k_n \phi \cot k_n \phi) - k_2 r^{k_n} (\csc^2 k_n \phi + \cot^2 k_n \phi).$$

(c2) The Hamiltonian

$$H_{nc2} = T_n + U_{nc2}(r, \phi),$$

$$U_{nc2} = k_0 r^{n-1} + r^{2k_n} \left[\left(\frac{k_1}{\cos^2 k_n \phi} \right) + k_2 \left(\frac{\sin k_n \phi}{\cos^2 k_n \phi} \right) \right], \quad (12)$$

is similar to the previous one H_{nc1} but in this case is the existence of the second Runge–Lenz integral what is preserved

$$J_{c2} = p_\phi^2 + 2 \left[\left(\frac{k_1}{\cos^2 k_n \phi} \right) + k_2 \left(\frac{\sin k_n \phi}{\cos^2 k_n \phi} \right) \right],$$

$$J_{c3} = P_1 p_\phi + k_0 \sin(k_n \phi) + 2k_1 r^{k_n} (\sec k_n \phi \tan k_n \phi) + k_2 r^{k_n} (\sec^2 k_n \phi + \tan^2 k_n \phi).$$

5.2. Hamiltonian H_{nd}

The Hamiltonian

$$H_{nd} = T_n + U_{nd}(r, \phi),$$

$$U_{nd} = k_0 r^{n-1} + r^{k_n/2} (k_1 \cos k_n(\phi/2) + k_2 \sin k_n(\phi/2)), \quad (13)$$

that generalizes the Hamiltonians \mathcal{H}_1 and \mathcal{H}_2 obtained in [1] and mentioned in the Introduction (they correspond to $n = 3$ and $n = -1$). It possesses the following two independent constants of motion J_{d2} and J_{d3}

$$J_{d2} = P_1 p_\phi - k_0 \cos(k_n \phi) + \frac{k_1}{r^{k_n/2}} (\sin k_n \phi \sin k_n(\phi/2)) - \frac{k_2}{r^{k_n/2}} (\sin k_n \phi \cos k_n(\phi/2)),$$

$$J_{d3} = P_2 p_\phi + k_0 \sin(k_n \phi) + \frac{k_1}{r^{k_n/2}} (\cos k_n \phi \sin k_n(\phi/2)) - \frac{k_2}{r^{k_n/2}} (\cos k_n \phi \cos k_n(\phi/2)).$$

Both are of Runge–Lenz type.

6. Complex functions and superintegrability

We mention in the Introduction the existence of different approaches (Noether symmetries, Hidden symmetries, Lax pairs formalism, bi-Hamiltonian structures, H–J separability) for the study of Liouville integrability (or superintegrability). Now in this section we study the superintegrability of two of the Hamiltonians (H'_{na} related to the harmonic oscillator and H_{nd} related to the Kepler problem) already studied in the previous section but now making use of a rather different approach. The main idea is that the superintegrability can be related with the existence of certain complex functions with interesting Poisson brackets properties. This complex functions formalism has been recently studied in [22] for the Kepler problem in the Euclidean plane.

6.1. Hamiltonian H'_{na}

Let us first introduce the following real functions

$$M_{n1} = r^{2k_n} (r^2 p_r^2 - p_\phi^2) + \frac{2k_0}{r^{2k}} + \frac{2}{r^{k_n}} (k_1 \cos(k_n \phi) + k_2 \sin(k_n \phi)),$$

$$M_{n2} = 2r^{2n-1} p_r p_\phi + \frac{2}{r^{k_n}} (k_1 \sin(k_n \phi) - k_2 \cos(k_n \phi)),$$

and

$$N_{\phi 1} = \cos(2k_n \phi), \quad N_{\phi 2} = \sin(2k_n \phi).$$

Then if we denote by M_n and N_ϕ the complex functions

$$M_n = M_{n1} + i M_{n2}, \quad N_\phi = N_{\phi 1} + i N_{\phi 2},$$

we have

$$\frac{d}{dt} M_n = \{M_n, H'_{na}\} = i 2 \lambda_n M_n,$$

$$\frac{d}{dt} N_\phi = \{N_\phi, H'_{na}\} = i 2 \lambda_n N_\phi,$$

where the common factor λ_n is given by

$$\lambda_n = (n - 1)r^{2k_n} p_\phi.$$

This means that the function constructed by coupling M_n with N_ϕ is a constant of motion. This result is presented in the following proposition.

