



Large characteristically simple sections of finite groups

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Abstract

In this paper we prove that if G is a group for which there are k non-Frattini chief factors isomorphic to a characteristically simple group A , then G has a normal section C/R that is the direct product of k minimal normal subgroups of G/R isomorphic to A . This is a significant extension of the notion of crown for isomorphic chief factors.

Keywords Finite group · Maximal subgroup · Probabilistic generation · Primitive group · Crown

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1 Introduction and statement of results

All groups considered in this paper will be finite.

Gaschütz [8] introduced the notion of crown associated with a complemented chief factor of a soluble group G . Given a G -module A , he discovered an important section of G , called the A -crown of G , which is a completely reducible and homogeneous G -module and the length of its G -composition series is the number of complemented chief factors of G which are G -isomorphic to A in a given chief series of G . These crowns turn out to be complemented sections of G . Gaschütz applied his significant result to construct a characteristic conjugacy class of subgroups in every soluble group: the *prefrattini* subgroups. Later Hawkes [11] used this notion to define a closure operation for Schunck classes of finite soluble groups.

Lafuente [13] defined the crown associated with a non-Frattini chief factors of an arbitrary group G . His approach depends on a equivalence relation in the set of all chief factors of G , called G -connection, which is a natural extension of G -isomorphism. We say that two chief

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factors of G are G -connected (or G -equivalent) when they are G -isomorphic or there exists a normal subgroup N of G such that G/N is a primitive group of type 3 whose minimal normal subgroups are G -isomorphic to the given chief factors. It is clear that if two chief factors of G that are G -connected and non- G -isomorphic, then they are non-abelian and there is a primitive epimorphic image of G connecting them. Lafuente discovered the existence of some sections associated with the non-abelian chief factors with similar properties to Gaschütz's crowns (see [13–15]), and he also used this notion to defined in [13] a new closure operation of Schunck classes of arbitrary groups which allows us to discover new relations between Schunck classes and saturated formations. Later, Förster [6] used the crowns to give an alternative approach of the generalised Jordan–Hölder theorem, and Ezquerro and the first author [1] used them to introduce the prefrattini subgroups in every group.

Crowns are also important in probabilistic group theory. Hall [10] gave a formula for the probability $P_G(t)$ that t elements taken at random from a group G with a uniform probability distribution generate G and Gaschütz [7] developed a formula for the conditional probability $P_{G,N}(t)$ that a t -tuple generates G modulo a normal subgroup N , given that the corresponding elements of the quotient group G/N generate G/N . The concept of crown has become crucial in the work of Detomi and Lucchini [4] to obtain factorisations of $P_G(t)$. Given a monolithic primitive group L with a unique minimal normal subgroup A , for each positive integer k we can consider the direct product L^k of k copies of L and its subgroup

$$L_k = \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \dots \equiv l_k \pmod{A}\},$$

called in [5] the k th crown-based power of L . The crown-based powers play a key role in [5] to understand the groups that need more generators than their proper quotients.

There is a close relation between crowns and crown-based powers. In order to state it, we need to recall the definition of the primitive group associated with a chief factor.

Definition 1.1 (See [2, Definition 1.2.9]) Given a chief factor H/K of a group G , the primitive group $[H/K]*G$ associated with H/K in G is the semidirect product $[H/K](G/C_G(H/K))$ if H/K is abelian and the quotient group $G/C_G(H/K)$ if H/K is non-abelian.

Theorem 1.2 ([4, Proposition 9]) Let H/K be a non-Frattini chief factor of a finite group G and let C/R be its crown. Then G/R is isomorphic to a crown-based power L_k , where $L = [H/K]*G$ and k is the number of chief factors of G that are G -related to H/K in a given chief series of G .

The following result shows the relation between crowns and crown-based powers.

Theorem 1.3 ([4, Proposition 9]) Let H/K be a non-Frattini chief factor of a finite group G and let C/R be its crown. Then G/R is isomorphic to a crown-based power L_k , where $L = [H/K]*G$ and k is the number of chief factors of G that are G -related to H/K in a given chief series of G .

The main aim of this paper is to obtain an extension of the notion of crown for isomorphic chief factors, not necessarily related by G -connectedness. Our result will establish a relation between the number of non-Frattini chief factors isomorphic to a characteristically simple group A in a given chief series and the A -rank $r_A(G)$, defined as the largest number k such that G has a normal section that is the direct product of k non-Frattini chief factors of G that are isomorphic to A .

Theorem A Let A be a non-Frattini chief factor of a group G and suppose that in a given chief series of G there are k non-Frattini chief factors isomorphic to A . Then there exist two

normal subgroups C and R of G such that $R \leq C$ and C/R is isomorphic to a direct product of k minimal normal subgroups of G/R isomorphic to A .

