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MAXIMUM RECTILINEAR CONVEX SUBSETS[*]

HERNÁN GONZÁLEZ-AGUILAR[†], DAVID ORDEN[‡], PABLO PÉREZ-LANTERO[§],
DAVID RAPPAPORT[¶], CARLOS SEARA[‖], JAVIER TEJEL[#], AND JORGE URRUTIA[††]

Abstract. Let P be a set of n points in the plane. We consider a variation of the classical Erdős–Szekeres problem, presenting efficient algorithms with $O(n^3)$ running time and $O(n^2)$ space complexity that compute (1) a subset S of P such that the boundary of the rectilinear convex hull of S has the maximum number of points from P, (2) a subset S of P such that the boundary of the rectilinear convex hull of S has the maximum number of points from P and its interior contains no element of P, (3) a subset S of P such that the rectilinear convex hull of S has maximum area and its interior contains no element of P, and (4) when each point of P is assigned a weight, positive or negative, a subset S of P that maximizes the total weight of the points in the rectilinear convex hull of S. We also revisit the problems of computing a maximum area orthoconvex polygon and computing a maximum area staircase polygon, amidst a point set in a rectangular domain. We obtain new and simpler algorithms to solve both problems with the same complexity as in the state of the art.

Key words. Erdős–Szekeres problems, convex subsets, optimization, rectilinear convex hull

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1. Introduction. Let P be a point set in general position in the plane. A subset S of P with k elements is called a *convex k-gon* if the elements of S are the vertices of a convex polygon, and it is called a *convex k-hole* of P if the interior of the convex hull of S contains no element of P. The study of convex k-gons and convex k-holes

[†]Facultad de Ciencias, Universidad Autónoma de San Luis Potosí, México (hernan@fc.uaslp.mx).

[‡]Departamento de Física y Matemáticas, Universidad de Alcalá, Alcala de Henares, Madrid, 28871, Spain (david.orden@uah.es).

[§]Departamento de Matemática y Ciencia de la Computación, USACH, Santiago de Chile, Chile (pablo.perez.l@usach.cl).

[¶]School of Computing, Queen's University, Kingston, Canada (daver@cs.queensu.ca).

[‖]Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Spain (carlos.seara@upc.edu).

[#]Departamento de Métodos Estadísticos, IUMA, Universidad de Zaragoza, Zaragoza, 50009, Spain (jtejel@unizar.es).

[††]Instituto de Matemáticas, Universidad Nacional Autónoma de México, Coyoacán, México, D. F., 04510 (urrutia@matem.unam.mx).

of point sets started in a seminal paper by Erdős and Szekeres [9] in 1935. Since then, numerous papers about both the combinatorial and the algorithmic aspects of convex k-gons and convex k-holes have been published. The reader can consult the two survey papers about so-called Erdős–Szekeres type problems [7, 11].

There are recent papers studying the existence and number of convex k-gons and convex k-holes for finite point sets in the plane [1, 2, 3]. Papers dealing with the algorithmic complexity of finding largest convex k-gons and convex k-holes are, respectively, Chvátal and Kincsek [8] and Avis and Rappaport [5], which solve these problems in $O(n^3)$ time.

Erdős–Szekeres type problems have also been studied for colored point sets. Let P be a point set such that each of its elements is assigned a color, say red or blue. Bautista-Santiago et al. [6] studied the problem of finding a monochromatic subset S of P of maximum cardinality such that all of the elements of P contained in the convex hull of S have the same color. As a generalization, they also studied the problem in which each element of P has assigned a (positive or negative) weight. In this case, the goal is to find a subset S of P that maximizes the total weight of the points of P contained in the convex hull of S. Each of these problems was solved in $O(n^3)$ time and $O(n^2)$ space. In addition, their algorithm can easily be adapted to find a subset S of P such that the convex hull of S is empty and of maximum area in $O(n^3)$ time and $O(n^2)$ space.

In this paper, we study Erdős–Szekeres type problems under a variation of convexity known as *rectilinear convexity*, or *orthoconvexity*: Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in the plane in general position. A *quadrant* of the plane is the intersection of two open half-planes whose supporting lines are parallel to the x- and y-axes, respectively. We say that a quadrant Q is *P-free* if it does not contain any point of P. The *rectilinear convex hull* of P, denoted as $RCH(P)$, initially defined by Ottmann, Soisalon-Soininen, and Wood [14], is defined as

$$RCH(P) = \mathbb{R}^2 - \bigcup_{Q \text{ is } P\text{-free}} Q.$$

The rectilinear convex hull of a point set might be a simply connected set, yielding an intuitive and appealing structure (see Figure 1(a)). However, in other cases the
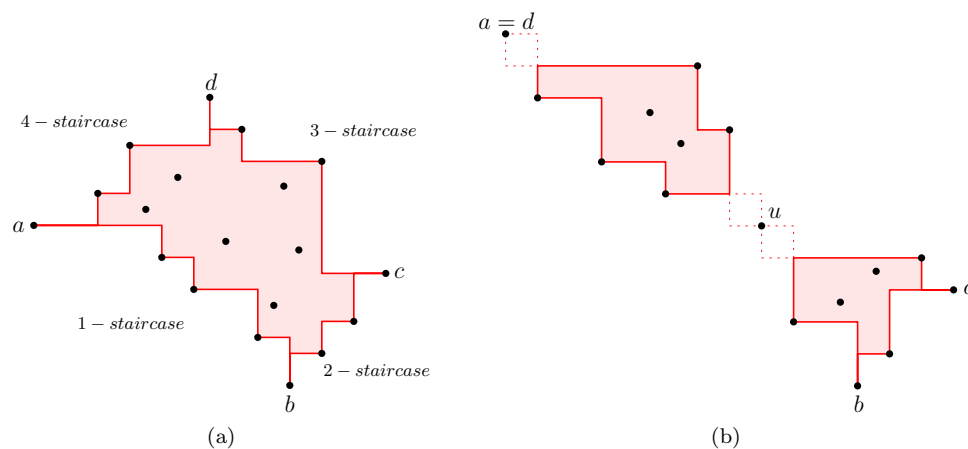

FIG. 1. (a) *A point set with a connected rectilinear convex hull.* (b) *A point set whose rectilinear convex hull is disconnected; two of its components are pinched points.*

rectilinear convex hull can have several connected components (see Figure 1(b)), some of which might be single points which we call *pinched* points. The *size* of $RCH(P)$ is the number of elements of P on the boundary of $RCH(P)$. The sizes of the rectilinear convex hulls in Figures 1(a) and 1(b) are, respectively, thirteen and twelve.

Alegría-Galicia et al. [4] gave an optimal $\Theta(n \log n)$-time and $O(n)$-space algorithm to compute the orientation of the coordinate axes such that the rectilinear convex hull of a set P of n points in the plane has minimum area. The reader can refer to the literature for other results related to rectilinear convexity [4, 10, 16]. In this paper, we present efficient algorithms for the following geometric optimization problems:

MaxRCH: Given a set P of n points in the plane, find a subset $S \subseteq P$ such that the size of $RCH(S)$ is maximized.

MaxEmptyRCH: Given a set P of n points in the plane, find a subset $S \subseteq P$ such that the interior of $RCH(S)$ contains no point of P and the size of $RCH(S)$ is maximized.

MaxAreaRCH: Given a set P of n points in the plane, find a subset $S \subseteq P$ such that the interior of $RCH(S)$ contains no point of P and the area of $RCH(S)$ is maximized.

MaxWeightRCH: Given a set P of n points in the plane, such that each $p \in P$ is assigned a (positive or negative) weight $w(p)$, find a subset $S \subseteq P$ that maximizes $\sum_{p \in P \cap RCH(S)} w(p)$.

In section 3, we give an $O(n^3)$-time $O(n^2)$-space algorithm to solve the MaxRCH problem. Then, in section 4 we show how to adapt this algorithm to solve the other three problems, each in $O(n^3)$ time and $O(n^2)$ space. The complexities of our algorithms are the same as the complexities of the best-known algorithms to solve these problems with the usual definition of convexity.

Besides presenting the first algorithms to solve the problems MaxRCH, MaxEmptyRCH, MaxAreaRCH, and MaxWeightRCH, we show that our techniques can be used to provide new algorithms to solve two additional problems, considered by Nandy, Mukhopadhyaya, and Bhattacharya [13] and Nandy and Bhattacharya [12].

MaxOrthoconvexPolygon: Given a set P of n points in the plane, contained in an axis-aligned rectangle $\mathcal{R}$ called the *domain*, find an orthoconvex polygon of maximum area that is contained in $\mathcal{R}$ and its interior contains no point of P, where a polygon is *orthoconvex* if its sides are axis-parallel and its intersection with any horizontal or vertical line is empty, or a line segment [16].

MaxStaircasePolygon: Given a set P of n points in the plane contained in an axis-aligned rectangle $\mathcal{R}$, find a staircase polygon of maximum area contained in $\mathcal{R}$ whose interior contains no point of P, where an orthoconvex polygon contained in $\mathcal{R}$ is called a *staircase polygon* if it contains two opposite corners of $\mathcal{R}$.

We show in section 5 how to solve the MaxOrthoconvexPolygon problem in $O(n^3)$ time and $O(n^2)$ space, and we give in section 6 an $O(n^2)$-time and $O(n^2)$-space algorithm to solve the MaxStaircasePolygon problem. The algorithms by Nandy, Mukhopadhyaya, and Bhattacharya [13] and Nandy and Bhattacharya [12] have the same time and space complexities as ours, but we believe that our solutions are simpler and easier to understand. Orthoconvex polygons (and in general isothetic polygons) have different applications in areas such as VLSI layout design or robotic visibility (see the references in [12] for more information regarding applications).

2. Some notation and definitions. A summary of the main notation used can be found at the end of the paper. For the sake of simplicity, we assume that all
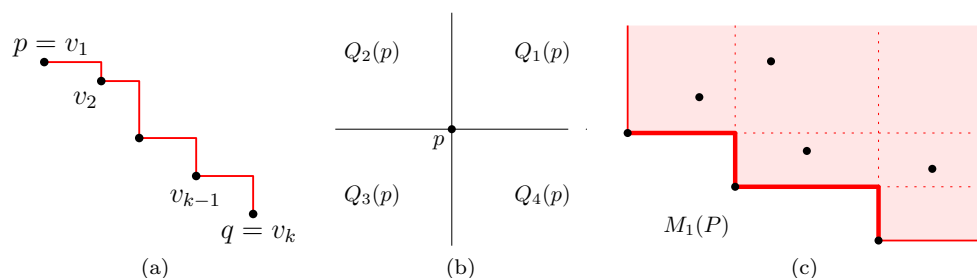
FIG. 2. (a) *A 1-staircase.* (b) *The definition of the sets* $Q_i(p)$. (c) *A 7-point set* P *and the set* $M_1(P)$. *The vertices of the boundary of* $M_1(P)$ *in* P *are the 1-extremal points of* P. *The thick polygonal line is the 1-staircase associated with* P.

point sets P considered in this paper are in *general position*, which means that no two points of P share the same x- or y-coordinate. Using a $O(n \log n)$-time preprocessing step, we can also assume when necessary that the points of a point set P are ordered by x-coordinate or y-coordinate. Given a point set P in the plane, we will use a, b, c, and d to denote the leftmost, bottommost, rightmost, and topmost points of P, respectively, unless otherwise stated. Note that a, b, c, and d are not necessarily different. In Figure 1(b), we have $a = d$.

Given a point p of the plane, let p_x and p_y denote the x- and y-coordinates of p, respectively.

DEFINITION 2.1. *For* $p, q \in \mathbb{R}^2$, $p \neq q$, *we write* $p \prec q$ *to denote that* $p_x < q_x$ *and* $p_y < q_y$, *and* $p \prec' q$ *to denote that* $p_x < q_x$ *and* $p_y > q_y$.

DEFINITION 2.2. *Let* $p, q \in \mathbb{R}^2$, *and consider a set* $S = \{v_1, \ldots, v_k\}$ *of* k *points such that* $v_1 = p$, $v_k = q$, *and* $v_i \prec' v_{i+1}$ *for* $i = 1, 2, \ldots, k - 1$. *A 1-staircase joining* p *to* q *is an orthogonal polygonal chain, such that two consecutive elements of* S *are joined by an elbow consisting of a horizontal segment followed by a vertical segment. For an illustration, see Figure* 2(a).

A 3-staircase joining p to q is defined in a similar way, but using elbows whose first segment is vertical. Analogously, we define 2- and 4-staircases, except that we require $v_i \prec v_{i+1}$. The first segment is vertical in the 2-staircase and horizontal in the 4-staircase. Points of S are called the vertices of the staircase.

Any point p in the plane defines four open axis-aligned quadrants $Q_i(p)$, $i = 1, 2, 3, 4$, as follows (see Figure 2(b)): $Q_1(p) = \{q \in \mathbb{R}^2 \mid p \prec q\}$, $Q_2(p) = \{q \in \mathbb{R}^2 \mid q \prec' p\}$, $Q_3(p) = \{q \in \mathbb{R}^2 \mid q \prec p\}$, and $Q_4(p) = \{q \in \mathbb{R}^2 \mid p \prec' q\}$. Given a point set P in the plane, for $i = 1, 2, 3, 4$, let

$$M_i(P) = \bigcup_{p \in P} \overline{Q_i(p)},$$

where $\overline{Q_i(p)}$ denotes the closure of $Q_i(p)$.

DEFINITION 2.3. *Given a point set* P *in the plane, the elements of* P *that belong to the boundary of* $M_i(P)$, *are called the (rectilinear)* i-extremal *points of* P *(see Figure* 2(c)).

Note that the i-extremal points of P are the vertices of a i-staircase connecting all of them. This i-staircase, that we call the *i-staircase associated with* P, is the part of the boundary of $M_i(P)$ that connects all the i-extremal points of P (see Figure 2(c)).

DEFINITION 2.4. *Given a point set P in the plane, for every $J \subseteq \{1, 2, 3, 4\}$, we say that $p \in P$ is J-extremal if p is j-extremal for every $j \in J$.*

DEFINITION 2.5. *Given a point set P in the plane, the rectilinear convex hull of P is the set*[1]

$$RCH(P) = \bigcap_{i=1,2,3,4} M_i(P).$$

Figure 1 shows some examples of rectilinear convex hulls. The boundary of $RCH(P)$ is (a part of) the union of the 1-, 2-, 3-, and 4-staircases associated with P. Observe that the endpoints of these four staircases are a, b, c, and d, a is $\{1, 4\}$-extremal, b is $\{1, 2\}$-extremal, c is $\{2, 3\}$-extremal, and d is $\{3, 4\}$-extremal. In Figure 1(b), as $a = d$, then a is $\{1, 3, 4\}$-extremal and the 4-staircase associated with P consists of only point a.

