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A two-level method for isogeometric discretizations based on multiplicative Schwarz iterations

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ABSTRACT

Isogeometric Analysis (IGA) is a computational technique for the numerical approximation of partial differential equations (PDEs). This technique is based on the use of spline-type basis functions, that are able to hold a global smoothness and allow to exactly capture a wide set of common geometries. The current rise of this approach has encouraged the search of fast solvers for isogeometric discretizations and nowadays this topic is receiving a lot of attention. In this framework, a desired property of the solvers is the robustness with respect to both the polynomial degree p and the mesh size h. For this task, in this paper we propose a two-level method such that a discretization of order p is considered in the first level whereas the second level consists of a linear or quadratic discretization. On the first level, we suggest to apply one single iteration of a multiplicative Schwarz method. The choice of the block-size of such an iteration depends on the spline degree p, and is supported by a local Fourier analysis (LFA). At the second level one is free to apply any given strategy to solve the problem exactly. However, it is also possible to get an approximation of the solution at this level by using an h-multigrid method. The resulting solver is efficient and robust with respect to the spline degree p. Finally, some numerical experiments are given in order to demonstrate the good performance of the proposed solver.

1. Introduction

The IGA technique was firstly introduced by Tom Hughes et al. in the seminal paper [1] in order to integrate the finite element method (FEM) with the computer aided geometric design. This analysis consists of using spline-type basis functions for the representation of the physical domain, as well as for the numerical approximation of the solution of PDEs. These functions are globally smooth providing up to C^{p-1} continuity of the solution, where p denotes the polynomial degree.

Given that the isogeometric discretizations of PDEs yield stiffness matrices whose number of non-zero entries per row grows as p is increased, the search of a robust solver with respect to the spline degree p is not an easy task. However, it is of great interest to obtain efficiently the solution of isogeometric discretizations when high spline degrees are considered. Firstly, in [2,3] a study of the computational efficiency of direct and iterative solvers for IGA, respectively, was performed, and since then, the design of iterative solvers for isogeometric discretizations has attracted a lot of attention. For example, a multilevel BPX-preconditioner was developed in [4] for isogeometric analysis. Beirão da Veiga et al. analyzed overlapping Schwarz methods for IGA in [5], whereas in [6] they studied BDDC preconditioners by introducing appropriate discrete norms. Algebraic multilevel iteration (AMLI) methods were applied for the isogeometric discretization of scalar second order elliptic problems in [7], and preconditioners based on fast solvers for the Sylvester equation were proposed in [8]. In the framework of multigrid techniques, different types of smoothers have been proposed to avoid the troubles encountered by standard relaxation procedures. In [9] a preconditioned Krylov smoother at the finest level was considered and in [10] the authors proposed a multigrid solver based on a mass matrix smoother. In both cases, an increase in the number of smoothing iterations was needed in order to obtain robustness with respect to the spline degree. To avoid the lack of robustness of the mass smoother, in [11] a new version of such a relaxation including a boundary correction was presented. However, the extension of that version to three dimensions was not clear, and therefore, in [12], the authors proposed a multigrid smoother based on an additive subspace correction technique. In such approach, a different smoother is applied to each of the subspaces: in the regular interior subspace a mass smoother is considered, whereas in the other subspaces they proposed to use relaxations which exploit the particular structure of the

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subspaces. Also p-multigrid methods have been applied for solving IGA. In [13] the authors apply a p-multigrid method based on an ILUT (Incomplete LU factorization based on a dual Threshold strategy) smoother and compare this approach with h-multigrid methods based on the same smoother. Recently, we have proposed in [14] a very simple robust and efficient geometric multigrid algorithm based on a $V(1,0)$-cycle with overlapping multiplicative Schwarz-type methods as smoothers for solving IGA. The key for the robustness of the algorithm with respect to the spline degree is the choice of larger blocks within the Schwarz smoother when the spline degree grows up.

The main contribution of this work is to propose a robust two-level method for solving a target isogeometric discretizacion of order p, such that a linear/quadratic discretization is considered at the second level depending on the parametrization of the physical domain. At the first level, we apply only one iteration of a suitable overlapping multiplicative Schwarz method. Then, a restriction operator is constructed via projection of the B-spline basis functions between the corresponding approximation spline spaces of the target degree p and $p=1$ or $p=2$. At this point, the prolongation operator is defined as the adjoint of the restriction operator. For solving exactly the system arising on the second level there exist well-known solution techniques. However, one can also obtain an approximation of the solution at the second level by using few steps of an iterative method. In this work, we propose to apply an h-multigrid on the coarse level. More concretely, one single iteration of a $F(1,1)$-cycle that uses a red-black Gauss-Seidel smoother provides very good results. Moreover, a further improvement of the algorithm can be achieved by using a more aggressive coarsening strategy. In addition to reduce the spline degree from p to 1 or 2, we propose to apply standard coarsening (from h to $2h$) from the first to the second level. The proposed two-level algorithm presents certain advantages over the geometric multigrid introduced in [14], since it provides identical performance but reducing the computational cost.

The proposed two-level method is theoretically studied by a local Fourier analysis. This analysis, introduced by Achi Brandt in [15,16], is the main quantitative analysis for the convergence of multilevel algorithms, and results in a very useful tool for the design of this type of methods. Moreover, in [17] it has been recently proved that under standard assumptions LFA is a rigorous analysis, providing the exact asymptotic convergence factors of the method. LFA has been successfully applied to isogeometric discretizations in [14] in order to analyze the convergence of an h-multigrid method based on multiplicative Schwarz smoothers. In particular, an analysis for any spline degree p and an arbitrary size of the blocks in the smoother is provided in such work. Here, such an analysis is used to choose for each spline degree p the block-size in the multiplicative Schwarz iteration on the first level that provides a robust two-level algorithm. Thus, this analysis theoretically supports the convergence of the proposed two-level method. Furthermore, LFA can be also performed to analyze the version of the algorithm in which we approximate the solution at the second level by using an h-multigrid method. In that case, a three-grid local Fourier analysis has to be considered in order to take into account the approximation on the second level instead of an exact solve. Finally, again a two-grid LFA is applied to support the enhancement of the algorithm by considering a standard coarsening strategy between the first and the second level.

It is not the first time that a two-level method is proposed for high-order discretizations. In the framework of discontinuous Galerkin (DG) methods, in [18] it was theoretically proved that a suitable additive Schwarz method provides uniform convergence with respect to all the discretization parameters, i.e. the mesh size, the polynomial order and the penalization coefficient appearing in the DG bilinear form. However, in such a work, the block-size of the appropriate additive Schwarz iteration is not provided and here we support its choice by a suitable local Fourier analysis.

The rest of the paper is structured as follows: In Section 2 a brief introduction to the isogeometric analysis is given. Also, we state here a model problem and the basics of B-splines and NURBS. Section 3 is devoted to the presentation of the proposed two-level method. The algorithm, together with its components, is introduced in Section 3.1; the approach in which an h-multigrid is applied on the coarse level is explained in Section 3.2; and finally an improvement of the two-level method based on an aggressive coarsening is presented in Section 3.3. In Section 4, we develop the corresponding LFA in order to support the design of our solver. We perform the LFA for the three versions of the method and we present the corresponding results. Section 5 presents a comparison between the proposed two-level algorithm and the h-multigrid method introduced in [14]. In Section 6 three numerical experiments show the good performance of the proposed two-level method. Finally, Section 7 summarizes the main results of this work and draws some conclusions.

