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New fractal functions on the sphere

Md. Nasim Akhtar[1,a], M. Guru Prem Prasad[2], M. A. Navascués[3], and R. N. Mohapatra[4]

[1] Department of Mathematics, Presidency University, 86/1, College Street, Kolkata 700 073, India
[2] Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, Assam 781039, India
[3] Departamento de Matemática Aplicada, Universidad de Zaragoza, Zaragoza, Spain
[4] Department of Mathematics, University of Central Florida, 4393 Andromeda Loop, Orlando, FL 32816, USA

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Abstract In this article, a family of continuous functions on the unit sphere $S \subseteq \mathbb{R}^3$ is considered as a generalization of spherical harmonics. The family is fractalized using a linear and bounded operator of functions on the sphere. Particular values of the scale vector in the iterated function system (IFS) may yield classical functions system on the sphere. We have shown that for different values of the scale vector in the IFS, Bessel sequences, frames, and Riesz bases can be established for the space $\mathcal{L}^2(S)$ of square integrable functions on the sphere.

1 Introduction

The notion of Fractal Interpolation Function (FIF) which relates new research fields in approximation theory, functional analysis, etc. was proposed in the nineteen eighties [1]. The graph of the aforementioned function is the attractor of an Iterated Function System (IFS). In constructive approximation, fractal functions form the basis for non-smooth functions. The FIFs generated from IFSs with free parameters known as scaling factors give more flexibility to fit complicated curves that exhibit some kind of self-similarity. To define a fractal interpolation function, a general IFS is constructed as mentioned below.

Let for any $r \in \mathbb{N}$, $\mathbb{N}_r = \{1, 2, \ldots, r\}$ and $\mathbb{N}_r^0 = \mathbb{N}_r \cup \{0\}$. Consider a set of interpolation points $\{(x_i, y_i) \in I \times \mathbb{R} \ : \ i \in \mathbb{N}_N^0\}$, $N > 2$, where $\Delta : x_0 < x_1 < \cdots < x_N$ is a partition of the closed interval $I = [x_0, x_N]$ and $y_i \in [h_1, h_2] \subset \mathbb{R}$ for $i \in \mathbb{N}_N^0$. Set $I_i = [x_{i-1}, x_i]$ for $i \in \mathbb{N}_N$ and $K = I \times [h_1, h_2]$. Let $L_i : I \to I_i, i \in \mathbb{N}_N$, be contraction homeomorphisms, such that

$$L_i(x_0) = x_{i-1}, \quad L_i(x_N) = x_i, \tag{1}$$

$$|L_i(c_1) - L_i(c_2)| \leq d|c_1 - c_2| \text{ for all } c_1 \text{ and } c_2 \text{ in } I, \tag{2}$$

for some $0 \leq d < 1$. Furthermore, let $F_i : K \to \mathbb{R}$, $i \in \mathbb{N}_N$, be given continuous functions, such that

$$F_i(x_0, y_0) = y_{i-1}, \quad F_i(x_N, y_N) = y_i, \tag{3}$$

$$|F_i(x, \xi_1) - F_i(x, \xi_2)| \leq |\alpha_i||\xi_1 - \xi_2| \tag{4}$$

for all $x \in I$ and for all ξ_1 and ξ_2 in $[h_1, h_2]$, for some $\alpha_i \in (-1, 1), i \in \mathbb{N}_N$. Define mappings $W_i : K \to I_i \times \mathbb{R}, i \in \mathbb{N}_N$ by

$$W_i(x, y) = (L_i(x), F_i(x, y)) \text{ for all } (x, y) \in K\ .$$

Then

$$\{K;\ W_i(x, y)\ :\ i \in \mathbb{N}_N\} \tag{5}$$

constitutes an IFS. Barnsley [1] proved that the IFS $\{K; W_i : i \in \mathbb{N}_N\}$ defined above has a unique attractor G, where G is the graph of a continuous function $g : I \to \mathbb{R}$ which obeys $g(x_i) = y_i$ for $i \in \mathbb{N}_N^0$. This function g is called a fractal interpolation function (FIF) or simply fractal function and it is the unique function satisfying the following fixed point equation:

$$g(x) = F_i(L_i^{-1}(x), g(L_i^{-1}(x))) \text{ for all } x \in I_i,\ i \in \mathbb{N}_N. \tag{6}$$

The widely studied FIFs so far are defined by the iterated mappings

$$L_i(x) = a_i\ x\ +\ d_i, \ \ F_i(x, y) = \alpha_i\ y\ + q_i(x), \ \ i \in \mathbb{N}_N, \tag{7}$$

where the real constants a_i and d_i are determined by the condition (1) as

$$a_i = \frac{(x_i - x_{i-1})}{(x_N - x_0)} \text{ and } d_i = \frac{(x_N x_{i-1} - x_0 x_i)}{(x_N - x_0)}, \tag{8}$$

[a] e-mail: nasim.iitm@gmail.com (corresponding author)

and $q_i(x)$'s are suitable continuous functions, such that the conditions (3) and (4) hold. For each i, α_i is a free parameter with $|\alpha_i| < 1$ and is called a vertical scaling factor of the transformation W_i. Then, the vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is called the scale vector of the IFS. If $q_i(x)$ is taken as linear, then the corresponding FIF is known as affine FIF (AFIF). Let $\mathcal{C}(I)$ denote the normed space of real-valued continuous functions on I endowed with the uniform norm $\|f\|_\infty = \sup\{|f(x)| : x \in I\}$. Let $f \in \mathcal{C}(I)$. Consider the case

$$q_i(x) = f(L_i(x)) - \alpha_i\, b(x), \tag{9}$$

where $b(x)$ is a continuous function, such that $b(x_0) = f(x_0)$ and $b(x_N) = f(x_N)$. Let f^α be the continuous map whose graph is the attractor of the IFS (7), (8), and (9). Then, the function f^α is called the α-fractal function associated with f with respect to the function $b(x)$ and the partition Δ according to the definition of Navascués in the reference [2]. The function f^α interpolates and approximates f. In [3,4], authors defined new approximation classes consisting of self-referential functions. The graph of the of the function f^α may have non-integer fractal dimensions [5,6]. For results on the fractal dimensions of different fractal functions, interested reader may see [4,7–14] and references therein. In [15], the authors introduces the novel notion of dimension preserving approximation for continuous functions defined on $[0,1]$ and initiates the study of it. From (6) and (9), f^α satisfies the following fixed point equation:

$$f^\alpha(x) = f(x) + \alpha_i\,(f^\alpha - b) \circ L_i^{-1}(x) \quad \text{for all } x \in I_i, i \in \mathbb{N}_N. \tag{10}$$

