Alexander polynomial
from the point of view of group homology

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Abstract

Knots are a commonplace element in our daily lives, whether in our shoelaces or our earphones cable. Knot theory is a branch of topology that tackles the study of knots as mathematical objects. In principle, the concept of knot in the field of mathematics is similar to our familiar understanding of it: a tied rope, with the exception that both ends are sticked together. Formally, a knot can be defined as an embedding of the circle S^1 in $\mathbb{R}^3$.

The basis of knot theory is rather intuitive, but endowing it with mathematical rigor is challenging. The paramount goal of knot theory is to determine whether two knots are equivalent, that is, whether one can be deformed into the other in a smooth way. Although there is an algorithm able to solve this problem, it has unknown complexity and is therefore not used in practice. Hence, knots are usually distinguished by knot invariants. A knot invariant is a mathematical structure, such as a polynomial or a group, that can be computed directly from the knot and remains the same when computed from equivalent knots.

The objective of this dissertation is to present the Alexander polynomial, a classical knot invariant discovered by James Waddell Alexander II in 1923. Several alternative definitions of this polynomial have been proposed, ranging from purely algebraic to purely geometric, but this dissertation will focus on the algebraic one.

In the first chapter, some pertinent results about algebra and algebraic topology will be reviewed. First, we will explain the concept of group presentation and state some relevant results about modules. Then, the fundamental group of a topological space will be defined. Finally, we will study some facts about covering spaces, ending with the definitions of group of deck transformations and cyclic covering, that will be needed later.

In the second chapter, a short introduction about CW-complexes (spaces that can be built inductively by attaching cells) and group homology will be offered. Briefly, homological theory is based on the concept of homology groups. Given any CW-complex X, the homology of X is a chain $\{H_n(X)\}_{n\geq 0}$ of Abelian groups. Intuitively, $H_n(X)$ measures the number of n-holes of X. This concept will be used to define group homology, which is a chain $\{H_n(G)\}_{n\geq 0}$ where G is a group instead of a CW-complex. Indeed, group homology will enable us to define the Alexander polynomial.

In the third chapter, knot theory will be introduced. In the first section, we will define in a mathematical way what a knot is and how we can represent it. In the second section, we will present our first knot invariant, the notion of knot group (given a knot K it is defined as $\pi_1(\mathbb{R}^3 \setminus K)$), plus an easy algorithm to compute it given a knot diagram.

Finally, in the last chapter, we will use the concepts previously introduced to define the Alexander polynomial. It will be defined in the following way: consider a knot K and a cyclic cover X of $\mathbb{R}^3 \setminus K$. Then, there is a deck transformation t acting on X. This transformation t acts on $H_1(X)$ and so we can consider $H_1(X)$ to be a module over the ring $\mathbb{Z}[t,t^{-1}]$. Certain polynomial associated to this module structure will be called the Alexander polynomial. But this definition makes it hard to compute the polynomial, so we define the concept of Fox derivatives (derivatives for words of group presentations) and use it to give a computational method for computing the Alexander polynomial of a knot. In the last section, we will explain a further method for finding the Alexander polynomial. The latter corresponds with one of the aforementioned alternative definitions of the Alexander polynomial.

In conclusion, the aim of this dissertation is to offer a brief introduction of homology, to relate these concepts with knot theory and, in particular, to build the Alexander polynomial. This allows us to use pure algebraic concepts to solve a topological problem.

Resumen

Los nudos son un elemento común en nuestra vida cotidiana, desde los cordones de nuestros zapatos hasta los cables enredados de unos auriculares. La teoría de nudos es una rama de la topología que se encarga de estudiar el objeto matemático de los nudos. En principio, un nudo en matemáticas no difiere mucho de la noción cotidiana: una cuerda anudada, con la excepción de que pegaremos los extremos para que no se pueda desatar. Formalmente, un nudo se define como un encaje de S^1 en $\mathbb{R}^3$.

Las bases de esta teoría son muy intuitivas, no obstante dotarlas de un rigor matemático es complejo. El objetivo principal de la teoría de nudos consiste en saber cuándo dos nudos son equivalentes, es decir, que se puede transformar el uno en el otro mediante deformaciones suaves. Aunque existe un algoritmo que resuelve el problema, este tiene complejidad desconocida, por lo que no es muy útil en la práctica. Así pues, se suelen utilizar invariantes como polinomios o grupos, que se pueden calcular para cualquier nudo y coinciden cuando los nudos son equivalentes.

El objetivo principal de este trabajo es presentar el Polinomio de Alexander, uno de los invariantes de nudos más clásicos. Fue descubierto en 1923 por James Waddell Alexander II y es uno de los primeros invariantes de nudos conocido. Existen varias formas equivalentes de definir este polinomio, desde definiciones puramente algebraicas a definiciones puramente geométricas y con superficies, pero en este trabajo nos centraremos en su definición algebraica.

En el primer capítulo, vamos a realizar un pequeño resumen de algunos resultados de álgebra y topología algebraica que necesitaremos en el desarrollo del trabajo. Primero, vamos a introducir el concepto de presentación de un grupo, un método que usaremos para representar grupos en general. A continuación, recordaremos qué es un módulo y algunos resultados importantes sobre estos. Finalmente, definiremos qué es el grupo fundamental de un espacio topológico y daremos algunos resultados acerca de espacios recubridores, centrándonos en los conceptos de cubiertas cíclicas y de grupo de superposiciones.

En el segundo capítulo, haremos una pequeña introducción a los conceptos de CW-complejo (espacios que se construyen por inducción pegando celdas) y homología de grupos. La teoría de la homología se fundamenta en el concepto de los grupos de homología. Dado un CW-complejo X, su homología es una cadena $\{H_n(X)\}_{n\geq 0}$ de grupos abelianos. Intuitivamente, $H_n(X)$ mide el número de agujeros de dimensión n del espacio X. Este concepto será el que usaremos para definir la homología de grupos, una cadena $\{H_n(G)\}_{n\geq 0}$ donde G es un grupo en vez de un CW-complejo. La homología de grupos nos permitirá definir el polinomio de Alexander.

En el tercer capítulo, ofreceremos una introducción a la teoría de nudos. En la primera sección, definiremos formalmente qué es un nudo y cómo podemos representarlo. En la segunda, definiremos nuestro primer invariante de nudos, llamado el grupo del nudo (dado un nudo K se define como $\pi_1(\mathbb{R}^3 \setminus K)$), y veremos un algoritmo muy sencillo con el que podremos calcularlo a partir de su diagrama.

Finalmente, en el último capítulo, juntaremos todos los conceptos que hemos estudiado en las sec-

ciones anteriores para poder definir el polinomio de Alexander. De forma resumida, el polinomio de Alexander se definirá de esta manera: sea K un nudo y X la cubierta cíclica de $\mathbb{R}^3 \setminus K$. Existe una superposición t actuando en $H_1(X)$, por lo que se puede considerar $H_1(X)$ como un módulo sobre el anillo $\mathbb{Z}[t,t^{-1}]$. Cierto invariante asociado a esta estructura de módulo será a lo que llamaremos el polinomio de Alexander. Con esta definición es muy difícil calcular el polinomio, así que introduciremos el concepto de derivadas de Fox (derivadas para palabras de presentaciones de grupos) y lo usaremos para dar un método sencillo para hallar el polinomio de Alexander de un nudo. En el último apartado, veremos de forma resumida otra de las múltiples definiciones del polinomio de Alexander, que resulta útil debido a que proporciona un método más sencillo para calcularlo.

En resumen, el objetivo de este trabajo es realizar una pequeña introducción a la teoría de la homología, y ver cómo podemos relacionar estos conceptos con la teoría de nudos. Esto nos permite utilizar métodos algebraicos para resolver un problema puramente topológico.

Contents

Abstract iii

Resumen v

1 Group Presentations, Covering Spaces and Modules 1
- 1.1 Group Presentations 1
- 1.2 Modules 2
- 1.3 Fundamental Group 3
- 1.4 Covering spaces 5

2 CW-Complexes, Homology and Group Homology. 7
- 2.1 CW-Complexes 7
- 2.2 Simplicial Homology 9
- 2.3 Group Homology 11
  - 2.3.1 Algebraic definition 11
  - 2.3.2 Topological definition 16

3 Introduction to Knot Theory. 17
- 3.1 First definitions 17
- 3.2 The Knot Group 19

4 The Alexander Polynomial. 21
- 4.1 The Algebraic Way 21
- 4.2 Fox Derivatives 22
- 4.3 The Combinatorial Way 24

Bibliography 27

Chapter 1

Group Presentations, Covering Spaces and Modules

1.1 Group Presentations

The objective of this section is to describe an efficient way to determine a group. To this end, we will need to define the notion of group presentations, which is the most intuitive and useful way to do this.

Definition 1.1.1. Let S be a non-empty set. A **word** in S is a finite sequence of symbols from $S \cup S^{-1}$ (that is an expression of the form $s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$, where $s_1, \dots, s_n \in S$ and $\varepsilon_i = \pm 1\ \forall\, i = 1, \dots, n$). We will say that n is the **length** of the word. The only word of zero length is the word with no elements, called the **empty word**.

Given a word in S, the operation of reduction is defined in the following way: for every $x \in S$ the expressions xx^{-1} and $x^{-1}x$ can be deleted. A word that can not be reduced it is called **reduced**. For example $a^3b^{-2}ba$ is not a reduced word but can be reduced to $a^3b^{-1}a$. It is easy to see that the relation given by $w_1 \equiv w_2$ if and only if w_1 and w_2 yield to the same reduced word is an equivalence relation. From this, one deduces that if we consider the operation of juxtaposition, it induces a well defined operation on the equivalence classes. Notice that the operation is associative and has a neutral element, which is the empty word, and will be denoted as 1.

Definition 1.1.2. Let S be a non-empty set. We define $F(S)$, the **free group generated by S**, as the group of reduced words of S with the operation of juxtaposition. The elements of S are the **generators** of $F(S)$ and the number of elements of S is called the **rank** of $F(S)$. The **free group of rank n** will be denoted as F_n.

Definition 1.1.3. Let G be a group and $S \subset G$ be a subset. We define the **normal closure** of S as the closure of $S^G = \{g^{-1}sg \mid g \in G, s \in S\}$ under the group operation and the operation of taking inverses. It will be denoted as $Ncl_G(S)$.

By definition $Ncl_G(S)$ is a normal subgroup. It is the smallest normal subgroup of G that contains S.

Definition 1.1.4. Let S be a non-empty set and R a subset of words in S. We define a **presentation** as:

$$\langle S \mid R \rangle = F(S) \big/ Ncl_{F(S)}(R)$$

Let G be a group. We say that $\langle S \mid R \rangle$ is a presentation of G if $G \cong \langle S \mid R \rangle$.

In general, understanding a group given by a presentation is complex, and there is no general procedure. But we can give some easy examples.

Examples 1.1.5. 1. The easiest example is $G = \langle x \mid x^n = 1 \rangle \cong \mathbb{Z}_n$.

2. Let $G = \langle x, y \mid x^2 = y^2 = (xy)^n = 1 \rangle$, it seems complicated, but notice that if we take $a = xy$ then $y = x^{-1}a = xa$, since $x = x^{-1}$. Interchanging y with a in the presentation, we have:

$$G = \langle x, a \mid x^2 = xaxa = a^n = 1 \rangle = \langle x, a \mid x^2 = a^n = 1, xax = a^{-1} \rangle$$

where, if $A = \langle a \rangle$ and $X = \langle x \rangle$, the presentation gives us that $A \cong \mathbb{Z}_n$ and $X \cong \mathbb{Z}_2$. Also, from the equation $xax = a^{-1}$, we have that $A \rtimes X$ and then $G \cong D_{2n}$ (where D_{2n} is the dihedral group of order $2n$, that is the group of symmetries of a regular n-gon).

3. In the same way, it can be seen that $G = \langle x, y \mid x^2 = y^2 = 1 \rangle \cong D_\infty$ (where D_∞ is the dihedral group of infinite order, that is the group of symmetries of $\mathbb{Z}$).

Let us prove an important theorem about group presentations that will be needed lately.