Also observe that $RCH(P)$ is disconnected when either the intersection of the complements $\mathbb{R}^2 \setminus M_1(P)$ and $\mathbb{R}^2 \setminus M_3(P)$ is not empty, as shown in Figure 1(b), or the intersection of the complements $\mathbb{R}^2 \setminus M_2(P)$ and $\mathbb{R}^2 \setminus M_4(P)$ is not empty. In other words, when either the 1- and 3-staircases associated with P cross or the 2- and 4-staircases associated with P cross.

DEFINITION 2.6. *Given a point set P in the plane, a pinched point u of $RCH(P)$ occurs when u is either $\{1, 3\}$-extremal, as shown in Figure 1(b), or $\{2, 4\}$-extremal.*

DEFINITION 2.7. *Given a point set P in the plane, the size of $RCH(P)$ is the number of points of P which are i-extremal for at least one $i \in \{1, 2, 3, 4\}$.*

From the definition of the staircases for P, the following observation is straightforward.

Observation 2.8. Assume that the concatenation of the four i-staircases associated with P is traversed counterclockwise. For two consecutive i-extremal points p and p', $Q_{i+2}(o)$ contains no element of P, where $i + 2$ is taken modulo 4 and $o = (p'_x, p_y)$ for $i = 1, 3$ or $o = (p_x, p'_y)$ for $i = 2, 4$.

DEFINITION 2.9. *Given two points $u \neq v$ in the plane and a point set P, $B(u, v)$ and $P(u, v) = P \cap B(u, v)$ will denote the smallest open axis-aligned rectangle containing u and v, and the set of points in P that belong to $B(u, v)$, respectively. If $u = v$, then we define $B(u, u)$ as point u.*

Note that u and v are two opposed corners of $B(u, v)$.

DEFINITION 2.10. *Given a point set P in the plane, we say that $RCH(P)$ is vertically separable if rectangles $B(a, d)$ and $B(b, c)$ are separated by a vertical line. The two examples shown in Figure 1 are vertically separable.*

Given a point set S, and a horizontal line ℓ, let S' be the image of S under a reflection around ℓ. The following lemma is key for our algorithms.

LEMMA 2.11. *Let P be a point set in the plane. For all $S \subseteq P$, $|S| \geq 2$, either $RCH(S)$ or $RCH(S')$ is vertically separable.*

Proof. Note that $d_x < b_x$ is necessary and sufficient for vertical separability of $B(a, d)$ and $B(b, c)$, that is, $RCH(S)$ is vertically separable. Suppose then that $b_x < d_x$, and let ℓ be a horizontal line. It is straightforward to see that, if we reflect the point set S around ℓ, then S becomes S' and we have that $RCH(S')$ is vertically separable. □

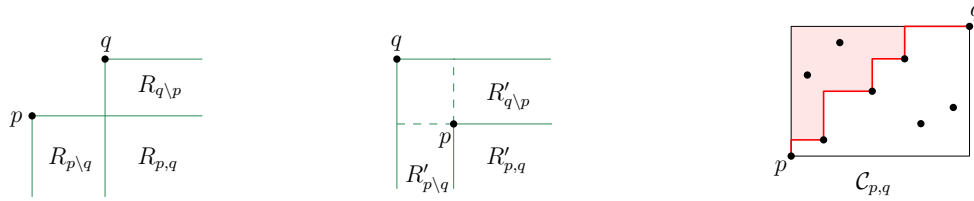[1]The notation $\mathcal{RH}(P)$ is also used for the rectilinear convex hull [4].

FIG. 3. *Left: Sets* $R_{p\setminus q}$, $R_{q\setminus p}$, $R_{p,q}$, $R'_{p\setminus q}$, $R'_{q\setminus p}$, *and* $R'_{p,q}$. *Right: Example of* $\mathcal{C}_{p,q}$.

In each of the problems MaxRCH, MaxEmptyRCH, MaxAreaRCH, and MaxWeightRCH, we will assume that the optimal subset $S \subseteq P$ is such that $RCH(S)$ is vertically separable.

To finish this section, we give one more definition.

DEFINITION 2.12. *Given a point set P in the plane, for every $p, q \in P$ such that $p \prec q$, we define $R_{p\setminus q}$, $R_{q\setminus p}$, and $R_{p,q}$ as the subsets of P in the regions $Q_4(p) \setminus \overline{Q_4(q)}$, $Q_4(q) \setminus \overline{Q_4(p)}$, and $Q_4(p) \cap Q_4(q)$, respectively (see Figure 3, left). For every $p, q \in P$ such that $q \prec' p$, we define $R'_{p\setminus q}$, $R'_{q\setminus p}$, and $R'_{p,q}$ as the subsets of P in the regions $Q_4(q) \cap Q_3(p)$, $Q_4(q) \cap Q_1(p)$, and $Q_4(p)$, respectively.*

Observe that if $r \in R_{p\setminus q}$, then $r \prec q$, if $r \in R_{q\setminus p}$, then $p \prec r$, and if $r \in R_{p,q}$, then $r \not\prec q$ and $p \not\prec r$.

3. Rectilinear convex hull of maximum size. In this section, we solve the MaxRCH problem. Given P, our goal is to combine four staircases in order to obtain a subset S of P whose rectilinear convex hull is of maximum size. All of this has to be done carefully, since the occurrence of pinched points may lead to overcounting.

Our algorithm to solve the MaxRCH problem proceeds in three steps: In the first step, we compute the 2-staircases of maximum size for every $p, q \in P$ such that $p \prec q$. In the second step, we compute what we call a *triple 1-staircase* and a *triple 3-staircase* of maximum sizes (yet to be defined). In the third and last step, we show how to combine a triple 1-staircase and a triple 3-staircase to solve the MaxRCH problem. In this step, we will make sure that the solution thus obtained is vertically separable. Our algorithm will run in $O(n^3)$ time and $O(n^2)$ space. We describe now in detail the steps of our algorithm.

The first step: For every $p, q \in P$ such that $p \prec q$ or $p = q$, let $\mathcal{C}_{p,q}$ be a 2-staircase with endpoints p and q of maximum size; see Figure 3, right. Let $C_{p,q}$ be the number of elements of P in $\mathcal{C}_{p,q}$. Note that $C_{p,q}$ equals the maximum number of 2-extremal points over all $S \cup \{p, q\}$ with $S \subseteq P(p, q)$. We can easily calculate $C_{p,q}$ for all $p, q \in P$ with $p \prec q$ or $p = q$ in $O(n^3)$ time and $O(n^2)$ space, using dynamic programming with the following recurrence:

$$(3.1) \qquad C_{p,q} = \begin{cases} 1 & \text{if } p = q, \\ \max\{1 + C_{r,q}\} \text{ over all } r \in P(p,q) \cup \{q\} & \text{if } p \neq q. \end{cases}$$

Using the elements $C_{p,q}$, it is a routine matter to determine $\mathcal{C}_{p,q}$, for any $p \prec q$.[2]

DEFINITION 3.1. *Given a point set $S \subseteq P$, we define the triple 1-staircase (resp., triple 3-staircase) associated with S as the concatenation of the 1-, 2-, and 3-staircases (resp., the 3-, 4-, and 1-staircases) associated with S.*

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[2]We note that using not so trivial methods, we can calculate all of the $C_{p,q}$'s in $O(n^2 \log n)$ time. However, this yields no improvement on the overall time complexity of our algorithms.
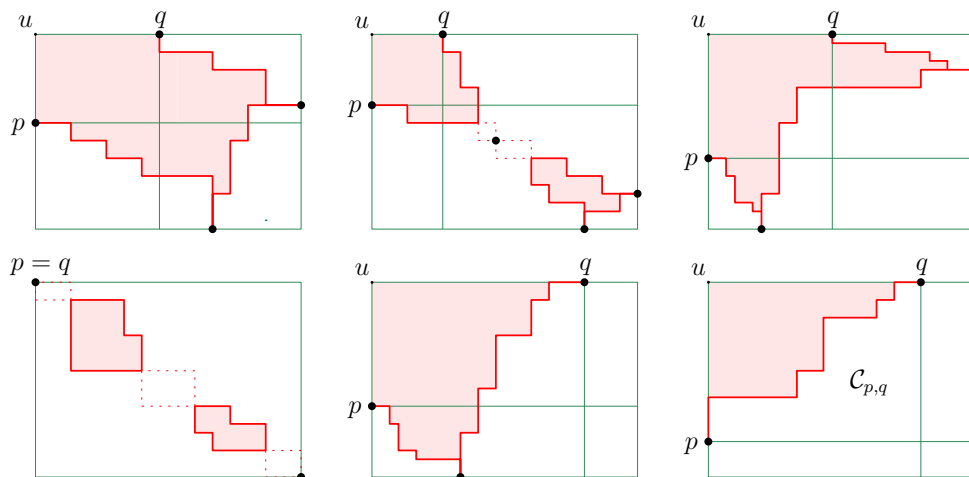
FIG. 4. *Examples of triple 1-staircases $\mathcal{T}_{p,q}$.*

The second step: In this step, our goal is to obtain a triple 1-staircase and a triple 3-staircase of maximum cardinality, starting and ending at some pairs of points of P. Triple staircases allow us to conveniently manage pinched points and disconnections of the rectilinear convex hull. Notice that the boundary of $M_1(S) \cap M_2(S) \cap M_3(S)$ (except for its two infinite rays) always belongs to the triple 1-staircase associated with S.

DEFINITION 3.2. *Consider $p, q \in P$ such that $p \prec q$ or $p = q$. We use $Z(p, q) = Q_4(u)$ to denote the fourth quadrant associated with u, where $u = (p_x, q_y)$, and $z(p, q) = Z(p, q) \cap P$ to denote the points of P in this quadrant.*

Let $\mathcal{T}_{p,q}$ be the triple 1-staircase of maximum cardinality among all subsets $S \cup \{p, q\}$ with $S \subseteq z(p, q)$. If $S' \subseteq z(p, q)$ is the set associated with $\mathcal{T}_{p,q}$, observe that $M_1(S' \cup \{p, q\}) \cap M_2(S' \cup \{p, q\}) \cap M_3(S' \cup \{p, q\})$ may contain points in $P(p, q)$, it may be disconnected, and it may have pinched points (see Figure 4). Note that p and q are always the endpoints of $\mathcal{T}_{p,q}$. Let $X_{p,q}$ denote the set of extreme vertices of $\mathcal{T}_{p,q}$ (that is, the set of 1-, 2- and 3-extremal points of $S' \cup \{p, q\}$), and let $T_{p,q}$ be the cardinality of $X_{p,q}$.

We calculate all of the $T_{p,q}$'s by dynamic programming using (3.2) and (3.3). We store all of the $T_{p,q}$'s in a table T. If $\alpha_{p,q} = 1$ when $p = q$, and $\alpha_{p,q} = 2$ when $p \neq q$, then

$$
(3.2) \qquad T_{p,q} = \max \begin{cases} C_{p,q} & \text{(A)}, \\ 1 + T_{r,q} \text{ over all } r \in R_{p \setminus q} & \text{(B)}, \\ 1 + T_{p,r} \text{ over all } r \in R_{q \setminus p} & \text{(C)}, \\ \alpha_{p,q} + T_{r,r} \text{ over all } r \in R_{p,q} & \text{(D)}, \\ \alpha_{p,q} + U_{p,r} \text{ over all } r \in R_{p,q} & \text{(E)}, \end{cases}
$$

where for every pair $p, r \in P$ such that $p \prec' r$

$$
(3.3) \qquad U_{p,r} = \max\{T_{r,s}\} \text{ over all } s \in R'_{p \setminus r}.
$$

Values $U_{p,r}$ are stored in a table U. The next lemma shows the correctness of this recurrence.
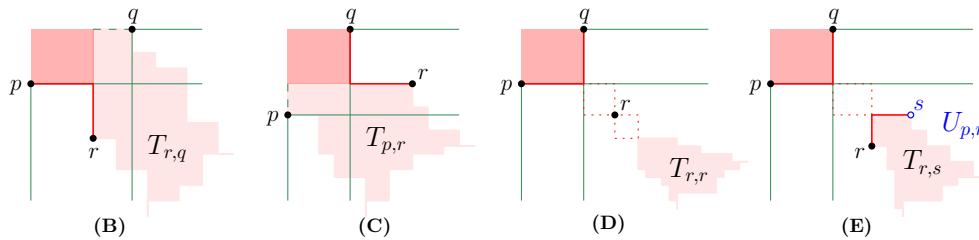
FIG. 5. *Cases in the recursive computation of $T_{p,q}$.*

LEMMA 3.3. *The recurrence* (3.2) *correctly calculates $T_{p,q}$, the size of $X_{p,q}$, in $O(n^3)$ time and $O(n^2)$ space.*

Proof. Let $\mathcal{T}_{p,q}$ be an optimal triple 1-staircase for a pair of points $p, q \in P$ such that $p \prec q$ and let $S' \subseteq z(p,q)$ be the point set associated with $\mathcal{T}_{p,q}$. In the counter-clockwise traversal of the triple 1-staircase, let p^- and q^- be the elements of P that follow and precede p and q, respectively. Hence, $\mathcal{T}_{p,q}$ can be obtained as an extension of $\mathcal{T}_{p,q^-}$, $\mathcal{T}_{p^-,q}$, or $\mathcal{T}_{p^-,q^-}$.

If $p^-, q^- \in B(p,q)$, -essarily $\mathcal{T}_{p,q}$ is a 2-staircase (the 1- and 3-staircases of $\mathcal{T}_{p,q}$ consist of only points p and q, resp.), so $T_{p,q} = C_{p,q}$ and case (A) is used to set $T_{p,q}$. Thus, we assume in the rest of the proof that at least one of p^- and q^- is not in $B(p,q)$. See Figure 5 for the cases.