From (10), it is easy to deduce the following inequality:

$$\|f^\alpha - f\|_\infty \le \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|f - b\|_\infty\ , \tag{11}$$

where $|\alpha|_\infty = \max\{|\alpha_i| : i \in \mathbb{N}_N\}$. For $\alpha = 0$, the fractal function f^α agrees with f. The theory of α-fractal function for different choices of $b(x)$ can be found, in [16–18]. In this article, we consider the case

$$b = Lf\ ,$$

where $L : \mathcal{C}(I) \to \mathcal{C}(I)$ is a linear and bounded operator with respect to the least square norm

$$\|f\|_{\mathcal{L}^2} = \left(\int_a^b |f|^2 \mathrm{d}x\right)^{1/2},$$

such that $Lf(x_0) = f(x_0)$, $Lf(x_N) = f(x_N)$ and $L \neq I_d$. The following result can be found in [19].

Theorem 1 (a) *The operator*

$$\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$$
$$f \hookrightarrow f^\alpha$$

is linear and bounded with respect to the $\mathcal{L}^2$-norm.
(b) *If $\alpha = 0$, $\mathcal{F}^\alpha$ is identity operator I_d.*
(c) *The following inequalities hold:*

$$\|\mathcal{F}^\alpha\|_2 \le 1 + \frac{|\alpha|_\infty}{1 - \|\alpha\|_\infty}\|I_d - L\|_2\ ,$$
$$\|I_d - \mathcal{F}^\alpha\|_2 \le \frac{|\alpha|_\infty}{1 - \|\alpha\|_\infty}\|I_d - L\|_2\ ,$$

where $\|T\|_2$ is the operator norm defined as

$$\|T\|_2 = \sup\{\|T(f)\|_{\mathcal{L}^2} : \|f\|_{\mathcal{L}^2} = 1, f \in \mathcal{C}(I)\}\ .$$

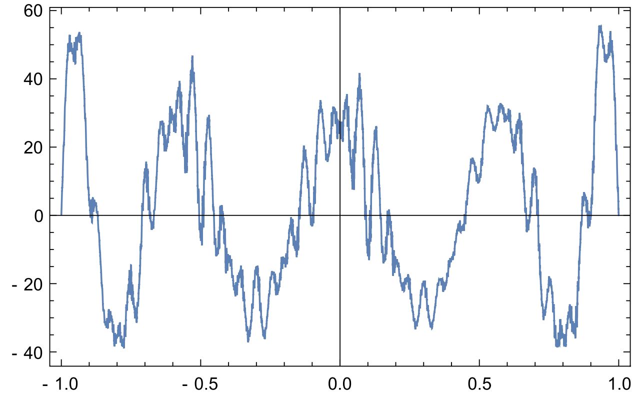

Fig. 1 Fractal function

Figure 1 represents the fractal map f^α for $f(x) = P_{10}^2(x)$, where $P_{10}^2(x)$ is the associated Legendre polynomial of orders 10, 2; $b(x) = f(x)v(x)$, where $v(x) = 4 - 3|x|$, $I = [-1, 1]$, $N = 10$, the sampling is uniform and $\alpha = (0.15, -0.2, 0.3, -0.15, 0.2, 0.3, -0.1, 0.1, -0.2, 0.2)$.

The fractal functions on the sphere are initially considered in [19]. In [20], the authors considered a family of continuous functions on the unit sphere $S \subseteq \mathbb{R}^3$ generalizing the spherical harmonics. In the present paper, the fractalization of the continuous functions on the unit sphere $S \subseteq \mathbb{R}^3$ is different from the existing ones. Therefore, the present paper can be considered as an amalgam of the articles [19,20], although some extensions are given in some cases.

2 Fractal functions on the sphere

Let us consider a family of continuous functions

$$u_{nm} : J = [-1, 1] \to \mathbb{R}\ ,$$

such that for any nonnegative integer m, the system of functions

$$\mathcal{U}_m = \{u_{nm}; n = p, p+1, \ldots\} \tag{12}$$

forms an orthogonal system in $\mathcal{C}(J)$ with respect to the inner product

$$\langle g, h\rangle = \int_{-1}^{1} g(t)h(t)\mathrm{d}t \, ,$$

where p is the least positive integer, such that

$$\frac{m}{2} \leq p. \tag{13}$$

Let

$$\big\{v_m: \ I = [0, 2\pi] \to \mathbb{R} |\ v_m \text{ is continuous and periodic}, \\ m = 0, 1, 2, \dots\big\}$$

be an orthonormal system in $\mathcal{C}(I)$ with respect to the inner product

$$\langle g, h\rangle = \int_{0}^{2\pi} g(t)h(t)\mathrm{d}t \, .$$

Let $P = (\varphi, \theta)$ represent a point on the unit sphere S. Let us define functions on the unit sphere S as

$$H_{nm}(\varphi, \theta) = u_{nm}(\cos\varphi)v_m(\theta), \\ (n = 0, 1, 2, \dots; m = 0, 1, 2, \dots, 2n). \tag{14}$$

On $\mathcal{L}^2(S)$, define the inner product

$$\langle F, G\rangle = \int_S F.G\mathrm{d}S \ \text{ for } \ F, G \in \mathcal{L}^2(S)$$

and the norm

$$\begin{aligned}\|F\|_{\mathcal{L}^2(S)} &= \Big(\langle F, F\rangle\Big)^{1/2} \\ &= \Big(\int_S |F|^2 \mathrm{d}S\Big)^{1/2} \ \text{for } F \in \mathcal{L}^2(S) \, .\end{aligned}$$

Lemma 1 *For any nonnegative integer m*

$$\langle H_{nm}, H_{rm}\rangle = 0, \ \ (n, r = p, p+1, p+2, \dots; n \neq r) \, ,$$

where p is defined as in (13).