Theorem 1.1.6. *(Von Dyck's) Let $G = \langle x_1, ..., x_n \mid r_1 = 1, ..., r_m = 1 \rangle$ and H be another group generated by the set $\{x_1, ..., x_n\}$ and satisfying the relations of G ($r_1 = 1, ..., r_m = 1$ in H). Then, there is an epimorphism from G to H.*

Proof. Let F be the free group generated by $\{x_1, ..., x_n\}$. It can be defined $\phi : F \to H$ as $\phi(x) = x \; \forall \, x \in F$. Now, let $R = Ncl_F(\{r_1, ..., r_m\})$ and $g \in R$. One can write g as a product of relations and its inverses. Therefore, $\phi(g) = g = 1$, because all the relations of G are satisfied in H. Therefore, $\overline{\phi} : G \cong F \big/ R \to H$ given by $\overline{\phi}(\overline{g}) = g$ is a well defined epimorphism. □

1.2 Modules

Modules can be understood as a generalization of vector spaces. A module is similar to a vector space, where instead of a field we work with a ring R. In this section, we will define what a module is, and state some pertinent properties of them that will be useful.

Definition 1.2.1. Let R be a ring. An **R-module** is a set M together with two binary operations:

1. an inner operation called **addition**:

$$\begin{aligned} M \times M &\longrightarrow M \\ (x, y) &\longmapsto x + y \end{aligned}$$

2. an external operation called **multiplication by scalars**:

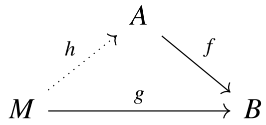
$$\begin{aligned} R \times M &\longrightarrow M \\ (a, x) &\longmapsto ax \end{aligned}$$

satisfying the following:

- $(M, +)$ is an Abelian group.
- Multiplication by scalars is distributive with respect to addition: for any $a, b \in R$ and $x, y \in M$ we have $a(x + y) = ax + ay$ and $(a + b)x = ax + bx$.

Definition 1.2.2. Let R be a ring and M an R-module. We say that $\{e_i\}_{i \in I} \subset M$ is a **basis** of M if for every $x \in M$ there exist a unique family $\{x_i\}_{i \in I} \subset R$ such that $x = \sum_{i \in I} x_i e_i$. If M has a basis, then it is said to be a **free R-module**.

Definition 1.2.3. Let R be a ring. An R-module M is called **projective** if for every surjective module homomorphism $f : A \to B$ and every module homomorphism $g : M \to B$ there exists a module homomorphism $h : M \to A$ such that $f \circ h = g$.



Lemma 1.2.4. *Free modules are projective.*

Proof. Let M be a free module with basis $\{e_i\}_{i \in I}$. Let $f : A \to B$ be a surjective module homomorphism and $g : M \to B$ be a module homomorphism. For every $i \in I$, let $b_i = g(e_i)$ and, since f is surjective, there exists $a_i \in A$ such that $f(a_i) = b_i$. Then, we can define $h : M \to A$ by $h(e_i) = a_i \; \forall \, i \in A$ and extend it by linearity to M. This is trivially a well defined homomorphism and $f(h(e_i)) = f(a_i) = b_i = g(e_i) \; \forall \, i \in I$, which proves that $f \circ h = g$. Therefore, M is a projective module. □

Let us state some well known results from modules that will be needed lately.

Theorem 1.2.5. *Let M be an $n \times m$ matrix with entries in a principal ideal domain R. Then, there exist $0 \le r \le n, m$ and two invertible matrices P and Q of size $n \times n$ and $m \times m$ respectively such that $PMQ = D$, where D is a diagonal matrix with diagonal elements $d_1, \dots, d_r, 0, \dots, 0$ such that $0 \neq d_1, \dots d_r \in R$ and $d_1 \mid d_2 \mid \cdots \mid d_r$. Moreover, r and the ideals $(d_r) \subset \cdots \subset (d_1)$ are uniquely determined by M.*

Let A be an $n \times m$ matrix with entries in a commutative ring R and $1 \le k \le n, m$. The ideal generated by the minors of order k of A will be denoted by $I_k(A)$.

Proposition 1.2.6. *In the conditions of the previous theorem $I_k(M) = I_k(D)$ for all $1 \le k \le n, m$.*

Theorem 1.2.7. *Let M be a finitely generated module over a principal ideal domain R. Then, M is isomorphic to a direct sum of cyclic modules:*

$$M \cong R/(p_1) \bigoplus \cdots \bigoplus R/(p_n)$$

where $p_i \in R \setminus R^*$ for $i = 1, ..., n$.

Definition 1.2.8. In the conditions of the previous theorem, let $k \in \{0, ..., n\}$ such that $p_1 = \cdots = p_k = 0$ and $p_{k+1}, ..., p_n \neq 0$. Then, $R/(p_1) \bigoplus \cdots \bigoplus R/(p_k) \cong R^k$ is called the **free part** of the module M and $R/(p_{k+1}) \bigoplus \cdots \bigoplus R/(p_n)$ is called the **torsion part** (denoted as $\text{tor}(M)$). M is said to be a **torsion module** if $\text{tor}(M) = M$ and a **torsion free module** if $\text{tor}(M) = 0$.

Definition 1.2.9. In the conditions of the previous theorem, we define the **order** of the module M as the ideal $(p_1 \dots p_n)$. It will be denoted as order(M).

It is immediate to see that the order function is well defined.

Lemma 1.2.10. *1. order$(M) = (1)$ if and only if M is the zero module.*

2. order$(M) \neq (0)$ if and only if M is a torsion module.

Proof. It follows immediately from the definition. □

1.3 Fundamental Group

Every topological space can be associated with a group called the fundamental group. This is important because it is a topological invariant, homeomorphic topological spaces have the same fundamental group. In this section, we are going to introduce it without providing the details and proofs. (They can be found in [8, Chapter 9])

Definition 1.3.1. Let X and Y be two topological spaces and $f, f' : X \to Y$ be two continuous functions. Then, f and f' are said to be **homotopic** if there exists a continuous map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = f'(x) \; \forall \, x \in X$. We will say that H is an **homotopy** between f and f'. If f and f' are homotopic it will be denoted as $f \cong f'$.

Definition 1.3.2. Two spaces X and Y are said to be of the same **homotopy type** if there exist two continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \cong 1_Y$ and $g \circ f \cong 1_X$, where 1_X and 1_Y are the identity maps.

Definition 1.3.3. A space X is said to be **contractible** if it has the homotopy type of a point.

Definition 1.3.4. Let X be a topological space and let $x_0, x_1 \in X$. A **path** from x_0 to x_1 is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. If $x_0 = x_1$ we say that γ is a **loop** based at x_0.

Definition 1.3.5. Let $\gamma, \gamma' : [0,1] \to X$ be two paths such that $\gamma(0) = \gamma'(0) = x_0$ and $\gamma(1) = \gamma'(1) = x_1$. Then, γ and γ' are said to be **homotopic** if there exists a continuous map $H : [0,1] \times [0,1] \to X$ such that $H(x,0) = \gamma(x)$, $H(x,1) = \gamma'(x)$, $H(0,t) = x_0$ and $H(1,t) = x_1$ $\forall\, x,t \in [0,1]$. We will say that H is an **homotopy** between γ and γ'. If γ and γ' are homotopic it will be denoted as $\gamma \cong_p \gamma'$.

Now we have easily the following result:

Lemma 1.3.6. *The relations $\cong$ and $\cong_p$ are equivalence relations.*

The equivalence class of γ with respect to $\cong_p$ will be denoted as $[\gamma]$.

Definition 1.3.7. If γ is a path from x_0 to x_1 and γ' is a path from x_1 to x_2 then the **product of paths** is defined as:

$$\left(\gamma * \gamma'\right)(x) = \begin{cases} \gamma(2x) & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \gamma'(2x-1) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

The product of paths defines an operation in the homotopy classes:

$$[\gamma] * [\gamma'] = [\gamma * \gamma']$$

From the definition one can easily prove the next theorem:

Theorem 1.3.8. *The operation $*$ has the following properties:*

- $([\gamma] * [\gamma']) * [\gamma'']$ *is defined if and only if* $[\gamma] * ([\gamma'] * [\gamma''])$ *is defined. In such case, both are equal.*
- *If e_y is the constant path in $y \in X$ ($e_y(x) = y\ \forall\, x \in [0,1]$) then:*
$$[\gamma] * [e_{x_1}] = [\gamma] \text{ and } [e_{x_0}] * [\gamma] = [\gamma]$$
- *Let γ be a path, we define the inverse path as $\overline{\gamma}(x) = \gamma(1-x)$. Then:*
$$[\gamma] * [\overline{\gamma}] = [e_{x_1}] \text{ and } [\overline{\gamma}] * [\gamma] = [e_{x_0}]$$

Notice that this operation does not gives us a group structure, since the product of two paths is not always defined. But if we restrict ourselves to loops based at x_0 we do not have this problem. This leads us to the following definition:

Definition 1.3.9. Let X be a topological space and $x_0 \in X$. We define the **fundamental group of** X **based at** x_0 as the group of homotopy classes of loops based at x_0 with the operation $*$. It is denoted by $\pi_1(X, x_0)$.

Example 1.3.10. Let α and β be two paths on $A \subset \mathbb{R}^n$, where A is a convex set. Then, $H(x,t) = (1-t)\alpha(x) + t\beta(x)$ is an homotopy between α and β called **straight line homotopy**. Therefore, $\pi_1(A, x)$ is the trivial group for every $x \in A$.

Remark 1.3.11. It can be seen that if the space X is path-connected then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for all $x_0, x_1 \in X$. Therefore, if X is path-connected, we can omit the base point of the fundamental group. In such case, it will be denoted as $\pi_1(X)$.

Definition 1.3.12. A topological space X is said to be **simply connected** if it is path-connected and $\pi_1(X)$ is the trivial group.

It can be proven that the fundamental group is an invariant of the homotopy type:

Theorem 1.3.13. *Two topological spaces with the same homotopy type have isomorphic fundamental groups.*

1.4 Covering spaces

In this section, we are going to introduce the notion of covering spaces and enumerate some of its important properties. It is a very useful tool, since it plays an important role in many branches of mathematics.

Definition 1.4.1. Let X,Y be topological spaces and $p: X \to Y$ be a continuous surjective map. An open set $U \subset Y$ is said to be **evenly covered** by p if $p^{-1}(U) = \dot{\bigcup} V_\alpha$, where $\{V_\alpha\}$ is a family of open sets of X such that $p|_{V_\alpha}: V_\alpha \to U$ is an homeomorphism. The collection $\{V_\alpha\}$ is called a partition of $p^{-1}(U)$ into **slices**.

Definition 1.4.2. Let X,Y be topological spaces and $p: X \to Y$ be a continuous surjective map. If for all $y \in Y$ there exists U, an open neighbourhood of y that is evenly covered by p, then p is called a **covering map**, and X is called a **covering space** of Y.

Example 1.4.3. The map $p: \mathbb{R} \to S^1$ given by $p(x) = (\cos(2\pi x), \sin(2\pi x))$ is a covering map.

Proof. Let $U_1 = \{(x,y) \in S^1 \mid x > 0\}$, $U_2 = \{(x,y) \in S^1 \mid x < 0\}$, $U_3 = \{(x,y) \in S^1 \mid y > 0\}$ and $U_4 = \{(x,y) \in S^1 \mid y < 0\}$. Then, $\{U_1, U_2, U_3, U_4\}$ is an open covering of S^1, hence it suffices to prove that these open sets are evenly covered by p. Let us prove that U_1 is evenly covered by p (since the rest is analogous). First, notice that $p^{-1}(U_1)$ consists on the points $x \in \mathbb{R}$ such that $\cos(2\pi x) > 0$, which implies $p^{-1}(U_1) = \bigcup_{n\in\mathbb{Z}} V_n$ where $V_n = \left(n - \frac{1}{4}, n + \frac{1}{4}\right)$. Therefore, if we prove that $p|_{V_n}: V_n \to U$ is an homeomorphism we will be done. Notice that $\sin(2\pi x)$ is strictly monotonic on V_n, which implies that p is injective on V_n. Furthermore, p carries $\overline{V}_n$ surjectively onto $\overline{U}$. Then, since $\overline{V}_n$ is compact and $\overline{U}$ is Hausdorff, $p|_{\overline{V}_n}$ is an homeomorphism from $\overline{V}_n$ to $\overline{U}$. In particular, $p|_{V_n}$ is an homeomorphism of V_n with U. □

Definition 1.4.4. Let $p: E \to B$ be a covering map. If E is simply connected then it is called a **universal covering space** of B and p is called a **universal cover**.