2. Isogeometric analysis

Let us consider the Poisson equation in a d-dimensional domain Ω with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

In (1), the physical domain Ω is the image of the so-called parametric domain $\widehat{\Omega} = (0,1)^d$ under a geometrical mapping $\mathbf{F} : \widehat{\Omega} \to \Omega$.
The variational formulation of our model problem (1) reads as follows: Find $u \in H_0^1(\Omega)$ such that $a(u,v) = (f,v), \forall v \in H_0^1(\Omega)$, where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \text{and} \quad (f,v) = \int_\Omega f v \, \mathrm{d}x.$$

The Galerkin approximation of the variational problem is given by: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{2}$$

where V_h is a finite dimensional space. In the isogeometric framework, V_h is a given space of splines whose global smoothness might vary depending on the refinement strategy [1]. In this work, we will consider spline spaces of degree p holding maximum continuity, that is, C^{p-1} regularity. These spaces are given by

$$S_h^p(\Omega) = \operatorname{span}\{N_i^p, i = 1, \dots, n_h^p\}, \tag{3}$$

where $N_i^p = \widehat{N}_i^p \circ \mathbf{F}^{-1}$ are the d-variate B-spline/NURBS basis functions defined on Ω, given in terms of the basis functions on the parametric domain $\widehat{N}_i^p$, which will be described below, and the corresponding geometrical mapping $\mathbf{F}$ previously mentioned, and n_h^p is the number of these basis functions. The solution of (2), which we will denote as u_h^p to emphasize the dependence on the spline degree p, can be expanded as a linear combination of the considered spline basis functions. That is,

$$u_h^p = \sum_{i=1}^{n_h^p} u_i^p N_i^p,$$

where the coefficients u_i^p are the control points associated to the numerical solution. These coefficients can be computed by solving the linear system $A_p u_p = b_p$, where the stiffness matrix is given by $A_p = (a_{i,j}) = (a(N_j^p, N_i^p))_{i,j=1}^{n_h^p}$ and the right hand side is $b_p = (f, N_i^p)_{i=1}^{n_h^p}$.

2.1. B-spline basis functions

Isogeometric analysis is based on B-spline basis functions that are constructed parametrically. Let us recall that the parametric domain is taken as $\widehat{\Omega} = (0,1)^d$, where d is the number of parametric directions. Then, a non-decreasing sequence of real numbers called knot vector is required to determine this parametrization on each direction. Thus, let us start by considering an open knot vector Ξ_p for the one-dimensional case

$$\Xi_p = \{\xi_1 = \ldots = \xi_{p+1} = 0 < \xi_{p+2} < \ldots \xi_{p+m} < 1 = \xi_{p+m+1} = \ldots = \xi_{2p+m+1}\}.$$

For the case $p = 0$, the univariate piecewise constant splines are step functions with support on the corresponding knot span. That is, for $i = 1, \ldots, m$,

$$\widehat{N}_i^0(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1}, \\ 0 & \text{otherwise}. \end{cases} \quad (4)$$

Then, for every pair (k, i) such that $1 \le k \le p$, $1 \le i \le m + 2p - k$, the basis functions $\widehat{N}_i^k : [0, 1] \to \mathbb{R}$ are given recursively by the Cox-de-Boor formula:

$$\widehat{N}_i^k(\xi) = \frac{\xi - \xi_i}{\xi_{i+k} - \xi_i} \widehat{N}_i^{k-1}(\xi) + \frac{\xi_{i+k+1} - \xi}{\xi_{i+k+1} - \xi_{i+1}} \widehat{N}_{i+1}^{k-1}(\xi), \quad (5)$$

in which fractions of the form $0/0$ are considered as zero. For more details, we refer to the reader to [19].
For higher spatial dimensions, that is $d > 1$, both parameter space and basis functions are built by tensorization. For instance, in the two-dimensional case, that we consider in this work, the B-spline basis functions are constructed over the tensor product of two knot-vectors

$$\Xi_p \times \Xi_p = \{(\xi, \eta), \xi \in \Xi_p, \eta \in \Xi_p\}.$$

Note that for the sake of simplicity we are taking the same spline degree p and the same set of knot vector, but this is not restrictive. Hence, a bivariate B-spline basis function $\widehat{N}_{i,j}^p$ is given by means of tensor product of two univariate B-spline basis functions:

$$\widehat{N}_{i,j}^p(\xi, \eta) = (\widehat{N}_i^p \otimes \widehat{N}_j^p)(\xi, \eta) = \widehat{N}_i^p(\xi)\widehat{N}_j^p(\eta).$$

Thus, a B-spline geometrical mapping is defined as an application $\mathbf{F} : \widehat{\Omega} \to \Omega$. In the case $d = 2$, this mapping is defined as follows

$$\mathbf{F}(\xi, \eta) = \sum_{i=1}^{m+p} \sum_{j=1}^{m+p} \mathbf{P}_{i,j} \widehat{N}_{i,j}^p(\xi, \eta), \quad (6)$$

where $\{\mathbf{P}_{i,j}\}_{i,j=1,\ldots,m+p}$ are the control points that determine the geometry of the physical domain.

2.2. Non-uniform rational B-spline (NURBS) basis functions

In order to capture a wider set of complex geometries that use to appear in practice, we also introduce the so-called non-uniform rational B-splines (NURBS). Hence, by using NURBS as basis functions the full potential of IGA can be exploited. In order to construct them, a set of weights $\omega_1, \ldots, \omega_{m+p}$ is also needed. Then, the i-th univariate NURBS basis function of polynomial degree p is given by

$$\widehat{R}_i^p(\xi) = \frac{\omega_i \widehat{N}_i^p(\xi)}{\sum_{j=1}^{m+p} \omega_j \widehat{N}_j^p(\xi)}.$$

In general, the two-dimensional NURBS basis functions cannot be constructed straightforwardly by tensorization since each weight is associated to each basis function. Hence, for $d = 2$ a net of weights $\omega_{i,j}$ is considered and these basis functions are given by

$$\widehat{R}_{i,j}^p(\xi, \eta) = \frac{\omega_{i,j} \widehat{N}_{i,j}^p(\xi, \eta)}{\sum_{k,l=1}^{m+p} \omega_{k,l} \widehat{N}_{k,l}^p(\xi, \eta)},$$

with $i, j = 1, \ldots, m + p$.

Note that NURBS geometrical mappings are defined as in (6), but using $\widehat{R}_{i,j}^p$ as basis functions.

3. Two-level method

In this work, we propose an algebraic two-level method for solving isogeometric discretizations of an arbitrary polynomial degree in an efficient and robust way. This two-level method considers the target polynomial degree on the fine level whereas the order of the approximation at the coarse level is as low as possible, dictated by the parametrization of the physical domain. In the following, in Section 3.1 we present the proposed two-level algorithm, specifying the components of the method. The problem on the coarse level can be exactly solved by using the preferred solver of the user, but it also can be approximated by using a suitable iterative method, for example using one iteration of a multigrid cycle as we will present in Section 3.2. This, however, is only a choice of the authors but other possibilities can be equally valid. Finally, in Section 3.3 we also show that a more aggressive coarse level can be used, improving the efficiency of the method.

3.1. Two-level algorithm

As previously mentioned, in this section we explain the proposed algorithm and we introduce its main components. Recall that this two-level method solves an isogeometric discretization with a target polynomial degree on the fine grid by using a linear/quadratic discretization on the coarse level. Let us denote with p and p_{low} the polynomial orders of the discretization at the fine and coarse level respectively. A general two-level algorithm for solving the system $A_p u_p = b_p$, where A_p denotes the isogeometric discretization of spline degree p, consists of the following:

1. Apply ν_1 steps of a suitable iterative method S_p to the initial approximation u_p^0 on the fine level:
$$u_p^k = u_p^{k-1} + S_p(b_p - A_p u_p^{k-1}), \quad k = 1, \ldots, \nu_1.$$
2. Compute the defect on the fine level $d_p = b_p - A_p u_p^{\nu_1}$ and restrict it to the coarse level by using the fine-to-coarse transfer operator
$$d_{p_{low}} = I_p^{p_{low}} d_p.$$
3. Compute the correction $e_{p_{low}}$ in the coarse level by solving the defect equation
$$A_{p_{low}} e_{p_{low}} = d_{p_{low}},$$
where $A_{p_{low}}$ denotes the isogeometric discretization of spline degree p_{low}.
4. Prolongate and update the correction to the fine level by means of the coarse-to-fine transfer operator
$$u_p^{\nu_1} = u_p^{\nu_1} + I_{p_{low}}^p e_{p_{low}}.$$
5. Apply ν_2 steps of the same iterative method S_p to the current approximation:
$$u_p^{\nu_1+k} = u_p^{\nu_1+k-1} + S_p(b_p - A_p u_p^{\nu_1+k-1}), \quad k = 1, \ldots, \nu_2.$$

Of course, the choice of the components of the algorithm is very important. Hence, let us describe in the following the choice of the iterative method applied on the fine level, that we will call smoother, and the construction of the inter-grid transfer operators for the proposed two-level method.