If $p^- \in R_{p \setminus q}$, then we have $p^- \prec q$ and p^- is not 3-extremal in $\mathcal{T}_{p^-,q}$. We use case (B) to find a point r that plays the role of p^- to compute the value of $T_{p,q}$. If $q^- \in R_{q \setminus p}$, then we have $p \prec q^-$ and q^- is a point that is not 1-extremal in $\mathcal{T}_{p,q^-}$. We use case (C) to find a point r that plays the role of q^- to compute the value of $T_{p,q}$.

Suppose now that p^- is in $R_{p,q}$ (a similar reasoning applies if q^- is in $R_{p,q}$). In this case, q^- cannot be in $B(p,q)$ and if $q^- \in R_{q \setminus p}$, then we use case (C). If $q^- \in R_{p,q}$, there are two cases to analyze: $p^- = q^-$ and $p^- \neq q^-$. If $p^- = q^-$, then we can use case (D) to find a point r that plays the role of p^- to compute the value of $T_{p,q}$. When $p^- \neq q^-$, we prove that $p^- \prec q^-$, and then case (E) can be used to find a pair of points r and s playing the roles of p^- and q^-, both in $R_{p,q}$, with maximum value $T_{r,s}$. By Observation 2.8, as p and p^- are consecutive 1-extremal points, then $Q_3(o)$ contains no element in S', where $o = (p_x^-, p_y)$. In particular, since q^- is in $R_{p,q}$, this implies that q^- cannot be in either $Q_2(p^-)$ or $Q_3(p^-)$. Moreover, as q and q^- are consecutive 3-extremal points, then $Q_1(o')$ contains no element in S' again by Observation 2.8, where $o' = (q_x, q_y^-)$. As a consequence, q^- cannot be in $Q_4(p^-)$, so we conclude that $q^- \in Q_1(p^-)$.

To compute tables T and U, we scan the elements of P from right to left. Each time an element $p \in P$ is encountered, we scan all of the $q \in P$ such that $p_x < q_x$, again from right to left. When $p \prec q$ we compute $T_{p,q}$, and when $p \prec' q$ we compute $U_{p,q}$. Each entry of T and U is determined in $O(n)$ time. Thus, U and T can be computed in overall $O(n^3)$ time and $O(n^2)$ space. Cases (A) to (D) are in $O(n)$. We charge the work done in case (E) to constructing table U, which can be done in $O(n)$ time per entry. Thus the entire complexity is in $O(n^3)$ time and $O(n^2)$ space. □

In a totally analogous way, we can calculate triple 3-staircases of maximum size. For $p \prec q$ or $p = q$, let $T'_{p,q}$ be the size of the triple 3-staircase $\mathcal{T}'_{p,q}$ of maximum cardinality among all subsets $S' \cup \{p, q\}$, where S' is now a subset of points of P in the quadrant $Q_2(v)$ with $v = (q_x, p_y)$. After rotating the coordinates by π, observe
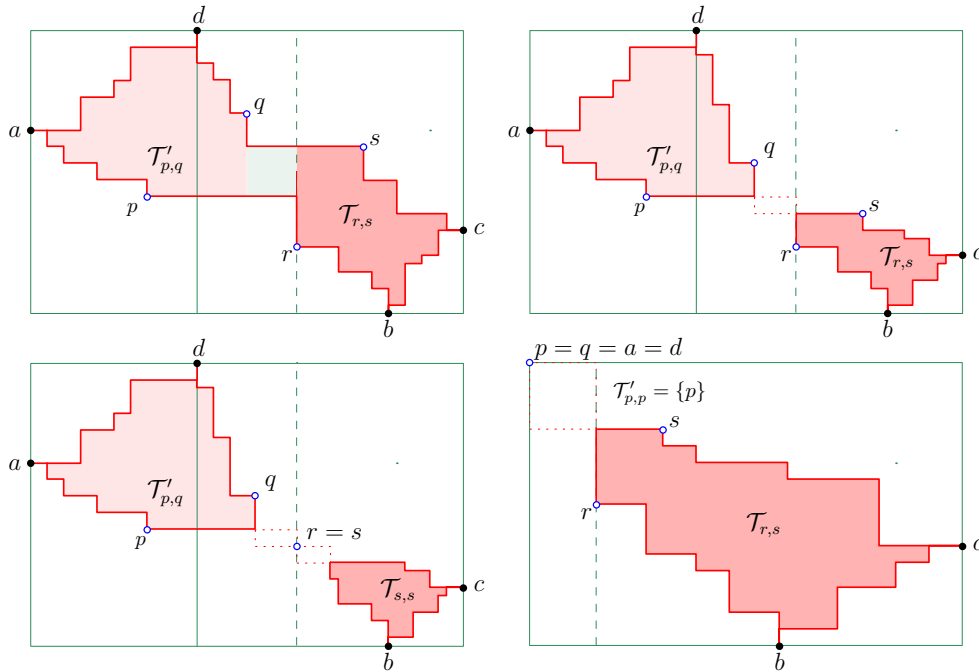
FIG. 6. *Examples of 4-separators* (p, q, r, s).

that the triple 3-staircase $\mathcal{T}'_{p,q}$ is the triple 1-staircase $\mathcal{T}_{q,p}$. Thus, by symmetry with the $T_{p,q}$'s, all the $T'_{p,q}$'s can also be calculated in $O(n^3)$ time and $O(n^2)$ space.

The third step: In this step, we show how to combine a triple 1-staircase and a triple 3-staircase to solve the MaxRCH problem. Recall that the solution must be vertically separable. Next we give the definition of a 4-separator and then we show that it is equivalent to vertical separability.

DEFINITION 3.4. *Let $S \subseteq P$ be any subset with $|S| \geq 2$. Given four (not necessarily distinct) extremal points $p, q, r, s \in S$, we say that the tuple (p, q, r, s) is a 4-separator of $RCH(S)$ if the following five conditions are satisfied: (1) $p \prec q$ or $p = q$; (2) $q \prec' r$; (3) $r \prec s$ or $r = s$; (4) p and r are consecutive points in the 1-staircase of S; and (5) s and q are consecutive points in the 3-staircase of S (see Figure 6).*

The following lemma shows the equivalence between the existence of 4-separators and vertical separability.

LEMMA 3.5. *$RCH(S)$ is vertically separable if and only if $RCH(S)$ has a 4-separator.*

Proof. Let us first assume that $RCH(S)$ is vertically separable, that is, rectangles $B(a, d)$ and $B(b, c)$ are separated by a vertical line. Recall that a, b, c, and d are the leftmost, bottommost, rightmost, and topmost points of S, respectively. Then, we can argue the following: If $a \prec d$, then S has at least one 1-extremal point to each side of the vertical line through d. Otherwise, if $a = d$, then S has at least one 1-extremal point to the right side. Thus, covering both cases, let p and r be the two consecutive 1-extremal points of S such that $p_x \leq d_x < r_x$. Now, given that $d_x < r_x$, we have that if r is also 3-extremal, thus a pinched point, then S has at least one 3-extremal

point to the left side of the vertical line through r. Otherwise, if r is not 3-extremal, then S has at least one 3-extremal point to each side of this line. Thus, we can define s and q as the two consecutive 3-extremal points of S such that $q_x < r_x \leq s_x$. It is straightforward to see now that (p, q, r, s) is a 4-separator of $RCH(S)$. Note that, when $p = a = d$, then necessarily $q = p = a = d$.

Assume now that (p, q, r, s) is a 4-separator of $RCH(S)$. We then have that $d \prec' q$ or $d = q$, and $r \prec' b$ or $r = b$. These conditions, together with $q \prec' r$, directly imply that rectangles $B(a, d)$ and $B(b, c)$ are separated by a vertical line, thus $RCH(S)$ is vertically separable.   □

Using 4-separators, we show how to find an optimal solution that is vertically separable. Among all subsets S of P such that $RCH(S)$ is vertically separable, let S_0 be a subset of P such that $RCH(S_0)$ has maximum size. Let (p, q, r, s) be a 4-separator of $RCH(S_0)$. The key observation is that the vertices of $\mathcal{T}'_{p,q} \cup \mathcal{T}_{r,s}$ are the set of extremal points of S_0. Note that $\mathcal{T}'_{p,q} \cap \mathcal{T}_{r,s} = \emptyset$ and $|RCH(S_0)|$, the size of $RCH(S_0)$, is $T'_{p,q} + T_{r,s}$.

Thus, we proceed as follows: For given $p, s \in P$ such that $p_x < s_x$, let $\mathcal{S}_{p,s}$ be the rectilinear convex hull of maximum size, among all subsets $S \subseteq P$ containing p and s such that there exist two points $q, r \in S$ with (p, q, r, s) being a 4-separator of $RCH(S)$. Let $S_{p,s}$ be the size of $\mathcal{S}_{p,s}$. Note that $S_{p,s} = T'_{p,q} + T_{r,s}$ for some 4-separator (p, q, r, s). Then, the following equations allow us to calculate $|RCH(S_0)|$ in $O(n^3)$ time and $O(n^2)$ space, as Theorem 3.6 proves:

$$(3.4) \qquad |RCH(S_0)| = \max\{S_{p,s}\} \text{ over all } p, s \in P \text{ such that } p_x < s_x,$$

where for each pair of points $p, s \in P$ with $p_x < s_x$

$$(3.5) \quad S_{p,s} = \max\{T'_{p,q} + T_{r,s}\} \text{ over all } q, r \in P \text{ such that } (p, q, r, s) \text{ is a 4-separator.}$$

THEOREM 3.6. *The* MaxRCH *problem can be solved in* $O(n^3)$ *time and* $O(n^2)$ *space.*

Proof. According to (3.4) and (3.5), we only need to show how to compute $S_{p,s}$ in linear time, for given p and s. Let Q_p be the set of all points $q \in P$ such that $q \prec' s$, and $p \prec q$ or $p = q$. Let Q_s be the set of all points $r \in P$ such that $p \prec' r$, and $r \prec s$ or $r = s$. Note that $Q_p \cap Q_s = \emptyset$ and that p and s belong to Q_p and Q_s, respectively, only when $p_y > s_y$. Let $L_{p,s}$ be the list of the elements of $Q_p \cup Q_s$ sorted by x-coordinate. Observe that if (p, q, r, s) is a 4-separator of $RCH(S_{p,s})$, then $r \in Q_s$, $q \in Q_p$, and q is the point $q^* \in Q_p$ from the beginning of $L_{p,s}$ to r such that $T'_{p,q^*} + T_{r,s} = \max\{T'_{p,q'} + T_{r,s}\}$ over all $q' \in Q_p$ from the beginning of $L_{p,s}$ to r.

We calculate $S_{p,s}$ by processing the elements of $L_{p,s}$ in order. For an element t in $L_{p,s}$, let q_t^* be the point in Q_p maximizing $T'_{p,q'}$ over all $q' \in Q_p$ from the beginning of $L_{p,s}$ to t (including t). When processing a point t, observe that if $t \in Q_p$, then q_t^* is either t or q_{t-1}^*. Otherwise, if $t \in Q_s$, then $q_t^* = q_{t-1}^*$. Moreover, if $t \in Q_s$, then we set $q = q_t^*$ and $r = t$, and consider (p, q, r, s) as a *feasible* 4-separator of $\mathcal{S}_{p,s}$. After processing the last element of $L_{p,s}$, among all the (linear number) feasible separators, we return the solution $S_{p,s}$ induced by the feasible separator that maximizes $T'_{p,q} + T_{r,s}$. Thus, $S_{p,s}$ can be calculated in $O(n)$ time, once tables T and T' have been constructed.   □

4. Maximum size/area empty rectilinear convex hulls and maximum weight rectilinear convex hull. In this section, we show how to adapt the algorithm of section 3 to solve the MaxEmptyRCH, the MaxAreaRCH, and the MaxWeightRCH
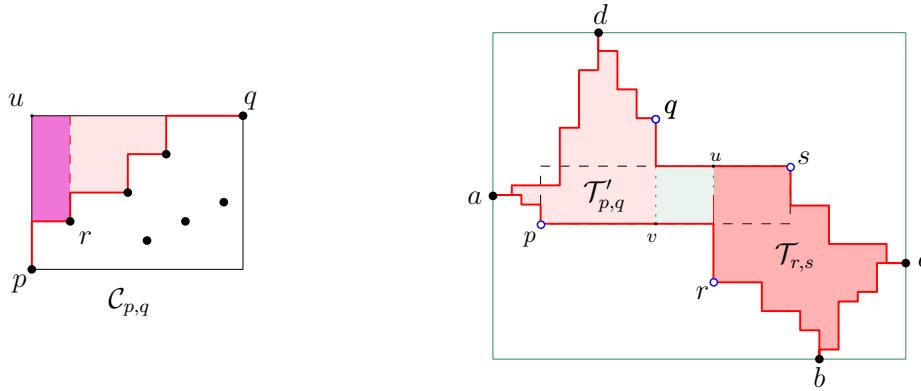
FIG. 7. *Left: Example of $\mathcal{C}_{p,q}$ when the 2-staircases must be empty. Right: Third step of the algorithm when the interior must be empty of elements of P.*

problems. The first observation is that, as the optimal solution for any of these problems is the rectilinear convex hull of a subset S of P, then Lemmas 2.11 and 3.5 hold. This implies that we can assume that rectangles $B(a,d)$ and $B(b,c)$ are separated by a vertical line, so a 4-separator exists for the optimal solution in any of the problems. As a consequence, the algorithms to solve these three problems follow the same scheme as the algorithm described in the previous section, and we only need to show how to adapt in each problem the calculation of the 2-staircases, the triple 1- and 3-staircases, and the rectilinear convex hulls $\mathcal{S}_{p,s}$ to fulfill the requirements on emptiness, area, or weight.

We start by solving the MaxEmptyRCH problem, we continue with the MaxAreaRCH problem, and we finish with the MaxWeightRCH problem.

4.1. Maximum size empty rectilinear convex hull. To solve the MaxEmptyRCH problem in $O(n^3)$ time and $O(n^2)$ space, we modify the steps of our previous algorithm. These modifications ensure that the "interiors" of the triple 1- and 3-staircases and the rectilinear convex hulls are empty. Recall that in this problem we are looking for a subset $S \subseteq P$ such that $RCH(S)$ has maximum size and there is no element of P in the interior of $RCH(S)$.