Definition 1.4.5. Let $p: E \to B$ and $p': E' \to B$ be two covering maps. They are said to be **equivalent** if there exists a homeomorphism $h: E \to E'$ such that $p = p' \circ h$.

Universal covers have the following universal property:

Theorem 1.4.6. *If $p: E \to B$ is a universal cover of B and $p': E' \to B$ is another covering map with E' connected, then there exists a covering map $f: E \to E'$ such that $p' \circ f = p$.*

Proof. Can be found in [8, Chapter 12]. □

This theorem implies that any two universal covers are equivalent. Then, one can speak about the universal cover of a space.

Example 1.4.7. Since $\mathbb{R}$ is simply connected (because it is a convex set) and the map of 1.4.3 is a covering map, then $\mathbb{R}$ is the universal covering space of S^1.

Theorem 1.4.8. *Let $p: E \to B$ and $p': E' \to B'$ be two covering maps. Then $p \times p': E \times E' \to B \times B'$ is a covering map.*

Proof. Let $(b,b') \in B \times B'$. As p and p' are covering maps, there exist U and U', neighbourhoods of b and b' respectively, such that they are evenly covered by p and p'. Let $\{V_\alpha\}$ and $\{V'_\beta\}$ be two partitions into slices of $p^{-1}(U)$ and $p'^{-1}(U')$ respectively. Then, $(p \times p')^{-1}(U \times U')$ is equal to the union of all open sets of the form $V_\alpha \times V'_\beta$ which are pairwise disjoint. Also, they are mapped homeomorphically onto $U \times U'$. Hence, $U \times U'$ is evenly covered by $p \times p'$. □

Example 1.4.9. The universal covering space of S^1 is $\mathbb{R}$. Therefore, by the previous theorem, the universal covering space of the torus $T = S^1 \times S^1$ is $\mathbb{R}^2$.

Definition 1.4.10. A **deck transformation** of a covering map $p : X \to Y$ is a homeomorphism $f : X \to X$ such that $p \circ f = p$.

The set of all deck transformations will be denoted as $Deck(X)$.

Proposition 1.4.11. *The set $Deck(X)$ forms a group with the operation of composition.*

Proof. First, notice that this set is closed under the operation, because if f and g are two deck transformations then $p \circ (f \circ g) = (p \circ f) \circ g = p \circ g = p$. Let us check the existence of neutral element, the associativity and the existence of inverses:

- The identity map $e : X \to X$ given by $e(x) = x \; \forall \, x \in X$ is the neutral element.
- The associativity is given by the associativity of the composition of maps.
- If $f \in Deck(X)$ then, since f is a homeomorphism, there exists an inverse f^{-1}. And $f^{-1} \in Deck(X)$, since $p \circ f^{-1} = p \circ f \circ f^{-1} = p$.

Hence, $Deck(X)$ is a group. □

Now, suppose that we have a continuous map $h : X \to Y$ such that $h(x_0) = y_0$. This fact will be denoted by $h : (X, x_0) \to (Y, y_0)$.

Definition 1.4.12. If $h : (X, x_0) \to (Y, y_0)$ is a continuous function, we define the **homomorphism induced by h** as $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $h_*([f]) = [h \circ f]$

Definition 1.4.13. A covering space X of Y is **regular** if $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(Y, y_0)$ for every $x_0 \in X$.

Definition 1.4.14. If H is a subgroup of a group G, the **normalizer** of H in G is defined as:

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

By definition, $N_G(H)$ is a normal subgroup. It is the largest subgroup of G in which H is normal.

Proposition 1.4.15. *Let Y be a path-connected and locally path-connected space and X a path-connected covering space of Y. If $H = p_*(\pi_1(X, x_0))$ and $G = \pi_1(Y, y_0)$, then:*

$$Deck(X) \cong N_G(H) \big/ H$$

In particular, $Deck(X) \cong \pi_1(Y, y_0) \big/ H$ if X is a regular cover.

Proof. See [2]. □

Proposition 1.4.16. *Let $p : X \to Y$ be a covering map. Then, $Deck(X)$ acts freely on X.*

Proof. Can be found at [2]. □

Definition 1.4.17. A covering space X of Y is a **cyclic covering** if it is a regular covering and $Deck(X)$ is a cyclic group. A cyclic cover is said to be finite if the cyclic group is finite and infinite in the other case.

Chapter 2

CW-Complexes, Homology and Group Homology.

2.1 CW-Complexes

CW-complexes are spaces that can be built inductively by "adding cells". The letters CW stand for C="Closure finite" (the closure of each cell meets only finitely many other cells) and W="Weak topology" (the topology of the space). To construct this spaces, we use the following procedure:

1. We take a set of points X_0, whose elements are called 0-cells.

2. Now, assume that we have X_{n-1} and let us construct X_n. We take a family $\{D^n_\alpha\}_{\alpha \in I}$ of n-discs (which will be called n-cells), and we add them via attaching maps

$$\phi_\alpha : S^{n-1}_\alpha \to X_{n-1}$$

   where $S^{n-1}_\alpha = \partial D^n_\alpha$ (the map attaches the boundary of an n-cell to an (n-1)-cell) and we set X_n to be the quotient of $\left(X_{n-1} \cup \left(\bigcup\limits_{\alpha \in I} D^n_\alpha \right) \right)$ by the equivalence relation $x \sim \phi_\alpha(x)\ \forall\, x \in S^{n-1}_\alpha$.

3. We can finish after a finite number of steps n, defining $X_m = \emptyset\ \forall\ m > n$, or we can continue infinitely. Then, we obtain the set $X = \bigcup\limits_{n=1}^{\infty} X_n$.

4. Finally, we give the weak topology to X: $A \subset X$ is an open set if and only if $A \cap X_n$ is open in $X_n\ \forall\, n \in \mathbb{N}$.

The space X is called a **cell complex** or a **CW-complex** and X_n is called the **n-skelleton** of X. If $X = X_n$ for some n, then X is said to be finite-dimensional and the smallest such n is the **dimension** of X.

Examples 2.1.1. 1. An easy example of CW-complex is when $X = X_1$. This is called a **graph**. It consists of vertices (0-cells) and edges (1-cells).

2. The sphere S^n can be seen as a CW-complex with just two cells, D^0 and D^n, where D^n is attached to D^0 by the constant map $S^{n-1} \to D^0$. This is equivalent to the fact that $S^n \cong D^n \big/ \partial D^n$

3. The real projective space $\mathbb{R}P^n$ can be seen as a CW-complex. By restricting $\mathbb{R}^{n+1} - \{0\}$ to S^n we can see $\mathbb{R}P^n \cong S^n/\!\sim$ where $x \sim y$ if and only if $x = \pm y$. This is equivalent to saying that $\mathbb{R}P^n$ is the quotient of a hemisphere D^n where the antipodal points of ∂D^n are identified. But notice that $\partial D^n = S^{n-1}$ with the antipodals points identified is $\mathbb{R}P^{n-1}$, hence $\mathbb{R}P^n$ can be obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell with the map $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. By induction on n, it follows that $\mathbb{R}P^n$ has CW-structure $D^1 \cup \cdots \cup D^n$.

Definition 2.1.2. Let $G = \langle x_1, \dots, x_n \mid r_1 = 1, \dots, r_m = 1 \rangle$ be a group given by its presentation. First, let us take one point and, for each generator x_i, we attach an oriented 1-cell labelled x_i starting and ending at the point. Then, we attach a 2-cell for each r_i, where the boundary of the 2-cell is glued along the 1-cells following the order of the letters of r_i (the orientation is respected for the letters with exponent 1 and inverted for the letters with exponent -1). This 2-dimensional complex will be denoted as X_G and it is called a **presentation complex** of G. It can be seen that $\pi_1(X_G) = G$.

Definition 2.1.3. Let $G = \langle x_1, \dots, x_n \mid r_1 = 1, \dots, r_m = 1 \rangle$ be a group given by its presentation. First, let us take a point for each element of G. Then, at each point $g \in G$ we insert for each generator x_i an oriented 1-cell labelled x_i joining g with gx_i. Finally, for every $g \in G$ we attach a 2-cell for each r_i, where the boundary is glued following the order of the letters of r_i (the orientation is respected for the letters with exponent 1 and inverted for the letters with exponent -1). This is called a **Cayley complex** of G and it is denoted as $\tilde{X}_G$. One can see that $\tilde{X}_G$ is the universal covering space of X_G.

Proposition 2.1.4. *For every group G its Cayley complex $\tilde{X}_G$ is simply connected.*

Proof. First, notice that $\tilde{X}_G$ is path-connected since every element of G is a product of generators, so there is a sequence of vertices joining each vertex to the identity vertex 1. To see that $\pi_1(\tilde{X}_G) = 0$, let us take a loop based on 1. This loop is homotopic to a loop consisting of a finite sequence of edges, corresponding to a word w which is equivalent to the identity (viewed as an element of G). Hence, we can write w as a product of relations and its inverses. Therefore, as an element of $\tilde{X}_G$, the loop can be seen as a product of loops such that each loop is of the following way: a path starting at 1 and ending at a point in the boundary of a 2-cell, followed by the boundary loop of this 2-cell, and finishing with the inverse path from the point of the boundary to 1. Such loops are trivially homotopic to the constant map at 1. This implies that the loop is homotopic to the constant loop. □

Definition 2.1.5. The **chain complex** of a CW-complex X is defined as

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X)$$

where $C_n(X)$ is the free abelian group generated by the set of n-cells and ∂_n is the **boundary map** (to see a description of it see [2, p 140]). In the next section we will define it in the particular case of simplicial complexes. We also define the **cycles** of the chain complex as $Z_n(X) = Ker(\partial_n)$ and the **boundaries** as $B_n(X) = Im(\partial_n)$

Definition 2.1.6. The **augmented chain complex** of a CW-complex X is defined as:

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0 = \varepsilon} \mathbb{Z} \to 0$$

where $\partial_0 = \varepsilon$ is the augmentation map, the map that sends each 0-cell to 1.

One can see that $\partial_n \partial_{n+1} = 0 \; \forall \; n \in \mathbb{N}$, which implies $B_{n+1}(X) \subseteq Z_n(X)$. A chain complex is said to be **exact** if $B_{n+1}(X) = Z_n(X)$ for all $n \in \mathbb{N}$.

Definition 2.1.7. The **homology** of a CW-complex X is a set of Abelian groups $\{H_n(X)\}_{n \geq 0}$ defined as:

$$H_n(X) = Z_n(X) \big/ B_{n+1}(X)$$

A CW-complex X is said to be **n-acyclic** if $H_k(X) = 0$ for $0 < k \leq n$ and $H_0(X) = \mathbb{Z}$. It is said to be **acyclic** if it is n-acyclic for all $n \geq 1$.
It can be proven that the homology is an invariant of the homotopy type:

Theorem 2.1.8. *(The Homotopy Axiom) Suppose that X and Y are two CW-complexes of the same homotopy type, then $H_n(X) \cong H_n(Y) \; \forall \; n \in \mathbb{N}$.*

Proof. See [11, Chapter 4]. □

Corolary 2.1.9. *If a CW-complex X is contractible, then it is acyclic.*

There is a way in which we can generalize the homology. The generalization consists on using chains of the form $\sum n_i\sigma_i$ where each σ_i is an n-simplex, but with the coefficients n_i lying in a fixed Abelian group A rather than in $\mathbb{Z}$. Such n-chains form an Abelian group denoted $C_n(X,A)$. Notice that one can define the homology groups $H_n(X,A)$ in the same way as before, they are called **homology groups with coefficients in A**.

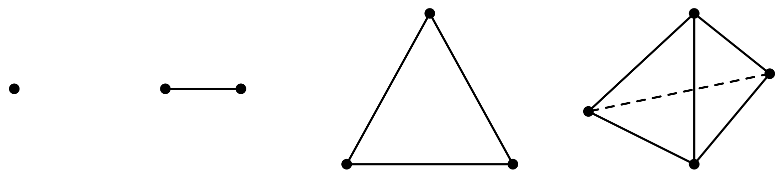
2.2 Simplicial Homology

In the previous section, we defined what a chain complex is, but we did not define the boundary map due to its complexity. In this section, we will define it in the particular case of simplicial homology.