Proof The inner product can be expressed in spherical coordinates as

$$\begin{aligned}\langle H_{nm}, H_{rm}\rangle &= \int_0^{2\pi}\int_0^{\pi} u_{nm}(\cos\varphi)u_{rm}(\cos\varphi) \\ &\qquad |v_m(\theta)|^2 \sin\varphi \mathrm{d}\varphi\mathrm{d}\theta \\ &= \Big(\int_0^{2\pi} |v_m(\theta)|^2 \mathrm{d}\theta\Big) \\ &\qquad \Big(\int_0^{\pi} u_{nm}(\cos\varphi)u_{rm}(\cos\varphi)\sin\varphi\mathrm{d}\varphi\Big).\end{aligned}$$

Since the second integral is zero if $n \neq r$, it follows that $\langle H_{nm}, H_{rm}\rangle = 0$. □

Lemma 2 *For any nonnegative integer m*

$$\|H_{nm}\|_{\mathcal{L}^2(S)} = \|u_{nm}\|_{\mathcal{L}^2(J)}, \ (n = p, p+1, p+2, \dots) \, .$$

Proof We have

$$\begin{aligned}\|H_{nm}\|^2_{\mathcal{L}^2(S)} &= \int_0^{2\pi}\int_0^{\pi} |u_{nm}(\cos\varphi)|^2 |v_m(\theta)|^2 \sin\varphi\mathrm{d}\varphi\mathrm{d}\theta \\ &= \Big(\int_{-1}^{1} |u_{nm}(t)|^2 \mathrm{d}t\Big)\Big(\int_0^{2\pi} |v_m(\theta)|^2 \mathrm{d}\theta\Big) \\ &= \|u_{nm}\|^2_{\mathcal{L}^2(J)},\end{aligned}$$

since the second integral is 1 according to the definition of $\{v_m\}$. Hence, the result follows. □

Let us define, for any nonnegative integer m

$$\mathcal{H}^j_m = \Big\{\sum_{n=p}^{j} \lambda_{nm} H_{nm}; \lambda_{nm} \in \mathbb{R}\Big\} \, ,$$

where p is the least integer, such that $\frac{m}{2} \leq p$. Note that

$$\mathcal{H}^j_m = \emptyset \text{ if } p > j \, .$$

Due to Lemma 1, $\{H_{nm}\}^j_{n=p}$ is an orthogonal basis for $\mathcal{H}^j_m$. Define

$$\mathcal{H}_m = \overline{\cup_{j=0}^{\infty}\mathcal{H}^j_m} = \overline{\mathrm{span}\{H_{nm} : n = p, p+1, \dots\}} \, .$$

If $\|u_{nm}\|_{\mathcal{L}^2(J)} = 1$, for all n, m, then $\{H_{nm}\}^{\infty}_{n=p}$ is an orthonormal Schauder basis for $\mathcal{H}_m$. In this article, we will fractalize the second function $v_m(\theta)$ in $H_{nm}(\varphi, \theta)$ as

$$H^{\alpha}_{nm}(\varphi, \theta) = u_{nm}(\cos\varphi)v^{\alpha}_m(\theta), \tag{15}$$

where $v^{\alpha}_m(\theta) = \mathcal{F}^{\alpha}(v_m(\theta))$, $\mathcal{F}^{\alpha}$ is the operator defined in Sect. 1. Now, consider

$$(\mathcal{H}^j_m)^{\alpha} = \Big\{\sum_{n=p}^{j} \lambda_{nm} H^{\alpha}_{nm}; \lambda_{nm} \in \mathbb{R}\Big\} \, .$$

Then, $\{H^\alpha_{nm}\}_{n=0}^{j}$ is an orthogonal basis for $(\mathcal{H}^j_m)^\alpha$. For instance

$$\begin{aligned}&\langle H^\alpha_{nm}(\varphi,\theta), H^\alpha_{rm}(\varphi,\theta)\rangle\\ &\quad=\int_0^{2\pi}\int_0^{\pi} u_{nm}(\cos\varphi)u_{rm}(\cos\varphi)\\ &\qquad|v^\alpha_m(\theta)|^2\sin\varphi \mathrm{d}\varphi\mathrm{d}\theta\\ &\quad=\left(\int_0^{\pi} u_{nm}(\cos\varphi)u_{rm}(\cos\varphi)\sin\varphi\mathrm{d}\varphi\right)\\ &\qquad\left(\int_0^{2\pi}|v^\alpha_m(\theta)|^2\mathrm{d}\theta\right)=0,\end{aligned}$$

since $\langle u_{nm}, u_{rm}\rangle = 0$ for $n \neq r$.

Lemma 3 *For* $n = p, p+1, p+2, \ldots$ *and* $m = 0, 1, 2, \ldots$

$$\begin{aligned}\|H^\alpha_{nm}\|_{\mathcal{L}^2(S)} &= \|u_{nm}\|_{\mathcal{L}^2(J)}\|v^\alpha_m\|_{\mathcal{L}^2(I)}\\ &\leq \|\mathcal{F}^\alpha\|_2\|H_{nm}\|_{\mathcal{L}^2(S)} .\end{aligned}$$

Proof For instance

$$\begin{aligned}\|H^\alpha_{nm}\|^2_{\mathcal{L}^2(S)} &= \int_0^{2\pi}\int_0^{\pi}|u_{nm}(\cos\varphi)|^2|v^\alpha_m(\theta)|^2\sin\varphi\mathrm{d}\varphi\mathrm{d}\theta\\ &=\left(\int_{-1}^{1}|u_{nm}(t)|^2\mathrm{d}t\right)\left(\int_0^{2\pi}|v^\alpha_m(\theta)|^2\mathrm{d}\theta\right)\\ &=\|u_{nm}\|^2_{\mathcal{L}^2(J)}\|v^\alpha_m\|^2_{\mathcal{L}^2(I)}\\ &\leq \|u_{nm}\|^2_{\mathcal{L}^2(J)}\|\mathcal{F}^\alpha\|^2_2\|v_m\|^2_{\mathcal{L}^2(I)}.\end{aligned}$$

Since $\{v_m\}_{m=0}^{\infty}$ is an orthonormal family, it follows that (Lemma 2):

$$\|H^\alpha_{nm}\|_{\mathcal{L}^2(S)} \leq \|\mathcal{F}^\alpha\|_2\|H_{nm}\|_{\mathcal{L}^2(S)} .$$

□

Theorem 2 *The operator*

$$\Theta^\alpha_m : \cup_{j=0}^{\infty}\mathcal{H}^j_m \to \mathcal{L}^2(S)\ ,$$
$$\sum_{n=p}^{j}\lambda_{nm}H_{nm} \hookrightarrow \sum_{n=p}^{j}\lambda_{nm}H^\alpha_{nm}$$

is linear and bounded.