Proposition 1. *Let us consider the following Hamiltonian with a position dependent mass $m = r^{2n}$*

$$H'_{na} = T_n + U'_{na}(r, \phi),$$

$$U'_{na} = \frac{k_0}{r^{2k_n}} + \frac{1}{r^{k_n}} (k_1 \cos(k_n \phi) + k_2 \sin(k_n \phi)).$$

Then, the complex function J_{23} defined as

$$J_{23} = M_n N_\phi^*$$

is a quadratic (complex) constant of motion.

Of course J_{23} determines two real first-integrals

$$J_{23} = J_2 + i J_3, \quad \{J_2, H'_{na}\} = 0, \quad \{J_3, H'_{na}\} = 0,$$

whose coordinate expressions turn out to be

$$J_2 = r^{2(n-1)} \left((r^2 p_r^2 - p_\phi^2) \cos(2k_n \phi) + (2r p_r p_\phi) \sin(2k_n \phi) \right) + \frac{2}{r^{2k_n}} k_0 \cos(2k_n \phi) + \frac{2}{r^{k_n}} (k_1 \cos(k_n \phi) - k_2 \sin(k_n \phi)),$$

$$J_3 = r^{2(n-1)} \left((r^2 p_r^2 - p_\phi^2) \sin(2k_n \phi) - (2r p_r p_\phi) \cos(2k_n \phi) \right) + \frac{2}{r^{2k_n}} k_0 \sin(2k_n \phi) + \frac{2}{r^{k_n}} (k_1 \sin(k_n \phi) + k_2 \cos(k_n \phi)).$$

Concerning the linear constant of motion J'_{a3} (obtained from an exact Noether symmetry), it determines the following Poisson brackets with J_2 and J_3

$$\{J'_{a3}, J_2\} = 4(n - 1)(k_0 J_3 + k_1 k_2),$$

$$\{J'_{a3}, J_3\} = -2(n - 1)(2k_0 J_2 + k_1^2 - k_2^2).$$

6.2. Hamiltonian H_{nd}

Let us denote by A_{nj} and $N_{\phi j}$, $j = 1, 2$, the following real functions

$$A_{n1} = r^{n-1} p_\phi^2 + k_0,$$

$$A_{n2} = \frac{1}{r^{k_n/2}} \left(r^{m_n} p_r p_\phi + k_1 \sin(k_n/2) \phi - k_2 \cos(k_n/2) \phi \right),$$

$$m_n = \frac{1}{2}(3n - 1),$$

and

$$N_{\phi 1} = \cos k_n \phi, \quad N_{\phi 2} = \sin k_n \phi.$$

Then we have the following properties

$$(i) \quad \frac{d}{dt} A_{n1} = \{A_{n1}, H_{nd}\} = (n - 1) \lambda_n A_{n2},$$

$$\frac{d}{dt} A_{n2} = \{A_{n2}, H_{nd}\} = -(n - 1) \lambda_n A_{n1},$$

$$(ii) \quad \frac{d}{dt} N_{\phi 1} = \{N_{\phi 1}, H_{nd}\} = -(n - 1) \lambda_n N_{\phi 2},$$

$$\frac{d}{dt} N_{\phi 2} = \{N_{\phi 2}, H_{nd}\} = (n - 1) \lambda_n N_{\phi 1},$$

where λ_n denotes the following function

$$\lambda_n = r^{2(n-1)} p_\phi. \tag{14}$$

Therefore, the two complex functions A_n and N_ϕ defined as

$$A_n = A_{n1} + i A_{n2}, \quad N_\phi = N_{\phi 1} + i N_{\phi 2},$$

satisfy the following Poisson bracket properties

$$\{A_n, H_{nd}\} = -i(n - 1) \lambda_n A_n, \quad \{N_\phi, H_{nd}\} = i(n - 1) \lambda_n N_\phi,$$

and consequently the Poisson bracket of the complex function $A_n N_\phi$ with the Kepler-related Hamiltonian H_{nd} vanishes

$$\begin{aligned} \{A_n N_\phi, H_{nd}\} &= \{A_n, H_{nd}\} N_\phi + A_n \{N_\phi, H_{nd}\} \\ &= (n - 1)(-i \lambda_n A_n) N_\phi + (n - 1) A_n (i \lambda_n N_\phi) = 0. \end{aligned}$$

We can summarize this result in the following proposition.