Definition 2.2.1. Let $n \in \mathbb{N}$. The **n-simplex** is defined as:

$$\Delta^n = \{(x_0,...,x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1 \text{ and } x_i \geq 0 \; \forall \; i = 0,...,n\}$$

If we have an n-simplex together with an order in the set of the vertices we will denote $\Delta^n = [x_0,...,x_n]$. This gives us an orientation for the n-simplex. A **face** of a n-simplex is a (n-1)-simplex obtained by removing one vertex from the set $[x_0,...,x_n]$. The first four n-simplex are the following:



Definition 2.2.2. A **Δ-complex** structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ with n depending on α and such that:

1. The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective and each point of X belongs to exactly one $Im(\sigma_\alpha|_{\mathring{\Delta}^n})$.
2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. The face of Δ^n is identified with Δ^{n-1} by a linear homeomorphism that preserves the ordering of the vertices.
3. The following topology is given to X: $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

Intuitively, X is built as a quotient space of a collection of disjoint simplices Δ^n_α, one for each σ_α, where the quotient space is obtained by identifying each face of an n-simplex Δ^n_α with the (n-1)-simplex Δ^{n-1}_β corresponding to the restriction σ_β of σ_α to the face in question.

From this point, the free abelian group generated by the set of n-simplices will be denoted as $\Delta_n(X)$.

Definition 2.2.3. Let X be a Δ-complex. The **boundary map** $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is defined $\forall \; n \geq 1$ as:

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \; \sigma_\alpha|_{[v_0,...,v_{i-1},v_{i+1},...,v_n]}$$

By convention, we will say that $\partial_0 = 0$.

Lemma 2.2.4. $\partial_{n-1}\partial_n = 0 \; \forall \; n \geq 1$.

Proof. By definition, we have:

$$\partial_{n-1}\partial_n(\sigma) = \sum_{i<j} (-1)^i(-1)^{j-1} \; \sigma|_{[v_0,...,v_{i-1},v_{i+1},...,v_{j-1},v_{j+1},...,v_n]} + \sum_{i>j} (-1)^i(-1)^j \; \sigma|_{[v_0,...,v_{j-1},v_{j+1},...,v_{i-1},v_{i+1},...,v_n]}$$

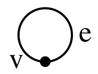
Notice that changing the indices in the second sum we get the same as in the first sum but with a minus sign. Then, both sums cancel and the result is proven. □

Definition 2.2.5. The **chain complex** of a Δ-complex X is defined as:

$$\dots \xrightarrow{\partial_{n+1}} \Delta_{n+1}(X) \xrightarrow{\partial_n} \Delta_n(X) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0$$

We also define the **cycles** of the chain complex as $Z_n^\Delta(X) = Ker(\partial_n)$, the **boundaries** as $B_n^\Delta(X) = Im(\partial_n)$ and the **simplicial homology groups** as $H_n^\Delta(X) = Z_n^\Delta(X) \big/ B_{n+1}^\Delta(X)$.

Examples 2.2.6. 1. $X = S^1$ can be seen as the Δ-complex with one vertex v and one edge e:



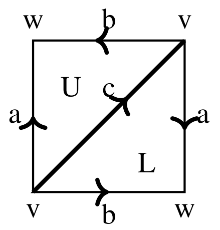
Let us compute the simplicial homology groups. First, notice that $\Delta_0(X)$ is generated by v and then it is isomorphic to $\mathbb{Z}$. The same applies for $\Delta_1(X)$, since it is generated by e. Also, since there are no n-simplices for $n \geq 2$, we have $\Delta_n(X) = 0 \ \forall \ n \geq 2$. For the boundaries it is trivial that $\partial_n = 0 \ \forall \ n \neq 1$. For the case $n = 1$ we have the following:

$$\begin{aligned} \partial_1 : \Delta_1(X) &\longrightarrow \Delta_0(X) \\ e &\longmapsto v - v = 0 \end{aligned}$$

Then, we have that $\partial_1 = 0$. Therefore, the simplicial homology groups are:

$$H_n^\Delta(S^1) = \begin{cases} \mathbb{Z} & \text{if } n = 0,1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

2. $X = \mathbb{R}P^2$ can be constructed by identifying the opposite edges of a square in the following way:



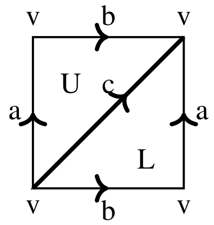
Notice that $\Delta_0(X) = \mathbb{Z}v \bigoplus \mathbb{Z}w$, $\Delta_1(X) = \mathbb{Z}a \bigoplus \mathbb{Z}b \bigoplus \mathbb{Z}c$ and $\Delta_2(X) = \mathbb{Z}U \bigoplus \mathbb{Z}L$. Let us compute the simplicial homology groups. First, notice that $\partial_n = 0 \ \forall \ n \geq 3$, which implies that $H_n^\Delta(X) = 0 \ \forall \ n \geq 3$. Now, it is easy to see that:

$$\begin{aligned} \partial_2 : \Delta_2(X) &\longrightarrow \Delta_1(X) \\ U &\longmapsto -a+b+c \\ L &\longmapsto a-b+c \end{aligned} \qquad \begin{aligned} \partial_1 : \Delta_1(X) &\longrightarrow \Delta_0(X) \\ a &\longmapsto w-v \\ b &\longmapsto w-v \\ c &\longmapsto 0 \end{aligned}$$

Then, ∂_2 is injective, which implies that $Ker(\partial_2) = 0$ and $H_2^\Delta(X) = 0$. Also, $Im(\partial_1)$ is generated by $w-v$, thus it is isomorphic to $\mathbb{Z}$ and $H_0^\Delta(X) = \mathbb{Z}$. Notice that $Ker(\partial_1)$ is generated by $a-b$ and c, which implies that it is isomorphic to $\mathbb{Z} \bigoplus \mathbb{Z}$. It is easy to see that $a-b+c$ and c also generates $Ker(\partial_1)$. Therefore, since $\{2c, a-b+c\}$ is a basis for $Im(\partial_2)$ (because $2c = (-a+b+c)+(a-b+c)$ and $\{-a+b+c, a-b+c\}$ is a basis for $Im(\partial_2)$) we have that $H_1^\Delta(X) = \mathbb{Z}_2$. Then, the simplicial homology groups are:

$$H_n^\Delta(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}_2 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

3. We can build $X = T$ by identifying the opposite edges of a square in the following way:



Notice that $\Delta_0(X) = \mathbb{Z}v$, $\Delta_1(X) = \mathbb{Z}a \bigoplus \mathbb{Z}b \bigoplus \mathbb{Z}c$ and $\Delta_2(X) = \mathbb{Z}U \bigoplus \mathbb{Z}L$. Let us compute the simplicial homology groups. First, notice that $\partial_n = 0\ \forall\ n \geq 3$ which implies $H_n^{\Delta}(X) = 0\ \forall\ n \geq 3$. Now, it is easy to see that:

$$\begin{aligned} \partial_2 : \Delta_2(X) &\longrightarrow \Delta_1(X) \\ U &\longmapsto a+b-c \\ L &\longmapsto a+b-c \end{aligned} \qquad\qquad \begin{aligned} \partial_1 : \Delta_1(X) &\longrightarrow \Delta_0(X) \\ a &\longmapsto 0 \\ b &\longmapsto 0 \\ c &\longmapsto 0 \end{aligned}$$

Therefore $\partial_1 = 0$, which implies $H_0^{\Delta}(X) = 0$. Also, we have that $Im(\partial_2)$ is generated by $a+b-c$ and, since $\{a, b, a+b-c\}$ is a basis for $\Delta_1(X)$, we have that $H_1^{\Delta}(X) = \mathbb{Z} \bigoplus \mathbb{Z}$. To compute $H_2^{\Delta}(X)$ notice that $Im(\partial_3) = 0$ which implies $H_2^{\Delta}(X) = Ker(\partial_2) = \mathbb{Z}$, since $Ker(\partial_2)$ is generated by $U - L$ (notice that for every $p, q \in \mathbb{Z}$ we have $\partial(pU + qL) = (p+q)(a+b-c) = 0$ if and only if $p = -q$). Then, the simplicial homology groups are:

$$H_n^{\Delta}(T) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z} \bigoplus \mathbb{Z} & \text{if } n = 0, 2 \\ 0 & \text{if } n \geq 2 \end{cases}$$

2.3 Group Homology

2.3.1 Algebraic definition

Definition 2.3.1. Let R be an associative ring with identity and M an R-module. A **resolution** of M is an exact sequence of R-modules (all the maps are homomorphisms and the image of each homomorphism is the kernel of the next)

$$\ldots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

If each F_i is free, then it is called a **free resolution**. If each F_i is projective, then it is called a **projective resolution**.

Lemma 2.3.2. *For every R-module M there exists a free module F and a surjection $f : F \to M$.*

Proof. First, notice that for every module M one can choose a set $\{\alpha_i\}_{i \in I}$ of generators of M. Now, let F be a free module generated by a basis such that there is a one-to-one corresponded between the elements of the basis and the generators of M (for example $F = \bigoplus_{i \in I} R\alpha_i$). Then, we can define $f : F \to M$ by sending every basis element of F to the corresponding generator of M given by the one-to-one correspondence. □

Proposition 2.3.3. *A free resolution of M exists for any module M.*

Proof. Let us construct the resolution by an inductive process:

- Choose a surjection $\varepsilon : F_0 \to M$, where F_0 is free.

- Choose a surjection $\sigma_1 : F_1 \to Ker(\varepsilon)$, were F_1 is free. For every $i \geq 2$ choose a surjection $\sigma_i : F_i \to Ker(\sigma_{i-1})$, where F_i is free.
- Define $\partial_1 = j_\varepsilon \circ \sigma_1$, where $j_\varepsilon : Ker(\varepsilon) \hookrightarrow F_0$ is the inclusion map. For every $i \geq 2$ define $\partial_i = j_{\sigma_{i-1}} \circ \sigma_i$, where $j_{\sigma_{i-1}} : Ker(\sigma_{i-1}) \hookrightarrow F_{i-1}$ is the inclusion map.

Notice that this is well defined by the previous lemma. This process gives us the free resolution. □

Corolary 2.3.4. *Any exact finite sequence of free/projective modules:*

$$F_i \xrightarrow{\partial_i} \dots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

can be completed to an exact free/projective resolution.

Proof. Apply the same process as in the previous proof, but starting at the step i. □

Example 2.3.5. A free module F admits the following free resolution:

$$0 \to F \xrightarrow{id} F \to 0$$

Let G be a group,the free $\mathbb{Z}$-module generated by the elements of G will be denoted by $\mathbb{Z}G$. Every element of $\mathbb{Z}G$ can be uniquely expressed as $\sum_{g\in G} a(g)g$ with $a : G \to \mathbb{Z}$ a function which is zero for all but a finite number of elements of G. The products of $\mathbb{Z}$ and G can be extended to a $\mathbb{Z}$-bilinear product $\mathbb{Z}G \times \mathbb{Z}G \to \mathbb{Z}G$. Then, $\mathbb{Z}G$ is a unital ring which is called the **integral group ring**.

Example 2.3.6. If G is an infinite cyclic group with t as a generator, then $\mathbb{Z}G$ has a $\mathbb{Z}$-basis $\{t^n\}_{n\in\mathbb{Z}}$. Hence, $\mathbb{Z}G \cong \mathbb{Z}[t, t^{-1}]$ which is the ring of Laurent polynomials.

Definition 2.3.7. A **G-module** consists of an Abelian group A together with a group homomorphism from G to the group of automorphisms of A. Equivalently, a G-module can be understood as an Abelian group A together with an action of G on A. Also, A can be seen as a $\mathbb{Z}G$-module.

Definition 2.3.8. Let G be a group and M a G-module. The **group of co-invariants of M**, denoted as M_G, is defined to be the quotient of M by the additive subgroup generated by the elements of the form $gm - m$ with $g \in G$ and $m \in M$.

