Approximate Trigonometric Series Expansions of some Bounded Solutions in the Constant Radial Acceleration Problem

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I. Introduction

In the last decades the studies on the dynamics of a spacecraft that moving in the gravitational field of a celestial body is subjected to some low-thrust propulsion system have attracted the attention of many researchers in Astrodynamics.

Here we will consider a particular case, the so called Tsien problem [1], in which the spacecraft moves in a circular Keplerian orbit (often referred to as the parking orbit) and after a given time $t_0 = 0$ it is subjected to a constant outward radial acceleration.

By using the integrals of energy and angular momentum this problem can be reduced to a two-dimensional scenario and denoting by r and θ the polar coordinates that give the position of the spacecraft in the plane of the orbit [1], the differential equations that define their motion are

$$\frac{d\theta}{dt} = \frac{\sqrt{\mu r_0}}{r^2}, \quad \frac{d^2 r}{dt^2} = \frac{\mu}{r^2}\left(\frac{r_0}{r} + \eta \frac{r^2}{r_0^2} - 1\right) \tag{1}$$

where r_0 is the radius of the parking orbit, μ the gravitational parameter and $\eta\mu/r_0^2$ the dimensionless

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propulsive acceleration ($\eta \geq 0$). The initial conditions at $t_0 = 0$ are

$$r(0) = r_0,\ r'(0) = 0,\ \theta(0) = 0,\ \theta'(0) = \sqrt{\mu/r_0^3}. \tag{2}$$

In the early paper of Tsien [1] this author showed that only in the case that the thurst parameter $\eta > 1/8$ the spacecraft escapes from the gravitational field of the celestial body in the sense that it attains a parabolic velocity. The time required to reach this parabolic velocity can be expressed in terms of elliptic functions. For $0 < \eta < 1/8$ the trajectory is bounded and the radius vector $r = r(t)$ oscillates between $r_0 = 1$ and some $r_A > 1$. Now the second equation of (1) can be integrated to give the time in terms of elliptic integrals of first and second kind that depends on r and θ. A complete derivation of these formulas has been given in the book of Battin [2], pp. 408 as well as in the papers [3] and [4].

A remarkable fact is that in most results presented in the literature the time is obtained as a function of the state variables whereas the natural would be the converse dependence. An exception is the paper of Izzo and Biscani [5] in which the state variables are expressed explicitly in terms of an anomaly related with the time by means of Weierstrass functions. Also Akella [6] has shown that the orbit evolution can be expressed in terms of the incomplete elliptic integral of the third kind.

In [7] by using a suitable set of state redundant variables together with a Sundman type time-regularization the authors derive explicit solutions of Tsien problem for all values of η. These analitical solutions have been used by these authors as test problems to compare several orbit propagation method.

Asymptotic solutions of Tsien problem have been derived in [8] and the case of an elliptic parking orbit has been considered in [9].

However, as remarked by Quarta and Mengali in [10] a transparent analytical description of the spacecraft trajectory for all radial thrust acceleration is not available. With the aim to get a more simple description of the motion in the orbital plane they introduce a new variable ρ by

$$\rho = \rho(\theta) = 1 - \frac{r_0}{r(\theta)}, \tag{3}$$

Then according to (1)(2) the functions $\rho = \rho(\theta)$ and $\theta = \theta(t)$ of Tsien problem satisfy the nonlinear set of differential equations

$$\frac{d^2\rho}{d\theta^2} + \rho = \frac{\eta}{(1-\rho)^2}, \quad \frac{d\theta}{dt} = \sqrt{\frac{\mu}{r_0^3}}(1-\rho)^2, \quad \rho \in [0,1), \tag{4}$$

together with the initial conditions

$$\rho(0) = 0, \quad \rho'(0) = \frac{d\rho}{d\theta}(0) = 1, \quad \theta(0) = 0. \tag{5}$$

Note that the first equation of (4) has the first integral

$$H(\rho', \rho) \equiv \frac{\rho'^2}{2} + \frac{\rho^2}{2} - \frac{\eta\rho}{(1-\rho)} = H(\rho'(0), \rho(0)) \tag{6}$$

and for the Tsien problem $H(\rho'(0), \rho(0)) = 1/2$.

The type of solution of (4)(5) depends on the value of the constant parameter η, so that $\rho = \rho(\theta; \eta)$ but to simplify the notation we will write simply $\rho = \rho(\theta)$.

First of all if $\eta > 1/8$, it follows from the first integral (6) that the solution $\rho = \rho(\theta)$ of (4) is a monotone increasing function of θ for all $\theta \geq 0$ and $\lim_{\theta\to+\infty} \rho(\theta) = 1$, and by (3) $r(t) \to +\infty$ when $t \to +\infty$. Then the trajectory becomes unbounded and the spacecraft escapes from the gravitational field of the attracting body.

If $\eta \in [0, 1/8)$ by using again (6) it can be seen that the corresponding orbit $\rho = \rho(\theta)$ of (4)–(5) oscillates between a minimum $\rho = \rho_P = 0$ and a maximum $\rho = \rho_A$ where ρ_A is the smaller root of the quadratic equation $\rho^2 - \rho + 2\eta = 0$, i.e.

$$\rho_A = \rho_- = \frac{1-q}{2} \quad \text{with } q = \sqrt{1-8\eta} > 0 \tag{7}$$

and it is an even periodic orbit of θ with half–period

$$\theta_A = \int_0^{\rho_A} \frac{\sigma\sqrt{1-\rho}\, d\rho}{\sqrt{\rho(\rho^2 - \rho + 2\eta)}} = \int_0^{\pi/2} \frac{2\sqrt{2}\sqrt{1-\rho_A \sin^2\phi}\, d\phi}{\sqrt{\cos^2\phi + q(1+\sin^2\phi)}} \tag{8}$$

where the last expression is to be used for the numerical calculation of θ_A for all $\eta \in [0, 1/8)$ to avoid the singularity of the first integral at both ends of the integration interval. Note that according to (8), $\theta_A = \theta_A(\eta)$ and when $\eta \to 1/8$ the half period $\theta_A \to +\infty$.

For $\eta = 1/8$ the unique solution $\rho = \rho(\theta)$ of Tsien problem (4)–(5), $\rho = \rho(\theta; 1/8) = \rho^*(\theta)$ satisfies $\rho^*(\theta) < 1/2$ and it is monotone increasing for all θ, and $\lim_{\theta\to\infty} \rho^*(\theta) = 1/2$. Further,

$\rho(\theta) = 1/2$ for all θ is a circular non-Keplerian orbit of (4) corresponding to $\eta = 1/8$ (see [11]). Hence $\rho^*(\theta)$ is a non periodic solution of Tsien problem that has $\rho = 1/2$ as a limit cycle.