3.1.1. Smoother

As relaxation procedure on the fine level, we propose the use of multiplicative Schwarz methods. These methods are a particular case of block-wise iterations which update simultaneously a set of unknowns at each time. They are based on a splitting of the grid into blocks that gives rise to local problems. There are many possibilities to construct these blocks. One can allow the blocks to overlap, giving rise to the class of overlapping block iterations, where smaller local problems are solved and combined via an additive or multiplicative Schwarz method. In this work, we consider multiplicative Schwarz iterations with maximum overlapping. Although this overlapping increases the computational cost of the method, it improves the convergence rates and thus a fewer number of iterations is required in order to reach the stopping criteria. A deep study of the computational cost of these smoothers was presented in [14].

More specifically, we can describe the multiplicative Schwarz iteration for solving the system $A_p u_p = b_p$ of size n in the following way. Let us denote as B_p^j the subset of unknowns involved in the $j-th$ block of size n_p, that is $B_p^j = \left\{u_{k_1}, \dots, u_{k_{n_p}}\right\}$ where k_i is the global index of the i-th unknown in the block. In order to construct the matrix to solve associated with such a block, that is $A_p^{B_p^j}$, we consider the projection operator from the vector of unknowns u_p to the vector of unknowns involved in the block. This results in a matrix $V_{B_p^j}$ of size $(n_p \times n)$, whose i-th row is the k_i-th row of the identity matrix of order n. Thus, matrix $A_p^{B_p^j}$ is obtained as $A_p^{B_p^j} = V_{B_p^j} A_p V_{B_p^j}^T$, and the iteration matrix of the multiplicative Schwarz method can be written as

$$\prod_{j=1}^{NB}\left(I - V_{B_p^j}^T (A_p^{B_p^j})^{-1} V_{B_p^j} A_p\right),$$

where NB denotes the number of blocks obtained from the splitting of the grid, which corresponds to the number of small systems that have to be solved in a relaxation step of the multiplicative Schwarz smoother. In our particular case of maximum overlapping, NB coincides with the number of grid-points and every block is related to a grid-point, involving that grid-point and its neighbours. Given an appropriate n_p for each polynomial degree p, the size of the blocks is given by $\sqrt[d]{n_p} \times \ldots \times \sqrt[d]{n_p}$ for the d-dimensional case. However, in this work we deal with the case $d = 2$, so square blocks of size $\sqrt{n_p} \times \sqrt{n_p}$ around each grid point are considered. More concretely, we will use the nine-, twenty five- and forty nine-point multiplicative Schwarz smoothers, depending on the spline degree p.
Our study will be carried out up to $p = 8$. Isogeometric discretizations with spline degree larger than $p = 8$, however, can be also solved by considering the proposed two-level approach based on a Schwarz iteration with a big enough number of unknowns within the blocks.

As it will be shown, by applying only one iteration of this smoother at the fine level we get a very simple and efficient solver. In order to obtain a robust solver with respect to the spline degree p, the size of the blocks will be chosen depending on the order of the discretization. In addition, we apply a nine-colour version of the considered Schwarz-type smoothers since these counterparts provide, in general, better convergence rates, see [14]. In order to apply the nine-coloured version of the smoother, first we split the grid into nine subgrids, each one associated to a different colour. This splitting can be seen in Fig. 1 (a), where each colour is represented by a number from 1 to 9. Then, we perform a sweep over the nine subgrids, updating simultaneously the unknowns within the blocks centered on the grid-points of the corresponding subgrid, in a lexicographic manner. If the coloured version of the 9-point (3×3) multiplicative Schwarz iteration is considered, in Fig. 1 (b) we illustrate the blocks of unknowns updated within the first sweep corresponding to the first colour. Then, we run over the grid-points associated to the subgrid corresponding to the second, third, …, and ninth colours. As an example, in Fig. 1 (c), we show the blocks of unknowns which are simultaneously updated within the sweep corresponding to the fifth colour.

3.1.2. Transfer operators

Another important point of our two-level method is the construction of the restriction and prolongation operators. After computing the residual on the fine level, we restrict it to the coarse level by means of an L^2 projection among spline spaces. On the fine level, the solution of (2) is given by $u_h^p = \sum_{j=1}^{n_h^p} u_j^p N_j^p$, where $\dim V_h^p = n_h^p$. Since the approximation of $u_h^p \in V_h^p$ is restricted by means of the restriction operator $I_p^{p_{low}} : V_h^p \to V_h^{p_{low}}$ to the space $V_h^{p_{low}}$, the resulting function $I_p^{p_{low}} u_h^p$ can be expanded as a linear combination of the spline basis functions of $V_h^{p_{low}}$. Consequently, there exists a vector of coefficients $u_{p_{low}} = \{u_j^{p_{low}}\}_{j=1}^{n_h^{p_{low}}}$ such that

$$I_p^{p_{low}} u_h^p = \sum_{j=1}^{n_h^{p_{low}}} u_j^{p_{low}} N_j^{p_{low}}. \tag{7}$$

In order to obtain the relationship among the coefficients $u_p = \{u_j^p\}_{j=1}^{n_h^p}$ and $u_{p_{low}}$, we test both the approximation on the fine level and its restricted term with every basis function spanning $V_h^{p_{low}}$. Thus, one gets the following system of equations:

$$\sum_{k=1}^{n_h^{p_{low}}} u_k^{p_{low}} \left(N_k^{p_{low}}, N_i^{p_{low}}\right) = \sum_{j=1}^{n_h^p} u_j^p \left(N_j^p, N_i^{p_{low}}\right), \quad \forall i = 1, \dots, n_h^{p_{low}}. \tag{8}$$

This system can also be described as follows,

$$M_{p_{low}}^{p_{low}} u_{p_{low}} = M_p^{p_{low}} u_p,$$

where

$$\left(M_{p_{low}}^{p_{low}}\right)_{i,j} = \int_\Omega N_i^{p_{low}} N_j^{p_{low}} \mathrm{d}x, \qquad \left(M_p^{p_{low}}\right)_{i,j} = \int_\Omega N_i^{p_{low}} N_j^p \mathrm{d}x.$$

Therefore, the restriction operator is given by $I_p^{p_{low}} = \left(M_{p_{low}}^{p_{low}}\right)^{-1} M_p^{p_{low}}$. Moreover, the prolongation operator is taken as its adjoint, that is, $I_{p_{low}}^p = \left(M_p^{p_{low}}\right)^T \left(M_{p_{low}}^{p_{low}}\right)^{-T}$. At this point, we approximate $\left(M_{p_{low}}^{p_{low}}\right)^{-1}$ by row-sum lumping in order to avoid the computation of this inverse matrix exactly. That is, $\left(M_{p_{low}}^{p_{low}}\right)^{-1}$ is replaced by a diagonal matrix such that its i-th element in the diagonal is given by $\left(\sum_{j=1}^{n_h^{p_{low}}} M_{p_{low}}^{p_{low}}(i,j)\right)^{-1}$.

Once introduced the components of the method, one iteration of our two-level algorithm is described in Algorithm 1.

Algorithm 1 Two-level algorithm: $\mathbf{u_p^0} \to \mathbf{u_p^1}$.

$u_p^1 = u_p^0 + S_p(b_p - A_p u_p^0)$	Apply **one step** of the **multiplicative Schwarz method** on the fine level.
$d_p = b_p - A_p u_p^1$	Compute the defect on the fine level.
$d_{p_{low}} = I_p^{p_{low}} d_p$	Restrict the defect to the coarse level.
$A_{p_{low}} e_{p_{low}} = d_{p_{low}}$	Compute the correction $e_{p_{low}}$ in the coarse level by solving the defect equation.
$u_p^1 = u_p^1 + I_{p_{low}}^p e_{p_{low}}$	Prolongate and update the correction to the fine level.