Intuitively, M_G is obtained from M by dividing by the G-action. This is a useful definition since it gives us a way to construct new sequences from a given one. Indeed, it preserves some nice properties:

- If $M'' \to M' \to M \to 0$ is an exact sequence of G-modules then $(M'')_G \to (M')_G \to (M)_G \to 0$ is also exact.
- If F is a free $\mathbb{Z}G$-module then F_G is a free $\mathbb{Z}$-module.

Now, we are in the conditions to give the algebraic definition of the homology of a group.

Definition 2.3.9. Let G be a group and let $\dots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. We define the **homology groups** of G by $H_n(G) = H_n(F_G)$ $\forall\, n \in \mathbb{N}$, where F_G is the chain complex obtained by taking co-invariants to our given resolution.

Remark 2.3.10. Although the definition of the homology groups is given just for projective resolutions, it its immediate to see that by 2.3.4 one can compute the homology up to degree i just with an exact finite resolution up to degree i.

The next step is to prove that the homology groups are well defined and give us an invariant. To prove this we will need a couple definitions and some technical results:

Definition 2.3.11. Let R be a ring. A **chain complex** over R is a pair (C,∂), where $C=\{C_n\}_{n\in\mathbb{Z}}$ is a sequence of R-modules and $\partial=\{\partial_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ is a family of R-module homomorphisms such that $\partial_{n-1}\partial_n = 0\ \forall\, n\in\mathbb{Z}$.

Definition 2.3.12. Let (C,∂) and (C',∂') be two chain complexes. Then:

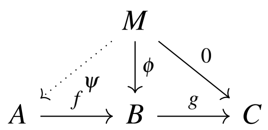
1. A **chain map** between the two chain complexes is a family of homomorphisms $f=\{f_n : C_n \to C'_n\}_{n\in\mathbb{Z}}$ such that $\partial'_n \circ f_n = f_{n-1}\circ\partial_n\ \forall\, n\in\mathbb{Z}$.

2. A **homotopy** between two chain maps f and g is a family of homomorphisms $h=\{h_n : C_n \to C'_{n+1}\}_{n\in\mathbb{Z}}$ such that $\partial'_n\circ h_n + h_{n-1}\circ\partial_n = f-g\ \forall\, n\in\mathbb{Z}$. It is said that f is **homotopic** to g if there is an homotopy from f to g. It is denoted as $f\cong g$.

Proposition 2.3.13. *Let (C,∂) and (C',∂') be two chain complexes and $f=\{f_n : C_n\to C'_n\}_{n\in\mathbb{Z}}$, $g=\{g_n : C'_n\to C_n\}_{n\in\mathbb{Z}}$ be two chain maps such that $f_n\circ g_n \cong id_{C'_n}$ and $g_n\circ f_n\cong id_{C_n}$ for every $n\in\mathbb{Z}$. Then, $H_n(C)\cong H_n(C')\ \forall\, n\in\mathbb{Z}$.*

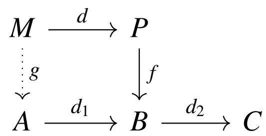
Proof. See [1]. □

Lemma 2.3.14. *Let R be a ring. An R-module M is projective if and only if for every exact sequence of R-modules $A\xrightarrow{f} B\xrightarrow{g} C$ and every module homomorphism $\phi : M\to B$ such that $\phi\circ g=0$ there exists a module homomorphism $\psi : M\to A$ such that $f\circ\psi=\phi$.*



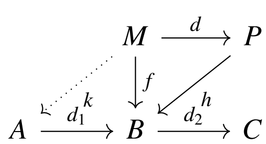
Proof. Notice that the definition of projective is equivalent to what we want to prove but with $C=0$ and $B=Ker(f)$. Then, the result follows immediately. □

Lemma 2.3.15. 1. *Given the following diagram:*



where $d_2\circ f\circ d=0$ and we want to find g such that $d_1\circ g=f\circ d$. If P is projective and the bottom row is exact, then such g exists.

2. *Given the following diagram:*



where $d_2\circ h\circ d = d_2\circ f$ and it is desired to find k such that $d_1\circ k + h\circ d = f$. If P is projective and the bottom row is exact, then such k exists.

Proof. 1. Apply the previous lemma with $\phi = f\circ d$.

2. Apply the previous lemma with $\phi = f - h\circ d$.

□

Lemma 2.3.16. *(Fundamental Lemma of Homological Algebra) Let (C,∂) and (C',∂') be two chain complexes and $r \in \mathbb{Z}$. Let $\{f_i : F_i \to F_i'\}_{i\leq r}$ satisfying $\partial_i' \circ f_i = f_{i-1} \circ \partial_i$ for $i \leq r$. If C_i is projective for $i > r$ and $H(C_i') = 0$ for $i \geq r$ then, the family $\{f_i\}_{i\leq r}$ extends to a chain map $f = \{f_i : C_i \to C_i'\}_{i\in\mathbb{Z}}$ which is unique up to homotopy.*

Proof. Let us construct f_i by induction on i. Assume that f_i has been defined for $i \leq n$ where $n \geq r$ and that $\partial_i' \circ f_i = f_{i-1} \circ \partial_i$ for $i \leq n$. Then we have a mapping problem:

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \vdots\, f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \end{array}$$

where $\partial_n' \circ f_n \circ \partial_{n+1} = f_{n-1} \circ \partial_n \circ \partial_{n+1} = 0$. Hence, applying the previous lemma, there exists f_{n+1} making the diagram commutative.
Now, suppose that $g = \{g_i : C_i \to C_i'\}_{i\in\mathbb{Z}}$ is another extension of f. We have to find an homotopy h between f and g. First, we can take trivially $h_i = 0$ for $i \leq r$. Let us construct the rest by induction. Suppose that h_i has been defined for $i \leq n$, where $n \geq r$, and that $\partial_{i+1}' \circ h_i + h_{i-1} \circ \partial_i = f_i - g_i\ \forall\ i \leq n$. Let $\tau_i = f_i - g_i\ \forall\ n \in \mathbb{Z}$, then we have the following mapping problem:

$$\begin{array}{ccccccc} & & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \\ & \swarrow^{h_{n+1}} & \downarrow \tau_{n+1} & \swarrow^{h_n} & \downarrow \tau_n & \swarrow^{h_{n-1}} & \\ C'_{n+2} & \xrightarrow{\partial'_{n+2}} & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & & \end{array}$$

with $\partial_{n+1}' \circ h_n \circ \partial_{n+1} = (\tau_n - h_{n-1} \circ \partial_n) \circ \partial_{n+1} = \tau_n \circ \partial_{n+1} = \partial_n' \circ \tau_{n+1}$. Therefore, the previous lemma can be applied to show that there exists h_{n+1} with $\partial_{n+2}' \circ h_{n+1} + h_n \circ \partial_{n+1} = \tau_{n+1}$. □

Proposition 2.3.17. *Let M be an R-module and $\ldots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \to 0$, $\ldots \xrightarrow{\partial_2'} F_1' \xrightarrow{\partial_1'} F_0' \xrightarrow{\varepsilon'} M \to 0$ be two projective resolutions of M, then there is a family of maps $\{f_i : F_i \to F_i'\}_{i\geq 0}$, unique up to homotopy, which satisfies $\varepsilon' \circ f_0 = \varepsilon$ and $\partial_i' \circ f_i = f_{i-1} \circ \partial_i$.*

Proof. The two projective resolutions can be seen as two chain complexes, where $F_{-1} = F_{-1}' = M$ and $F_i = F_i' = 0\ \forall\ i < -1$. Applying the fundamental lemma of homological algebra with $r = -1$ we are done. □

Theorem 2.3.18. *Let G be a group. Then, the homology groups $\{H_n(G)\}_{n\in\mathbb{N}}$ are unique up to isomorphism.*

Proof. Let $\ldots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, $\ldots \xrightarrow{\partial_2'} F_1' \xrightarrow{\partial_1'} F_0' \xrightarrow{\varepsilon'} \mathbb{Z} \to 0$ be two projective resolutions. Then, there exist two families of maps $\{f_i : F_i \to F_i'\}_{i\geq 0}$ and $\{g_i : F_i' \to F_i\}_{i\geq 0}$ satisfying the previous proposition. This implies that $\{g_i \circ f_i : F_i \to F_i\}_{i\geq 0}$ and $\{f_i \circ g_i : F_i' \to F_i'\}_{i\geq 0}$ are families of maps satisfying the previous proposition, where we are considering twice the same resolution. On the other hand, we have immediately that $\{id_{F_i} : F_i \to F_i\}_{i\geq 0}$ and $\{id_{F_i'} : F_i' \to F_i'\}_{i\geq 0}$ also satisfies the proposition. Hence, since the family of maps is unique up to homotopy, it follows that $g_i \circ f_i \cong id_{F_i}$ and $f_i \circ g_i \cong id_{F_i'}\ \forall\ i \geq 0$. Considering the group of co-invariants of the resolution, we have that $(g_i)_G \circ (f_i)_G \cong (g_i \circ f_i)_G \cong (id_{F_i})_G \cong id_{(F_i)_G}$ and $(f_i)_G \circ (g_i)_G \cong (f_i \circ g_i)_G \cong (id_{F_i'})_G \cong id_{(F_i')_G}$. Applying 2.3.13 the result is proven. □

Example 2.3.19. Let us compute the homology groups of the finite cyclic group $G = \langle x \mid x^n = 1\rangle$. First, notice that if $e = 1 + x + \ldots + x^{n-1}$ then $(1-x)e = 1 - x^n = 0$. Hence, we can define the following resolution of period 2 of $\mathbb{Z}$:

$$\ldots \xrightarrow{\partial_4 = e\cdot} \mathbb{Z}G \xrightarrow{\partial_3=(1-x)\cdot} \mathbb{Z}G \xrightarrow{\partial_2 = e\cdot} \mathbb{Z}G \xrightarrow{\partial_1=(1-x)\cdot} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\forall\ \alpha \in \mathbb{Z}G\ \ \partial_i(\alpha) = e\alpha$ if i is odd and $\partial_i(\alpha) = (1-x)\alpha$ if i is even. To see that it is a projective resolution we just need to check that the sequence is exact, since it is trivially projective for $\mathbb{Z}G$ being free. Let us prove that the sequence is exact:

1. First, let us show that if $\alpha = \sum_{j=0}^{n-1} a_j x^j \in \mathbb{Z}G$ is such that $(1-x)\alpha = 0$ then $\alpha = e\beta$ with $\beta \in \mathbb{Z}G$:
$$0 = (1-x)\alpha = (1-x)\sum_{j=0}^{n-1} a_j x^j = \sum_{j=0}^{n-1} a_j x^j - \sum_{j=0}^{n-1} a_j x^{j+1} = a_0 - a_{n-1} + \sum_{j=1}^{n-1}(a_j - a_{j-1})x^j$$
where the last equality holds since $1 = x^n$. Therefore, we have that $a_0 = a_{n-1}$ and $a_j = a_{j-1}\ \forall\ j = 1,\ldots,n-1$. Then, $a_0 = a_1 = \cdots = a_{n-1}$ which implies that $\alpha = \sum_{j=0}^{n-1} a_j x^j = \sum_{j=0}^{n-1} a_0 x^j = a_0 e$.

2. Next, let us show that if $\alpha = \sum_{j=0}^{n-1} a_j x^j \in \mathbb{Z}G$ is such that $e\alpha = 0$ then $\alpha = (1-x)\beta$ with $\beta \in \mathbb{Z}G$:
Notice that $(1-x)e = 0$, which implies $e = ex = ex^2 = \cdots = ex^{n-1}$. Therefore:
$$0 = e\alpha = e\sum_{j=0}^{n-1} a_j x^j = \sum_{j=0}^{n-1} a_j e x^j = \sum_{j=0}^{n-1} a_j e$$
This means that $\sum_{j=0}^{n-1} a_j = 0$, which implies that α has a root in $x = 1$.