Here we will consider $\rho = \rho(\theta)$ solutions of (4)–(5) with $\eta \in [0, 1/8)$ so that $\rho = \rho(\theta)$ is an analytic and periodic function of θ with period $2\theta_A$. Note that this periodicity of $\rho(\theta)$ with respect to θ does not implies that the orbit of the Tsien problem $\rho = \rho(\theta(t)), \theta = \theta(t)$ is periodic with respect to t. Only if $2\theta_A$ is a rational multiple of 2π the orbit starting from the initial conditions (5) arrives to the same initial point.

Furthermore, since $\rho = \rho(\theta)$ it is an even function of θ it can be expressed as a cosine series Fourier expansion of type

$$\rho(\theta) = \frac{c_0}{2} + \sum_{j=1}^{\infty} c_j \cos\left(\frac{j\pi\theta}{\theta_A}\right) \tag{9}$$

with

$$c_j = c_j(\eta) = \frac{1}{\theta_A} \int_0^{\theta_A} \rho(\theta) \cos\left(\frac{j\pi\theta}{\theta_A}\right) d\theta, \quad j = 0, 1, \ldots \tag{10}$$

Note that as a consequence of Riemann–Lebesgue Lemma [12] if $\rho = \rho(\theta) \in \mathcal{C}^{(m-1)}[0, \theta_A]$ the Fourier coefficients (10) satisfy

$$\lim_{k \to +\infty} k^m c_k = 0 \tag{11}$$

i. e. the coefficients of (9) decay very fast with k. In other words the coefficients c_k of high wave numbers corresponding to rapidly oscillating waves must be very small. This holds for all $\eta \in [0, \eta^* = 1/8)$ however when the values of η are close to $1/8$ since the half period θ_A tends to ∞ this convergence is slower. Furthermore the N-th partial sums of (9)

$$S_N(\rho(\theta)) = \frac{c_0}{2} + \sum_{j=1}^{N} c_j \cos\left(\frac{j\pi\theta}{\theta_A}\right) \tag{12}$$

converge uniformly to $\rho = \rho(\theta)$ in the sense that the following bound holds

$$|\rho(\theta) - S_N(\rho(\theta))| \leq K \frac{\ln(N)}{N^m} w(2\pi/N) \quad \theta \in [0, \theta_A] \tag{13}$$

with a constant K, where w is the modulus of continuity of $\rho^{(m)}(\theta)$. Therefore the uniform convergence of the partial sums is also very fast.

In our case since the solution of (4)–(5) $\rho = \rho(\theta)$ is an analytic function for all $\eta \in [0, \eta^* = 1/8)$ in the interval $[0, \theta_A]$ the above results (11), (13) hold for all non negative integer m and clearly a partial Fourier sum (12) with a few terms, particularly for η not very close to 1/8, gives an accurate representation of the solution of (4)–(5). Note that the smooth function $\rho(\theta)$ changes slowly and the according to (9) the coefficients c_n of high wave numbers corresponding to rapidly oscillating waves must be very small.

Since a direct calculation of the coefficients c_j of (10) only can be carried out numerically after non trivial computations, Quarta and Mengali have proposed in [10] some approximations to (9) that with a good accuracy provide a transparent expression of the orbit and avoid the use of elliptic integrals. In particular the simplest expression $S_1(\theta) = (\rho_A/2)\,(1 - \cos(\pi\theta/\theta_A))$ satisfies the end conditions $S_1(0) = \rho(0) = 0$ and $S_1(\theta_A) = \rho(\theta_A) = \rho_A$. Also higher order approximations of type $S_n(\theta) = \sum_{j=0}^{n} b_j \, \cos(j\pi\theta/\theta_A)$ have been derived in [10] by using a least squares fitting approach.

The aim of this note is to propose an alternative approach to obtain accurate approximations to the solution $\rho = \rho(\theta; \eta)$ of (4)–(5) up to any order. These approximate solutions are obtained as cosine series Hermite interpolating polynomials of $\rho(\theta; \eta)$ that satisfy the differential equation at both ends of the semi–period $[0, \theta_A]$ up to any order. It is shown that such polynomials are very accurate approximations to $\rho(\theta; \eta)$ even for values of the parameter η close to to $\eta^* = 1/8$ when $\theta_A \to +\infty$. Moreover these approximations of $\rho(\theta; \eta)$ together with the second equation of (4) will allow us to obtain a Kepler's type equation relating the polar angle θ with the time.

II. Trigonometrically fitted approximations of the orbit

For a given $\eta \in [0, \eta^* = 1/8)$ and a non negative integer n we consider the cosine trigonometric polynomials

$$T_n(\theta) = T_n(\theta; \eta) = \sum_{j=0}^{2n+1} \beta_j \, \cos\left(\frac{j\pi\theta}{\theta_A}\right) \tag{14}$$

that are intended to approximate the solution $\rho = \rho(\theta; \eta)$ of Eqs. (4) and (5).

Clearly, the form (14) is motivated by the Fourier series expansion of the exact solution although it is not a truncation of this series expansion. To determine the $(2n + 2)$ coefficients of (14) we will use Hermite interpolatory conditions at both ends of the θ–interval $[0, \theta_A]$. The initial conditions

(4) imply that $T_n(0) = 0$, $T_n'(0) = 0$ and on the other hand, the symmetry implies that all odd order derivatives of T_n at both ends of the interval $[0, \theta_A]$ vanish, then we must include only even order derivatives at both ends

$$\begin{aligned} T_n^{(2j)}(0) &= \rho^{(2j)}(0) \quad j = 1, \ldots, k_1, \\ T_n^{(2j)}(\theta_A) &= \rho^{(2j)}(\theta_A) \quad j = 1, \ldots, k_2, \end{aligned} \tag{15}$$

with k_1 and k_2 such that $k_1 + k_2 = 2n + 2$.

Here we will consider only symmetric interpolants that will be defined by the conditions

$$T_n^{(2j)}(0) = \rho^{(2j)}(0), \quad T_n^{(2j)}(\theta_A) = \rho^{(2j)}(\theta_A), \quad j = 0, \ldots, n \tag{16}$$

and this trigonometrically fitted interpolant will be referred to as the interpolant of order n.

First of all observe that T_n depends linearly on the $(2n+2)$ coefficients $\beta_j, j = 0, \ldots, 2n+1$ and the linear system (15) in these unknowns is non singular and therefore this interpolant is well defined for any $\eta \in [0, \eta^* = 1/8)$.