Notice that it results in a very simple algorithm since only one single iteration of a multiplicative overlapping Schwarz method is applied on the fine level.

3.2. Approximation of the coarse level problem

Although there is an open choice for the solver at the coarse level, instead of solving exactly the coarse problem, it can also be approximated by using a suitable iterative method. In this work, we apply one $F(1,1)$-cycle that uses a red-black Gauss-Seidel iteration as smoother. The multigrid F-cycle is a hybrid algorithm between the cheap V-cycle and the expensive W-cycle (see [20]). Our numerical experiments show that one iteration of such an h-multigrid method is enough to ensure a good convergence rate. This choice will be theoretically supported by a suitable local Fourier analysis, which will be explained in Section 4.

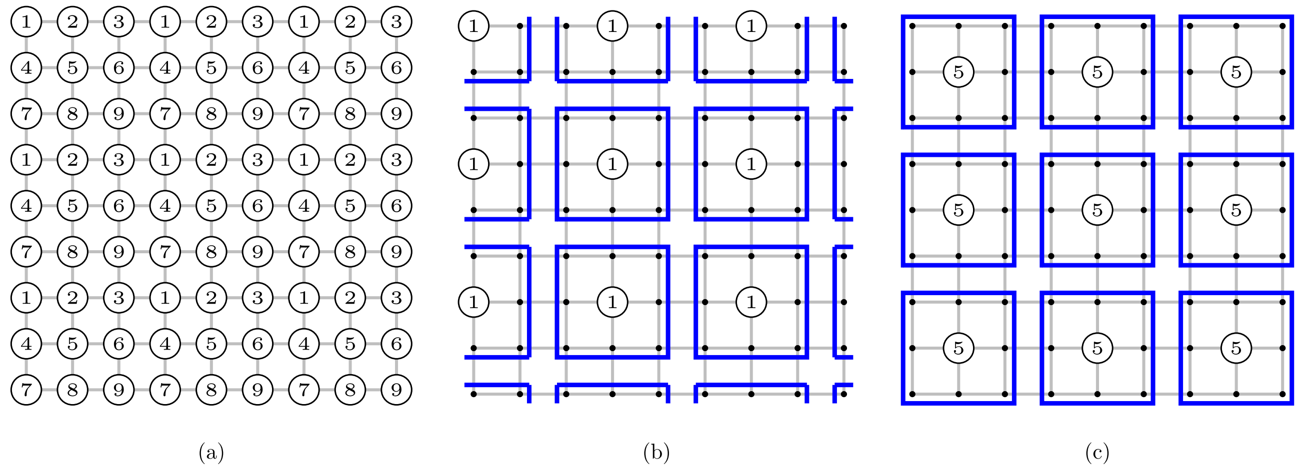


Fig. 1. Coloured version of the 9-point (3×3) multiplicative Schwarz iteration. (a) Numbering of the nine colours used within the coloured version of the smoother. (b) Blocks of unknowns updated within the sweep corresponding to the first colour. (c) Blocks of unknowns updated within the sweep corresponding to the fifth colour.

3.3. Improvement of the algorithm

A further improvement of the algorithm can be achieved by using a more aggressive coarsening strategy. More concretely, we can take a discretization with p_{low} and a mesh size $H = 2h$ as the coarse level. Thus, the computational cost is reduced and the performance of the solver is improved without any significant effect on the convergence factors. Again, local Fourier analysis is able to theoretically support this approach, as we will see in Section 4.

4. Local Fourier analysis

In this section we apply a local Fourier analysis pursuing different objectives. First, we use this analysis to theoretically support the proposed two-level algorithm and in particular the choice of the size of the blocks for the multiplicative Schwarz iteration depending on the spline degree p. In addition, in order to support the use of the h-multigrid as approximation on the coarse level, we apply a three-grid Fourier analysis, and as it will be shown very similar convergence rates to the case of the two-level with an exact solve on the coarse level are obtained. Finally, again a two-grid LFA is used to support the improvement of the algorithm presented in Section 3.3. We restrict our analysis to the case $d = 2$, but this analysis can be applied to all dimensional cases. Given that LFA assumes a regular mesh, uniform and open knot vectors are required for the discretization. That is, the internal knots are equally spaced.

4.1. Basics of LFA

Local Fourier analysis (LFA) is based on the Fourier transform theory, assuming that any grid function defined on an infinite grid $\mathcal{G}_h$ can be decomposed as a "formal" linear combination of complex exponential functions, $\varphi_h(\theta, \mathbf{x}) = e^{\imath\theta\mathbf{x}/h}$ with $\mathbf{x} \in \mathcal{G}_h$ and $\theta \in \Theta := (-\pi, \pi]^2$, known as Fourier modes. In particular such decomposition of the error function is considered and LFA studies how the operators involved in the multilevel method act on these Fourier components, and in particular on the so-called Fourier space $\mathcal{F}(\mathcal{G}_h) := \text{span}\{\varphi_h(\theta, \mathbf{x}) \,|\, \theta \in \Theta\}$.

Here, we study the two-level method previously introduced by using this analysis. With this purpose, we define the error propagation operator of the two-level method, $T_p^{p_{low}}$, which relates the error in the iteration $m+1$, e^{m+1}, with the error in the previous iteration, e^m, that is,

$$e^{m+1} = T_p^{p_{low}} e^m = (I - I_{p_{low}}^{p} A_{p_{low}}^{-1} I_p^{p_{low}} A_p) S_p e^m. \tag{9}$$

In the previous expression, A_p and $A_{p_{low}}$ correspond to the IGA discrete operators of order p and p_{low}, respectively; $I_{p_{low}}^{p}$ and $I_p^{p_{low}}$ are the inter-grid transfer operators, and S_p represents the multiplicative Schwarz iteration which is applied within the two-level method. It is easy to see that the Fourier modes are eigenfunctions of all the operators involved in the two-level method. Notice that, in this case, the transfer operators between levels do not couple Fourier modes unlike the inter-grid transfer operators within the standard h-multigrid method. Thus, the Fourier symbol of the error transfer operator for $\theta \in \Theta$ is given by,

$$\widetilde{T}_p^{p_{low}}(\theta) = (\widetilde{I}(\theta) - \widetilde{I}_{p_{low}}^{p}(\theta)\widetilde{A}_{p_{low}}^{-1}(\theta)\widetilde{I}_p^{p_{low}}(\theta)\widetilde{A}_p(\theta))\widetilde{S}_p(\theta),$$

where $\widetilde{I}(\theta)$ denotes the symbol of the identity operator, $\widetilde{I}_{p_{low}}^{p}(\theta)$ and $\widetilde{I}_p^{p_{low}}(\theta)$ denote the Fourier symbols of the prolongation and restriction operators, $\widetilde{A}_{p_{low}}(\theta)$ and $\widetilde{A}_p(\theta)$ are the symbols of the discrete operators $A_{p_{low}}$ and A_p respectively, and $\widetilde{S}_p(\theta)$ denotes the Fourier symbol of the smoother S_p applied to the discretization given on the fine level. Thus, in order to compute this expression, the Fourier symbol of the discrete operators, inter-grid transfer operators, and smoothers are required. Since the Fourier modes are eigenfunctions of any discrete operator A which can be described by a stencil with coefficients a_κ, $\kappa \in \mathcal{I}$ (where $\mathcal{I}$ is the set of indexes defining the stencil), it is satisfied that $A\varphi_h(\theta, \mathbf{x}) = \widetilde{A}(\theta)\varphi_h(\theta, \mathbf{x})$, with $\widetilde{A}(\theta) = \sum_{\kappa \in \mathcal{I}} a_\kappa e^{\imath\theta\cdot\kappa}$ being the Fourier symbol of operator A (see [21]). In our case, the symbol of the discrete operator can be obtained by computing the Hadamard product of the stencil of the chosen discretization and the corresponding matrix of Fourier modes, and adding up all the elements of the resulting matrix. For instance, for each $\theta = (\theta_1, \theta_2) \in \Theta$, the discrete operator of a quadratic B-spline discretization A_2 and the corresponding matrix of Fourier modes F_2 are given by