3. To show exactness we have to show that $Im(\partial_1) = Ker(\varepsilon)$. If $\alpha \in Im(\partial_1)$ then exists $\beta \in \mathbb{Z}G$ such that $\alpha = (1-x)\beta$ which implies $\varepsilon(\alpha) = \varepsilon((1-x)\beta) = 0$. Therefore, $\alpha \in Ker(\varepsilon)$ and we have $Im(\partial_1) \subset Ker(\varepsilon)$. On the other hand, if $\alpha \in Ker(\varepsilon)$ then α as a polynomial in x has a root in $x = 1$, so exists $\beta \in \mathbb{Z}G$ such that $\alpha = (1-x)\beta$. This implies that $\alpha \in Im(\partial_1)$ and so $Ker(\varepsilon) \subset Im(\partial_1)$.

4. Let us show that $Im(\partial_i) = Ker(\partial_{i-1})$ for i odd. If $\alpha \in Im(\partial_i)$ then exists $\beta \in \mathbb{Z}G$ such that $\alpha = (1-x)\beta$ which implies $\partial_{i-1}(\alpha) = e(1-x)\beta = 0$. Therefore, $\alpha \in Ker(\partial_{i-1})$ and we have $Im(\partial_i) \subset Ker(\partial_{i-1})$. If $\alpha \in Ker(\partial_{i-1})$, then $e\alpha = 0$ which implies $\alpha = (1-x)\beta$. Therefore, $\alpha \in Im(\partial_i)$ and $Ker(\partial_{i-1}) \subset Im(\partial_i)$.

5. Finally let us show that $Im(\partial_i) = Ker(\partial_{i-1})$ for i even. If $\alpha \in Im(\partial_i)$ then exists $\beta \in \mathbb{Z}G$ such that $\alpha = e\beta$ which implies $\partial_{i-1}(\alpha) = (1-x)e\beta = 0$. Therefore, $\alpha \in Ker(\partial_{i-1})$ and we have $Im(\partial_i) \subset Ker(\partial_{i-1})$. If $\alpha \in Ker(\partial_{i-1})$, then $(1-x)\alpha = 0$ which implies $\alpha = e\beta$. Therefore, $\alpha \in Im(\partial_i)$ and $Ker(\partial_{i-1}) \subset Im(\partial_i)$.

Hence, the sequence is exact. Now, let us compute the chain of co-invariants of the resolution. Notice that $(\mathbb{Z}G)_G = \mathbb{Z}G \big/ \langle m - mg \mid g \in G, m \in \mathbb{Z}G\rangle$. Thus, since $1 \in \mathbb{Z}G$ and $x \in G$ we have $\bar{x} = \bar{1}$. Therefore $(\mathbb{Z}G)_G = \mathbb{Z}$ and then the chain is:
$$\ldots \xrightarrow{(\partial_4)_G} \mathbb{Z} \xrightarrow{(\partial_3)_G} \mathbb{Z} \xrightarrow{(\partial_2)_G} \mathbb{Z} \xrightarrow{(\partial_1)_G} \mathbb{Z} \xrightarrow{(\varepsilon)_G} \mathbb{Z} \to 0$$
For computing the maps notice that $(\partial_i)_G(1) = (\partial_i(1))_G$ and then $(\partial_i)_G(1) = (1-x)_G = 0$ if i is odd and $(\partial_i)_G(1) = (e)_G = n$ if i is even, moreover $(\varepsilon)_G(1) = (\varepsilon(1))_G = (1)_G = 1$. Then, we can compute the homologies of the chain and we have:
$$H_n(G) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}_n & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

2.3.2 Topological definition

Definition 2.3.20. A **G-CW-complex** is a CW-complex together with an action of G on X which permutes the cells. We say that the G-complex is **free** if the action of G freely permutes the cells of X ($g\sigma \neq \sigma \;\forall\, e \neq g \in G$).

If X is a contractible space then, by 2.1.9, we have that:

$$\dots \overset{\partial_{n+1}}{\to} C_n(X) \overset{\partial_n}{\to} C_{n-1}(X) \overset{\partial_{n-1}}{\to} \dots \overset{\partial_2}{\to} C_1(X) \overset{\partial_1}{\to} C_0(X) \overset{\partial_0=\varepsilon}{\to} \mathbb{Z} \to 0$$

is an exact chain, which leads us to the following proposition:

Proposition 2.3.21. *Let X be a contractible free G-complex. The augmented chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Definition 2.3.22. Let Y be a G-CW-complex satisfying the following conditions:

1. Y is connected.
2. $\pi_1(Y) = G$.
3. The universal cover X of Y is contractible.

This space Y is called an **Eilenberg-MacLane complex of type (G,1)** or simply **K(G,1)-complex**.

Example 2.3.23. Let us consider the Abelian group $G = \mathbb{Z}^2$ and let $T = S^1 \times S^1$ be the torus. Let us see that T is a $K(G,1)$-complex. The fact that T is connected is trivial and, by 1.4.9, the universal cover of T is $\mathbb{R}^2$ which is contractible. The fact that $\pi_1(T) = G$ it is also true, but we are not going to prove it (see [8]).

As a consequence of the previous proposition we have:

Proposition 2.3.24. *If Y is a $K(G,1)$-complex then the augmented cellular chain complex of the universal cover of Y is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Notice that, by 1.4.16, G acts freely on Y.

Proposition 2.3.25. *Let X be a G-complex. Then $C_n(X/G) \cong C_n(X)_G \;\forall\, n \in \mathbb{N}$.*

Proof. Let us consider the projection map $\phi : C_n(X) \to C_n(X/G)$. Passing to the quotient it induces a well defined map $\overline{\phi} : C_n(X)_G \to C_n(X/G)$. Now, notice that $C_n(X)$ is a free Abelian group, and hence it is a $\mathbb{Z}$-module, which implies that $C_n(X)_G$ has a $\mathbb{Z}$-basis with one basis element for each orbit. In a similar way $C_n(X/G)$ has a $\mathbb{Z}$-basis with one basis element for each orbit and $\overline{\phi}$ maps each basis element to a basis element. Then, $\overline{\phi}$ is a well defined isomorphism by construction. □

Now, we are in the conditions of giving the topological definition of the homology of a group.

Proposition 2.3.26. *If Y is a $K(G,1)$-complex, then $H_n(G) \cong H_n(Y) \;\forall\, n \in \mathbb{N}$.*

Proof. Let Y be a $K(G,1)$-complex with universal cover X then, by 2.3.24, $\{C_n(X)\}_{n\in\mathbb{N}}$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Therefore, by 2.3.25, $C_n(Y) \cong C_n(X)_G \;\forall\, n \in \mathbb{N}$, which proves the result. □

Remark 2.3.27. This is what is called the topological definition of the homology of a group. We have derived it from the algebraic definition, but there are some books that do it the other way round.

Example 2.3.28. If $G = \mathbb{Z}^2$ then the torus is a $K(G,1)$-complex. Hence, in order to know the homology groups of G, we just need to know the simplicial homology groups of the torus, which have been computed in 2.2.6:

$$H_n(G) = \begin{cases} \mathbb{Z} & \text{if } n = 0,2 \\ \mathbb{Z}^2 & \text{if } n = 1 \\ 0 & \text{if } n \geq 3 \end{cases}$$

This fact can be generalized for $G = \mathbb{Z}^n$. In fact, one can see that $H_i(G) = \mathbb{Z}^{\binom{n}{i}} \;\forall\, i \in \mathbb{N}$ (see [1]).

Chapter 3

Introduction to Knot Theory.

3.1 First definitions

In this section, we will define what a knot is and how it can be represented. Since the proofs of all the facts are very easy, but quite long and technical, they will be omitted. Details and proofs of this section can be found at [4, Chapter 2].

Definition 3.1.1. Let $\{p_1,...,p_n\}$ be a set of distinct points, the union of the segments $[p_1,p_2]$, $[p_2,p_3]$,...,$[p_{n-1},p_n]$,$[p_n,p_1]$ is called a **closed polygonal curve**. If each segment intersects exactly two other segments, intersecting only at the endpoints, then the curve is said to be **simple**.

Definition 3.1.2. A **knot** is a simple polygonal closed curve in $\mathbb{R}^3$. The elements of $\{p_1,...,p_n\}$ are called the vertices of the knot.

Definition 3.1.3. An **oriented knot** consists of a knot $\{p_1,...,p_n\}$ with an orientation given by the order of the vertices. Two orientations are considered to be equivalent if they differ by a cyclic permutation.

It is immediate to notice that a knot can only have two non equivalent orientations.

Remark 3.1.4. Although knots are defined as simple polygonal closed curves, we will regard them as smooth curves.

Now, we need to know how to draw knots, since they are three-dimensional objects. Let us consider the projection map $\mathbb{R}^3 \to \mathbb{R}^2$ given by $(x,y,z) \longmapsto (x,y)$. One possibility is to represent a knot as its image under the projection. But this creates some problems (for example, there might be multiple lines that project to the same points of $\mathbb{R}^2$). To avoid these problems, some results will be needed.

Definition 3.1.5. A knot projection is called **regular** if there are not three points in the knot that project to the same point of $\mathbb{R}^2$.

Definition 3.1.6. Two knots K_1 and K_2 are **equivalent** if there exists a continuous map $h : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $h(K_1,0) = K_1$ and $h(K_2,1) = K_2$. The map h is called a **isotopy** between K_1 and K_2.

The main objective of knot theory is to study whether two knots are equivalent or not. From now on, we will not distinguish between equivalent knots. It can be proven the following result:

Theorem 3.1.7. *Let K be a knot. There exists a knot K' such that it is equivalent to K and K' has a regular projection.*

A knot will be represented by a regular projection. This is called a **knot diagram**. The points in the diagram which correspond to double points in the projection are called **crossing points**. The lines that are drawn in the knot are called **arcs**. In each crossing point there are two arcs of the knot, one is called **overcrossing**, while the other is called **undercrossing**. By convention, the undercrossing will be drawn as a broken line and the overcrossing will be drawn as a continuous line.

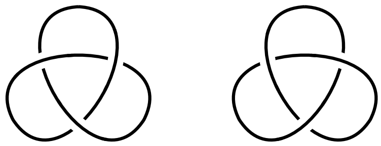
Theorem 3.1.8. *Let K and J be two knots with regular projections. If they have the same knot diagrams then they are equivalent.*

The proof of this theorem is very technical and of no interest, so it will be omitted . This implies that knot diagrams and knots can be used equivalently. In order to study whether two knots are equivalent, we will only study their projection.

Examples 3.1.9. Now, let us see the first examples of knots:

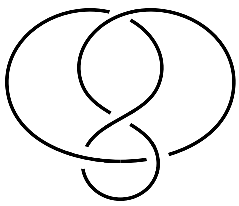
1. The easiest example of a knot is S^1. This is called the trivial knot. It is easy to see that it is the only knot that can be embedded in $\mathbb{R}^2$ (Every knot that can be embedded in $\mathbb{R}^2$ is equivalent to the trivial one).

2. The easiest examples of non trivial knots are:



these are known as the **right-trefoil knot** and **left-trefoil knot** respectively. These are the only two knots with three crossings.

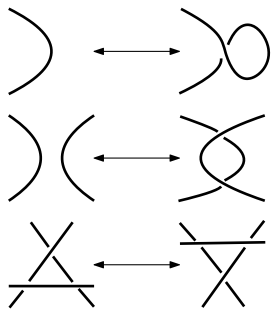
3. The only knot with four crossings is:



This is called the **eight knot**.

Now, we are interested in knowing when a transformation does not affect the knot. the basic idea is to observe that there are only three transformations that one can do, and every other transformation is a composition of the three basic ones (we will not prove this fact).

Definition 3.1.10. We define the **Reidemeister moves** as the following:



Theorem 3.1.11. *(Reidemeister) Two knots are equivalent if and only if there exists a sequence of Reidemeister moves that transforms the first into the second.*

3.2 The Knot Group

Definition 3.2.1. Let K be a knot. We define the **knot group** of K as $\pi_1(\mathbb{R}^3 \setminus K)$.

Notice that we do not need to indicate the base point in the fundamental group, since for every knot K the space $\mathbb{R}^3 \setminus K$ is trivially path-connected.