For the computation of the even order derivatives of $\rho(\theta)$ we use the differential equation (4) so that in the recursive computation of successive derivatives, ρ'' is replaced by $-\rho + \eta(1-\rho)^{-2}$ and then we get the derivatives up to any order as a functions of ρ and ρ'. First of all, according to the initial conditions (3), to obtain these derivatives at the left end $\theta = 0$ we substitute $(\rho, \rho') \to (0, 0)$. In particular for the first orders we get

$$\begin{aligned} \rho(0) &\equiv L_0 = 0, \\ \rho^{(2)}(0) &\equiv L_2 = \eta, \\ \rho^{(4)}(0) &\equiv L_4 = 2\eta^2 - \eta, \\ \rho^{(6)}(0) &\equiv L_6 = 22\eta^3 - 4\eta^2 - \eta, \\ \rho^{(8)}(0) &\equiv L_8 = \eta\,(584\eta^3 - 120\eta^2 + 6\eta - 1). \end{aligned} \tag{17}$$

In general $\rho^{(2k)}(0)$ is a polynomial in the thrust parameter η of degree k and integer coefficients.

For the right end $\theta = \theta_A$, similarly substituting $(\rho, \rho') \to (\rho_A, 0)$ we have

$$\begin{aligned} \rho(\theta_A) &\equiv R_0 = \rho_A, \\ \rho^{(2)}(\theta_A) &\equiv R_2 = (\rho_A^2 - 1)/(4(3-\rho_A)), \\ \rho^{(4)}(\theta_A) &\equiv R_4 = (\rho_A^2 - 1)(5 - 10\rho_A + \rho_A^2)/(4(\rho_A - 3)^3), \\ \rho^{(6)}(\theta_A) &\equiv R_6 = (\rho_A^2 - 1)(11 + 136\rho_A - 146\rho_A^2 - 16\rho_A^3 - \rho_A^4)/(4(\rho_A - 3)^5). \end{aligned} \tag{18}$$

The expressions $\rho^{(2k)}(\theta_A)$ are continuous rational functions in ρ_A.

For the explicit computation of the $(2n+2)$ coefficients $\beta_0, \beta_1, \ldots \beta_{2n+1}$ of $T_n(\theta)$ in the case of symmetric interpolants defined by the condition

$$T_n^{(2j)}(0) = L_{2j}, \quad T_n^{(2j)}(\theta_A) = R_{2j}, \quad j = 0, \ldots, n, \tag{19}$$

observe that these equations can be written equivalently as two sets of $(n+1)$ equations in the form

$$\frac{(-1)^j}{2\,w^{2j}}\left(T_n^{(2j)}(0) + T_n^{(2j)}(\theta_A)\right) = (-1)^j \left(\frac{L_{2j} + R_{2j}}{2\,w^{2j}}\right) \equiv S_{2j}, \quad j = 0, 1, \ldots, n \tag{20}$$

$$\frac{(-1)^j}{2\,w^{2j}}\left(T_n^{(2j)}(0) - T_n^{(2j)}(\theta_A)\right) = (-1)^j \left(\frac{L_{2j} - R_{2j}}{2\,w^{2j}}\right) \equiv S_{2j+1}, \tag{21}$$

where $w = \pi/\theta_A$.The set (20) defines the $(n+1)$ coefficients $\beta_0, \beta_2, \ldots \beta_{2n}$ by the linear system

$$\begin{aligned}
&\beta_0 + \beta_2 + \ldots + \beta_{2n} = S_0 \\
&\beta_2\ (2^2) + \ldots + \beta_{2n}\ (2n)^2 = S_2 \\
&\ldots\ldots\ldots\ldots\ldots \\
&\beta_2\ 2^{(2n)} + \ldots + \beta_{2n}\ (2n)^{(2n)} = S_{2n}
\end{aligned} \tag{22}$$

whereas the set (21) defines the $(n+1)$ coefficients $\beta_1, \beta_3, \ldots \beta_{2n+1}$ by the linear system

$$\begin{aligned}
&\beta_1 + \beta_3 + \ldots + \beta_{2n+1} = S_1 \\
&\beta_1\ (1^3) + \beta_3 (3^3) + \ldots + \beta_{2n+1}\ (2n+1)^2 = S_3 \\
&\ldots\ldots\ldots\ldots\ldots \\
&\beta_1\ 1^{(2n)} + +\beta_3\ (3)^{(2n)} \ldots + \beta_{2n+1}\ (2n+1)^{(2n)} = S_{2n+1}
\end{aligned} \tag{23}$$

Clearly, Eq. (22) is a non singular linear system in the unknowns even coefficients $\beta_0, \beta_2, \ldots \beta_{2n}$ and similarly for (23) in the odd coefficients. Hence, for all integer $n \geq 0$ there is a uniquely defined symmetric interpolant T_n with coefficients depending on ρ_A and η.

For the first order approximation

$$T_1(\theta) = \sum_{j=0}^{3} \beta_j\ \cos(jw\theta), \tag{24}$$

with $w = \pi/\theta_A$, we have the system

$$\left.\begin{aligned} \beta_0 + \beta_2 &= S_0 \\ \beta_2 (2^2) &= S_2 \\ \beta_1 + \beta_3 &= S_1, \\ \beta_1\ (1^2) + \beta_3\ (3^2) &= S_3 \end{aligned}\right\} \tag{25}$$

with

$$S_0 = (L_0 + R_0)/2, \qquad S_1 = (L_0 - R_0)/2,$$
$$S_2 = (-1)(L_2 + R_2)/(2\ w^2), \quad S_3 = (-1)(L_2 - R_2)/(2\ w^2).$$

Then the coefficients β_j are

$$\begin{aligned} \beta_0 &= -\frac{(\sigma-1)\left[(\sigma-1)^2\theta_A^2 + 16\pi^2(\sigma+1)\right]}{64\pi^2(\sigma+1)} \\ \beta_2 &= -\frac{(\sigma-1)\left(\left(\sigma^2+6\sigma+1\right)\theta_A^2 - 36\pi^2(\sigma+1)\right)}{128\pi^2(\sigma+1)} \\ \beta_1 &= \frac{(\sigma-1)^3\theta_A^2}{64\pi^2(\sigma+1)} \\ \beta_3 &= \frac{(\sigma-1)\left[\left(\sigma^2+6\sigma+1\right)\theta_A^2 - 4\pi^2(\sigma+1)\right]}{128\pi^2(\sigma+1)} \end{aligned}$$

For the sake of completeness we compare the above trigonometrically fitted approximation $T_1(\theta)$ with the least squares approximation $QM_2(\theta)$ given by Quarta and Mengali in [10] for the values $\eta = 1/80$, $\eta = 1/16$ and $\eta = 19/160$. This approximation is given by

$$QM_2(\theta) = \rho_A \left[\frac{1}{2}\left(1 - \cos\left(\frac{\pi\theta}{\hat{\theta}_A}\right)\right) + b_2\left(\cos\left(\frac{2\pi\theta}{\hat{\theta}_A}\right) - 1\right)\right]$$

where b_2, $\hat{\theta}$ are obtained from Table 1 for several values of η.