Proposition 2. *Let us consider the Kepler-related Hamiltonian H_{nd} with $p_{dm} m_n = 1/r^{2n}$*

$$H_{nd} = T_n + U_{nd}(r, \phi),$$

$$U_{nd} = k_0 r^{k_n} + r^{k_n/2} (k_1 \cos(k_n/2) \phi + k_2 \sin(k_n/2) \phi).$$

Then, the complex function J_{23} defined as

$$J_{23} = A_n N_\phi$$

is a quadratic (complex) constant of motion.

Of course J_{23} determines two real first-integrals

$$J_{23} = \text{Re}(J_{23}) + i \text{Im}(J_{23}), \quad \{\text{Re}(J_{23}), H_{nd}\} = 0,$$

$$\{\text{Im}(J_{23}), H_{nd}\} = 0,$$

whose coordinate expressions turn out to be

$$\text{Re}(J_{23}) = J_{d2}, \quad \text{Im}(J_{23}) = J_{d3}.$$

That is, the two real functions $\text{Re}(J_{23})$ and $\text{Im}(J_{23})$ are just the two components of the pdm-version of the two-dimensional Laplace-Runge-Lenz vector.

Summarizing, we have got two interesting properties. First, the superintegrability of the pdm-deformed version H_{nd} of the Kepler problem is directly related with the existence of two complex functions (A_n and N_ϕ) whose Poisson brackets with the Hamiltonian H_{nd} are proportional, with a common complex factor, to themselves; and second, the two components of the pdm-deformed version of the Laplace–Runge–Lenz vector appear as the real and imaginary parts of the complex first-integral of motion. Remark that N_ϕ is a complex function of constant modulus one, while the modulus of A_n is a polynomial of degree four in the momenta that is just the sum of the squares of J_{d2} and J_{d3}

$$A_n A_n^* = J_{d2}^2 + J_{d3}^2.$$

7. Final comments

We have studied the superintegrability of Hamiltonian systems with a pdm $m_n = r^{2n}$, $n \neq 1$, and we have proved that the particular Hamiltonians previously obtained in [1] are just very particular cases of the systems here obtained. We have made use of the properties of Killing vectors as the starting point of our approach, and we have proved that the Hamiltonians so obtained can be considered as pdm-deformations of the classical Euclidean superintegrable systems with potentials V_a and V_b (related with the harmonic oscillator), and V_c and V_d (both related with the Kepler problem). This result clearly reinforces the importance of these four potentials since, although defined in an Euclidean geometry, they are directly related with superintegrable systems with a nonEuclidean metric (this close relation between superintegrable Hamiltonians with and without pdm was already considered in [9]).

Integrability and superintegrability on spaces of constant and nonconstant curvature is a matter recently studied by several authors (see, e.g. [23–26] and references therein). Nevertheless, in differential geometric terms a pdm global factor means that the configuration space Q is endowed with a conformal metric (a nonEuclidean space but with conformal equivalence to the Euclidean one) and in this case we have the additional property that the pdm is a function dependent only of the radial variable. This is probably the main reason for the existence of a so close relation between the pdm Hamiltonians we have obtained and the four Euclidean superintegrable systems mentioned in Section 2. Moreover, and concerning that existence of curvature, we recall that in two dimensions the Riemann tensor R_{abcd} only has one independent component which can be taken R_{1212}

$$R_{1212} = \frac{1}{2} (\partial_2 \partial_1 g_{21} - \partial_2^2 g_{11} + \partial_1 \partial_2 g_{12} - \partial_1^2 g_{22}) - g_{ef} (\Gamma_{11}^e \Gamma_{22}^f - \Gamma_{12}^e \Gamma_{21}^f).$$

In this case (with $g_{11} = 1/r^{2n}$ and $g_{22} = 1/r^{2n-2}$) the result is $R_{1212} = 0$. So the configuration space for the Hamiltonian H_n (that is, $Q = \mathbb{R}^2$ with the line element ds_n^2) is in fact a flat manifold.