The first step: For a pair of points $p, q \in P$ such that $p \prec q$ or $p = q$, we say that the 2-staircase associated with a subset S of $P(p,q)$ is *empty* (recall that $P(p,q)$ is the set of points in P that belong to the rectangle $(B(p,q))$ if no point of P is in the interior of $B(p,q) \cap M_2(S \cup \{p,q\})$; see Figure 7, left.

Let $\mathcal{C}_{p,q}$ be the empty 2-staircase of maximum cardinality over all subsets $S \cup \{p,q\}$ with $S \subseteq P(p,q)$, and let $C_{p,q}$ be the size of $\mathcal{C}_{p,q}$. Observe that if u is the point (p_x, q_y) and $r \in P$ is the vertex of the 2-staircase that follows p, then $P(r,u) = \emptyset$. Thus, values $C_{p,q}$ can be computed using the following recurrence:
(4.1)
$$C_{p,q} = \begin{cases} 1 & \text{if } p = q, \\ \max\{1 + C_{r,q}\} \text{ over all } r \in P(p,q) \cup \{q\} \text{ such that } P(r,u) = \emptyset & \text{if } p \neq q. \end{cases}$$

As q and u are on the same horizontal line, $P(q,u)$ is not defined. In this case, we assume that $P(q,u)$ is the empty set. Using standard range counting techniques [15], we can preprocess the grid G generated by the vertical and horizontal lines through the elements of P in $O(n^2)$ time and space, so that for every pair of vertices u, v of

G we can query the number of points in $P(u,v)$ in $O(1)$ time. Thus, we can decide whether $P(u,v) = \emptyset$ in $O(1)$ time. Therefore, values $C_{p,q}$ can be calculated in $O(n^3)$ time and $O(n^2)$ space.

The second step: For every $p, q \in P$ such that $p \prec q$ or $p = q$, we say that the triple 1-staircase $\mathcal{T}$ corresponding to a subset S of $z(p,q)$ is empty (recall that $z(p,q)$ is the set of points in P that belong to $Z(p,q)$, the <u>fourth</u> quadrant associated with point (p_x, q_y)) if the (disconnected) region $\mathcal{O}_{\mathcal{T}} = \overline{Z(p,q)} \cap M_1(S \cup \{p,q\}) \cap M_2(S \cup \{p,q\}) \cap M_3(S \cup \{p,q\})$ associated with $\mathcal{T}$ contains no element of P. Please refer to Figure 4, where the shaded areas correspond to $\mathcal{O}_{\mathcal{T}}$. Let $\mathcal{T}_{p,q}$ be the empty triple 1-staircase of maximum size among all subsets $S \cup \{p,q\}$ with $S \subseteq z(p,q)$. Let $E(p,q,r)$ denote the interior of $B(p,q) \cap Q_2(r)$ if $p \neq q$, and the empty set if $p = q$. In Figure 5 examples of open rectangles $E(p,q,r)$ are shown as shaded rectangles. We show how to compute $T_{p,q}$, the cardinality of the set of extreme vertices of $\mathcal{T}_{p,q}$, using the following equations that are similar to (3.2) and (3.3):

$$(4.2) \qquad T_{p,q} = \max \begin{cases} C_{p,q} & \text{(A)}, \\ 1 + T_{r,q} \text{ over all } r \in R_{p \setminus q} \ : \ P \cap E(p,q,r) = \emptyset & \text{(B)} \\ 1 + T_{p,r} \text{ over all } r \in R_{q \setminus p} \ : \ P \cap E(p,q,r) = \emptyset & \text{(C)} \\ \alpha_{p,q} + T_{r,r} \text{ over all } r \in R_{p,q} \ : \ P \cap E(p,q,r) = \emptyset & \text{(D)} \\ \alpha_{p,q} + U_{p,r} \text{ over all } r \in R_{p,q} \ : \ P \cap E(p,q,r) = \emptyset & \text{(E)} \end{cases}$$

where for every pair $p, r \in P$ such that $p \prec' r$

$$(4.3) \qquad U_{p,r} = \max\{T_{r,s}\} \text{ over all } s \in R'_{p \setminus r}.$$

In case (A), $\mathcal{O}_{\mathcal{T}_{p,q}}$ is empty as $C_{p,q}$ is an empty 2-staircase. Equation (4.2) is obtained from (3.2) by further constraining r in the cases from (B) to (E) to satisfy $P \cap E(p,q,r) = \emptyset$. This guarantees that the interior of $\mathcal{O}_{p,q}$ is empty of elements of P for all p, q. Verifying that $P \cap E(p,q,r) = \emptyset$ can be decided in $O(1)$ time by using a range counting query. The proof of correctness of (4.2) and (4.3) follows the same steps as in Lemma 3.3. Hence, computing the new table T can be done in $O(n^3)$ time and $O(n^2)$ space. By symmetry, values in $T'_{p,q}$, the sizes of the empty triple 3-staircases $\mathcal{T}'_{p,q}$, can also be calculated in $O(n^3)$ time and $O(n^2)$ space.

The third step: For given $p, s \in P$ such that $p_x < s_x$, let $\mathcal{S}_{p,s}$ be the empty rectilinear convex hull of maximum size, among all subsets $S \subseteq P$ containing p and s such that $RCH(S)$ is empty and there exist two points $q, r \in S$ with (p,q,r,s) being a 4-separator of $RCH(S)$. Let $S_{p,s}$ denote the size of $\mathcal{S}_{p,s}$. To compute $S_{p,s}$ we have to distinguish whether $p \prec' s$ or $p \prec s$. If $p \prec' s$ (see Figure 6, top-right, or any in the bottom), then $S_{p,s} = \max\{T'_{p,q} + T_{r,s}\}$ over all 4-separators (p,q,r,s). Otherwise, if $p \prec s$ (see Figure 7, right), then we must ensure that each 4-separator (p,q,r,s) satisfies the emptiness of the rectangle $B(u,v)$ (that is, $P(u,v) = \emptyset$), where $u = (r_x, s_y)$ and $v = (q_x, p_y)$.

If S_0 is a subset of P such that $RCH(S_0)$ is empty, vertically separable, and of maximum size, the new equations to compute $|RCH(S_0)|$ are

$$(4.4) \qquad |RCH(S_0)| = \max\{S_{p,s}\} \text{ over all } p, s \in P \text{ with } p_x < s_x,$$

where for each pair of points $p, s \in P$ such that $p_x < s_x$
(4.5)
$$S_{p,s} = \begin{cases} \max\{T'_{p,q} + T_{r,s}\} \ : \ (p,q,r,s) \text{ is a 4-separator,} & p \prec' s, \\ \max\{T'_{p,q} + T_{r,s}\} \ : \ (p,q,r,s) \text{ is a 4-separator, } P(u,v) = \emptyset, & p \prec s. \end{cases}$$

THEOREM 4.1. *The* MaxEmptyRCH *problem can be solved in* $O(n^3)$ *time and* $O(n^2)$ *space.*

Proof. Again, we only need to show that given points p and s $S_{p,s}$ can be computed in linear time. When $p \prec' s$, we argue as in the proof of Theorem 3.6. However, when $p \prec s$, we need to only consider 4-separators such that $P(u, v) = \emptyset$. Let $Q_{p,s} = \{t \in P : p \prec t \prec s\}$, which satisfies $Q_{p,s} \cap Q_p = \emptyset$ and $Q_{p,s} \cap Q_s = \emptyset$. Recall that $Q_p \cap Q_s = \emptyset$ and that, when $p \prec s$, $Q_p = \{q \in P : p \prec q, q \prec' s\}$ and $Q_s = \{r \in P : r \prec s, p \prec' r\}$. Let $L_{p,s}$ be the list of the points of $Q_p \cup Q_s \cup Q_{p,s}$ sorted by x-coordinate. Assuming that we have already sorted P by x-coordinate, $L_{p,s}$ is obtained in $O(n)$ time.

As before, we calculate $S_{p,s}$ by processing the elements of $L_{p,s}$ in order. For an element t in $L_{p,s}$, let q_t^* be the point in Q_p maximizing $T'_{p,q'}$ over all $q' \in Q_p$ from the beginning of $L_{p,s}$ to t (including t) subject to there being no elements of $Q_{p,s}$ in $L_{p,s}$ from q' to t. When processing a point $t \in Q_{p,s}$, we set $q_t^* = \mathbf{nil}$ denoting that q_t^* is undefined. Observe that, when processing a point $t \in Q_p$, if $q_{t-1}^* = \mathbf{nil}$, then $q_t^* = t$, and if $q_{t-1}^* \neq \mathbf{nil}$, then q_t^* is either t or q_{t-1}^*. When processing a point $t \in Q_s$, then $q_t^* = q_{t-1}^*$, and if $q_{t-1}^* \neq \mathbf{nil}$, then we set $q = q_t^*$ and $r = t$, and consider (p, q, r, s) as a feasible 4-separator of $\mathcal{S}_{p,s}$. Note that for this 4-separator we have $P(u, v) = \emptyset$. After processing all elements in $L_{p,s}$, $S_{p,s}$ is determined by a feasible 4-separator that maximizes $T'_{p,q} + T_{r,s}$. □

4.2. Maximum area empty rectilinear convex hull. In the MaxAreaRCH problem, we determine an empty rectilinear convex hull of maximum area. To solve this problem, we proceed as in the previous subsection. The only difference is that we sum areas in all of our recurrences, instead of counting points. Given a bounded set $Z \subset \mathbb{R}^2$, we denote the area of Z as $\mathsf{Area}(Z)$.

Now, $\mathcal{C}_{p,q}$, $\mathcal{T}_{p,q}$, $\mathcal{T}'_{p,q}$, and $\mathcal{S}_{p,q}$ are as described in section 4.1, with the difference that they maximize area instead of maximizing size. The areas are defined as follows. If $S = \{p, v_2, \ldots, v_{k-1}, q\}$ is the set of vertices of an empty 2-staircase, we define the area of this staircase as $\mathsf{Area}(B(p, q) \cap M_2(S))$. For an empty triple 1-staircase or an empty triple 3-staircase $\mathcal{T}$, its area is the area of its associated region $\mathcal{O}_{\mathcal{T}}$. The area of a rectilinear convex hull is the area of its interior.

The first step: For a pair of points $p, q \in P$ such that $p \prec q$ or $q = p$, we compute $C_{p,q}$, the area of $\mathcal{C}_{p,q}$, using the following recurrence, which is a variant of (4.1) maximizing area:

$$(4.6) \quad C_{p,q} = \begin{cases} 0 & \text{if } p = q, \\ \max_{r \in P(p,q) \cup \{q\}} \{\mathsf{Area}(B(r,u)) + C_{r,q}\} : P(r,u) = \emptyset & \text{if } p \neq q, \end{cases}$$

where $u = (p_x, q_y)$. As $B(q, u)$ is not defined, we set $\mathsf{Area}(B(q, u)) = 0$.

The second step: For every $p, q \in P$ such that $p \prec q$ or $p = q$, let $T_{p,q}$ be the area of $\mathcal{T}_{p,q}$. All $T_{p,q}$'s can be calculated in $O(n^3)$ time and $O(n^2)$ space using the following equations, which are variants of (4.2) and (4.3) maximizing area (recall that if $p = q$, then rectangle $E(p, q, r) = \emptyset$, so its area is 0):

$$(4.7)$$

$$T_{p,q} = \max \begin{cases} C_{p,q} & \text{(A)}, \\ \mathsf{Area}(E(p,q,r)) + T_{r,q} \text{ over all } r \in R_{p \setminus q} : P \cap E(p,q,r) = \emptyset & \text{(B)} \\ \mathsf{Area}(E(p,q,r)) + T_{p,r} \text{ over all } r \in R_{q \setminus p} : P \cap E(p,q,r) = \emptyset & \text{(C)} \\ \mathsf{Area}(E(p,q,r)) + T_{r,r} \text{ over all } r \in R_{p,q} : P \cap E(p,q,r) = \emptyset & \text{(D)} \\ \mathsf{Area}(E(p,q,r)) + U_{p,r} \text{ over all } r \in R_{p,q} : P \cap E(p,q,r) = \emptyset & \text{(E)}, \end{cases}$$

where for every pair $p, r \in P$ such that $p \prec' r$

$$(4.8) \qquad\qquad U_{p,r} = \max\{T_{r,s}\} \text{ over all } s \in R'_{p \setminus r}.$$

The areas $T'_{p,q}$ for the empty triple 3-staircases $\mathcal{T}'_{p,q}$ can be calculated in a similar way.

The third step: Let $S_{p,s}$ be the area of $\mathcal{S}_{p,s}$. Recall that, for given $p, s \in P$ such that $p_x < s_x$, $\mathcal{S}_{p,s}$ is the empty rectilinear convex hull of maximum area, among all subsets $S \subseteq P$ containing p and s such that $RCH(S)$ is empty and there exist two points $q, r \in S$ with (p, q, r, s) a 4-separator of $RCH(S)$. Observe that if $p \prec' s$, then $S_{p,s} = T'_{p,q} + T_{r,s}$ for some 4-separator (p, q, r, s). Otherwise, if $p \prec s$, then $S_{p,s} = \mathsf{Area}(B(u,v)) + T'_{p,q} + T_{r,s}$ for some 4-separator (p, q, r, s), subject to $P(u,v) = \emptyset$, where $u = u(r) = (r_x, s_y)$ and $v = v(q) = (q_x, p_y)$ (see Figure 7, right).