$$A_2 = \begin{pmatrix} \frac{-1}{360} & \frac{-7}{180} & \frac{-1}{12} & \frac{-7}{180} & \frac{-1}{360} \\ \frac{-7}{180} & \frac{-13}{90} & \frac{1}{30} & \frac{-13}{90} & \frac{-7}{180} \\ \frac{-1}{12} & \frac{1}{30} & \frac{11}{10} & \frac{1}{30} & \frac{-1}{12} \\ \frac{-7}{180} & \frac{-13}{90} & \frac{1}{30} & \frac{-13}{90} & \frac{-7}{180} \\ \frac{-1}{360} & \frac{-7}{180} & \frac{-1}{12} & \frac{-7}{180} & \frac{-1}{360} \end{pmatrix},$$

$$F_2 = \begin{pmatrix} e^{-2\imath\theta_1} e^{-2\imath\theta_2} & e^{-\imath\theta_1} e^{-2\imath\theta_2} & e^{-2\imath\theta_2} & e^{\imath\theta_1} e^{-2\imath\theta_2} & e^{2\imath\theta_1} e^{-2\imath\theta_2} \\ e^{-2\imath\theta_1} e^{-\imath\theta_2} & e^{-\imath\theta_1} e^{-\imath\theta_2} & e^{-\imath\theta_2} & e^{\imath\theta_1} e^{-\imath\theta_2} & e^{2\imath\theta_1} e^{-\imath\theta_2} \\ e^{-2\imath\theta_1} & e^{-\imath\theta_1} & 1 & e^{\imath\theta_1} & e^{2\imath\theta_1} \\ e^{-2\imath\theta_1} e^{\imath\theta_2} & e^{-\imath\theta_1} e^{\imath\theta_2} & e^{\imath\theta_2} & e^{\imath\theta_1} e^{\imath\theta_2} & e^{2\imath\theta_1} e^{\imath\theta_2} \\ e^{-2\imath\theta_1} e^{2\imath\theta_2} & e^{-\imath\theta_1} e^{2\imath\theta_2} & e^{2\imath\theta_2} & e^{\imath\theta_1} e^{2\imath\theta_2} & e^{2\imath\theta_1} e^{2\imath\theta_2} \end{pmatrix},$$

and then, the Fourier symbol $\widetilde{A}_2(\theta)$ is given as follows:

$$\widetilde{A}_2(\theta) = \sum_{i=1}^{5}\sum_{j=1}^{5} A_2(i,j)F_2(i,j).$$

Note that the size of the matrix of Fourier modes is determined by the size of the stencil of the discretization. In fact, the Fourier modes in F_p are centred at the element $((n+1)/2,(n+1)/2)$, where $n\times n$ is the size of the stencil of the discrete operator A_p.

The Fourier symbols of the inter-grid transfer operators, $\widetilde{I}_p^{p_{low}}(\theta)$ and $\widetilde{I}_{p_{low}}^{p}(\theta)$, are obtained in the same way, but taking into account that the Fourier modes depend on the related positions among the grid-functions considered for the two levels.

Finally, the Fourier symbols of the multiplicative Schwarz smoothers considered in this work, $\widetilde{S}_p(\theta)$, can be found in [14], where they are carefully detailed.

In this way, the asymptotic convergence factor of the two-level method can be estimated by the following expression

$$\rho_{2g} = \sup_{\theta\in\Theta} |\widetilde{T}_p^{p_{low}}(\theta)|. \tag{10}$$

In order to support the approximation approach by multigrid method on the coarse level given in Section 3.2, we have to take into account a smoothing effect at the second level and, by means of a standard coarsening on the mesh size h, a third level whose discretization corresponds to the spline space $S_{2h}^{p_{low}}((0,1)^2)$. Thus, a three-grid analysis is required and a smoother $S_{p_{low}}$ is considered. For this purpose, we introduce the error propagation matrix $M_{p,h}^{p_{low},2h}$ as follows:

$$M_{p,h}^{p_{low},2h} = (I - I_{p_{low}}^{p}(I-(M_{p_{low},h}^{p_{low},2h}))A_{p_{low}}^{-1}I_p^{p_{low}}A_p)S_p,$$

where $M_{p_{low},h}^{p_{low},2h}$ is the two-grid operator between the second and third levels, that is,

$$M_{p_{low},h}^{p_{low},2h} = S_{p_{low}}^{\nu_2}(I - I_{2h}^{h}A_{p_{low},2h}^{-1}I_h^{2h}A_{p_{low}})S_{p_{low}}^{\nu_1},$$

with I_{2h}^{h} and I_h^{2h} the standard inter-grid transfer operators between the grids of size h and $2h$. In addition, ν_1 and ν_2 denote the number of pre- and post-smoothing steps of the smoother $S_{p_{low}}$ on the second level.

In this case, in the transition from the second to the third level, some Fourier modes are coupled. Hence, we split the Fourier components into high- and low-frequency components on $\mathcal{G}_h$. The low-frequency Fourier components are those associated with frequencies belonging to $\Theta_{2h} = (-\pi/2,\pi/2]^2$. Then, each low-frequency $\theta^{00} = (\theta_1^{00},\theta_2^{00}) \in \Theta_{2h}$ is coupled with three high frequencies θ^{11}, θ^{10}, θ^{01}, given by $\theta^{ij} = \theta^{00} - (i\, sign(\theta_1^{00}), j\, sign(\theta_2^{00}))\pi,\ i,j=0,1$, giving rise to the so-called spaces of $2h$-harmonics:

$$\mathcal{F}^2(\theta^{00}) = span\left\{\varphi_h(\theta^{00},\cdot),\varphi_h(\theta^{11},\cdot),\varphi_h(\theta^{10},\cdot),\varphi_h(\theta^{01},\cdot)\right\},\ \text{with } \theta^{00}\in\Theta_{2h}.$$

Based on this decomposition of the Fourier space in terms of the subspaces of $2h$-harmonics, the spectral radius of the three-grid operator can be computed as follows:

$$\rho_{3g} = \rho(M_{p,h}^{p_{low},2h}) = \sup_{\theta^{00}\in\Theta_{2h}} \rho(\widetilde{M}_{p,h}^{p_{low},2h}(\theta^{00})).$$

Finally, in order to analyze the improved version of the two-grid algorithm given in Section 3.3, we apply a two-grid LFA in which from the fine to the coarse levels we reduce the polynomial degree from p to p_{low} and also we double the grid-size from h to $2h$. This two-grid analysis couples Fourier modes as explained before, and the corresponding error transfer operator is given by:

$$T_{p,h}^{p_{low},2h} = (I - I_{p_{low},2h}^{p,h}A_{p_{low},2h}^{-1}I_{p,h}^{p_{low},2h}A_p)S_p,$$

where the transfer operators between the spaces $S_h^p((0,1)^2)$ and $S_{2h}^{p_{low}}((0,1)^2)$, that is $I_{p_{low},2h}^{p,h}$ and $I_{p,h}^{p_{low},2h}$, are obtained by composition of $I_p^{p_{low}}, I_{p_{low}}^{p}$ and the transfer operators I_h^{2h}, I_{2h}^{h} between spline spaces with equal spline degree but different mesh size h and $2h$. From this expression, the asymptotic convergence factor of the improved two-level method can be estimated by the following expression:

$$\rho_{2g}^{ag} = \sup_{\theta^{00}\in\Theta_{2h}} \rho(\widetilde{T}_{p,h}^{p_{low},2h}(\theta^{00})). \tag{11}$$

Table 1
Two-level (ρ_{2g}) convergence factors predicted by LFA together with the asymptotic convergence factors obtained numerically (ρ_h), for different values of the spline degree p. In this case, the second level is a linear discretization with the same mesh size h considered for the first level.