We finalize with the following questions for future work.

- It is natural to suppose the existence of superintegrable systems with a position dependent mass but with higher order constants of motion. We recall that the Euclidean potentials V_a and V_c admit two generalizations

$$V_{ttw}(r, \phi) = \frac{1}{2} \omega_0^2 r^2 + \frac{1}{2r^2} \left(\frac{\alpha}{\cos^2(m\phi)} + \frac{\beta}{\sin^2(m\phi)} \right),$$

$$V_{pww}(r, \phi) = -\frac{g}{r} + \frac{1}{2r^2} \left(\frac{\alpha}{\cos^2(m\phi)} + \frac{\beta}{\sin^2(m\phi)} \right),$$

that are superintegrable but with higher order constants of motion [27–29]. So, the existence of superintegrable systems similar to these two Euclidean systems but with a pdm of the form m_j is a matter to be studied. The higher superintegrability of the potentials V_{ttw} and V_{pww} has been studied making use of different techniques (at both the classical and the quantum levels); here we point out the existence of a method that make use of products of complex functions [30,31]; probably this method can also be applied to the study of the m_j -dependent case.

- Concerning the complex functions formalism presented in Section 6, we mention that it was proved in [22] that it is related with the existence of quasi-bi-Hamiltonian structures. So the existence of these structures (bi-Hamiltonian or quasi-bi-Hamiltonian) for systems with a pdm is also a matter to be studied making use of the properties of these complex functions.
- Finally, the study of quantum systems with a pdm is a matter highly studied in these last years. First the quantization of these systems is not an easy matter (because the problem of order in the quantization of the kinetic term) and second, it seems that some of these pdm systems belong to the family of Hamiltonians with an exactly solvable Schrödinger equation. Therefore, the quantum study of all these pdm Hamiltonians is also an interesting matter to be studied.

Acknowledgements

This work has been supported by the research projects MTM-2012-33575 (MICINN, Madrid) and DGA-E24/1 (DGA, Zaragoza).

References

- [1] W. Szuminski, A.J. Maciejewski, M. Przybylska, Note on integrability of certain homogeneous Hamiltonian systems, *Phys. Lett. A* 379 (45–46) (2015) 2970–2976.
- [2] J.J. Morales-Ruiz, *Differential Galois Theory and Non-integrability of Hamiltonian Systems*, Progress in Mathematics, vol. 179, Birkhäuser Verlag, Basel, 1999.
- [3] J.J. Morales-Ruiz, J.P. Ramis, A note on the non-integrability of some Hamiltonian systems with a homogeneous potential, *Methods Appl. Anal.* 8 (1) (2001) 113–120.
- [4] K. Nakagawa, H. Yoshida, A necessary condition for the integrability of homogeneous Hamiltonian systems with two degrees of freedom, *J. Phys. A, Math. Theor.* 34 (11) (2001) 2137–2148.
- [5] C. Chanu, L. Degiovanni, R.G. McLenaghan, Geometrical classification of Killing tensors on bidimensional flat manifolds, *J. Math. Phys.* 47 (2006) 073506.
- [6] J.T. Horwood, R.G. McLenaghan, R.G. Smirnov, Hamilton–Jacobi theory in three-dimensional Minkowski space via Cartan geometry, *J. Math. Phys.* 50 (2009) 053507.
- [7] T.I. Fris, V. Mandrosov, Y.A. Smorodinsky, M. Uhler, P. Winternitz, On higher symmetries in quantum mechanics, *Phys. Lett.* 16 (1965) 354–356.
- [8] P. Tempesta, P. Winternitz, J. Harnad, W. Miller, G. Pogossyan, M. Rodriguez (Eds.), *Superintegrability in Classical and Quantum Systems*, CRM Proc. Lecture Notes, vol. 37, Amer. Math. Soc., Providence, RI, 2004.
- [9] M.F. Rañada, Superintegrable deformations of superintegrable systems: quadratic superintegrability and higher-order superintegrability, *J. Math. Phys.* 56 (2015) 042703.
- [10] A.P. Fordy, A note on some superintegrable Hamiltonian systems, arXiv:1601.03079, 2016.
- [11] P.G.L. Leach, V.M. Gorringe, A conserved Laplace–Runge–Lenz-like vector for a class of three-dimensional motions, *Phys. Lett. A* 133 (6) (1988) 289–294.
- [12] A. Holas, N.H. March, A generalisation of the Runge–Lenz constant of classical motion in a central potential, *J. Phys. A* 23 (5) (1990) 735–749.
- [13] P.G.L. Leach, G.P. Flessas, Generalisations of the Laplace–Runge–Lenz vector, *J. Nonlinear Math. Phys.* 10 (3) (2003) 340–423.
- [14] J.F. Cariñena, M.F. Rañada, M. Santander, The Kepler problem and the Laplace–Runge–Lenz vector on spaces of constant curvature and arbitrary signature, *Qual. Theory Dyn. Syst.* 7 (1) (2008) 87–99.
- [15] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, Hamiltonian systems admitting a Runge–Lenz vector and an optimal extension of Bertrand’s theorem to curved manifolds, *Commun. Math. Phys.* 290 (3) (2009) 1033–1049.
- [16] H. White, On a class of dynamical systems admitting both Poincaré and Laplace–Runge–Lenz vectors, *Nuovo Cimento B* 125 (1) (2010) 7–25.