Given that $\mathsf{Area}(B(u,v))$ depends on both r and q, using the inclusion/exclusion principle, we can then calculate $S_{p,s}$ as

$$S_{p,s} = T'_{p,q} + \mathsf{Area}(B(v,s)) + T_{r,s} + \mathsf{Area}(B(p,u)) - \mathsf{Area}(B(p,s)).$$

Since p and s are fixed, note that $U(p,q,s) = T'_{p,q} + \mathsf{Area}(B(v(q),s))$ depends only on q and $V(p,r,s) = T_{r,s} + \mathsf{Area}(B(p,u(r))) - \mathsf{Area}(B(p,s))$ depends only on r. Each of these two values can be computed in $O(1)$ time, once T and T' have been computed in the second step. If S_0 is a subset of P such that $RCH(S_0)$ is empty, vertically separable and of maximum area, the new equations to compute $RCH(S_0)$ are

$$(4.9) \qquad\qquad |RCH(S_0)| = \max\{S_{p,s}\} \text{ over all } p, s \in P \text{ with } p_x < s_x,$$

where for each pair of points $p, s \in P$ such that $p_x < s_x$
(4.10)
$$S_{p,s} = \begin{cases} \max_{\{4\text{-separators } (p,q,r,s)\}}\{T'_{p,q} + T_{r,s}\}, & p \prec' s, \\ \max_{\{4\text{-separators } (p,q,r,s)\}}\{U(p,q,s) + V(p,r,s))\} : P(u(r), v(q)) = \emptyset, & p \prec s. \end{cases}$$

THEOREM 4.2. *The* MaxAreaRCH *problem can be solved in* $O(n^3)$ *time and* $O(n^2)$ *space.*

Proof. The proof follows the proof of Theorem 4.1. The only difference is that, when processing an element t in $L_{p,s}$, q_t^* is the point in Q_p maximizing $T'_{p,q'} + \mathsf{Area}(B(v(q'),s))$, instead of maximizing $T'_{p,q'}$. After processing all elements in $L_{p,s}$, $S_{p,s}$ is determined by a feasible 4-separator that maximizes $T'_{p,q} + \mathsf{Area}(B(v,s)) + T_{r,s} + \mathsf{Area}(B(p,u)) - \mathsf{Area}(B(p,s))$. ☐

4.3. Maximum weight rectilinear convex hull. In the MaxWeightRCH problem, each input point p of P comes with a (positive or negative) weight $w(p)$. We determine a subset $S \subseteq P$ such that $RCH(S)$ has maximum weight, that is, such that $\sum_{p \in P \cap RCH(S)} w(p)$ is maximized.

The algorithm to solve this problem combines the ideas of the previous algorithms and follows the same steps, however, now we add weights. We define $\mathsf{Weight}(Z) = \sum_{p \in P \cap Z} w(p)$ as the weight of a region $Z \subset \mathbb{R}^2$. Using the same range counting techniques [15] as in section 3, we can preprocess the grid G generated by the vertical and horizontal lines through the elements of P in $O(n^2)$ time and space, so that for every pair of vertices u, v of G we can query $\mathsf{Weight}(B(u,v)) = \sum_{p \in P(u,v)} w(p)$ in $O(1)$ time, for any rectangle $B(u,v)$.

Now, $\mathcal{C}_{p,q}$, $\mathcal{T}_{p,q}$, $\mathcal{T}'_{p,q}$, and $\mathcal{S}_{p,q}$ are as described in section 3, except that weight is maximized. The weights are defined as follows. If $S = \{p, v_2, \ldots, v_{k-1}, q\}$ is the set of

vertices of a 2-staircase, its weight is defined as $w(p) + w(q) + \mathsf{Weight}(B(p,q) \cap M_2(S))$. Note that the weights of all points in S are included in this formula. For a triple 1-staircase or a triple 3-staircase $\mathcal{T}$, its weight is the addition of the weights of the points of P that appear on the boundary or in the interior of $\mathcal{O}_\mathcal{T}$, the region associated with $\mathcal{T}$. Finally, the weight of a rectilinear convex hull is the addition of the points of P on the boundary or the interior of the rectilinear convex hull.

The first step: If $C_{p,q}$ is the weight of $\mathcal{C}_{p,q}$, for a pair of points $p, q \in P$ such that $p \prec q$ or $q = p$, all $C_{p,q}$'s can be computed in $O(n^3)$ time and $O(n^2)$ space using the following recurrence:

$$(4.11) \qquad C_{p,q} = \begin{cases} w(p) & \text{if } p = q, \\ w(p) + \max_{r \in P(p,q) \cup \{q\}} \{\mathsf{Weight}(B(r,u)) + C_{r,q}\} & \text{if } p \neq q, \end{cases}$$

where $u = (p_x, q_y)$. We set $\mathsf{Weight}(B(q,u)) = 0$ as $B(q,u)$ is not defined.

The second step: If $T_{p,q}$ is the weight of $\mathcal{T}_{p,q}$, for every $p, q \in P$ such that $p \prec q$ or $p = q$, then all $T_{p,q}$'s (and, by symmetry, all $T'_{p,q}$'s) can be calculated in $O(n^3)$ time and $O(n^2)$ space using the following equations, where $\alpha_{p,q} = w(p)$ if $p = q$, and $\alpha_{p,q} = w(p) + w(q)$ if $p \neq q$:

$$(4.12) \qquad T_{p,q} = \max \begin{cases} C_{p,q} & \text{(A)}, \\ w(p) + \mathsf{Weight}(E(p,q,r)) + T_{r,q} \text{ over all } r \in R_{p \setminus q} & \text{(B)}, \\ w(q) + \mathsf{Weight}(E(p,q,r)) + T_{p,r} \text{ over all } r \in R_{q \setminus p} & \text{(C)}, \\ \alpha_{p,q} + \mathsf{Weight}(E(p,q,r)) + T_{r,r} \text{ over all } r \in R_{p,q} & \text{(D)}, \\ \alpha_{p,q} + \mathsf{Weight}(E(p,q,r)) + U_{p,r} \text{ over all } r \in R_{p,q} & \text{(E)}, \end{cases}$$

where for every pair $p, r \in P$ such that $p \prec' r$

$$(4.13) \qquad\qquad U_{p,r} = \max\{T_{r,s}\} \text{ over all } s \in R'_{p \setminus r}.$$

The third step: For given $p, s \in P$ such that $p_x < s_x$, let $S_{p,s}$ be the weight of $\mathcal{S}_{p,s}$. Using similar reasoning as in the previous subsection, one can show that, if S_0 is a subset of P such that $RCH(S_0)$ is vertically separable of maximum weight, the following equations calculate $|RCH(S_0)|$:

$$(4.14) \qquad |RCH(S_0)| = \max\{S_{p,s}\} \text{ over all } p, s \in P \text{ such that } p_x < s_x,$$

where for each pair of points $p, s \in P$ such that $p_x < s_x$

$$(4.15) \qquad S_{p,s} = \begin{cases} \max_{\{4\text{-separators } (p,q,r,s)\}}\{T'_{p,q} + T_{r,s}\}, & p \prec' s, \\ \max_{\{4\text{-separators } (p,q,r,s)\}}\{U(p,q,s) + V(p,r,s))\}, & p \prec s. \end{cases}$$

Now $U(p,q,s)$ is defined as $T'_{p,q} + \mathsf{Weight}(B(v(q),s))$, and $V(p,r,s)$ is defined as $T_{r,s} + \mathsf{Weight}(B(p,u(r))) - \mathsf{Weight}(B(p,s))$ with $u = u(r) = (r_x, s_y)$ and $v = v(q) = (q_x, p_y)$.

The proof of the next theorem is a straightforward adaptation of the previous arguments.

THEOREM 4.3. *The* MaxWeightRCH *problem can be solved in* $O(n^3)$ *time and* $O(n^2)$ *space.*
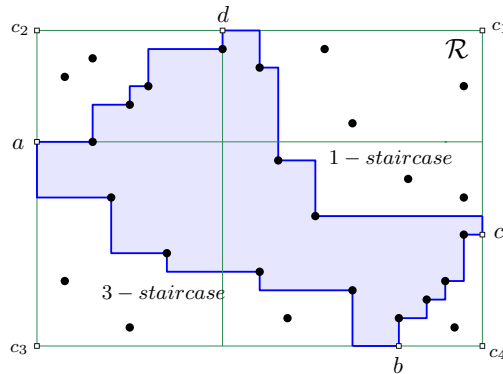
FIG. 8. *The maximum area orthoconvex polygon.*

5. Maximum area orthoconvex polygon. Let $\mathcal{R}$ be an axis-aligned rectangle in the plane (usually called domain) and let $P \subset \mathcal{R}$ be a set of n points in general position. In the MaxOrthoconvexPolygon problem, we look for an orthoconvex polygon $\mathcal{OC} \subset \mathcal{R}$ of maximum area containing no element of P in its interior. Recall that a polygon is orthoconvex if its sides are axis-parallel and its intersection with any horizontal or vertical line is empty, or a line segment. $\mathcal{OC}$ is bounded by four staircases, determined by some points of P (see Figure 8). Observe the differences between an optimal solution for the MaxAreaRCH problem and an optimal solution for the MaxOrthoconvexPolygon problem (see Figures 6 and 8). In both cases, the boundary of the solutions is defined by four staircases, but in the second case the 1- and the 3-staircases (and the 2- and the 4-staircases) interchange their roles. Thus, the techniques previously explained must be adapted to this new situation.

As in the previous sections, our algorithm to solve the MaxOrthoconvexPolygon problem is divided into three steps. In the first step, the algorithm calculates empty staircases. In the second step, the algorithm computes empty orthoconvex polygons of maximum area bounded by three staircases. Finally, in the third step, we combine some of these empty orthoconvex polygons to find an optimal solution. The two main differences in relation to the previous algorithms are that we use 2-separators (yet to be defined) instead of 4-separators in the third step, and that the empty orthoconvex polygons used to find the optimal solution must also be defined for the orthogonal projections of the points of P onto the sides of $\mathcal{R}$. This results in recurrences that are a bit more elaborate in the two first steps. We give some definitions, before describing the algorithm.

DEFINITION 5.1. *We use c_1, c_2, c_3, and c_4 to denote the top-right, top-left, bottom-left, and bottom-right vertices of $\mathcal{R}$, respectively. Let $\bar{P}_t$ (resp., $\bar{P}_\ell$, $\bar{P}_b$, $\bar{P}_r$) be the orthogonal projections of the points in P onto the top (resp., left, bottom, right) side of $\mathcal{R}$, and let $\bar{P} = \bar{P}_t \cup \bar{P}_\ell \cup \bar{P}_b \cup \bar{P}_r \cup \{c_1, c_2, c_3, c_4\}$.*

Observe that for an optimal solution $\mathcal{OC}$, the boundary of $\mathcal{OC}$ always shares four segments with $\mathcal{R}$, each of them on one different side of $\mathcal{R}$ (see Figure 8). Let d (resp., a, b, c) be the leftmost (resp., topmost, rightmost, bottommost) vertex of $\mathcal{OC}$ on the top (resp., left, bottom, right) side of $\mathcal{OC}$, as shown in Figure 8. Since these four points belong to $\bar{P}$, we have to define structures whose endpoints belong to $\bar{P}$ in the different steps of our algorithm, as we explain later.
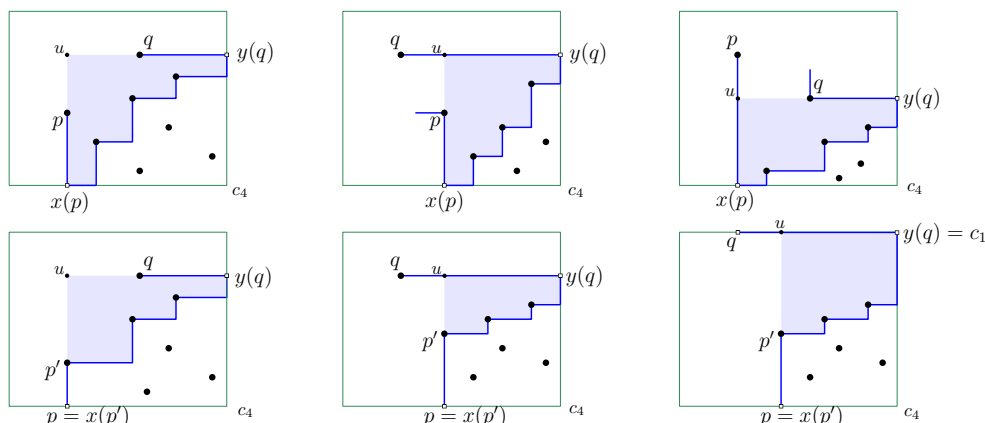
FIG. 9. *Examples of $D_{p,q}$.*

DEFINITION 5.2. *For a point $p \in P$, let $x(p)$ and $y(p)$ be the orthogonal projections of p onto the bottom and right sides of $\mathcal{R}$, respectively. Note that $x(p) \in \bar{P}_b$ and $y(p) \in \bar{P}_r$.*

For an optimal solution $\mathcal{OC}$, Lemma 2.11 also holds. Therefore, we can assume again that rectangles $B(a,d)$ and $B(b,c)$ are separated by a vertical line, so $\mathcal{OC}$ is vertically separable. We now describe how to find an optimal solution of the Max-OrthoconvexPolygon problem that is vertically separable.

The first step: In this step, we build a table D whose entries $D_{p,q}$ contain areas that are associated with some empty 4-staircases. Let p and q be a pair of points such that $p \in P \cup \bar{P}_b$ and $q \in P \cup \bar{P}_t \cup \{c_2\}$. If $p \in P$ and $q \not\prec p$, then $D_{p,q}$ is the area of the set $B(x(p), y(q)) \setminus M_4(P(x(p), y(q)))$. If $p \in \bar{P}_b$, p' is the point of P such that $x(p') = p$ and $q \not\prec p'$, then $D_{p,q}$ is the area of the set $B(p, y(q)) \setminus M_4(P(p, y(q))) \cup \{p'\}$. Figure 9 shows some examples of $D_{p,q}$.