	9p Schwarz		25p Schwarz		49p Schwarz	
	ρ_{2g}	ρ_h	ρ_{2g}	ρ_h	ρ_{2g}	ρ_h
$p=2$	0.1234	0.1212	0.0813	0.0752	0.0604	0.0725
$p=3$	0.2150	0.2141	0.0874	0.0854	0.0622	0.0712
$p=4$	0.4581	0.4558	0.1294	0.1466	0.0697	0.0852
$p=5$	0.7095	0.7058	0.2690	0.2847	0.1001	0.1215
$p=6$	0.8786	0.8756	0.4549	0.4555	0.1909	0.2113
$p=7$	0.9576	0.9573	0.6623	0.6601	0.3260	0.3284
$p=8$	0.9868	0.9851	0.8278	0.8146	0.4885	0.4764

4.2. Local Fourier analysis results

Next, we show some LFA results to demonstrate the good performance of the proposed two-level method. Firstly, we consider a linear discretization as the second level, that is, $p_{low}=1$. In Table 1, the two-level convergence factors predicted by LFA, ρ_{2g}, are shown together with the asymptotic convergence factors, ρ_h, obtained numerically for different values of the spline degree p varying from $p=2$ to $p=8$. The asymptotic convergence factors are obtained numerically by solving problem (1) with a zero right-hand side and a random initial guess. We consider the 9-point, 25-point and 49-point multiplicative Schwarz iterations at the first level. It can be seen from Table 1 that the factors predicted by LFA match very accurately the asymptotic convergence factors numerically obtained, and therefore the LFA results in a very useful tool to analyze the performance of the method. It is also observed from the table that choosing an appropriate multiplicative Schwarz smoother for each polynomial degree p, we obtain a robust solver with respect to p. This choice of the size of the blocks in the relaxation depending on the spline degree is done taking into account the two-grid convergence factors provided by the LFA, as well as the computational cost of the algorithm. In particular, we choose blocks of size 3×3 (9-point Schwarz smoother) for the cases $p=2,3,4$, blocks of size 5×5 (25-point Schwarz smoother) for the cases $p=5,6$ and blocks of size 7×7 (49-point Schwarz smoother) for spline degree $p=7,8$. For a more detailed explanation of how to choose the size of the blocks of the multiplicative Schwarz relaxations for different values of p, in terms of the LFA results and the computational cost, we refer the reader to [14].

Next, we present some LFA results in order to support the approach proposed in Section 3.2. In this case, one single iteration of a $F(1,1)$-cycle using red-black Gauss-Seidel as smoother is considered to approximate the problem on the coarse level. Thus, in order to analyze such approximation, we need to use the three-grid local Fourier analysis introduced in Section 4.1. In Table 2, we show the three-grid convergence factors (ρ_{3g}) provided by LFA. One can observe that the predictions provided by the three-grid LFA match very well with the two-grid convergence factors predicted by the analysis for the two-level algorithm (with exact solve on the coarse level) shown in Table 1.

Finally, we want to analyze the improvement of the algorithm presented in Section 3.3. In order to do this, we need to consider that in the second level of the algorithm we now assume a grid-size $2h$ in addition of the reduction of the spline degree to p_{low}. Again, LFA is able to support this approach by using a two-grid analysis. In Table 3, the two-level convergence factors provided by this analysis (see expression in (11)) are shown.

Table 2
Three-level (ρ_{3g}) convergence factors predicted by LFA, for different values of the spline degree p.

	9p Schwarz	25p Schwarz	49p Schwarz
$p=2$	0.1281	0.0847	0.0624
$p=3$	0.2144	0.0920	0.0690
$p=4$	0.4566	0.1290	0.0733
$p=5$	0.7078	0.2676	0.0986
$p=6$	0.8773	0.4549	0.1909
$p=7$	0.9569	0.6591	0.3174
$p=8$	0.9864	0.8250	0.4734

Table 3
Two-grid (ρ_{2g}^{ag}) convergence factors predicted by LFA for different values of the spline degree p, for the improved version of the algorithm.

	9p Schwarz	25p Schwarz	49p Schwarz
$p=2$	0.1723	0.1137	0.0837
$p=3$	0.2145	0.1152	0.0863
$p=4$	0.4566	0.1290	0.0874
$p=5$	0.7078	0.2676	0.0986
$p=6$	0.8773	0.4549	0.1909
$p=7$	0.9569	0.6591	0.3174
$p=8$	0.9864	0.8250	0.4734

Table 4
Two-level (ρ_{2g}^{ag}) convergence factors predicted by LFA together with the asymptotic convergence factors obtained numerically (ρ_h), for different values of the spline degree p. In this case, the second level is a quadratic discretization with mesh size $H=2h$ considered for the second level.

	9p Schwarz		25p Schwarz		49p Schwarz	
	ρ_{2g}^{ag}	ρ_h	ρ_{2g}^{ag}	ρ_h	ρ_{2g}^{ag}	ρ_h
$p=3$	0.2144	0.2125	0.0700	0.0776	0.0550	0.0596
$p=4$	0.4566	0.4539	0.1290	0.1453	0.0522	0.0641
$p=5$	0.7078	0.7088	0.2676	0.2807	0.0986	0.1181
$p=6$	0.8773	0.8707	0.4549	0.4477	0.1909	0.1999
$p=7$	0.9569	0.9549	0.6591	0.6640	0.3174	0.3235
$p=8$	0.9864	0.9833	0.8250	0.8304	0.4734	0.4646

In order to support this approach for different values of p_{low}, a two-grid analysis has been carried out such that an isogeometric discretization with grid-size $H=2h$ and $p_{low}=2$ is considered at the second level. Hence, in Table 4 the two-grid convergence factors ρ_{2g}^{ag} provided by this analysis are shown together with the ones obtained experimentally with our multigrid codes (ρ_h).

Given that this last approach is more efficient and does not deteriorate the performance of the two-level method introduced before, this will be the strategy used in the numerical experiments section.

5. Comparison between the proposed two-level algorithm and *h*-multigrid

In this section, we compare the performance of the proposed two-level algorithm and the h-multigrid method introduced in [14]. In that paper, an h-multigrid method based on multiplicative Schwarz smoothers was proved to be a robust solver with respect to the polynomial degree up to $p=8$. The strategy to choose the size of the block within the smoother, depending on the polynomial degree, is the one that we adopted here for the fine level in the proposed two-level method. The difference, however, is that in the h-multigrid the multiplicative Schwarz relaxation is applied on every grid in the hierarchy, whereas here we only apply such an iteration on the fine level, and from the second level we approximate the solution of a linear/quadratic problem by using a single $F(1,1)$-cycle based on a standard red-black smoother. Although the most expensive part of both algorithms is the application of the multiplicative Schwarz iteration on the fine target grid, we can show that the two-level algorithm proposed in this work improves the performance of the h-multigrid method since it considerably reduces the computational cost at the second and coarser levels, resulting in an improvement of the previous approach.

First, we want to show that the convergence factor obtained by the two analyzed methods is almost the same. For this purpose, in Fig. 2, we compare the Fourier symbol of the two versions of the two-level algorithm proposed in this work, by considering on the second level $p_{low}=1$ and mesh size (a) $H=h$ or (b) $H=2h$, respectively, together with (c) the Fourier symbol of the two-grid operator corresponding to the h-multigrid, with the same polynomial degree p on the first and second levels and standard coarsening. These Fourier symbols are displayed for two different spline degrees on the fine grid: $p=3$ on the top row and $p=7$ on the bottom row. Thus, with the help of the LFA we can observe that the three approaches provide very similar asymptotic convergence factors.

The main difference lies in the computational cost of the methods. It is obvious that choosing $H=2h$ instead of $H=h$ on the second level of the two-level algorithm proposed here reduces the cost. Thus, this approach is the one that we are going to compare with the h-multigrid method introduced in [14]. Since the improvement of the proposed two-level algorithm is more significative for high-order discretizations, we consider spline degrees ranging from $p=4$ to $p=8$ for the comparison presented in Fig. 3. In such a figure, we show the ratios of the CPU times required to solve the Poisson equation on a square domain applying the proposed two-level algorithm using at the second level a single $F(1,1)$-cycle based on red-black Gauss-Seidel smoothing and the CPU times required to solve the same problem applying the h-multigrid method presented in [14]. These ratios are shown for different numbers of refinement levels on the horizontal axis. We can observe that, for most of the considered spline degrees, the CPU times obtained with the solver proposed in this work are close to a 25% faster than those obtained with the h-multigrid method. Thus, we can state that the two-level strategy proposed here clearly improves the performance of the h-multigrid solver.