Table 1 Coefficients of b_2, $\hat{\theta}$ and η of $QM_2(\theta)$

η	b_2	$\hat{\theta}_A$
1/80	-4.3357×10^{-5}	1.01325602218917
1/16	-1.8657×10^{-3}	1.09017029950805
19/160	-3.1444×10^{-2}	1.42575249853646

In Figures 1–2, 3–4 and 5–6 we display the defects of the differential equation (4)

$$\delta_d(\theta) = T_1^{(2)}(\theta) + T_1(\theta) - \eta\ (1 - T_1(\theta))^{-1} \tag{26}$$

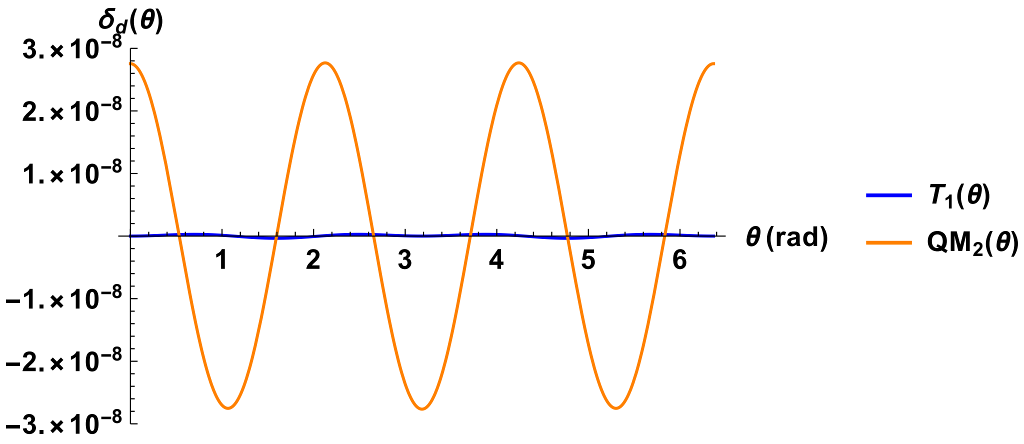

Fig. 1 Defects of the differential equation for the first order interpolant $T_1(\theta)$ and the approximation $QM_2(\theta)$ given by Quarta and Mengali for $\eta = 1/80$, $\eta^* = 1/10$.

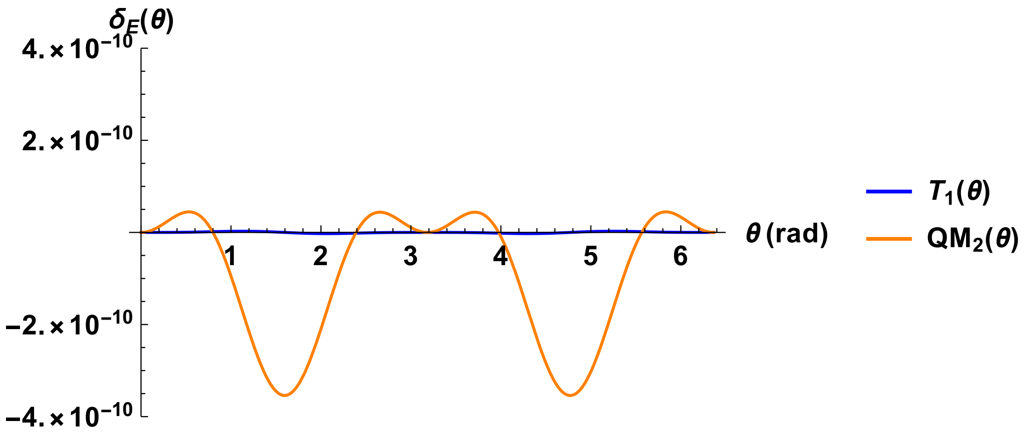

Fig. 2 Defects of the energy integral for the first order interpolant $T_1(\theta)$ and the approximation $QM_2(\theta)$ given by Quarta and Mengali for $\eta = 1/80$, $\eta^* = 1/10$.

and the first integral of the energy (4)

$$E(\rho(\theta)) \equiv \rho'(\theta)^2 + \rho(\theta)^2 - 2\eta\,(1 - \rho(\theta))^{-1} = \text{constant} \tag{27}$$

given by

$$\delta_E(\theta) = E(T_1(\theta)) - E(T_1(0)) \tag{28}$$

for the values of $\eta = 1/80$, $1/16$ and $19/160$ respectively.

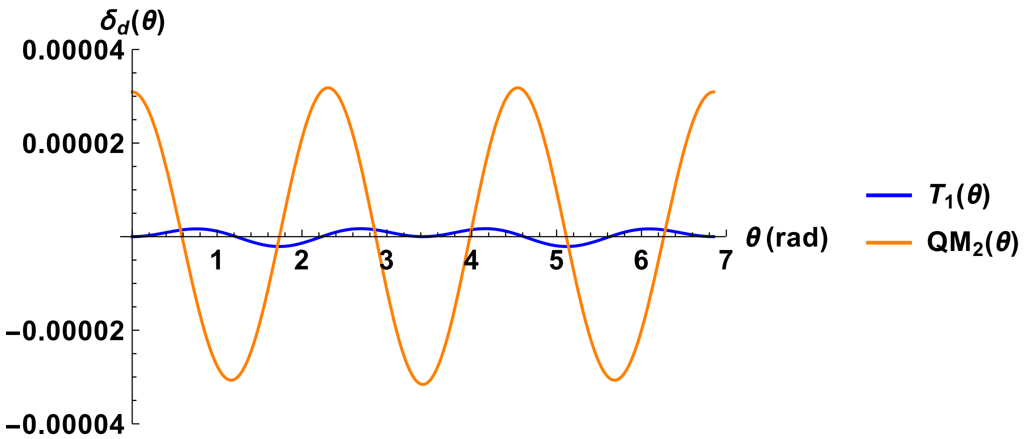

Fig. 3 Defects of the differential equation for the first order interpolant $T_1(\theta)$ and the approximation $QM_2(\theta)$ given by Quarta and Mengali for $\eta = 1/16$, $\eta^* = 1/2$.