Proof The linearity of Θ^α_m is obvious. For boundedness, due to the orthogonality of $\{H_{nm}\}_{n=p}^{j}$ and $\{H^\alpha_{nm}\}_{n=p}^{j}$, we get

$$\begin{aligned}&\left\|\Theta^\alpha_m\left(\sum_{n=p}^{j}\lambda_{nm}H_{nm}\right)\right\|^2_{\mathcal{L}^2(S)}\\ &\quad=\sum_{n=p}^{j}\|\lambda_{nm}H^\alpha_{nm}\|^2_{\mathcal{L}^2(S)},\\ &\quad\text{using Lemma 3,}\\ &\quad\leq\sum_{n=p}^{j}\|\mathcal{F}^\alpha\|^2_2\|\lambda_{nm}H_{nm}\|^2_{\mathcal{L}^2(S)}\\ &\quad=\|\mathcal{F}^\alpha\|^2_2\|\sum_{n=p}^{j}\lambda_{nm}H_{nm}\|^2_{\mathcal{L}^2(S)}.\end{aligned}$$

Consequently, $\|\Theta^\alpha_m\|_2 \leq \|\mathcal{F}^\alpha\|_2$. □

Since $\cup_{j=0}^{\infty}\mathcal{H}^j_m$ is dense in $\mathcal{H}_m$, the operator Θ^α_m can be extended to $\mathcal{H}_m$ preserving the norm. Let us denote the extension as $\overline{\Theta}^\alpha_m$. By linearity and continuity

$$\overline{\Theta}^\alpha_m\left(\sum_{n=p}^{\infty}\lambda_{nm}H_{nm}\right)=\sum_{n=p}^{\infty}\lambda_{nm}H^\alpha_{nm}.$$

Theorem 3 [21] (Vitali's completeness criterion) *Let* $\{\phi_n\}_{n=1}^{\infty}$ *be an othonormal sequence of functions in* $\mathcal{L}^2(a,b)$ *where* a, b *are finite. Then,* (ϕ_n) *is complete in* $\mathcal{L}^2(a,b)$ *if and only if*

$$\sum_n\left(\int_a^r\phi_n\right)^2 = r - a$$

for every $r \in (a,b)$.

Corollary 1 *Let* (a,b) *be a finite or infinite interval of* $\mathbb{R}$, *let* g *belong to* $\mathcal{L}^2_w(a,b)$, $g \neq w$, *where* w *is a positive continuous weight function, and let* (ϕ_n) *be an orthonormal sequence in* $\mathcal{L}^2_w(a,b)$. *Then,* (ϕ_n) *is complete in* $\mathcal{L}^2_w(a,b)$ *(equivalently* $(\phi_n\sqrt{w})$ *is complete in* $\mathcal{L}^2(a,b)$*) if and only if*

$$\sum_n\left|\int_a^r\phi_n(x)g(x)w(x)\mathrm{d}x\right|^2=\int_a^r|g(x)|^2w(x)\mathrm{d}x$$

for every r *in* (a,b).

From here on, let us assume that for any nonnegative integer $m = 01, 2, \ldots$, the system $\mathcal{U} = \cup_{m=0}^{\infty}\mathcal{U}_m$ given in (12) is orthonormal, that is

$$\int_{-1}^{1}u_{nm}(x)u_{jr}(x)\mathrm{d}x = \delta_{nj}\delta_{mr} \Rightarrow ||u_{nm}||_{\mathcal{L}^2(J)} = 1 \tag{16}$$

and $\mathcal{U}_m$ forms a complete system in $\mathcal{L}^2(J)$, where m is the least integer, such that $\frac{m}{2} \leq p$. Also, assume that $\{v_m\}_{m=0}^{\infty}$ is a complete orthonormal system in $\mathcal{L}^2(0,2\pi)$. Then, by Vitali's completeness criterion (Theorem 3)

$$\sum_{m=0}^{\infty}\left(\int_0^{\theta} v_m(\theta')\mathrm{d}\theta'\right)^2 = \theta \text{ for every } \theta \in (0,2\pi). \tag{17}$$

Lemma 4 *For any nonnegative integer m*

$$\sum_{n=p}^{\infty}\left(\int_0^{\varphi} u_{nm}(\cos\varphi')\sin\varphi'\mathrm{d}\varphi'\right)^2 = 1-\cos\varphi$$

for every $\varphi \in (0,\pi)$.

Proof For any nonnegative integer m, the completeness of the system $\mathcal{U}_m$ in $\mathcal{L}^2(J)$ implies the completeness of the system $\{u_{nm}(\cos\varphi)|\ n = p, p+1, p+2, \dots\}$ in $\mathcal{L}^2(0,\pi)$. Then, by modified Vitali's criterion Corollary 1, taking $g=1$

$$\sum_{n=p}^{\infty}\left(\int_0^{\varphi} u_{nm}(\cos\varphi')\sin\varphi'\mathrm{d}\varphi'\right)^2 = \int_0^{\varphi} 1^2.\sin\varphi'\mathrm{d}\varphi' = 1-\cos\varphi\ .$$

□

The Vitali's completeness criterion for the functions on the sphere is the following.

Lemma 5 *Let S denote the unit sphere with* (φ,θ) *as usual spherical polar coordinates. Let* $\{f_n\}$ *be a set of functions which are orthonormal over S, that is*

$$\int_S f_n f_m = \delta_{nm}\ .$$

The orthonormal sequence (f_n) *is complete in* $\mathcal{L}^2(S)$ *if and only if*

$$\sum_n\left[\int_0^{\theta}\int_0^{\varphi} f_n(\varphi',\theta')\sin\varphi'\mathrm{d}\varphi'\right]^2 = \theta(1-\cos\varphi)$$

for every $\theta \in (0,2\pi)$ *and every* $\varphi \in (0,\pi)$.