- [17] U. Ben-Yaacov, Laplace–Runge–Lenz symmetry in general rotationally symmetric systems, *J. Math. Phys.* 51 (2010) 122902.
- [18] I. Marquette, Generalized MICZ-Kepler system, duality, polynomial, and deformed oscillator algebras, *J. Math. Phys.* 51 (2010) 102105.
- [19] A.G. Nikitin, Laplace–Runge–Lenz vector with spin in any dimension, *J. Phys. A, Math. Theor.* 47 (37) (2014) 375201.
- [20] I.A. Bizyaev, A.V. Borisov, I.S. Mamaev, Superintegrable generalizations of the Kepler and Hook problems, *Regul. Chaotic Dyn.* 19 (3) (2014) 415–434.
- [21] D.M. Fradkin, Three-dimensional isotropic harmonic oscillator and $SU(3)$, *Am. J. Phys.* 33 (1965) 207–211.
- [22] J.F. Cariñena, M.F. Rañada, Quasi-bi-Hamiltonian structures of the 2-dimensional Kepler problem, *SIGMA* 12 (2016) 010.
- [23] E.G. Kalnins, J.M. Kress, J. Miller, G.S. Pogosyan, Complete sets of invariants for dynamical systems that admit a separation of variables, *J. Math. Phys.* 43 (2002) 3592–3609.
- [24] E.G. Kalnins, J.M. Kress, P. Winternitz, Superintegrability in a two-dimensional space of nonconstant curvature, *J. Math. Phys.* 43 (2002) 970–983.
- [25] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, A maximally superintegrable system on an n -dimensional space of nonconstant curvature, *Phys. D* 237 (4) (2008) 505–509.
- [26] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni, Superintegrable oscillator and Kepler systems on spaces of nonconstant curvature via the Stäckel transform, *SIGMA* 7 (2011) 048.
- [27] F. Tremblay, A.V. Turbiner, P. Winternitz, An infinite family of solvable and integrable quantum systems on a plane, *J. Phys. A, Math. Theor.* 42 (2009) 242001.
- [28] F. Tremblay, A.V. Turbiner, P. Winternitz, Periodic orbits for an infinite family of classical superintegrable systems, *J. Phys. A, Math. Theor.* 43 (2010) 015202.
- [29] S. Post, P. Winternitz, An infinite family of superintegrable deformations of the Coulomb potential, *J. Phys. A, Math. Theor.* 43 (2010) 222001.
- [30] M.F. Rañada, A new approach to the higher-order superintegrability of the Tremblay–Turbiner–Winternitz system, *J. Phys. A, Math. Theor.* 45 (2012) 465203.
- [31] M.F. Rañada, Higher order superintegrability of separable potentials with a new approach to the Post–Winternitz system, *J. Phys. A, Math. Theor.* 46 (2013) 125206.