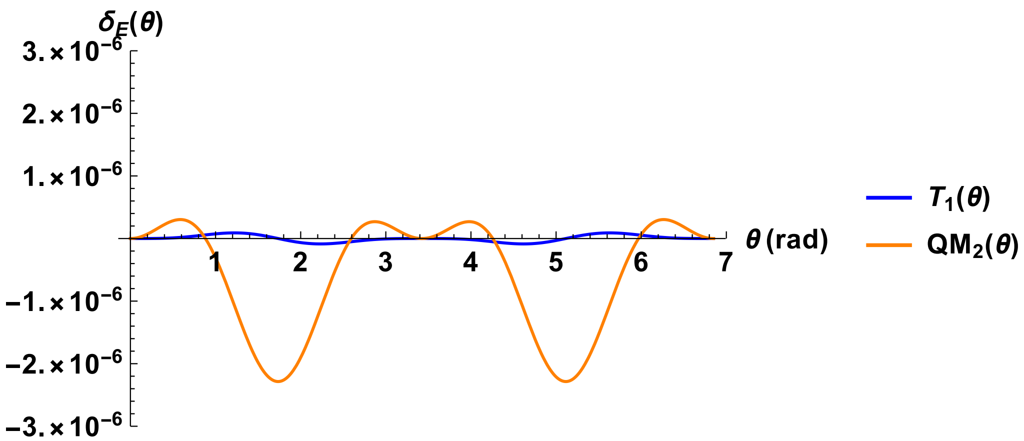

Fig. 4 Defects of the energy integral for the first order interpolant $T_1(\theta)$ and the approximation $QM_2(\theta)$ given by Quarta and Mengali for $\eta = 1/16$, $\eta^* = 1/2$.

For the second order approximation

$$T_2(\theta) = \sum_{j=0}^{5} \beta_j \ \cos(jw\theta), \tag{29}$$

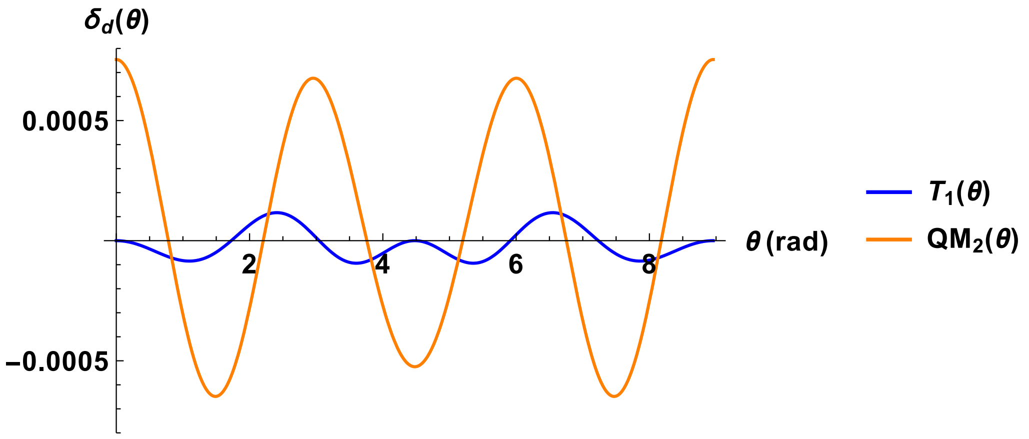

Fig. 5 Defects of the differential equation for the first order interpolant $T_1(\theta)$ and the approximation $QM_2(\theta)$ given by Quarta and Mengali for $\eta = 19/160$, $\eta^* = 95/100$.

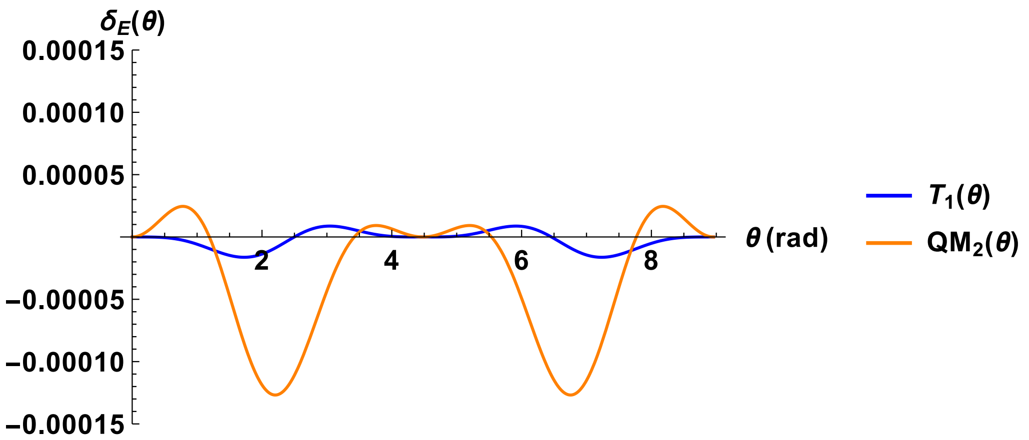

Fig. 6 Defects of the energy integral for the first order interpolant $T_1(\theta)$ and the approximation $QM_2(\theta)$ given by Quarta and Mengali for $\eta = 19/160$, $\eta^* = 95/100$.

with $w = \pi/\theta_A$, the conditions (22),(23) to determine the coefficients β_j of (29) are

$$\left.\begin{aligned}
&\beta_0 + \beta_2 + \beta_4 = S_0 \\
&\beta_2\ 2^2 + \beta_4\ 4^2 = S_2 \\
&\beta_2\ 2^4 + \beta_4\ 4^4 = S_4 \\
&\beta_1 + \beta_3 + \beta_5 = S_1 \\
&\beta_1 + \beta_3\ 3^2 + \beta_5\ 5^2 = S_3 \\
&\beta_1 + \beta_3\ 3^4 + \beta_5\ 5^4 = S_5
\end{aligned}\right\}$$

that possess the solution

$$\beta_0 = \tfrac{1}{64}(64S_0 - 20S_2 + S_4), \qquad \beta_1 = \tfrac{1}{192}(225S_1 - 34S_3 + S_5), \quad \beta_2 = \tfrac{1}{48}(16S_2 - S_4),$$

$$\beta_3 = \tfrac{1}{128}(-25S_1 + 26S_3 - S_5), \quad \beta_4 = \tfrac{1}{192}(-4S_2 + S_4), \qquad \beta_5 = \tfrac{1}{384}(9S_1 - 10S_3 + S_5).$$