For every pair p, q, $M_4(P(x(p), y(q))$ (or $M_4(P(p, y(q))) \cup \{p'\}$) can easily be calculated in $O(n)$ time, so also $D_{p,q}$. Therefore, table D can be filled in $O(n^3)$ time.[3]

The second step: Let $p \in P \cup \bar{P}_b \cup \{c_4\}$, $q \in P \cup \bar{P}_t \cup \{c_2\}$, and $u = (p_x, q_y)$. We define the table O with entries $O_{p,q}$ equal to the area of the orthoconvex empty polygon $\mathcal{O}_{p,q}$ of maximum area, with some restrictions on p, q, and u. Let us suppose first that $p \in P$. The value $O_{p,q}$ is the area of a maximum orthoconvex empty polygon with input points $P \cap Q_4(u)$ and domain $\mathcal{R} \cap \overline{Q_4(u)}$ such that the polygon contains p, q, and u if $p \prec q$ (see Figure 10, top-left), contains p and u if $q \prec' p$ (see Figure 10 top-middle), and contains q and u if $p \prec' q$ (see Figure 10 top-right). As we will see in Lemma 5.3, we can calculate each entry $O_{p,q}$ with $p \in P$, using the following recurrence:

$$(5.1) \qquad O_{p,q} = \max \begin{cases} D_{p,q} & \text{(A)}, \\ \mathsf{Area}(B(u,r)) + O_{r,q} \text{ over all } r \in N_{p,q} & \text{(B)}, \\ \mathsf{Area}(B(u,r)) + O_{p,r} \text{ over all } r \in N_{q,p} & \text{(C)}, \end{cases}$$

---

[3] This approach of defining the area as $B(x(p), y(q)) \setminus M_4(P(x(p), y(q)))$ can also be used in the first step of sections 4.1 and 4.2, as an alternative way to compute table C.
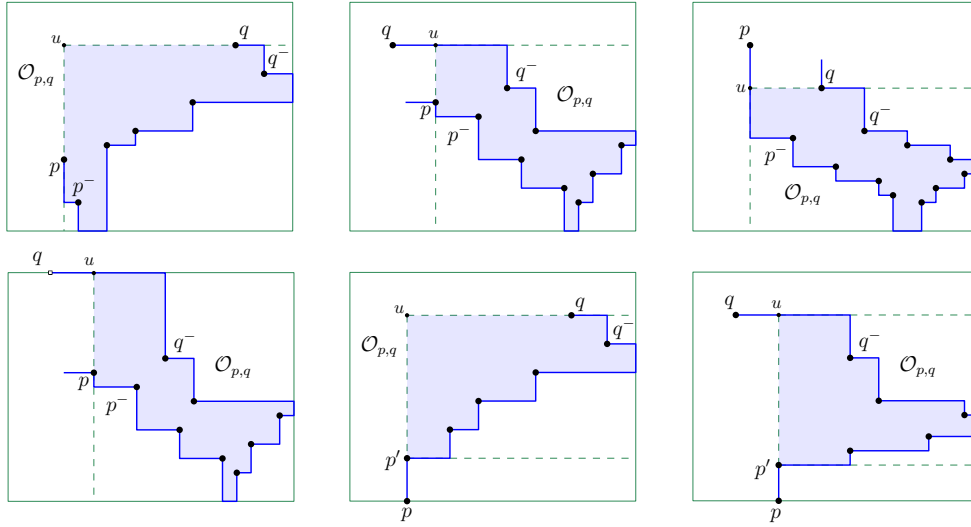
FIG. 10. *Examples of $\mathcal{O}_{p,q}$.*

where

$$N_{p,q} = \begin{cases} r \in P \cap Q_4(p) : P(u,r) = \emptyset & \text{if } p \prec q \text{ or } q \prec' p, \\ r \in P \cap Q_4(u) : P(u,r) = \emptyset & \text{if } p \prec' q, \end{cases}$$

and

$$N_{q,p} = \begin{cases} r \in P \cap Q_4(q) : P(u,r) = \emptyset & \text{if } p \prec q \text{ or } p \prec' q, \\ r \in P \cap Q_4(u) : P(u,r) = \emptyset & \text{if } q \prec' p. \end{cases}$$

Suppose that $p \in \bar{P}_b$ and let p' be the point of P such that $x(p') = p$. In this case, $O_{p,q}$ is the area of a maximum orthoconvex empty polygon with input points $P \cap Q_4(u) \cap Q_1(p')$ and domain $\mathcal{R} \cap \overline{Q_4(u)} \cap \overline{Q_1(p')}$, such that the polygon contains p', q, and u if $p' \prec q$ (see Figure 10 bottom-middle) and contains p' and u if $q \prec' p'$ (see Figure 10 bottom-right). We calculate each entry $O_{p,q}$ with $p \in \bar{P}_b$, using the following recurrence:

$$(5.2) \qquad O_{p,q} = \max \begin{cases} D_{p,q} & \text{(A)}, \\ \mathsf{Area}(B(u,r)) + O_{p,r} \text{ over all } r \in N'_{q,p} & \text{(C)}, \end{cases}$$

where

$$N'_{q,p} = \begin{cases} r \in P \cap Q_4(q) \cap Q_1(p') : P(u,r) = \emptyset & \text{if } p' \prec q, \\ r \in P \cap Q_4(u) \cap Q_1(p') : P(u,r) = \emptyset & \text{if } q \prec' p'. \end{cases}$$

Finally, suppose that $p = c_4$. For any point q, an orthoconvex polygon in region $\mathcal{R} \cap \overline{Q_4(u)}$ that contains p and u degenerates to the segment up. Thus, we define $O_{p,q} = 0$ in this case. The following lemma proves the correctness of (5.1) and (5.2).

LEMMA 5.3. *The previous recurrences correctly calculate all the $O_{p,q}$ values in $O(n^3)$ time and $O(n^2)$ space.*

Proof. Assume first that $p \in P$. Given $\mathcal{O}_{p,q}$, let $p^- \in P \cup \bar{P}_b$ be the vertex of $\mathcal{O}_{p,q}$ that follows p (or u when $p \notin \mathcal{O}_{p,q}$) in the anticlockwise traversal of the boundary
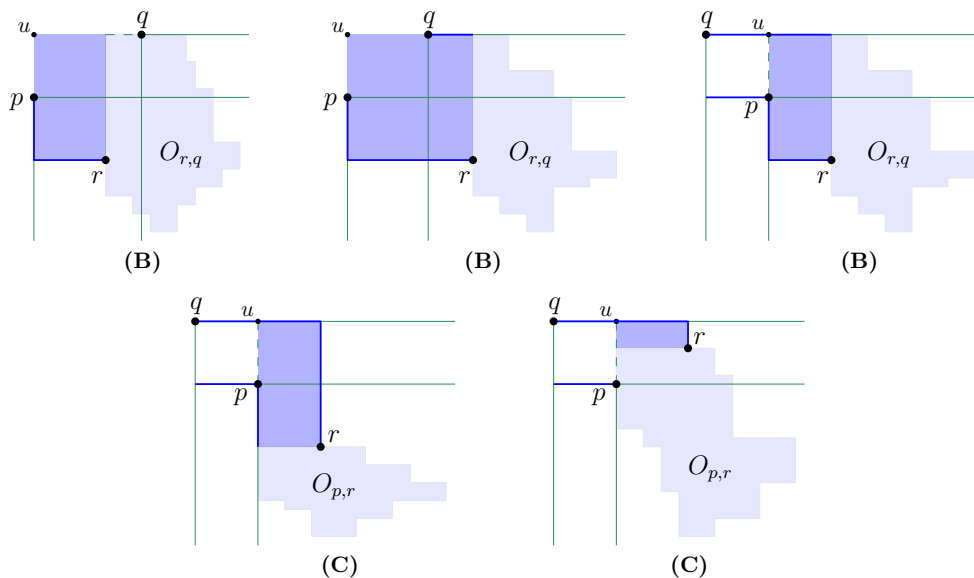
FIG. 11. *Some cases in the recursive computation of $O_{p,q}$, $p \in P$.*

of $\mathcal{O}_{p,q}$ (see Figure 10). Similarly, let $q^- \in P \cup \bar{P}_r \cup \{c_1\}$ be the vertex of $\mathcal{O}_{p,q}$ that precedes q (or u when $q \notin \mathcal{O}_{p,q}$) in the counterclockwise traversal of the boundary of $\mathcal{O}_{p,q}$ (see Figure 10). Note that we are excluding the points of $\bar{P}_t$ in the definition of q^-, although some of them could be a vertex of $\mathcal{O}_{p,q}$ when $q \in \bar{P}_t$ (see Figure 10 bottom-left). Also note that q^- can be c_1 only if $q \in \bar{P}_t$.

If $p^- \in \bar{P}_b$ and $q^- \in \bar{P}_r \cup \{c_1\}$, then $O_{p,q} = D_{p,q}$, and this fits into case (A) of (5.1). Otherwise, p^- or q^- (or both) belongs to P. Note that if $p^- \in P$, then necessarily $p^- \in N_{p,q}$, as $P(u,r) = \emptyset$ and p^- is in either $Q_4(p)$ (if $p \prec q$ or $q \prec' p$) or $Q_4(u)$ (if $p \prec' q$). Similarly, if $q^- \in P$, then $q^- \in N_{q,p}$. In the first case we have $O_{p,q} = \mathsf{Area}(B(u,p^-)) + O_{p^-,q}$, and letting $r = p^-$ this fits into case (B) of (5.1) (see Figure 11). In the second case we have $O_{p,q} = \mathsf{Area}(B(u,q^-)) + O_{p,q^-}$, and letting $r = q^-$ this fits into case (C) of (5.1).

The case in which $p \in \bar{P}_b$ is similar to prove, only using point q^-. If $q^- \in \bar{P}_r \cup \{c_1\}$, then $O_{p,q} = D_{p,q}$. If $q^- \in P$, then $q^- \in N'_{q,p}$ and $O_{p,q} = \mathsf{Area}(B(u,q^-)) + O_{p,q^-}$. Letting $r = q^-$ this fits into case (C) of (5.2). Hence, (5.1) and (5.2) correctly calculate all the $O_{p,q}$'s. □

To finish this step, let $\mathcal{O}'_{p,q}$ denote the orthoconvex polygon $\mathcal{O}_{q,p}$ after rotating the coordinates by π. Let $O'_{p,q}$ denote the area of $\mathcal{O}'_{p,q}$. Note that by symmetry, all the $O'_{p,q}$ values can also be calculated in $O(n^3)$ time.

The third step: We show now that by combining some orthoconvex polygons $\mathcal{O}_{p,q}$ and $\mathcal{O}'_{p,q}$, we can find a vertically separable orthoconvex polygon $\mathcal{OC} \subset \mathcal{R}$ of maximum area, whose interior does not contain elements of P. To this end, we next define a 2-separator. Let $p \in P \cup \bar{P}_b \cup \{c_4\}$ and $q \in P \cup \bar{P}_t \cup \{c_2\}$.

DEFINITION 5.4. *We say that (p,q) is a 2-separator of $\mathcal{OC}$ if (1) p belongs to the 3-staircase of $\mathcal{OC}$; (2) q belongs to the 1-staircase of $\mathcal{OC}$; (3) $q \prec' p$; and (4) $\mathcal{OC}$ is the union of the pairwise interior-disjoint polygons, $\mathcal{O}'_{p,q}$, $\overline{B(p,q)}$ and $\mathcal{O}_{p,q}$, where $\overline{B(p,q)}$ is the closure of the open rectangle $B(p,q)$ (see Figure 12).*
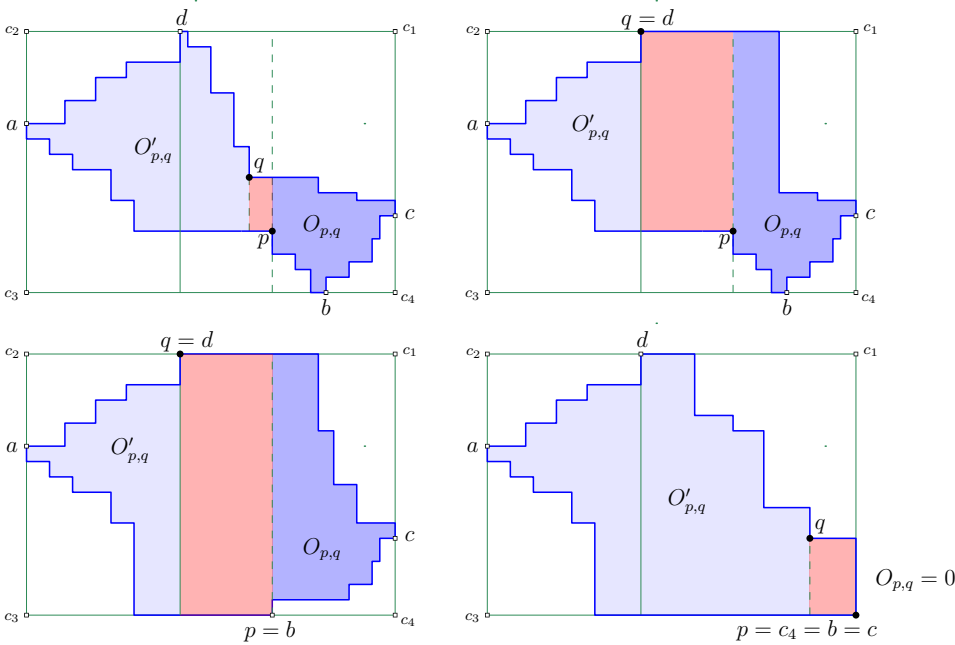
FIG. 12. *Third step of the algorithm:* 2-*separators* (p, q).

This definition implies that if the optimal solution $\mathcal{OC}$ has a 2-separator (p, q), then

$$\text{Area}(\mathcal{OC}) = O'_{p,q} + \text{Area}(B(p,q)) + O_{p,q}.$$

The next lemma establishes the equivalence between vertical separability and 2-separators.

LEMMA 5.5. $\mathcal{OC}$ *is vertically separable if and only if* $\mathcal{OC}$ *has a* 2-*separator.*

Proof. Let $\mathcal{OC} \subset \mathcal{R}$ be an orthoconvex polygon of maximum area that is vertically separable, i.e., rectangles $B(a,d)$ and $B(b,c)$ are separated by a vertical line. If there exist elements of P to the right of the vertical line ℓ_d through d that are vertices of the 3-staircase of $\mathcal{OC}$, then let p be the leftmost of them. Otherwise, let $p = b$. If there exist elements of P between ℓ_d and the vertical line through p that are vertices of the 1-staircase of $\mathcal{OC}$, then let q be the rightmost of them. Otherwise, let $q = d$. For the definitions of p and q, refer to Figure 12. It is easy to see that (p, q) is a 2-separator of $\mathcal{OC}$.