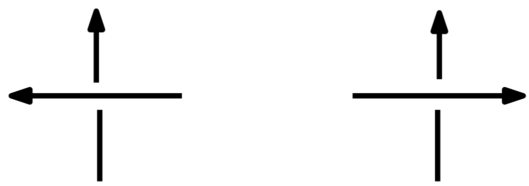
Proposition 3.2.2. *Equivalent knots have isomorphic fundamental group.*

Proof. Let K_1 and K_2 be two equivalent knots. It is trivial to see that $\mathbb{R}^3 \setminus K_1$ and $\mathbb{R}^3 \setminus K_2$ are homeomorphic spaces, which implies that their fundamental groups are isomorphic. □

Hence, the knot group is a knot invariant. However, the converse is not true, since there exist non-equivalent knots with isomorphic fundamental groups. Then, the knot group will be useful to distinguish when two knots are not equivalent, but it will be of no help to show when two regular projections give equivalent knots.

Now, let us give a procedure to find a presentation for the knot group of a given knot.

Definition 3.2.3. Given an oriented knot each of the crossing points is locally of one of the following two ways:



These are called **left-handed crossings** and **right-handed crossings** respectively. We call them **oriented crossings**.

In an oriented crossing b, we will denote the overcrossing arc as $o(b)$ and the two parts of the undercrossing arc as $u(b)$ and $u(b)+1$ respectively.

Proposition 3.2.4. *Let K be an oriented knot, $A = \{a_1, \dots, a_n\}$ be the set of all arcs and $B = \{b_1, \dots, b_n\}$ be the set of all crossing points. Then, the knot group of K is given by the following presentation:*

$$\langle a_1, \dots, a_n \mid r(b_1) = 1, \dots, r(b_n) = 1 \rangle$$

where

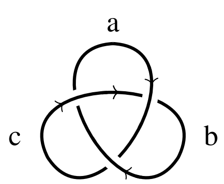
$$r(b) = \begin{cases} (u(b)+1)o(b)u(b)^{-1}o(b)^{-1} & \text{if } b \text{ is right-handed} \\ o(b)(u(b)+1)o(b)^{-1}u(b)^{-1} & \text{if } b \text{ is left-handed} \end{cases}$$

Proof. Can be found in [3]. □

This is called the **Wirtinger presentation** of the knot group.

Remark 3.2.5. It can be proven that one of the relations of the Wirtinger presentation can be deleted, leaving us with exactly the same group.

Examples 3.2.6.
1. Let us find the Wirtinger presentation of the trefoil knot. First, we need to give an orientation for the knot and to label each arc with a letter.



Now, we have to compute the three words corresponding to each of the intersection points. Notice that all three intersection points are right-handed and thus the equations we get are:

$$aca^{-1}b^{-1} = 1,\ cbc^{-1}a^{-1} = 1,\ bab^{-1}c^{-1} = 1$$

Hence, the Wirtinger presentation for the knot group is:

$$\langle a,b,c \mid aca^{-1} = b, cbc^{-1} = a, bab^{-1} = c \rangle$$

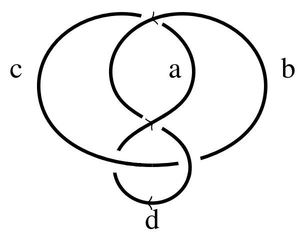
In our presentation $c = bab^{-1}$, which implies that we can delete the letter c from the presentation by substituting it by its value. Then, the new presentation is:

$$\langle a,b \mid abab^{-1}a^{-1} = b, bab^{-1}bba^{-1}b^{-1} = a \rangle = \langle a,b \mid abab^{-1}a^{-1} = b, baba^{-1}b^{-1} = a \rangle$$

It is easy to check that both words are the same, and we can delete one of them. Hence, our final presentation is:

$$\langle a,b \mid aba = bab \rangle$$

2. Let us find the Wirtinger presentation of the eight knot. First, we need to give an orientation for the knot and to label each arc with a letter.



Now, we have to compute the four words corresponding to each one of the intersection points. Notice that two of the intersection points are right-handed and two are left-handed. Thus, the equations are:

$$acd^{-1}c^{-1} = 1,\ bdc^{-1}d^{-1} = 1,\ ada^{-1}b^{-1} = 1,\ bcb^{-1}a^{-1} = 1$$

Hence, the Wirtinger presentation for the knot group is:

$$\langle a,b,c,d \mid acd^{-1}c^{-1} = 1,\ bdc^{-1}d^{-1} = 1,\ ada^{-1}b^{-1} = 1,\ bcb^{-1} = a \rangle$$

In our presentation $a = bcb^{-1}$ and we can delete the letter a from the presentation by substituting it by its value. Then, the new presentation is:

$$\langle b,c,d \mid bcb^{-1}cd^{-1}c^{-1} = 1,\ bdc^{-1}d^{-1} = 1,\ bcb^{-1}dbc^{-1}b^{-1}b^{-1} = 1 \rangle =$$

$$= \langle b,c,d \mid bcb^{-1}cd^{-1}c^{-1} = 1,\ b = dcd^{-1},\ bcb^{-1}dbc^{-1}b^{-1}b^{-1} = 1 \rangle$$

In the same way $b = dcd^{-1}$ in our presentation. Therefore, we can delete the letter b from the presentation by substituting is value. The new presentation is:

$$\langle c,d \mid dcd^{-1}cdc^{-1}d^{-1}cd^{-1}c^{-1} = 1,\ dcd^{-1}cdc^{-1}d^{-1}ddcd^{-1}c^{-1}dc^{-1}d^{-1}dc^{-1}d^{-1} = 1 \rangle =$$

$$= \langle c,d \mid dcd^{-1}cdc^{-1}d^{-1}cd^{-1}c^{-1} = 1,\ dcd^{-1}cdc^{-1}dcd^{-1}c^{-1}dc^{-1}c^{-1}d^{-1} = 1 \rangle$$

It is easy to see that the second word is equivalent to $d^{-1}cdc^{-1}dcd^{-1}c^{-1}dc^{-1} = 1$. With this, we have trivially that both words are the same and we can delete one. Then, the final presentation is:

$$\langle c,d \mid dcd^{-1}cd = cdc^{-1}dc \rangle$$

Remark 3.2.7. Although this method works for any knot, it does not have any geometric interpretation. It is very hard to understand the geometry of the knot with its Wirtinger presentation.

Chapter 4

The Alexander Polynomial.

The Alexander polynomial is a knot invariant which assigns a polynomial to each knot. There are several ways of defining this polynomial. Our main objective is to do it in the algebraic way, which is the rationale behind the previous chapters, but we will also see another definition which will be interesting due to the simplicity of the computations.

4.1 The Algebraic Way

First of all, let us state a couple of algebraic facts that will be needed.

Definition 4.1.1. A ring R is said to be **Noetherian** if every ideal I of R is finitely generated.

Definition 4.1.2. Let R be a ring and M an R-module. Then, M is said to be **Noetherian** if every submodule of M is finitely generated.

Lemma 4.1.3. *If R is a Noetherian ring, it is a Noetherian R-module.*

Lemma 4.1.4. *Let R be a ring, M an R-module and N a submodule of M. Then, if N and M/N are Noetherian, so is M.*

Proof. Let $\{y_1,\dots,y_n\}$ be a generating family of N and $\{z_1+N,\dots,z_m+N\}$ be a generating family of M/N. Let $x\in M$, then $x+N\in M/N$, so there exists $a_1,\dots,a_m\in R$ such that $x+N=\sum_{i=1}^{m}a_i(z_i+N)=\left(\sum_{i=1}^{m}a_iz_i\right)+N$. Since $x-\sum_{i=1}^{m}a_iz_i\in N$ there exists $b_1,\dots,b_n\in R$ such that $x-\sum_{i=1}^{m}a_iz_i=\sum_{i=1}^{n}b_iy_i$. This implies that $x=\sum_{i=1}^{m}a_iz_i+\sum_{i=1}^{n}b_iy_i$. Hence, $\{y_1,\dots,y_n,z_1,\dots,z_m\}$ is a generating family for M, so M is finitely generated. □

Proposition 4.1.5. *Let R be a ring and M an R-module. If R is Noetherian and M is finitely generated then M is Noetherian.*

Proof. Let us proceed by induction on the number of elements that generates M:
If M is generated by 1 element, then $\exists\, v\in R$ such that $M=Rv$, which implies that $\phi: R\to M$ given by $\phi(r)=rv$ is an epimorphism. Therefore, by the first isomorphism theorem $R/Ker(\phi)\cong Im(\phi)\cong M$. Then, M is Noetherian, as it is the quotient of the Noetherian module R.
If M is generated by n elements $v_1,\dots,v_n$ then, by our induction hypothesis, if M' is the module generated by $v_1,...,v_{n-1}$, it is Noetherian. Let N be a submodule of M, we have to prove that it is finitely generated. Let us define $T=N\cap M'$. Then, by the second isomorphism theorem:

$$N/T\cong N/N\cap M'\cong (N+M')/M'$$

The last one is a submodule of $M/M' \cong \langle v_n \rangle$, which is cyclic, and so it is Noetherian. Then, N/T is finitely generated. T is a submodule of the module M', so it is finitely generated by the induction hypothesis. Hence, by the previous lemma, N is finitely generated, which proves the induction step. □

Let us consider the following situation. Let X be a CW-complex and $\tilde{X}$ an infinite cyclic covering of X, determined by some homomorphism of the fundamental group $\pi_1(X)$ onto an infinite cyclic group Π generated by an element t. Then, Π acts as a group of covering transformations of the infinite complex $\tilde{X}$ and $\tilde{X}/\Pi \cong X$ (in the case of knots this homeomorphism is well defined because of Wirtinger presentation and 1.1.6, which can be applied because in Π all the x_i are substituted with t and so the relations are trivially true).
Now, let us choose a coefficient field F, and consider the chain complex $\{C_n(\tilde{X},F)\}_{n\geq 0}$ and its homology groups $\{H_n(\tilde{X},F)\}_{n\geq 0}$. The infinite cyclic group Π of covering transformations operates on these groups. Hence, one can think of $C_n(\tilde{X},F)$ and $H_n(\tilde{X},F)$ as vector spaces over F, or as a modules over $F\Pi$.
Notice that $C_n(\tilde{X},F)$ is free and finitely generated over $F\Pi$ with one generator for each n-cell of X. Then, since $F\Pi$ is Noetherian, by the previous proposition $H_n(\tilde{X},F)$ is also finitely generated over $F\Pi$.

Definition 4.1.6. Any generator of the ideal order($H_1(\tilde{X},F)$) is called the **Alexander polynomial** of $\pi_1(X)$ associated to the covering.

Definition 4.1.7. The **Alexander polynomial**, associated to the covering given by the map considered above, of a knot K is the Alexander polynomial of $\pi_1(\mathbb{R}^3 \setminus K)$. It will be denoted as Δ_K

Remark 4.1.8. Notice that the previous definition is ambiguous, since the Alexander polynomial can differ up to a factor of $\pm t^k$. Hence, by convention, we will say that the Alexander polynomial is the one with no negative powers and positive independent term.

4.2 Fox Derivatives

The definition of the Alexander polynomial of a knot is complicated and does not give us any hint on how to compute it. The aim of this section is to offer a solution to this problem. In order to do so, a special derivative for words and group presentations will be introduced.

Definition 4.2.1. Suppose that w_1 and w_2 are words and $x_1,...,x_n$ are letters. The **Fox derivative** of a word is defined with the following rules:

1. $\frac{\partial}{\partial x_i}(x_i) = 1$
2. $\frac{\partial}{\partial x_i}(x_j) = 0$ if $i \neq j$
3. $\frac{\partial}{\partial x_i}(1) = 0$
4. $\frac{\partial}{\partial x_i}(x_i^{-1}) = -x_i^{-1}$
5. $\frac{\partial}{\partial x_i}(w_1 w_2) = \frac{\partial}{\partial x_i}(w_1) + w_1 \frac{\partial}{\partial x_i}(w_2)$

Example 4.2.2. Let us compute the Fox derivative of $xyxy^{-1}x^{-1}y^{-1}$ with respect to x:

$$\frac{\partial}{\partial x}(xyxy^{-1}x^{-1}y^{-1}) = 1 + x\frac{\partial}{\partial x}(yxy^{-1}x^{-1}y^{-1}) = 1 + xy\frac{\partial}{\partial x}(xy^{-1}x^{-1}y^{-1}) = 1 + xy + xyx\frac{\partial}{\partial x}(y^{-1}x^{-1}y^{-1}) =$$

$$= 1 + xy + xyxy^{-1}\frac{\partial}{\partial x}(x^{-1}y^{-1}) = 1 + xy - xyxy^{-1}x^{-1} + xyxy^{-1}x^{-1}\frac{\partial}{\partial x}(y^{-1}) = 1 + xy - xyxy^{-1}x^{-1}$$

Now, we are going to propose a method to compute the Alexander polynomial with this tool. First, let us give a definition and then we will state and prove the main theorem.