Proof See [22]. □

Theorem 4 *The family*

$$\left\{H_{nm}:\ n=0,1,2,\dots;m=0,1,2,\dots,2n\right\} = \left\{H_{nm}:\ n=p,p+1,p+2,\dots;m=0,1,2,\dots\right\}$$

form an orthonormal complete system of $\mathcal{L}^2(S)$.

Proof Recall that $H_{nm}(\varphi,\theta) = u_{nm}(\cos\varphi)v_m(\theta)$. Due to Lemmas 1, 2, and (16), the family $\{H_{nm} : n = 0,1,2,\dots;m = 0,1,2,\dots,2n\}$ is orthonormal. Let $\Phi_n(S)$ be the orthonormal sequence of functions of the above system. Then

$$\sum_{n=0}^{\infty}\left[\int_0^{\theta}\mathrm{d}\theta'\int_0^{\varphi}\Phi_n(\varphi',\theta')\sin\varphi'\mathrm{d}\varphi'\right]^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{2n}\left[\int_0^{\theta} v_m(\theta')\mathrm{d}\theta'\int_0^{\varphi} u_{nm}(\cos\varphi')\sin\varphi'\mathrm{d}\varphi'\right]^2$$
$$= \sum_{m=0}^{\infty}\left[\int_0^{\theta} v_m(\theta')\mathrm{d}\theta'\right]^2\sum_{n=p}^{\infty}\left[\int_0^{\varphi} u_{nm}(\cos\varphi')\sin\varphi'\mathrm{d}\varphi'\right]^2,$$

inverting the order of integration on the right-hand side of the last inequality. Using Lemma 4 along with (17)

$$\sum_{n=0}^{\infty}\left[\int_0^{\theta}\mathrm{d}\theta'\int_0^{\varphi}\Phi_n(\varphi',\theta')\sin\varphi'\mathrm{d}\varphi'\right]^2 = \theta(1-\cos\varphi)\ .$$

Therefore, by Lemma 5, it follows that $\{H_{nm} :\ n = 0,1,2,\dots;m = 0,1,2,\dots,2n\}$ forms a complete system in $\mathcal{L}^2(S)$. □

Theorem 5 *For any* $f \in \mathcal{L}^2(S)$, *the operator* $\Theta^{\alpha} : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$ *defined by*

$$\Theta^{\alpha}(f) = \sum_{n=0}^{+\infty}\sum_{m=0}^{2n} c_{nm}H_{nm}^{\alpha}$$

if

$$f = \sum_{n=0}^{+\infty}\sum_{m=0}^{2n} c_{nm}H_{nm}$$

is linear and bounded.

Proof Let us consider the mapping

$$\mathcal{G}^{\alpha} : \mathrm{span}(H_{nm}) \to \mathcal{L}^2(S)$$

defined by

$$\mathcal{G}^{\alpha}(f) = \mathcal{G}^{\alpha}\left(\sum_{n=0}^{N}\sum_{m=j}^{2n} c_{nm}H_{nm}\right) = \sum_{n=0}^{N}\sum_{m=j}^{2n} c_{nm}H_{nm}^{\alpha}.$$

The operator $\mathcal{G}^\alpha$ is linear. If one consider $f \in$ span (H_{nm}), then $f \in \mathcal{H}_j$ for j sufficiently large, and

$$\mathcal{G}^\alpha(f) = \overline{\Theta}_j^\alpha(f)$$
$$||\mathcal{G}^\alpha(f)||_{\mathcal{L}^2(S)} = ||\Theta_j^\alpha(f)||_{\mathcal{L}^2(S)} \le ||\mathcal{F}^\alpha||_2||f||.$$

Consequently

$$||\mathcal{G}^\alpha||_2 \le ||\mathcal{F}^\alpha||_2$$

and $\mathcal{G}^\alpha$ is bounded and linear. Since $\overline{\text{span}}(H_{nm}) = \mathcal{L}^2(S)$, $\mathcal{G}^\alpha$ can be extended to $\mathcal{L}^2(S)$ preserving the norm. Let us denote the extension by $\Theta^\alpha : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$. The linearity and continuity of Θ^α imply that

$$\Theta^\alpha(f) = \sum_{n=0}^{+\infty}\sum_{m=0}^{2n} c_{nm}H_{nm}^\alpha$$

if

$$f = \sum_{n=0}^{+\infty}\sum_{m=0}^{2n} c_{nm}H_{nm} .$$

Moreover, $||\Theta^\alpha||_2 \le ||\mathcal{F}^\alpha||_2$. □

3 Fractal basis for $\mathcal{L}^2(S)$

In this section, we will consider the linear bounded operator $L : \mathcal{C}(I) \to \mathcal{C}(I)$ with respect to least square norm given in Sect. 1. Let L_H be defined on the basic elements as

$$L_H(H_{nm})(\varphi,\theta) = u_{nm}(\cos\varphi)L(v_m(\theta)). \tag{18}$$

Then, by linearity, it can be extended $L_H : \mathcal{H}_m^j \to \mathcal{L}^2(S)$, such that $\|L_H\|_2 \le \|L\|_2$. In Theorem 5, the operator L_H can be extended to $L_S : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$, such that $\|L_S\|_2 = \|L_H\|_2 \le \|L\|_2$.