Let (p, q) be a 2 separator of $\mathcal{OC}$. Since p belongs to the 3-staircase, we have $p = b$ or $p \prec' b$. Similarly, since q belongs to the 1-staircase, we have $q = d$ or $d \prec' q$. Given that $q \prec' p$, we have by transitivity that $d \prec' b$, which implies $d_x < b_x$. Therefore, rectangles $B(a,d)$ and $B(b,c)$ are separated by a vertical line, so $\mathcal{OC}$ is vertically separable. □

Following Lemma 5.5, the algorithm to find $\mathcal{OC}$ is simple: For every $p \in P \cup \bar{P}_b \cup \{c_4\}$ and $q \in P \cup \bar{P}_t \cup \{c_2\}$, we compute the orthoconvex polygon $\mathcal{OC}_{p,q}$ of maximum area (equal to $O'_{p,q} + \text{Area}(B(p,q)) + O_{p,q}$), such that (p, q) is a 2-separator of $\mathcal{OC}_{p,q}$. The optimal solution $\mathcal{OC}$ will be one of these $\mathcal{OC}_{p,q}$ values. Summarizing over all 2-separators (p, q) with $p \in P \cup \bar{P}_b \cup \{c_4\}$ and $q \in P \cup \bar{P}_t \cup \{c_2\}$,

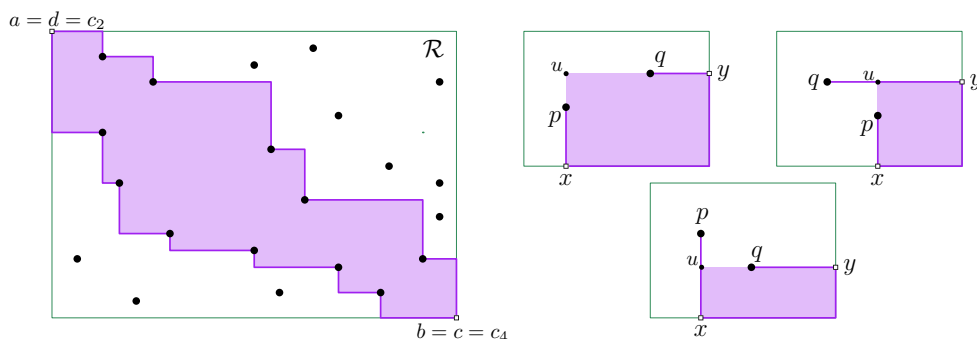$$(5.3) \qquad \text{Area}(\mathcal{OC}) = \max\{O'_{p,q} + \text{Area}(B(p,q)) + O_{p,q}\}.$$

FIG. 13. *A maximum area empty staircase polygon, and the three cases for $D_{p,q}$.*

This last step requires $O(n^2)$ time, once all the $O_{p,q}$ values and all the $O'_{p,q}$ values have been calculated. Thus, we have obtained a new $O(n^3)$ time and $O(n^2)$ space algorithm for the MaxOrthoconvexPolygon problem which is simpler than the one obtained by Nandy, Mukhopadhyaya, and Bhattacharya [13].

THEOREM 5.6 (Nandy, Mukhopadhyaya, and Bhattacharya [13]).   *The* Max-OrthoconvexPolygon *problem can be solved in $O(n^3)$ time and $O(n^2)$ space.*

6. Empty staircase polygon with the largest area. In this section, we further extend the applications of our techniques above to show that finding an empty staircase polygon with the largest area amidst the n-point set $P \subset \mathcal{R}$ (i.e., the MaxStaircasePolygon problem) can be done in $O(n^2)$ time and space. Recall that given a domain $\mathcal{R}$ and a point set $P \subset \mathcal{R}$, a maximum area empty staircase polygon is an orthoconvex polygon contained in $\mathcal{R}$ with no point of P in the interior, that includes two opposed corners of $\mathcal{R}$ as vertices. See Figure 13, left.

The algorithm in this section consists of two steps only: In the first step, some empty rectangles are defined. In the second step, we build some empty staircase polygons of maximum area for some pairs of points. One of these polygons will be the optimal solution for the MaxStaircasePolygon problem.

The first step: Let $\hat{P} = P \cup \{c_2\}$. For every $p, q \in \hat{P}$ such that $q \not\prec p$, we redefine $D_{p,q}$ to denote the $\mathsf{Area}(B(x(p), y(q)))$ when the open rectangle $B(x(p), y(q))$ does not contain elements of P (see Figure 13, right) and $D_{p,q} = -\infty$ otherwise. Recall that $u = (p_x, q_y)$. Let D be the table containing all of the values $D_{p,q}$. Note that we do not need to explicitly compute D, since each entry can be computed in $O(1)$ time on demand.

The second step: Consider a new table O, whose entry $O_{p,q}$ is the area of a maximum area empty staircase polygon $\mathcal{O}_{p,q}$ defined for (1) $p = q = c_2$; (2) $p, q \in P$, $p \prec q$, $P(p,q) = \emptyset$; and (3) $p, q \in \hat{P}$, $q \prec' p$ or $p \prec' q$. Recall that these empty staircase polygons $\mathcal{O}_{p,q}$ are built with input points $P \cap Q_4(u)$ and domain $\mathcal{R} \cap \overline{Q_4(u)}$ such that the polygon contains p, q, and u if $p \prec q$, contains p and u if $q \prec' p$, and contains q and u if $p \prec' q$. See Figure 14 for some examples.

When $p \prec q$, note that $\mathcal{O}_{p,q}$ always contains $B(p,q)$ in its interior, hence necessarily $P(p,q) = \emptyset$ so that $\mathcal{O}_{p,q}$ is defined. This is the reason why condition $P(p,q) = \emptyset$ was added to item (2) in the definition. This condition has more implications when calculating all the values $\mathcal{O}_{p,q}$. Suppose that $p \prec q$ and take a point $r \in R_{p \setminus q}$ (see Figure 15(b), where r_p in the figure plays the role of r). If $\mathcal{O}_{p,q}$ is obtained from $\mathcal{O}_{r,q}$ and $B(u,r)$, then necessarily $P(u,r) = \emptyset$, implying that $P(p,r) = \emptyset$. On the
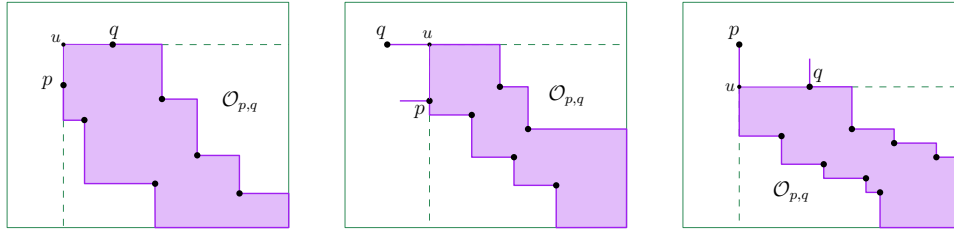
FIG. 14. *Some examples of $\mathcal{O}_{p,q}$.*

other hand, in $\mathcal{O}_{r,q}$ the set $P(q,r)$ must be necessarily empty. By adding these two restrictions $P(u,r) = P(q,r) = \emptyset$ when calculating $\mathcal{O}_{p,q}$, we reduce the number of points r to consider, as there exists only one point r in $R_{p \setminus q}$ such that $P(p,r)$ and $P(q,r)$ are empty at the same time.

Using a detailed case analysis, the following recurrence including restrictions on the emptiness of several subsets, allows us to calculate all the values $O_{p,q}$ in $O(n^2)$ time:

$$(6.1) \qquad O_{p,q} = \max \begin{cases} D_{p,q} & \text{(A)}, \\ \mathsf{Area}(B(u,r)) + O_{r,q} \text{ over all } r \in M_{p,q} & \text{(B)}, \\ \mathsf{Area}(B(u,r)) + O_{p,r} \text{ over all } r \in M_{q,p} & \text{(C)}, \end{cases}$$

where

$$M_{p,q} = \begin{cases} r \in P \cap Q_4(p) : P(u,r) = P(q,r) = \emptyset & \text{if } p \prec q, \\ r \in P \cap Q_4(p) : P(u,r) = \emptyset & \text{if } q \prec' p, \\ r \in P \cap Q_4(u) : P(u,r) = P(q,r) = \emptyset & \text{if } p \prec' q \end{cases}$$

and

$$M_{q,p} = \begin{cases} r \in P \cap Q_4(q) : P(u,r) = P(p,r) = \emptyset & \text{if } p \prec q, \\ r \in P \cap Q_4(u) : P(u,r) = P(p,r) = \emptyset & \text{if } q \prec' p, \\ r \in P \cap Q_4(q) : P(u,r) = \emptyset & \text{if } p \prec' q. \end{cases}$$

We remark that item (2) $p, q \in P$, $p \prec q$, $P(p,q) = \emptyset$ could be replaced by (2) $p, q \in P$, $p \prec q$, as the previous recurrence assigns the value $-\infty$ to $O_{p,q}$ (meaning that $\mathcal{O}_{p,q}$ does not exist) when $p \prec q$ and $P(p,q) \neq \emptyset$. In the definition of $M_{p,q}$ and $M_{q,p}$ when $p \prec q$, observe that for a point $r \in M_{p,q} \cup M_{q,p}$ we have $P(u,r) = P(q,r) = P(p,r) = P(p,q) = \emptyset$. In particular, this implies that if $P(p,q) \neq \emptyset$, then $M_{p,q} \cup M_{q,p} = \emptyset$. Thus, when applying (6.1) to the pair $p \prec q$ with $P(p,q) \neq \emptyset$, only case (A) applies, as $M_{p,q} \cup M_{q,p} = \emptyset$. Since $B(x(p), y(q))$ contains points of P, then $O_{q,p} = D_{q,p} = -\infty$ by definition.

The correctness of (6.1) follows in a similar way as (5.1): If a point $p^- \in P$ follows p (or u) in the 3-staircase of $\mathcal{O}_{p,q}$, then $\mathcal{O}_{p,q}$ is obtained from $\mathcal{O}_{p^-,q}$ and $B(u,p^-)$ (case (B)). In the same way, if a point $q^- \in P$ precedes q (or u) in the 1-staircase of $\mathcal{O}_{p,q}$, then $\mathcal{O}_{p,q}$ is obtained from $\mathcal{O}_{p,q^-}$ and $B(u,q^-)$ (case (C)). If such points do not belong to P, then $\mathcal{O}_{p,q}$ is $B(x(p), y(p))$ (case (A)).

Observe that O_{c_2,c_2} is the area of the maximum-area empty staircase polygon amidst P, so we only need to show that (6.1) calculates all the values $O_{p,q}$ in $O(n^2)$ time, to solve the MaxStaircasePolygon problem with this complexity.

For every $p \in \hat{P}$, let $L_i(p)$, $i \in \{1,2,3,4\}$, be the sequence of the points $t \in \hat{P} \cap Q_i(p)$ sorted by increasing x-coordinate such that $P(p,t) = \emptyset$. Since P is already sorted by x-coordinate, the sequence $L_i(p)$ can be computed in $O(n)$ time for each point p, thus $L_i(p)$ is computed in $O(n^2)$ time and space.
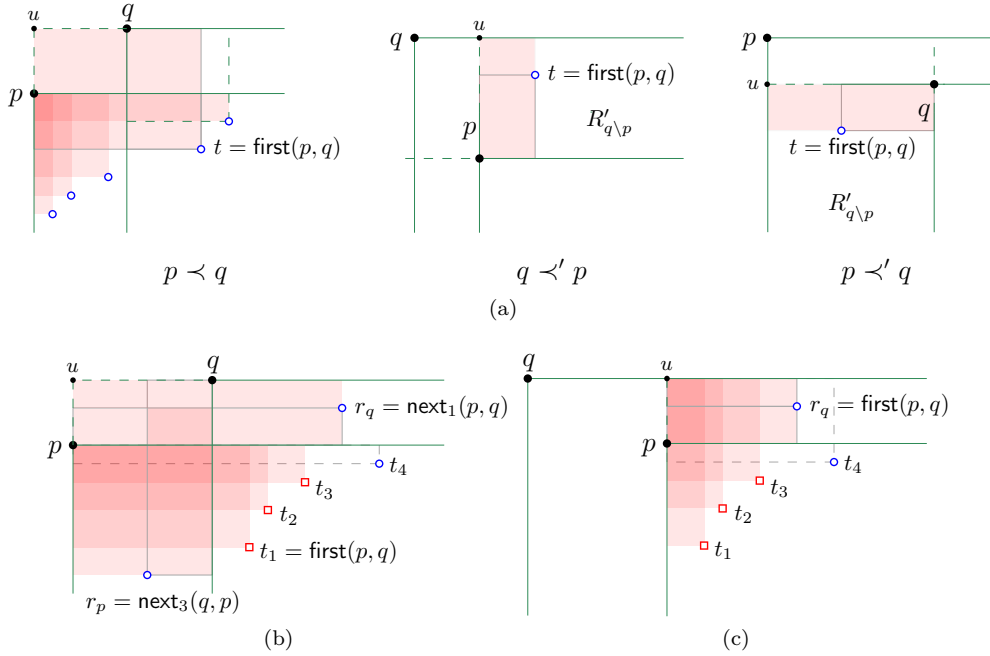
FIG. 15. (a) *The definition of* $\mathsf{first}(p,q)$. *(b) The case* $p \prec q$ *in the proof of Lemma* 6.1, *where* $r_p \neq \mathbf{nil}$ *and* $r_q \neq \mathbf{nil}$: *The points of* $L_4(p)$ *that are in* $R_{p,q}$ *are* $t_1 = \mathsf{first}(p,q)$, t_2, t_3, *and* t_4, *and* $R_{p,q} \cap M_{p,q} = R_{p,q} \cap M_{q,p} = \{t_1, t_2, t_3\}$. *Note that* t_4 *is not present in this set because* $r_q \in P(q,t_4)$. *(c) The case* $q \prec' p$ *in the proof of Lemma* 6.1, *where* $r_q \neq \mathbf{nil}$: *The points of* $L_4(p)$ *that are in* $R_{p,q} \cap M_{p,q} = R_{p,q} \cap M_{q,p}$ *are* $\{t_1, t_2, t_3\}$.

Let $K \subset \hat{P} \times \hat{P}$ be the set of point pairs (p,q) such that the entry $O_{p,q}$ is defined. The next lemma shows that for a pair $(p,q) \in K$, the complexity of calculating the sets $M_{p,q}$ and $M_{q,p}$ is linear in their size, after an $O(n^2)$-time preprocessing.