Definition 4.2.3. Let K be a knot and $\langle x_1,...,x_n \mid r_1 = 1,...,r_m = 1\rangle$ a presentation of its group knot. The **Fox matrix** of the presentation is defined as the following:

$$A = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

that is a matrix with entries in the set of words generated by the set $\{x_1,...,x_n\}$.

Because of 3.2.5, it can be assumed that $m \leq n-1$ in the definition of the Fox matrix.

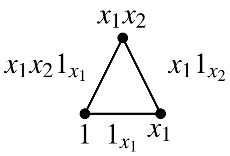
Theorem 4.2.4. *Let K be a knot and A a Fox matrix of some presentation of the group knot of K. If the matrix obtained by substituting $x_1,...,x_n$ with t is denoted as A_t then:*

$$\Delta_K = \gcd\{(n-1) - minors\ of\ A_t\}$$

Proof. Let $G = \langle x_1,...,x_n \mid r_1 = 1,...,r_m = 1\rangle$ be the knot group of K and let us consider the Cayley complex of the knot group $\tilde{X}_G$ and its chain complex:

$$\ldots C_2(X) \cong \bigoplus_{i=1}^{m} FG_{r_i} \xrightarrow{\partial_2} C_1(X) \cong \bigoplus_{i=1}^{n} FG_{x_i} \xrightarrow{\partial_1} C_0(X) \cong FG$$

Let us study the boundary maps. For every generator we will denote by 1_{x_i} the identity element of FG_{x_i}. Also, for every relation we will denote by 1_{r_i} the identity element of FG_{r_i}. First, we have that $\partial_1(1_{x_i}) = x_i - 1$ for every $i = 1,...,n$. To study ∂_2 we claim that its matrix, seen as a linear map, is exactly the Fox matrix of the presentation. In order to do this, one easily checks that if r_j is a relation then $\partial_2(1_{r_j}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(r_j)1_{x_i}$. Let us see an example: geometrically its easy to see that if if $r = x_1x_2x_1$ then $\partial_2(r) = x_1x_21_{x_1} + x_11_{x_2} + 1_{x_1} = (x_1x_2+1)1_{x_1} + x_11_{x_2} = \frac{\partial}{\partial x_1}(r)1_{x_1} + \frac{\partial}{\partial x_2}(r)1_{x_2}$.



The same can be done easily with every word. With this, the matrix of the linear map ∂_2 in the basis $\{1_{r_i}\}_{i=1}^{m}$ and $\{1_{x_i}\}_{i=1}^{n}$ is exactly the Fox matrix of the presentation.
The next step is to notice that we have to work in the cyclic cover, which is to take co-invariants by H, where $H = Ker(\overline{\phi})$ and $\overline{\phi} : G \to \Pi$ is the map that gives us the desired isomorphism. That is, to substitute all the generators with t, the generator of Π . Our new chain complex is:

$$\cdots \bigoplus_{i=1}^{m} F\left[t,t^{-t}\right] \xrightarrow{\partial_2} \bigoplus_{i=1}^{n} F\left[t,t^{-t}\right] \xrightarrow{\partial_1} F\left[t,t^{-t}\right]$$

Therefore, the matrix of the map ∂_2 written in row form in the basis $\{1_{r_i}\}_{i=1}^{m}$ and $\{1_{x_i}\}_{i=1}^{n}$ is A_t. Notice that $Ker(\partial_1)$ has as a basis $\{1_{x_i} - 1_{x_n}\}_{i=1}^{n-1}$, let us study the matrix of ∂_2 restricted to $Ker(\partial_1)$. Recall that $\partial_1 \circ \partial_2 = 0$, thus:

$$0 = (\partial_1 \circ \partial_2)(1_{r_j}) = \partial_1 \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(r_j)1_{x_i} \right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(r_j)\partial_1(1_{x_i}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(r_j)(t-1)$$

this implies that $\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(r_j) = 0$. From this, one deduces that $\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(r_j)1_{x_j} = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(r_j)(1_{x_j} - 1_{x_n})$. Therefore, the matrix of ∂_2 in the basis $\{1_{r_i}\}_{i=1}^{m}$ and $\{1_{x_i} - 1_{x_n}\}_{i=1}^{n-1}$ is A_t with the last column deleted.

By 1.2.5, there exist regular matrices P and Q and $0 \leq r \leq n-1$ such that $D = PA_tQ$ with D a diagonal matrix with diagonal elements $d_1, \dots, d_r, d_{r+1} = 0 \dots, d_{n-1} = 0$ and such that $d_1 \mid \cdots \mid d_r$. Therefore, applying 1.2.6 we have:

$$\gcd\{(n-1)-\text{minors of } A_t\} = \gcd\{(n-1)-\text{minors of } D\} = d_1 \dots d_{n-1}$$

hence, without loss of generality, it can be assumed that $A_t = D$. Since D is a diagonal matrix, we have that $Im(\partial_2) \cong Rd_1 \bigoplus \cdots \bigoplus Rd_{n-1}$.
By definition, the Alexander polynomial is $\text{order}\big(Ker(\partial_1)/Im(\partial_2)\big)$ and as $Ker(\partial_1) \cong R \bigoplus \overset{n-1}{\dots} \bigoplus R$ we have:

$$Ker(\partial_1)/Im(\partial_2) \cong R/Rd_1 \bigoplus \cdots \bigoplus R/Rd_{n-1}$$

Therefore, $\text{order}\big(Ker(\partial_1)/Im(\partial_2)\big) = d_1 \dots d_{n-1}$, which is what we wanted to prove. □

Therefore, we have a procedure to find the Wirtinger presentation of a knot and a procedure to find the Alexander polynomial given its presentation. Then, the whole problem is reduced to a computational problem.

Examples 4.2.5. 1. Let us find the Alexander polynomial of the trefoil knot. By 3.2.6, the Wirtinger presentation of the knot is $\langle a, b \mid aba = bab \rangle$. We need to compute the Fox matrix of the presentation. Notice that the Fox derivatives of $r_1 = abab^{-1}a^{-1}b^{-1}$ are:

$$\frac{\partial}{\partial a}(abab^{-1}a^{-1}b^{-1}) = 1 + ab - abab^{-1}a^{-1} \quad \frac{\partial}{\partial b}(abab^{-1}a^{-1}b^{-1}) = a - abab^{-1} - abab^{-1}a^{-1}b^{-1}$$

Hence, $A = \begin{bmatrix} 1 + ab - abab^{-1}a^{-1} & a - abab^{-1} - abab^{-1}a^{-1}b^{-1} \end{bmatrix}$ and thus $A_t = \begin{bmatrix} 1 + t^2 - t & t - t^2 - 1 \end{bmatrix}$ Therefore, applying the previous theorem:

$$\Delta_K(t) = t^2 - t + 1$$

2. Let us find the Alexander polynomial of the eight knot. By 3.2.6, the Wirtinger presentation of the knot is $\langle c, d \mid dcd^{-1}cd = cdc^{-1}dc \rangle$. Let us compute the Fox derivatives of $r_1 = dcd^{-1}cdc^{-1}d^{-1}cd^{-1}c^{-1}$:

$$\frac{\partial}{\partial c}(dcd^{-1}cdc^{-1}d^{-1}cd^{-1}c^{-1}) = d + dcd^{-1} - dcd^{-1}cdc^{-1} + dcd^{-1}cdc^{-1}d^{-1} - dcd^{-1}cdc^{-1}d^{-1}cd^{-1}c^{-1}$$

$$\frac{\partial}{\partial d}(dcd^{-1}cdc^{-1}d^{-1}cd^{-1}c^{-1}) = 1 - dcd^{-1} + dcd^{-1}c - dcd^{-1}cdc^{-1}d^{-1} - dcd^{-1}c - dcd^{-1}cdc^{-1}d^{-1}cd^{-1}$$

Then, $A_t = \begin{bmatrix} -t^2 + 3t - 1 & t^2 - 3t + 1 \end{bmatrix}$. Therefore, applying the previous theorem:

$$\Delta_K(t) = t^2 - 3t + 1$$

4.3 The Combinatorial Way

In this section, we are going to explain a combinatorial procedure to compute the Alexander polynomial, although we are not going to see that it is equivalent to the algebraic definition. To begin with, let us state a couple of results that we will need for computing the polynomial.

Definition 4.3.1. A **planar graph** is a graph that can be embedded in $\mathbb{R}^2$.

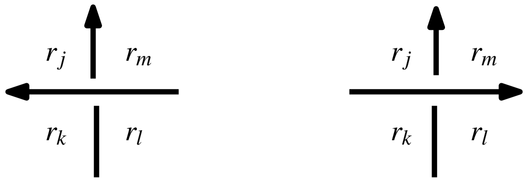
Proposition 4.3.2. *For every knot diagram there exists an associated planar graph.*

Proof. We need to consider the graph with vertices and edges the crossing points and the arcs respectively. This is trivially a planar graph. □

Theorem 4.3.3. *(Euler's formula) Let G be a connected planar graph with V vertices, E edges and F faces (were a face is a region enclosed by edges and vertices). Then* $V - E + F = 2$.

Proof. See [9]. □

Then, if we have an oriented diagram of a knot K with n crossing points labelled as $c_1, \dots, c_n$, it is immediate to see that the associated planar graph has n vertices and $2n$ edges. Therefore, by the Euler's formula, there are $n+2$ faces. This means that every knot K with n crossing points divides $\mathbb{R}^2$ into $n+2$ regions (we will label them by $r_1, r_2, \dots, r_{n+2}$).
Now, for each crossing point we will define an equation in the following way. First, if we have a crossing point c with a given labelling of the regions:



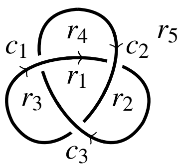
for the left-handed crossing we define the equation $c(t) = tr_j - tr_k + r_l - r_m = 0$ and for the right-handed we define the equation $c(t) = tr_m - tr_l + r_k - r_j = 0$. Doing this for every crossing point, a system of n equations with $n+2$ variables is obtained, which will be represented with a $n \times (n+2)$ matrix. The next step is to choose two neighbouring regions r_p and r_q and delete their respective columns from the matrix. This process gives us a $n \times n$ matrix.

Definition 4.3.4. The $n \times n$ matrix M obtained with this procedure is called an **Alexander matrix** of the knot K.

Theorem 4.3.5. *Let K be a knot and M an Alexander matrix of K. Then* $\Delta_K = \pm t^k \det(M)$, *where the* $\pm t^k$ *is chosen such that the smallest degree term of* Δ_K *is a positive constant.*

Proof. Can be found at [5]. □

Example 4.3.6. Let us find the Alexander polynomial of the trefoil knot. First of all, we need to choose an orientation for the knot and to label the crossings and the regions of the plane:



The next step is to find the equations for each crossing point. Notice that the three crossing points are right-handed and thus:

$$c_1(t) = tr_4 - tr_1 + r_3 - r_5 = 0, \; c_2(t) = tr_2 - tr_1 + r_4 - r_5 = 0, \; c_3(t) = tr_3 - tr_1 + r_2 - r_5 = 0$$

Hence, we have the following matrix:

$$\begin{bmatrix} -t & 0 & 1 & t & -1 \\ -t & t & 0 & 1 & -1 \\ -t & 1 & t & 0 & -1 \end{bmatrix}$$

Since r_4 and r_5 are adjacent regions we can delete the fourth and fifth columns of the matrix. Then:

$$M = \begin{bmatrix} -t & 0 & 1 \\ -t & t & 0 \\ -t & 1 & t \end{bmatrix}$$

hence, $\det(M) = -t^3 + t^2 - t = -t(t^2 - t + 1)$ which implies that $\Delta_K(t) = t^2 - t + 1$.

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