As above, we compare the trigonometrically fitted approximation $T_2(\theta)$ with the least squares approximation $QM_3(\theta)$ given by Quarta and Mengali in [10]

$$QM_3(\theta) = \rho_A \left[\frac{1}{2}\left(1 - \cos\left(\frac{\pi\theta}{\hat{\theta}_A}\right)\right) + \tilde{b}_2 \left(\cos\left(\frac{2\pi\theta}{\hat{\theta}_A}\right) - 1\right) + \tilde{b}_3 \left(\cos\left(\frac{3\pi\theta}{\hat{\theta}_A}\right) - \cos\left(\frac{\pi\theta}{\hat{\theta}_A}\right)\right)\right]$$

for $\eta = 1/80$, $\eta = 1/16$ and $\eta = 19/160$ given in Table 2

Table 2 Coefficients of $\tilde{b}_i$, $\hat{\theta}$ and η of $QM_3(\theta)$

η	$\tilde{b}_2$	$\tilde{b}_3$	$\hat{\theta}_A$
1/80	-4.3356×10^{-5}	1.3805×10^{-7}	1.01325602218917
1/16	-1.8640×10^{-3}	3.1705×10^{-5}	1.09017029950805
19/160	-3.1375×10^{-2}	4.2249×10^{-4}	1.42575249853646

A similar comparison can be carried out for other values of η with the coefficients b_j provided by the Table 1 of [10].

In Figures 7–8, 9–10 and 11–12 we display the defects of the differential equation and the the energy integral of $T_2(\theta)$ and $QM_3(\theta)$ for the values of $\eta = 1/80$, 1/16 and 19/160 respectively. Moreover we give here the values of the coefficients of the trigonometrical polynomial for $\eta = 1/16$ that show their rapid decrease

$$\beta_0 = 0.0734963, \qquad \beta_1 = -0.0732279, \qquad \beta_2 = -0.000272905,$$

$$\beta_3 = 4.63496\,10^{-6}, \quad \beta_4 = -1.04411\,10^{-7}, \quad \beta_5 = 2.55838\,10^{-9}.$$

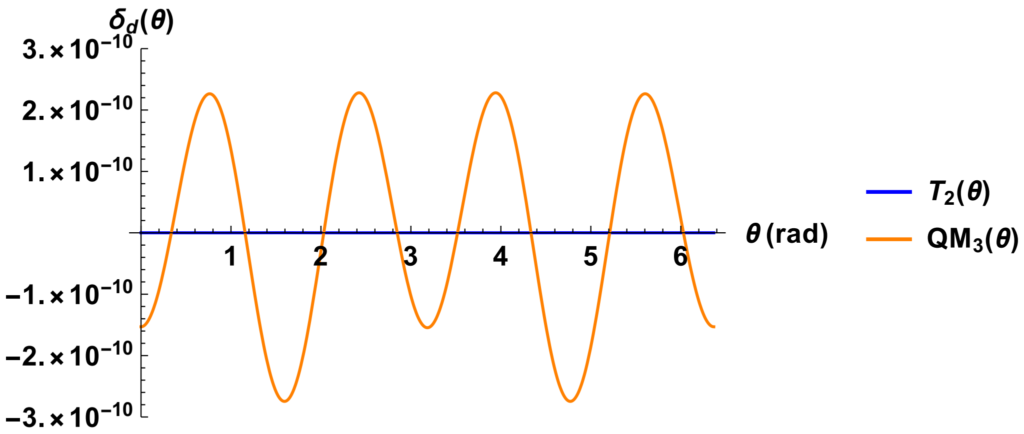

Fig. 7 Defects of the differential equation for the first order interpolant $T_2(\theta)$ and the approximation $QM_3(\theta)$ given by Quarta and Mengali for $\eta = 1/80$, $\eta^* = 1/10$.

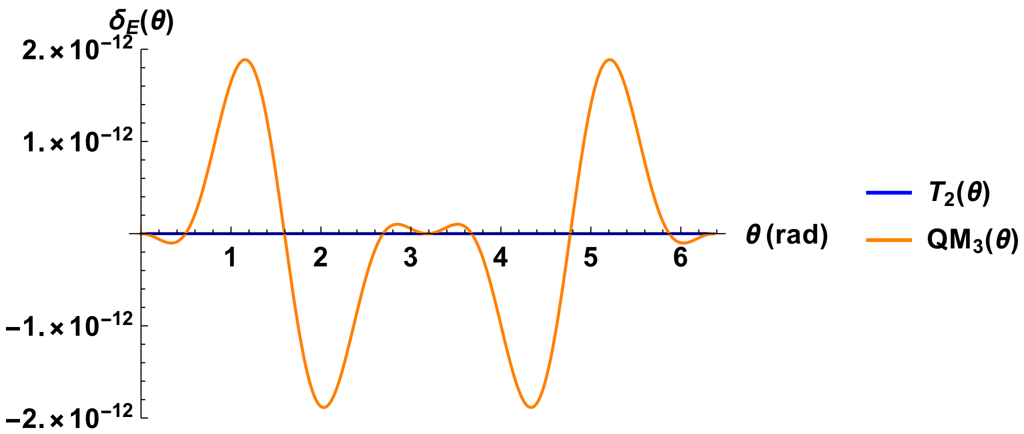

Fig. 8 Defects of the energy integral for the first order interpolant $T_2(\theta)$ and the approximation $QM_3(\theta)$ given by Quarta and Mengali for $\eta = 1/80$, $\eta^* = 1/10$.

As remarked above when the value of the thrust parameter η closes to the critical value $\eta^* = 1/8$ the accuracy of $T_2(\theta)$ decreases. In Figures 11–12 we display the defect(θ) and energy(θ) for $\eta = 0.95$, $\eta^* = 19/160$, and also the values of the the coefficients of the corresponding trigonometrical polynomial $T_2(\theta)$ that do not show their rapid decrease

$$\beta_0 = 0.206275, \quad \beta_1 = -0.194264, \quad \beta_2 = -0.0121868,$$
$$\beta_3 = 0.000166625, \quad \beta_4 = 9.65595 \times 10^{-6}, \quad \beta_5 = -1.39145 \times 10^{-6}.$$

The conclusion is that the errors of $QM_3(\theta)$ are about the same order that the $T_1(\theta)$ although