LEMMA 6.1. *For each* $(p,q) \in K$, *the sets* $M_{p,q}$ *and* $M_{q,p}$ *can be computed, respectively, in* $O(|M_{p,q}|)$ *and* $O(|M_{q,p}|)$ *time, after an* $O(n^2)$-*time preprocessing.*

Proof. We give an $O(n^2)$-time preprocessing to compute some special points that we call $\mathsf{first}(p,q)$. For every $t \in L_i(p)$, let $\mathsf{next}_i(p,t)$ be the point that goes after t in $L_i(p)$. If t is the last element, then $\mathsf{next}_i(p,t) = \mathbf{nil}$. We define $\mathsf{first}(p,q)$ for the next pairs $p, q \in \hat{P}$ as follows (see Figure 15(a)):

- $p \prec q$ and $P(p,q) = \emptyset$: Let $\mathsf{first}(p,q)$ be the leftmost point t in $L_4(p)$ such that $q \prec' t$ and $P(q,t) = \emptyset$. Note that q is in $L_1(p)$ and, by simultaneously traversing $L_1(p)$ and $L_4(p)$, we can compute $\mathsf{first}(p,q)$ in $O(n)$ time for fixed p and all q such that $p \prec q$.
- $q \prec' p$: Let $\mathsf{first}(p,q)$ be the leftmost point t in $R'_{q\backslash p}$, which ensures $P(u,t) = P(p,t) = \emptyset$. Recall that when $q \prec' p$, $R'_{q\backslash p}$ is the subset of P in the region $Q_4(q) \cap Q_1(p)$. Note that t is in $L_1(p)$ and, by simultaneously traversing the y-ordering of $\hat{P}$ and $L_1(p)$, we can compute $\mathsf{first}(p,q)$ in $O(n)$ time for fixed p and all $q \prec' p$.
- $p \prec' q$: Let $\mathsf{first}(p,q)$ be the topmost point t in $R'_{q\backslash p}$, which ensures $P(u,t) = P(q,t) = \emptyset$. Recall that when $p \prec' q$, $R'_{q\backslash p}$ is the subset of P in the region $Q_4(p) \cap Q_3(q)$. Note that t is in $L_3(q)$ and, by simultaneously traversing the

x-ordering of $\hat{P}$ and $L_3(q)$, we can compute $\mathsf{first}(p,q)$ in $O(n)$ time for fixed q and all $p \prec' q$.

In any of the above three cases, if the point t does not exist, then $\mathsf{first}(p,q) = \textbf{nil}$. Observe that we can compute $\mathsf{first}(p,q)$ for all p,q such that $\mathsf{first}(p,q)$ is defined in $O(n^2)$ time overall.

We now show how to compute $M_{p,q}$ and $M_{q,p}$ in $O(|M_{p,q}|)$ and $O(|M_{q,p}|)$ time, respectively. Consider the case where $p \prec q$, in which $P(p,q) = \emptyset$. Recall that for a point $r \in M_{p,q} \cup M_{q,p}$, we have $P(u,r) = P(q,r) = P(p,r) = P(p,q) = \emptyset$. Let $r_p = \mathsf{next}_3(q,p)$ and $r_q = \mathsf{next}_1(p,q)$, and assume without loss of generality that $r_p \neq \textbf{nil}$ and $r_q \neq \textbf{nil}$ (see Figure 15(b)).

If there exists a point $r \in R_{p \backslash q} \cap M_{p,q}$, then the condition $P(p,r) = P(q,r) = \emptyset$ implies $r = r_p$. Similarly, $r \in R_{q \backslash p} \cap M_{q,p}$ implies $r = r_q$. Now, observe that $P(u,r) = \emptyset$ also implies that $R_{p,q} \cap M_{p,q} = R_{p,q} \cap M_{q,p}$. Moreover, if there are elements in these two equal sets, then they appear in $L_4(p)$ as consecutive elements, from $\mathsf{first}(p,q)$ to the right, until the last element in $L_4(p)$ to the left of the vertical line passing through r_q. Then, all of the observations together with the fact that we check $P(u,r) = P(p,r) = P(q,r) = \emptyset$ in $O(1)$ time, allow us to calculate $M_{p,q}$ and $M_{q,p}$ in $O(|M_{p,q}|)$ and $O(|M_{q,p}|)$ time, respectively. Note that r_p and $\mathsf{first}(p,q)$ are consecutive in $L_4(p)$.

Suppose now that $q \prec' p$; see Figure 15(c). Let $r_q = \mathsf{first}(p,q)$ and assume without loss of generality that $r_q \neq \textbf{nil}$. If there exists a point $r \in R'_{q \backslash p} \cap M_{q,p}$, then the condition $P(u,r) = P(p,r) = \emptyset$ implies $r = r_q$. Furthermore, the condition $P(u,r) = \emptyset$ implies $R'_{p,q} \cap M_{p,q} = R'_{p,q} \cap M_{q,p}$, where the elements of this set form a prefix of $L_4(p)$ until the last element in $L_4(p)$ to the left of the vertical line passing through r_q. Hence, we can calculate $M_{p,q}$ and $M_{q,p}$ in $O(|M_{p,q}|)$ and $O(|M_{q,p}|)$ time, respectively. The case $p \prec' q$ is symmetric. □

The following lemma proves that the number of triples (p,q,r) such that $O_{p,q}$ is defined and that $r \in M_{p,q} \cup M_{q,p}$ is $O(n^2)$.

LEMMA 6.2. $\sum_{(p,q) \in K} (|M_{p,q}| + |M_{q,p}|) \in O(n^2)$.

Proof. $\sum_{(p,q) \in K} (|M_{p,q}| + |M_{q,p}|)$ counts the number of triples (p,q,r) such that $O_{p,q}$ is defined and $r \in M_{p,q} \cup M_{q,p}$. If $p \prec q$ with $P(p,q) = \emptyset$, then we have (1) if $r \in R_{p \backslash q} \cap M_{p,q}$, then p and r are consecutive in $L_3(q)$; (2) if $r \in R_{p,q} \cap M_{p,q} = R_{p,q} \cap M_{q,p}$, then p and q are consecutive in $L_2(r)$; (3) if $r \in R_{q \backslash p} \cap M_{q,p}$, then q and r are consecutive in $L_1(p)$. Since there are n choices for a point $s \in P$, and for each s there are at most $O(n)$ choices for two consecutive points in any $L_i(s)$, there are at most $O(n^2)$ triples (p,q,r) with $p \prec q$.

Consider $q \prec' p$, and define $U = \{(s,t) \in \hat{P} \times P \mid s \prec' t\}$. If $r \in R'_{q \backslash p} \cap M_{q,p}$, we charge the triplet (p,q,r) to $(q,p) \in U$. Condition $P(u,r) = P(p,r) = \emptyset$ implies that r is unique for the combination q,p. Otherwise, if $r \in R'_{p,q} \cap M_{p,q} = R'_{p,q} \cap M_{q,p}$, then we charge the triplet (p,q,r) to $(q,r) \in U$, where $P(u,r) = \emptyset$ implies that p is unique for the combination q,r. Then, each element of U is charged to at most two triples, and this ensures that there are at most $O(|U|) = O(n^2)$ triples for $q \prec' p$. Counting for the case $p \prec' q$ is symmetric. Thus the lemma follows. □

The combination of (6.1) and Lemmas 6.1 and 6.2 allows us to calculate table O in $O(n^2)$ time. We can also find an empty staircase polygon with the largest area O_{c_2,c_2} in $O(n^2)$ time and space. Thus, we have given a new and simpler algorithm for the MaxStaircasePolygon problem.

THEOREM 6.3 (Nandy and Bhattacharya [12]). *The* MaxStaircasePolygon *problem can be solved in $O(n^2)$ time and space.*

To conclude, we remark that the approach given in this section cannot be applied to the MaxOrthoconvexPolygon problem described in section 5. For example, when $p \prec q$, necessarily $P(p,q) = \emptyset$ in an empty staircase polygon. However, in an empty orthoconvex polygon this property is not true anymore, as the 4-staircase of the polygon can cross $B(p,q)$ using points belonging to $P(p,q)$. Therefore, the conditions $P(u,r) = P(q,r) = \emptyset$ and $P(u,r) = P(p,r) = \emptyset$ cannot be added to the definition of $N_{p,q}$ and $N_{q,p}$, respectively, because otherwise they would imply that $P(p,q) = \emptyset$.

Summary of notation. The following table summarizes the main notation used. Horizontal lines denote a change of (sub)section implying a change on the meaning of a notation.

Notation	Description	Page
P	Point set in general position in the plane	145
a, b, c, d	Leftmost, bottommost, rightmost, and topmost points of P	148
p_x, p_y	x- and y-coordinates of a point p	148
$p \prec q$	Denotes that $p_x < q_x$ and $p_y < q_y$	148
$p \prec' q$	Denotes that $p_x < q_x$ and $p_y > q_y$	148
$Q_1(p)$	Open axis-aligned quadrant $\{q \in \mathbb{R}^2 \mid p \prec q\}$	148
$Q_2(p)$	Open axis-aligned quadrant $\{q \in \mathbb{R}^2 \mid q \prec' p\}$	148
$Q_3(p)$	Open axis-aligned quadrant $\{q \in \mathbb{R}^2 \mid q \prec p\}$	148
$Q_4(p)$	Open axis-aligned quadrant $\{q \in \mathbb{R}^2 \mid p \prec' q\}$	148
$M_i(P)$	Union $\bigcup_{p \in P} \overline{Q_i(p)}$	148
$RCH(P)$	Rectilinear convex hull of P, defined as $\bigcap_{i=1,2,3,4} M_i(P)$	149
$B(u,v)$	Smallest open axis-aligned rectangle containing u and v	149
$P(u,v)$	Set of points in P that belong to $B(u,v)$, i.e., $P \cap B(u,v)$	149
$R_{p\backslash q}$	For $p \prec q$, subset of P in the region $Q_4(p) \setminus \overline{Q_4(q)}$	150
$R_{q\backslash p}$	For $p \prec q$, subset of P in the region $Q_4(q) \setminus \overline{Q_4(p)}$	150
$R_{p,q}$	For $p \prec q$, subset of P in the region $Q_4(p) \cap Q_4(q)$	150
$R'_{p\backslash q}$	For $q \prec' p$, subset of P in the region $Q_4(q) \cap Q_3(p)$	150
$R'_{q\backslash p}$	For $q \prec' p$, subset of P in the region $Q_4(q) \cap Q_1(p)$	150
$R'_{p,q}$	For $q \prec' p$, subset of P in the region $Q_4(p)$	150
$\mathcal{C}_{p,q}$	For $p \prec q$ or $p = q$, a 2-staircase with endpoints p and q of maximum size	150
$C_{p,q}$	Number of elements of P in $\mathcal{C}_{p,q}$	150
$Z(p,q)$	Fourth quadrant associated with $u = (p_x, q_y)$	151
$z(p,q)$	Points of P in $Z(p,q)$, i.e., $= Z(p,q) \cap P$	151
$\mathcal{T}_{p,q}$	Triple 1-staircase of maximum cardinality among all $S \cup \{p,q\}$ with $S \subseteq z(p,q)$	151
$X_{p,q}$	Set of extreme vertices of $\mathcal{T}_{p,q}$	151
$T_{p,q}$	Cardinality of $X_{p,q}$	151
T	Table storing values $T_{p,r}$	151
$U_{p,r}$	$\max\{T_{r,s}\}$ over all $s \in R'_{p\backslash r}$	151
U	Table storing values $U_{p,r}$	151
$\mathcal{T}'_{p,q}$	Triple 3-staircase of maximum cardinality	152
$T'_{p,q}$	Size of $\mathcal{T}'_{p,q}$	152
$\mathcal{S}_{p,s}$	For $p_x < s_x$, rectilinear convex hull of maximum size	154
$S_{p,s}$	Size of $\mathcal{S}_{p,s}$	154
$\mathcal{C}_{p,q}$	Empty 2-staircase of maximum cardinality	155
$\mathcal{O}_{\mathcal{T}}$	Region $\overline{Z(p,q)} \cap M_1(S \cup \{p,q\}) \cap M_2(S \cup \{p,q\}) \cap M_3(S \cup \{p,q\})$	156
$\mathcal{T}_{p,q}$	Empty triple 1-staircase of maximum size	156
$E(p,q,r)$	Interior of $B(p,q) \cap Q_2(r)$ if $p \neq q$, the empty set if $p = q$	156
$T_{p,q}$	Cardinality of the extreme vertices of $\mathcal{T}_{p,q}$	156
$\mathcal{T}'_{p,q}$	Empty triple 3-staircase	156
$\mathcal{S}_{p,s}$	For $p_x < s_x$, empty rectilinear convex hull of maximum size	156

$\mathcal{C}_{p,q}$	As in the previous appearance, but maximizing area instead of size	157
$\mathcal{T}_{p,q}$	As in the previous appearance, but maximizing area instead of size	157
$\mathcal{T}'_{p,q}$	As in the previous appearance, but maximizing area instead of size	157
$\mathcal{S}_{p,q}$	As in the previous appearance, but maximizing area instead of size	157
$T_{p,q}$	Area of $\mathcal{T}_{p,q}$	157
$S_{p,s}$	Area of $\mathcal{S}_{p,s}$	158
$\mathcal{C}_{p,q}$	As in the first appearance, but maximizing weight	158
$\mathcal{T}_{p,q}$	As in the first appearance, but maximizing weight	158
$\mathcal{T}'_{p,q}$	As in the first appearance, but maximizing weight	158
$\mathcal{S}_{p,q}$	As in the first appearance, but maximizing weight	158
$C_{p,q}$	Weight of $\mathcal{C}_{p,q}$	159
$T_{p,q}$	Weight of $\mathcal{T}_{p,q}$	159
$S_{p,s}$	Weight of $\mathcal{S}_{p,s}$	159
$\mathcal{R}$	Axis-aligned rectangle in the plane	160
c_1, c_2, c_3, c_4	Top-right, top-left, bottom-left, and bottom-right vertices of $\mathcal{R}$, respectively	160
$\bar{P}_t, \quad \bar{P}_\ell, \quad \bar{P}_b, \\ \bar{P}_r$	Orthogonal projections of P onto the (resp.) top, left, bottom, right side of $\mathcal{R}$	160
$\bar{P}$	Union $\bar{P}_t \cup \bar{P}_\ell \cup \bar{P}_b \cup \bar{P}_r \cup \{c_1, c_2, c_3, c_4\}$	160
$D_{p,q}$	Areas associated with some empty 4-staircases	161
$\mathcal{O}_{p,q}$	Orthoconvex empty polygon of maximum area	161
$O_{p,q}$	Area of $\mathcal{O}_{p,q}$	161
$\mathcal{O}_{p,q}$	Maximum-area empty staircase polygon	165
$L_i(p)$	Sequence of the points $t \in \hat{P} \cap Q_i(p)$ such that $P(p,t) = \emptyset$	166

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