Lemma 6

$$\|\Theta^\alpha(H_{nm}) - H_{nm}\|_{\mathcal{L}^2(S)} \le |\alpha|_\infty \|\Theta^\alpha(H_{nm}) - L_S(H_{nm})\|_{\mathcal{L}^2(S)}.$$

Proof Note that

$$\Theta^\alpha(H_{nm}) = H_{nm}^\alpha .$$

Therefore

$$\begin{aligned}\|\Theta^\alpha(H_{nm}) - H_{nm}\|^2_{\mathcal{L}^2(S)} &= \|H^\alpha_{nm} - H_{nm}\|^2_{\mathcal{L}^2(S)}\\ &= \int_0^{2\pi}\int_0^\pi u^2_{nm}(\cos\varphi)|v^\alpha_m(\theta) - v_m(\theta)|^2 \sin\varphi \mathrm{d}\varphi \mathrm{d}\theta\\ &= \|u_{nm}\|^2_{\mathcal{L}^2(J)}\|v^\alpha_m - v_m\|^2_{\mathcal{L}^2(I)}.\end{aligned}$$

However, from (10), for $f = v_m$ and $b = Lf$

$$\|v_m^\alpha - v_m\|^2_{\mathcal{L}^2(I)} = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} |\alpha_i|^2 |(v_m^\alpha - Lv_m)\circ L_i^{-1}(x)|^2 \mathrm{d}x .$$

By changing of variable $\widetilde{x} = L_i^{-1}(x)$, it follows that:

$$\begin{aligned}\|v_m^\alpha - v_m\|^2_{\mathcal{L}^2(I)} &= \sum_{i=1}^N a_i|\alpha_i|^2\int_0^{2\pi}|(v_m^\alpha - Lv_m)(\widetilde{x})|^2\mathrm{d}\widetilde{x}\\ &= \sum_{i=1}^N a_i|\alpha_i|^2\|v_m^\alpha - Lv_m\|^2_{\mathcal{L}^2(I)}\\ &\le |\alpha|^2_\infty\|v_m^\alpha - Lv_m\|^2_{\mathcal{L}^2(I)}\sum_{i=1}^N a_i\\ &= |\alpha|^2_\infty\|v_m^\alpha - Lv_m\|^2_{\mathcal{L}^2(I)},\end{aligned}$$

since $\sum_{i=1}^N a_i = 1$. Therefore

$$\|\Theta^\alpha(H_{nm}) - H_{nm}\|^2_{\mathcal{L}^2(S)} \le |\alpha|^2_\infty \|u_{nm}\|^2_{\mathcal{L}^2(I)}\|v_m^\alpha - Lv_m\|^2_{\mathcal{L}^2(I)} .$$

On the other hand

$$\begin{aligned}&\|\Theta^\alpha(H_{nm}) - L_S(H_{nm})\|^2_{\mathcal{L}^2(S)}\\ &= \|H^\alpha_{nm} - L_S(H_{nm})\|^2_{\mathcal{L}^2(S)}\\ &= \int_0^{2\pi}\int_0^\pi u^2_{nm}(\cos\varphi)|v^\alpha_m(\theta) - L(v_m(\theta))|^2\sin\varphi\mathrm{d}\varphi\mathrm{d}\theta\\ &= \|u_{nm}\|^2_{\mathcal{L}^2(J)}\|v^\alpha_m - Lv_m\|^2_{\mathcal{L}^2(I)}.\end{aligned}$$

Hence the proof. □

Lemma 7 *For any* $f \in \mathcal{L}^2(S)$

$$\|\Theta^\alpha(f) - f\|_{\mathcal{L}^2(S)} \le |\alpha|_\infty \|\Theta^\alpha(f) - L_S f\|_{\mathcal{L}^2(S)} \tag{19}$$

and

$$\|\Theta^\alpha(f) - f\|_{\mathcal{L}^2(S)} \le \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|Id - L_S\|_2\|f\|_{\mathcal{L}^2(S)}. \tag{20}$$

Proof For any $f \in \mathcal{L}^2(S)$, let us consider a sequence X_k in $\mathcal{H}_{m_k}$, such that $f = \lim X_k$ with respect to the $\mathcal{L}^2$-norm (such sequence exists due to Theorem 5). Also, continuity of L_S implies that $L_S f = \lim L_S X_k$. Due to the continuity of Θ^α and the norm, it follows that:

$$\|\Theta^\alpha(f) - f\|^2_{\mathcal{L}^2(S)} = \lim_{k\to\infty}\|\Theta^\alpha(X_k) - X_k\|^2_{\mathcal{L}^2(S)}. \tag{21}$$

Now, for $X \in \mathcal{H}_m$

$$X = \sum_{n=p}^{\infty} \lambda_{nm} H_{nm}.$$

Note that $\{H_{nm}^{\alpha} - H_{nm}\}_{n=p}^{\infty}$ is orthogonal. For instance

$$\begin{aligned}&\langle H_{nm}^{\alpha} - H_{nm}, H_{rm}^{\alpha} - H_{rm}\rangle \\ &\quad = \int_0^{2\pi}\int_0^{\pi} u_{nm}(\cos\varphi) u_{rm}(\cos\varphi) \\ &\qquad |v_m^{\alpha}(\theta) - v_m|^2 \sin\varphi \mathrm{d}\varphi \mathrm{d}\theta \\ &\quad = \left(\int_0^{\pi} u_{nm}(\cos\varphi) u_{rm}(\cos\varphi)\sin\varphi \mathrm{d}\varphi\right) \\ &\qquad \left(\int_0^{2\pi} |v_m^{\alpha}(\theta) - v_m|^2 \mathrm{d}\theta\right) \\ &\quad = 0,\end{aligned}$$

since the first integral is zero. Therefore

$$\begin{aligned}\|\Theta^{\alpha}(X) - X\|_{\mathcal{L}^2(S)}^2 &= \sum_{n=p}^{\infty} |\lambda_{nm}|^2 \|H_{nm}^{\alpha} - H_{nm}\|_{\mathcal{L}^2(S)}^2 \\ &\le \sum_{n=p}^{\infty} |\alpha|_{\infty}^2 |\lambda_{nm}|^2 \|H_{nm}^{\alpha} \\ &\quad - L_S H_{nm}\|_{\mathcal{L}^2(S)}^2,\end{aligned}$$

using Lemma 6. Also, $H_{nm}^{\alpha} - L_S H_{nm}, H_{rm}^{\alpha} - L_S H_{rm}$ are orthogonal for $n \neq r$, due to the orthogonality of u_{nm}, u_{rm} for $n \neq r$. Therefore

$$\|\Theta^{\alpha}(X) - X\|_{\mathcal{L}^2(S)}^2 \le |\alpha|_{\infty}^2 \|\Theta^{\alpha}(X) - L_S X\|_{\mathcal{L}^2(S)}^2. \tag{22}$$