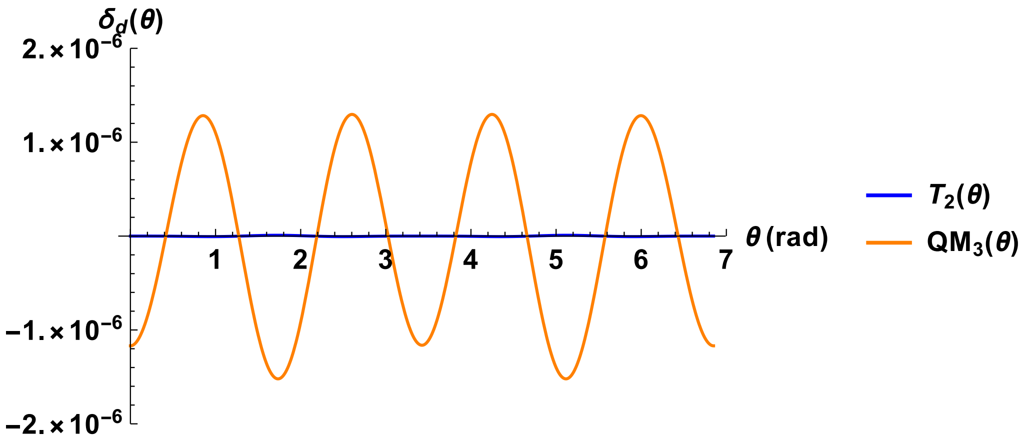

Fig. 9 Defects of the differential equation for the first order interpolant $T_2(\theta)$ and the approximation $QM_3(\theta)$ given by Quarta and Mengali for $\eta = 1/16$, $\eta^* = 1/16$.

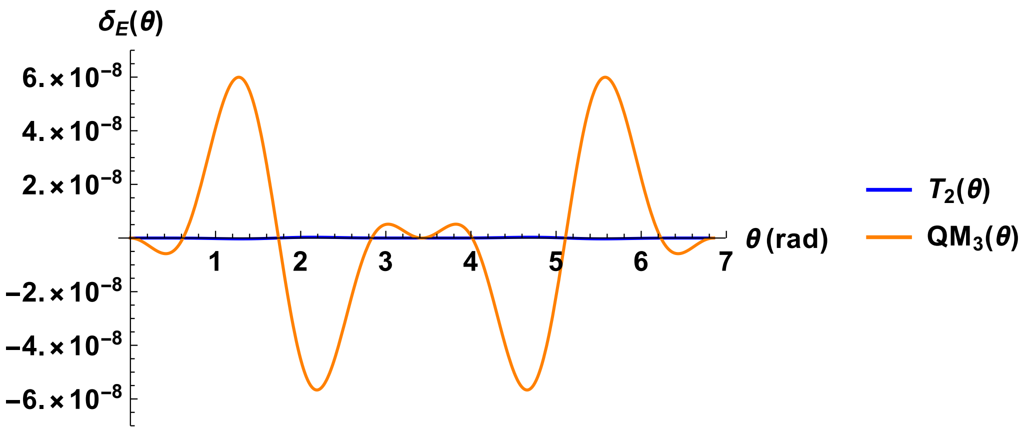

Fig. 10 Defects of the energy integral for the first order interpolant $T_2(\theta)$ and the approximation $QM_3(\theta)$ given by Quarta and Mengali for $\eta = 1/16, \eta^* = 1/16$.

the adjustment with a least squares procedure makes it slightly better. On the other hand $T_2(\theta)$ is clearly superior to $QM_3(\theta)$ as can be seen in Figures 13–14 for $\eta = 19/160$. Similar behaviour occurs with other values of η. In any case it is worth to remark that there is an analytical expression for the coefficients of $T_1(\theta)$ that makes it available for any $\eta \in [0, 1/8)$ whereas in the case of $QM_3(\theta)$ these must be derived for any particular value of η with least squares approach.

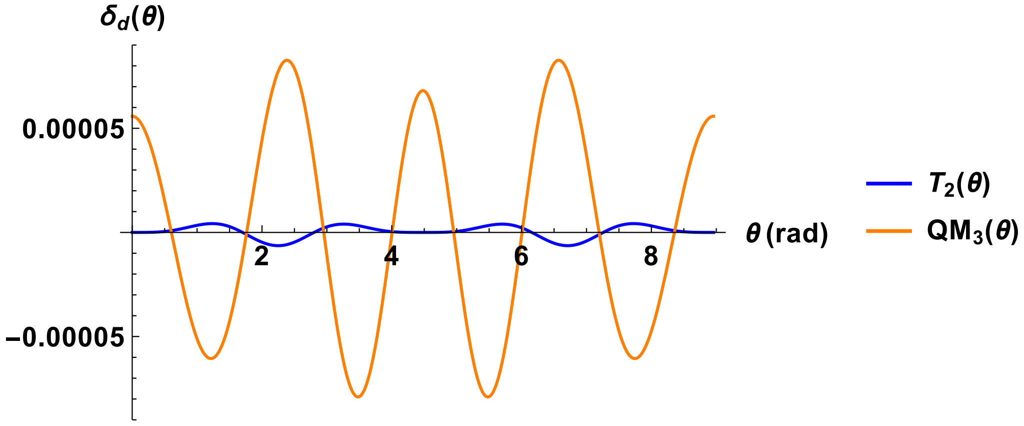

Fig. 11 Defects of the differential equation for the first order interpolant $T_2(\theta)$ and the approximation $QM_3(\theta)$ given by Quarta and Mengali for $\eta = 19/160$, $\eta^* = 95/100$.

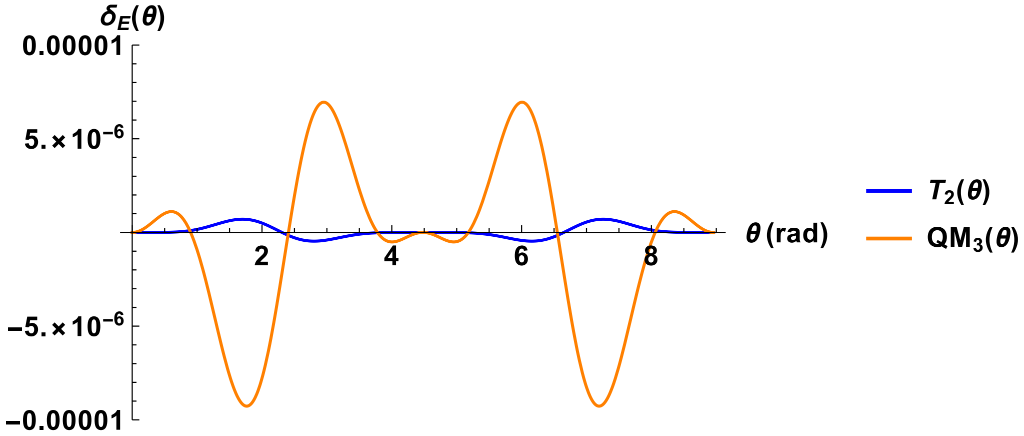

Fig. 12 Defects of the energy integral for the first order interpolant $T_2(\theta)$ and the approximation $QM_3(\theta)$ given by Quarta and Mengali for $\eta = 19/160$, $\eta^* = 95/100$.