6. Numerical experiments

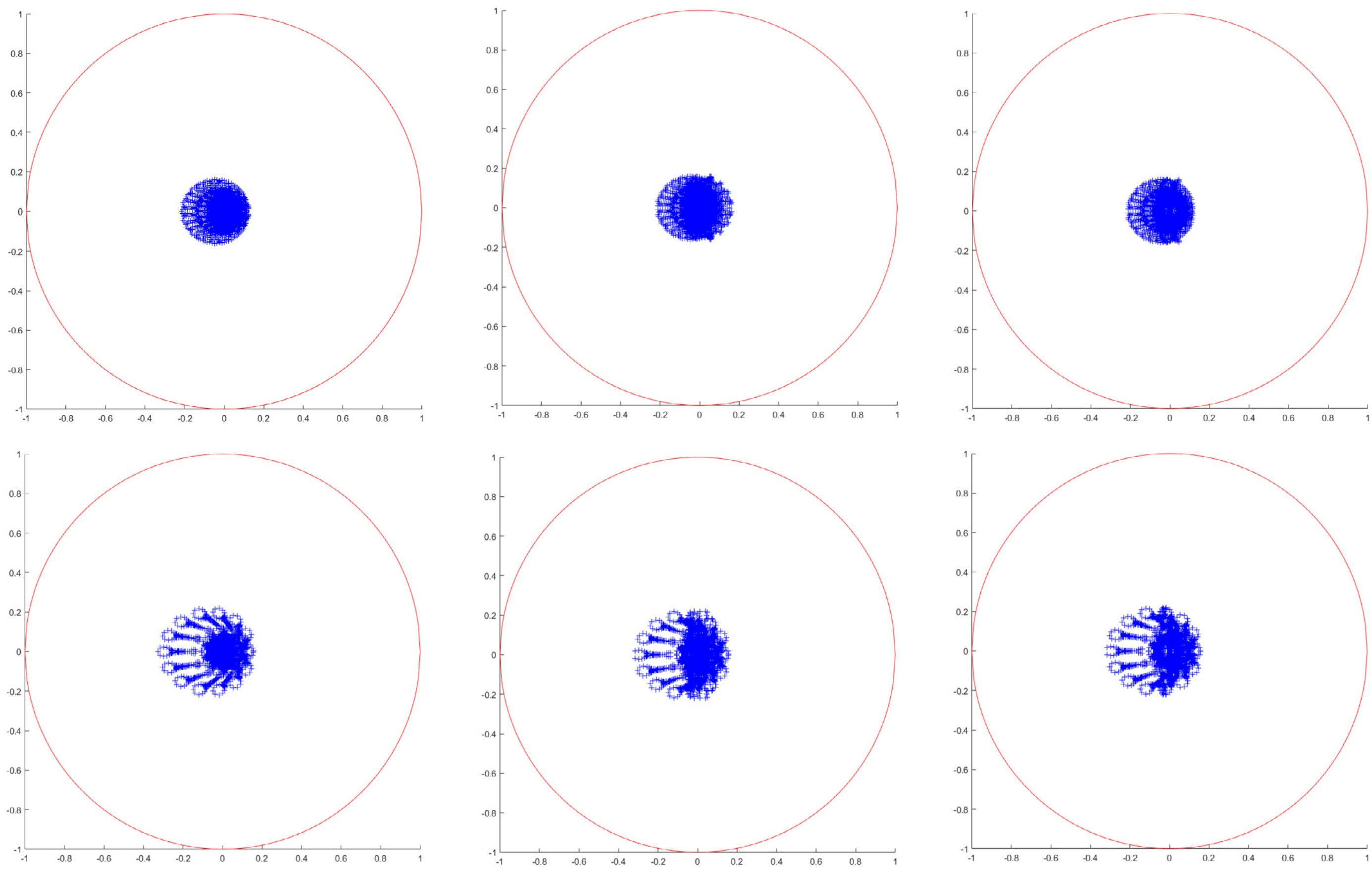
In order to support the robustness and efficiency of the proposed two-level method, we have considered three different numerical experiments. In the first one, we deal with a bidimensional problem on a square domain and then we consider two bidimensional problems defined in physical domains with nontrivial geometries, namely, a quarter annulus and a unit disk, respectively. For the first numerical experiment we consider B-splines as basis functions and $p_{low}=1$, whereas for the second and third numerical experiments NURBS are used in order to exactly describe the geometry for the considered domains and therefore $p_{low}=2$ is considered.

As it was mentioned in Section 3, we consider only one step of the coloured version of the multiplicative Schwarz method at the fine level. Instead of solving exactly at the coarse level, we follow the approximation strategy proposed in Section 3.2 with the improvement introduced in Section 3.3. In all the experiments, the coarsest grid is composed of 2×2 elements. Moreover, the initial guess is taken as a random vector and the stopping criterion for our two-level solver is set to reduce the initial residual by a factor of 10^{-8}. All the methods have been implemented in our in-house Fortran code, and the numerical computations have been carried out on an hp pavilion laptop 15-cs0008ns with a Core i7-8550U with 1,80 GHz and 16 GB RAM, running Windows 10.

6.1. Square domain

Now, let us apply our two-level method based on overlapping multiplicative Schwarz iterations on a two-dimensional problem defined on a square domain $\Omega=(0,1)^2$. We consider the following problem:

$$\begin{cases} -\Delta u = 2\pi^2 \sin(\pi x)\sin(\pi y), & (x,y)\in\Omega, \\ u(x,y)=0, & (x,y) \text{ on } \partial\Omega. \end{cases}$$



(a) (b) (c)

Fig. 2. Distribution of eigenvalues. (a) Two-level operator considering on the second level $p_{low} = 1$ and mesh size $H = h$, (b) Two-level operator considering on the second level $p_{low} = 1$ and mesh size $H = 2h$, and (c) Two-grid operator corresponding to the h-multigrid with standard coarsening. These distributions of eigenvalues are shown for spline degrees $p = 3$ on the top row, and $p = 7$ on the bottom row.

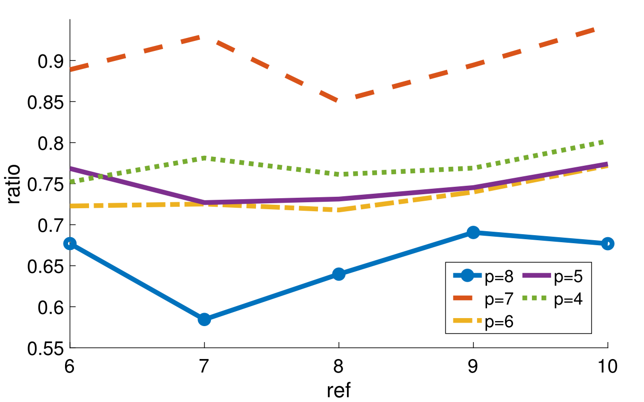


Fig. 3. Ratios of the CPU times required to solve the Poisson equation on a square domain applying the two-level algorithm proposed here and the CPU times required to solve the same problem applying the h-multigrid method given in [14], for spline degrees ranging from $p = 4$ to $p = 8$.

For this numerical experiment, we consider the spline space given in (3) for different degrees ranging from $p = 2$ until $p = 8$. In addition, we consider a linear discretization for the coarse level and the size of the blocks is chosen depending on the spline degree. We choose blocks of size 3×3 for the cases $p = 2, 3, 4$, blocks of size 5×5 for the cases $p = 5, 6$ and blocks of size 7×7 for spline degree $p = 7, 8$.

In Table 5, we show the number of iterations (it) and the cpu time (cpu) in seconds needed to reach the stopping criterion for several mesh sizes and different spline degrees $p = 2, \ldots, 8$. We observe that in both cases the iteration numbers are robust with respect to the size of the grid h and the spline degree p. With these results, we can conclude that our two-level method provides an efficient and robust solver for B-spline isogeometric discretizations.