Using it in (21) for $X = X_k$, it follows that:

$$\begin{aligned}\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)}^2 &= \lim_{k\to\infty} \|\Theta^{\alpha}(X_k) - X_k\|_{\mathcal{L}^2(S)}^2 \\ &\le |\alpha|_{\infty}^2 \lim_{k\to\infty} \|\Theta^{\alpha}(X_k) - L_S X_k\|_{\mathcal{L}^2(S)}^2 \\ &= |\alpha|_{\infty}^2 \|\Theta^{\alpha}(f) - L_S f\|_{\mathcal{L}^2(S)}^2\end{aligned}$$

and therefore

$$\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)} \le |\alpha|_{\infty} \|\Theta^{\alpha}(f) - L_S f\|_{\mathcal{L}^2(S)} .$$

For the second inequality

$$\begin{aligned}\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)} &\le |\alpha|_{\infty} \|\Theta^{\alpha}(f) - L_S f\|_{\mathcal{L}^2(S)} \\ &\le |\alpha|_{\infty} \big(\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)} \\ &\quad + \|f - L_S f\|_{\mathcal{L}^2(S)}\big)\end{aligned}$$

and the result follows. □

Proposition 1 *If $|\alpha|_{\infty} < \|L\|_2^{-1}$, then Θ^{α} is injective and its range is closed.*

Proof From (19), with $\|L_S\|_2 \le \|L\|_2$, it follows that:

$$\begin{aligned}&\|f\|_{\mathcal{L}^2(S)} - \|\Theta^{\alpha}(f)\|_{\mathcal{L}^2(S)} \le |\alpha|_{\infty} \big(\|\Theta^{\alpha}(f)\|_{\mathcal{L}^2(S)} \\ &\quad + \|L\|_2 \|f\|_{\mathcal{L}^2(S)}\big) .\end{aligned}$$

Therefore

$$\|f\|_{\mathcal{L}^2(S)} \le \frac{1 + |\alpha|_{\infty}}{1 - |\alpha|_{\infty}\|L\|_2} \|\Theta^{\alpha}(f)\|_{\mathcal{L}^2(S)}. \tag{23}$$

If $\Theta^{\alpha}(f) = 0$, then $f = 0$, and consequently, $\Theta^{\alpha}(f)$ is injective. To show that the range of Θ^{α} is closed, consider a convergent sequence $\Theta^{\alpha}(f_n)$, such that $\Theta^{\alpha}(f_n) \to g$. Since the sequence $\Theta^{\alpha}(f_n)$ is convergent, it is also Cauchy, and therefore, according to (23), f_n is also a Cauchy sequence. As a consequence, f_n is convergent in a Banach space. If $f_n \to f$ as $n \to \infty$. Then, the continuity of Θ^{α} implies that

$$\Theta^{\alpha}(f) = \lim_{n\to\infty} \Theta^{\alpha}(f_n) = g .$$

Therefore, g belongs to range of Θ^{α}, and hence, range of Θ^{α} is closed. □

The treatise [23] is a good reference for the basic definitions used in the sequel.

Definition 1 Let H be a Hilbert space. A sequence $(x_k) \subset H$ is a Bessel sequence in H if there exists a constant $B > 0$, such that for all $x \in H$

$$\sum_{k=0}^{\infty} |\langle x, x_k\rangle|^2 \le B\|x\|^2.$$

Proposition 2 *For any scale vector α with $|\alpha|_{\infty} < 1$, (H_{nm}^{α}) is a Bessel sequence.*

Proof For any $f \in \mathcal{L}^2(S)$

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{2n} |\langle f, H_{nm}^{\alpha}\rangle|^2 &= \sum_{n=0}^{\infty}\sum_{m=0}^{2n} |\langle f, \Theta^{\alpha}(H_{nm})\rangle|^2 \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{2n} |\langle (\Theta^{\alpha})^*(f), H_{nm}\rangle|^2,\end{aligned}$$

where $(\Theta^{\alpha})^*$ is the adjoint operator of Θ^{α}. Applying Parseval identity to the orthonormal basis H_{nm}, it follows that:

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{2n} |\langle f, H_{nm}^{\alpha}\rangle|^2 &= \|(\Theta^{\alpha})^*(f)\|_{\mathcal{L}^2(S)}^2 \\ &\le \|\Theta^{\alpha}\|_2^2 \|f\|_{\mathcal{L}^2(S)}^2,\end{aligned} \tag{24}$$

since $\|(\Theta^\alpha)^*\|_2^2 = \|\Theta^\alpha\|_2^2$. Therefore, (H^α_{nm}) is a Bessel sequence with Bessel constant $B = \|\Theta^\alpha\|_2^2$. □

Definition 2 A sequence (x_k) in a Hilbert space H is a frame if there exist numbers $A, B > 0$, such that for all $x \in H$, we have

$$A\|x\|^2 \leq \sum_{k=0}^{\infty} |\langle x, x_k\rangle|^2 \leq B\|x\|^2. \tag{25}$$

Definition 3 A sequence (x_k) in a Hilbert space H is a frame sequence if it is a frame for its closed span $[x_k] = \overline{\text{span}}(x_k)$.

Proposition 3 *If* $|\alpha|_\infty < \|L\|_2^{-1}$, (H^α_{nm}) *is a frame sequence.*

Proof In the proof of Proposition 2, for any $g \in \mathcal{L}^2(S)$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{2n} |\langle g, H^\alpha_{nm}\rangle|^2 \leq \|\Theta^\alpha\|_2^2 \|g\|^2_{\mathcal{L}^2(S)} .$$

Therefore, right-hand inequality of (25) holds for $B = \|\Theta^\alpha\|_2^2$.