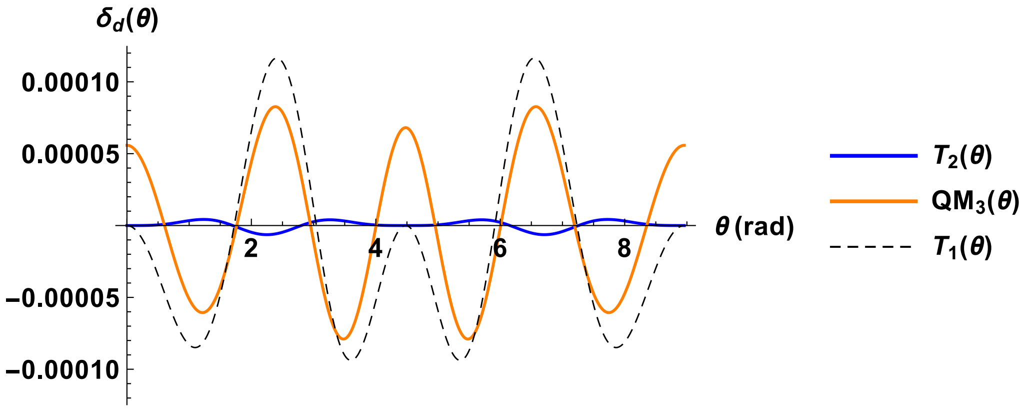

Fig. 13 Defects of the differential equation for the first order interpolants $T_1(\theta)$, $T_2(\theta)$ and the approximation $QM_3(\theta)$ given by Quarta and Mengali for $\eta = 19/160$, $\eta^* = 95/100$.

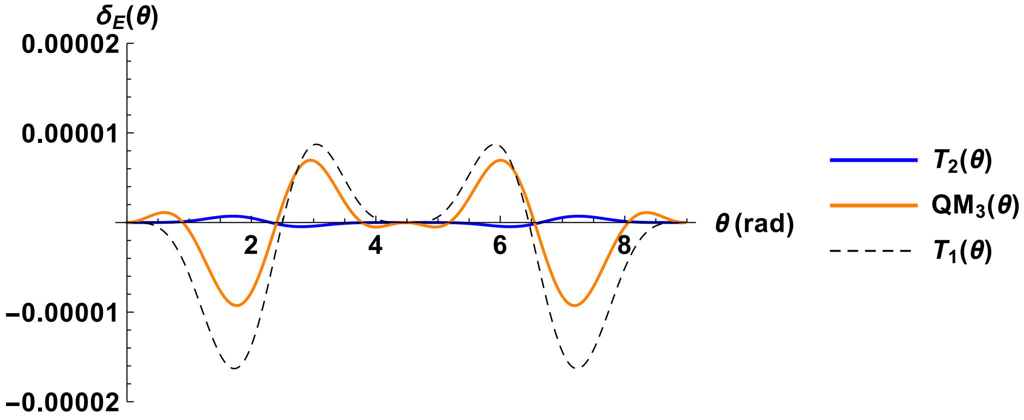

Fig. 14 Defects of the energy integral for the first order interpolants $T_1(\theta)$, $T_2(\theta)$ and the approximation $QM_3(\theta)$ given by Quarta and Mengali for $\eta = 19/160$, $\eta^* = 95/100$.

III. Time approximation

After obtaining a trigonometrical polynomial $T_n(\theta)$ approximating the orbit $\rho = \rho(\theta)$ we need to relate the polar angle θ with the physical time t. From the angular momentum integral we have

$$\frac{d\theta}{dt} = \frac{\sqrt{\mu r_0}}{r^2} = \sqrt{\frac{\mu}{r_0^3}}\,(1-\rho)^2 = \sqrt{\frac{\mu}{r_0^3}}\,\frac{\eta}{\rho'' + \rho},$$

hence,

$$(\rho'' + \rho)\; d\theta = \eta\sqrt{\frac{\mu}{r_0^3}}\; dt,$$

and by integration, the time t to reach a given θ satisfies

$$\eta\,\sqrt{\frac{\mu}{r_0^3}}\; t = \rho'(\theta) + \int_0^{\theta} \rho(\theta)\; d\theta.$$

In the case of periodic orbits when $\rho \simeq T_n$, the right hand side can be approximated by

$$I_n(\theta) = T_n'(\theta) + \int_0^{\theta} T_n(\theta)\; d\theta = \beta_0\;\theta + \sum_{j=1}^{2n+1} \beta_j \left(\frac{1 - j^2 w^2}{jw}\right)\, \sin(jw\theta),$$

where $\beta_j = \beta_j(\eta)$ are trigonometrically fitted coefficients of T_n and the relation

$$\eta\,\sqrt{\frac{\mu}{r_0^3}}\; t = I_n(\theta)$$

can be considered as a Kepler type equation to determine θ for each value of t.

Note that for $\theta = \theta_A$, since $w = \pi/\theta_A$, $I_n(\theta_A) = \beta_0\;\theta_A$ and

$$t_A = \frac{\beta_0 \theta_A}{\eta}\left(\frac{\mu}{r_0^3}\right)^{-1/2}.$$

IV. Conclusions

A new technique has been proposed to derive trigonometrically fitted approximations to the periodic solutions in the constant, outward radial acceleration problem sometimes referred to as Tsien problem [1]. It can be considered as an alternative solution to the one given by Quarta and Mengali in [10]. A remarkable property of these periodic orbits is that the function $\rho(\theta) = 1 - r(0)/r(\theta)$ is an even periodic solution of a nonlinear second order nonlinear analytical differential equation, and therefore the coefficients of their Fourier series have an spectral convergence to zero. This implies that our trigonometric polynomials $T_n(\theta)$ that mimic the Fourier expansion up to any order share a high accuracy that is uniform in $\theta > 0$. The coefficients of $T_n(\theta) \simeq \rho(\theta)$ are computed by Hermite interpolation up of the solution of the differential equation at both ends of a semi–period. Taking into account the differential equation this process can be derived recursively up to any order. Two criteria have been proposed to test the quality of these approximate solutions: the defect of differential equation satisfied by $\rho(\theta)$ and also a first integral of this equation and results of some numerical experiments are presented to show the quality of these approximations in the first orders. Further, by using $T_n(\theta)$ an approximation to the relation of the time t and the polar angle $t = t(\theta)$ is obtained. Such a relation can be viewed as a Kepler's type equation. In conclusion the proposed solution provides an approximate representation of the periodic solutions of $\rho = \rho(\theta)$ as a trigonometric polynomial with coefficients easily calculable with a good accuracy that is uniform for all θ.

Acknowledgments

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