6.2. Quarter annulus

For the second experiment, our goal is to apply the two-level method to a two-dimensional problem defined in a nontrivial geometry. Thus, we set as physical domain the quarter of an annulus,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid r^2 < x^2 + y^2 < R^2, x, y > 0\},$$

where $r = 0.3$, $R = 0.5$. Hence, we consider the solution of the Poisson problem in such domain with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\Delta u = f(x, y), & (x, y) \in \Omega, \\ u(x, y) = 0, & (x, y) \text{ on } \partial\Omega, \end{cases}$$

where $f(x, y)$ is such that the exact solution is

$$u(x, y) = \sin(\pi x)\sin(\pi y)(x^2 + y^2 - r^2)(x^2 + y^2 - R^2).$$

In order to construct this computational domain, the use of quadratic NURBS basis functions is required. Thus, we consider discretizations of degree $p = 3, \ldots, 8$ with maximal smoothness for the fine level whereas the quadratic discretization is used at the coarse level. In this case, we compare the performance of the multigrid method (MG) proposed in

Table 5
Square domain problem. Number of the proposed two-level method iterations (it) and computational time (cpu) necessary to reduce the initial residual in a factor of 10^{-8}, for different mesh-sizes h and for different values of the spline degree p, using the most appropriate coloured multiplicative Schwarz smoother for each p.

	Colour 9p Schwarz						Colour 25p Schwarz				Colour 49p Schwarz			
	$p=2$		$p=3$		$p=4$		$p=5$		$p=6$		$p=7$		$p=8$	
Grid	it	cpu	it	cpu	it	cpu	it	cpu	it	cpu	it	cpu	it	cpu
64^2	5	0.04	5	0.06	7	0.09	4	0.15	5	0.21	4	0.64	4	0.65
128^2	5	0.12	5	0.15	7	0.25	4	0.38	5	0.57	4	1.57	4	1.62
256^2	5	0.40	5	0.48	7	0.79	4	1.14	5	1.71	4	4.20	4	4.39
512^2	5	1.46	5	1.76	7	2.95	4	3.90	5	5.85	4	12.49	4	13.01
1024^2	5	5.85	5	7.01	7	11.71	4	14.59	5	21.74	4	41.42	4	43.09

Table 6
Quarter annulus problem. Number of the proposed two-level method iterations (it) necessary to reduce the initial residual in a factor of 10^{-8}, for different mesh-sizes h and for different values of the spline degree p, using the most appropriate coloured multiplicative Schwarz smoother for each p.

Grid	C. 9p Schwarz				C. 25p Schwarz				C. 49p Schwarz			
	$p=3$		$p=4$		$p=5$		$p=6$		$p=7$		$p=8$	
	DS	MG	DS	MG	DS	MG	DS	MG	DS	MG	DS	MG
32^2	5	5	8	7	4	4	6	6	4	3	4	5
64^2	5	6	8	8	4	4	6	6	4	4	5	5
128^2	6	6	8	8	4	4	6	6	4	4	5	5
256^2	6	6	8	7	4	4	6	6	4	4	5	5

Section 3.3 with a two-level based on a direct solver (DS) at the second level. For this purpose, in Table 6 we show the number of iterations needed to reach the stopping criterion for several mesh sizes and different spline degrees $p = 3, \dots, 8$. We observe that the use of the mentioned MG at the coarse level provides almost the same number of iterations than the version with the direct solver. Thus, we can conclude that our two-level method results in an efficient and robust solver also for NURBS discretizations.

6.3. Unit disk domain

For the last experiment, we apply our two-level method to another non-trivial planar geometry. We consider the unit disk ($r = 1$) as physical domain:

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

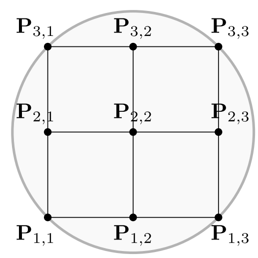
In this case, we consider the same Poisson problem with homogeneous Dirichlet boundary conditions such that the exact solution is given by

$$u(x, y) = (x^2 + y^2 - r^2)\sin(\pi x)\sin(\pi y).$$

The geometry of the unit disk Ω is described by using a quadratic NURBS surface ($p, q = 2$) with knot vector $\Xi = \{0, 0, 0, 1, 1, 1\}$ in both directions. In Fig. 4 (a), we show the control mesh required to parametrize the unit disk. In addition, we provide the control points together with the corresponding weights in Fig. 4 (b).

Again, a quadratic discretization is used at the coarse level whereas discretizations of spline degree ranging from $p = 3$ to $p = 8$ that hold global smoothness C^{p-1} are considered for the fine level. In order to support the good performance of the strategy proposed in Section 3.3, we compare its performace with a two-level method based on a direct solver at the second level. Thus, in Table 7, we show the number of iterations needed to reach the stopping criterion for several mesh sizes and different spline degrees $p = 3, \dots, 8$.

We can observe that the number of iterations required to reach the stopping criterion on both approaches are almost the same. Finally, the results presented in Table 7 also show that our solver is robust with respect to the spline degree p when NURBS discretizations are considered.



(a)

i	j	$\mathbf{P}_{i,j}$	$\omega_{i,j}$
1	1	$(-1/\sqrt{2}, -1/\sqrt{2})$	1
1	2	$(0, -1/\sqrt{2})$	$1/\sqrt{2}$
1	3	$(1/\sqrt{2}, -1/\sqrt{2})$	1
2	1	$(-1/\sqrt{2}, 0)$	$1/\sqrt{2}$
2	2	$(0, 0)$	1
2	3	$(1/\sqrt{2}, 0)$	$1/\sqrt{2}$
3	1	$(-1/\sqrt{2}, 1/\sqrt{2})$	1
3	2	$(0, 1/\sqrt{2})$	$1/\sqrt{2}$
3	3	$(1/\sqrt{2}, 1/\sqrt{2})$	1

(b)

Fig. 4. Example of the quadratic NURBS transformation of the unit disk: (a) control mesh and (b) control points $\mathbf{P}_{i,j}$ and their corresponding weights $\omega_{i,j}$.

Table 7
Unit disk problem. Number of the proposed two-level method iterations (it) necessary to reduce the initial residual in a factor of 10^{-8}, for different mesh-sizes h and for different values of the spline degree p, using the most appropriate coloured multiplicative Schwarz smoother for each p.

	C. 9p Schwarz				C. 25p Schwarz				C. 49p Schwarz			
	$p=3$		$p=4$		$p=5$		$p=6$		$p=7$		$p=8$	
Grid	DS	MG	DS	MG	DS	MG	DS	MG	DS	MG	DS	MG
32^2	2	2	4	4	2	2	3	3	2	2	3	3
64^2	2	3	4	4	2	2	3	3	2	2	2	2
128^2	2	3	4	5	2	2	3	3	2	2	2	2
256^2	2	4	4	5	2	2	3	3	2	2	2	2

7. Conclusions

In this work, we propose a two-level method for solving isogeometric discretizations of an arbitrary polynomial degree in an efficient and robust way. The algorithm considers the target polynomial degree on the fine level and a linear or quadratic approximation on the coarse level dictated by the parametrization of the physical domain. In this method, only one iteration of an appropriate multiplicative Schwarz method is applied on the fine level, and the coarse level can be exactly solved by using well-known techniques for solving linear and quadratic discretizations. The user can choose the preferred approach on the coarse level, but here we propose to approximate the coarse problem by using one single iteration of a suitable h-multigrid. In particular, we apply one $F(1,1)$-cycle based on a red-black Gauss-Seidel smoother. An enhancement of the performance of the solver is obtained if we apply a standard coarsening strategy from the first to the second level by considering a grid of size h on the fine level and a coarse grid-size of $2h$. The good convergence results of the proposed method are theoretically supported by two- and three-grid local Fourier analysis and also they are demonstrated by means of three numerical experiments. Furthermore, we show that the proposed two-level algorithm improves the performance of a successful h-multigrid method recently proposed in [14] for solving the same type of isogeometric discretizations.

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