If $|\alpha|_\infty < \|L\|_2^{-1}$, then due to Proposition 1, Θ^α is injective with closed range. Then, range of Θ^α, $rg(\Theta^\alpha)$ is a Hilbert space, since it is a closed subspace of a Hilbert space $\mathcal{L}^2(S)$. Consequently, $(\Theta^\alpha)^{-1}$ is well defined, linear, and bounded as Θ^α (see, e.g., Theorem 3.5.3, [24]). Therefore, $\Theta^\alpha \circ (\Theta^\alpha)^{-1}$ is the identity operator on $rg(\Theta^\alpha)$

$$[H^\alpha_{nm}] = \overline{\text{span}}(H^\alpha_{nm}) \subseteq rg(\Theta^\alpha) .$$

However, for any $g \in [H^\alpha_{nm}]$

$$g = \left((\Theta^\alpha)^{-1}\right)^* \circ (\Theta^\alpha)^*(g),$$

and thus

$$\|g\|^2_{\mathcal{L}^2(S)} \leq \|(\Theta^\alpha)^{-1}\|_2^2 \|(\Theta^\alpha)^*(g)\|^2_{\mathcal{L}^2(S)}, \tag{26}$$

since

$$\|\left((\Theta^\alpha)^{-1}\right)^*\|_2 = \|(\Theta^\alpha)^{-1}\|_2 .$$

As in the proof of Proposition 2

$$\|(\Theta^\alpha)^*(g)\|^2_{\mathcal{L}^2(S)} = \sum_{n=0}^{\infty}\sum_{m=0}^{2n} |\langle g, H^\alpha_{nm}\rangle|^2. \tag{27}$$

Using it in (26)

$$\|g\|^2_{\mathcal{L}^2(S)} \leq \|(\Theta^\alpha)^{-1}\|^2 \sum_{n=0}^{\infty}\sum_{m=0}^{2n} |\langle g, H^\alpha_{nm}\rangle|^2 .$$

Denoting

$$A = \|(\Theta^\alpha)^{-1}\|^{-2} ,$$

it follows that:

$$A\|g\|^2_{\mathcal{L}^2(S)} \leq \sum_{n=0}^{\infty}\sum_{m=0}^{2n} |\langle g, H^\alpha_{nm}\rangle|^2 .$$

This completes the proof. □

Definition 4 A sequence (x_k) in a Hilbert space H is a Riesz sequence if there exist $k_1, k_2 > 0$, such that for any $(\lambda_k) \in l^2$

$$k_1 \sum_{k=0}^{\infty} |\lambda_k|^2 \leq \|\sum_{k=0}^{\infty} \lambda_k x_k\|^2 \leq k_2 \sum_{k=0}^{\infty} |\lambda_k|^2. \tag{28}$$

Proposition 4 *If* $|\alpha|_\infty < \|L\|_2^{-1}$, (H^α_{nm}) *is a Riesz sequence.*

Proof If $(c_{nm}) \in l^2$, let us define for $f \in \mathcal{L}^2(S)$

$$f = \sum_{n=0}^{\infty}\sum_{m=0}^{2n} c_{nm} H_{nm} .$$

Then, due to Parseval's equality

$$\|f\|^2_{\mathcal{L}^2(S)} = \sum_{n=0}^{\infty}\sum_{m=0}^{2n} |c_{nm}|^2 .$$

Also

$$\begin{aligned}\|\Theta^\alpha(f)\|^2_{\mathcal{L}^2(S)} &= \|\sum_{n=0}^{\infty}\sum_{m=0}^{2n} c_{nm} H^\alpha_{nm}\|^2_{\mathcal{L}^2(S)}\\ &\leq \|\Theta^\alpha\|_2^2 \|f\|^2_{\mathcal{L}^2(S)}\\ &= k_2 \sum_{n=0}^{\infty}\sum_{m=0}^{2n} |c_{nm}|^2,\end{aligned}$$

where $k_2 = \|\Theta^\alpha\|_2^2$.
For the left inequality in (28), let

$$k_1 = \frac{1 - \|L\|_2 |\alpha|_\infty}{1 + |\alpha|_\infty} .$$

If $|\alpha|_\infty < \|L\|_2^{-1}$ then from (23), it follows that:

$$k_1 \|f\|^2_{\mathcal{L}^2(S)} \leq \|\Theta^\alpha(f)\|^2_{\mathcal{L}^2(S)} = \sum_{n=0}^{\infty}\sum_{m=0}^{2n} \|c_{nm} H^\alpha_{nm}\|^2_{\mathcal{L}^2(S)} .$$

Hence, (H^α_{nm}) is a Riesz sequence. □

Definition 5 A sequence (x_k) in a Hilbert space H is a Riesz basis for H if it is the image of an orthonormal basis for H under a topological isomorphism. In other words, if there is an orthonormal basis (e_k) for H and a topological isomorphism T, such that $Te_k = x_k$ for all k.

Lemma 8 *If L is a bounded and linear operator from a Banach space into itself, such that $\|I - L\| < 1$, then L^{-1} exists and is bounded.*

Theorem 6 *If $|\alpha|_\infty < (1 + \|I - L\|_2)^{-1}$ then (H^α_{nm}) is Riesz basis for $\mathcal{L}^2(S)$.*

Proof It is easy to check that the extension of the operator $Id - L$ to $\mathcal{L}^2(S)$ is $Id - L_S$ with $\|Id - L\|_2 = \|Id - L_S\|_2$.

If $|\alpha|_\infty < (1 + \|I - L\|_2)^{-1}$, then

$$\frac{1}{1 - |\alpha|_\infty} < \frac{1 + \|I - L_S\|_2}{\|I - L_S\|_2} ,$$

$$\frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L_S\|_2 < 1 ,$$

Then, due to (20)

$$\|I - \Theta^\alpha\|_2 < 1 .$$

According to Lemma 8, the operator Θ^α is an isomorphism, and hence, (H^α_{nm}) is Riesz basis. □

4 Conclusions

Maps on the sphere are crucial to understand and visualize many phenomena of the nature as meteorology, geodesy, oceanography, etc., and it is therefore essential to extend the family of the standard functions on this surface. To this end, we have constructed a set of square integrable functions that generalize the classical spherical harmonics. This process is done by means of α-fractal functions associated with one of their factors. The graph of these maps is the invariant attractor of an iterated function system and owns a fractal structure. Accordingly, the methodology used provides a method to define non-smooth functions on the sphere.

The process of fractalization of harmonics is made also through an operator defined on the space of square integrable functions $\mathcal{L}^2(S)$, where S represents the unit sphere. The transformed functions constitute a system of maps parameterized by a scale vector typical of α-fractal functions. According to its magnitude, one obtains different spanning sets of functions on the sphere. Taking the scale small enough, we have constructed a family of Riesz bases of $\mathcal{L}^2(S)$, that contains the classical bases as a particular case. As the fractal functions are non differentiable in general, the smoothness of the maps employed in a specific approximation problem becomes optional. And this is a major innovation in the field of function theory.

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