The proof of this result depends on the following property of monolithic primitive groups.

Theorem B *If G is a primitive group with a unique minimal normal subgroup B , then G/B has no chief factors isomorphic to B .*

We thank one of the anonymous referees for informing us of a result (see Theorem 3.3 below) that has allowed us to present Theorem B in its broadest generality.

We bring the paper to a close with a consequence of Theorem A.

Corollary 1.4 *If a group G has k non-Frattini chief factors in a given chief series isomorphic to a characteristically simple group A , then $r_A(G) = k$.*

2 Preliminaries

In this short section, we recall the definition of the precrowns and the crown associated with a non-Frattini chief factor and their main properties.

Definition 2.1 Let H/K be a supplemented chief factor of a group G . Assume that M is a maximal subgroup of G supplementing H/K in G such that G/M_G is a monolithic primitive group. We say that the chief factor $\text{Soc}(G/M_G) = HM_G/M_G$ is the *precrown* of G associated with M and H/K , or simply a *precrown* of G associated with H/K .

Note that if C/R is the precrown of G associated with the maximal subgroup M and the supplemented chief factor H/K of G , then G/R is the primitive quotient group G/M_G of G associated with M .

Definition 2.2 Let H/K be a non-Frattini chief factor of a group G . Let \mathcal{E} denote the set of all cores M_G of all maximal subgroups M of G such that the quotient G/M_G is a monolithic primitive group and M supplements chief factors G -connected to H/K , let

$$R = \bigcap \{N \mid N \in \mathcal{E}\},$$

and let $C^* = C_G^*(H/K)$. We say that the factor C^*/R is the *crown* of G associated with H/K .

Here $C_G^*(H/K) = HC_G(H/K)$ denotes the *inneriser* of a chief factor H/K of G . The crown associated with a supplemented chief factor H/K of G possesses the following properties.

Theorem 2.3 (see [2, Theorem 1.3.2]) *Let C^*/R be the crown of G associated to the supplemented chief factor H/K . Then $C^*/R = \text{Soc}(G/R)$. Furthermore,*

1. every minimal normal subgroup of G/R is a supplemented chief factor of G which is G -connected to H/K , and
2. no supplemented chief factor of G over C^* or below R is G -connected to H/K .

Unless otherwise stated, we will follow the notation of the books [2,3]. Detailed information about primitive groups and chief factors, crowns, and precrowns of a group can be found in [2, Chapter 1].

3 Proofs of the theorems

We begin by proving Theorem B. In order to show that in a primitive group of type 2 the only chief factor isomorphic to the socle is the socle itself, we use the following elementary result.

Lemma 3.1 *Let n be a natural number. Then 2^n does not divide $n!$.*

Proof Suppose that the result is false. Note that 2^1 does not divide $1!$. Suppose that n is the smallest natural number such that 2^n divides $n!$. We have that $n > 1$ and that n must be even. Hence 2^n divides the product

$$2 \cdot 4 \cdot 6 \cdot 8 \cdots n = 2^{n/2}(n/2)!.$$

Therefore $2^{n/2}$ divides $(n/2)!$. This contradicts the minimality of n . □

Theorem 3.2 *Let G be a monolithic primitive group in which $B = \text{Soc}(G)$ is non-abelian. Then G/B has no chief factors isomorphic to B .*

Proof Let $B = S_1 \times \cdots \times S_n$ be the decomposition of B as a product of isomorphic non-abelian simple groups $S_i \cong S$, $1 \leq i \leq n$. Let $Y = \bigcap_{i=1}^n N_G(S_i)$. By [2, Remarks 1.1.40 (13)], G is isomorphic to a subgroup of $X \wr P_n$, which is in turn isomorphic to a subgroup of $\text{Aut}(S) \wr \text{Sym}(n) \cong \text{Aut}(S^n)$, where P_n is a transitive subgroup of the symmetric group $\text{Sym}(n)$ and $X = N_G(S_1)/C_G(S_1)$ is isomorphic to a subgroup of $\text{Aut}(S)$ containing the inner automorphism group. Hence we can assume that G is in fact a subgroup of $W = \text{Aut}(S) \wr \text{Sym}(n)$. Let M be the intersection of G with the base group of W , then $M/B = (G \cap W^\sharp)/B$ is isomorphic to a subgroup of $\text{Aut}(S)^\sharp/B \cong (\text{Out } S)^\sharp$, which is a soluble group by the Schreier conjecture, whose validity has been checked with the classification of finite simple groups (see [12, page 151]).

Assume now that there exists a chief factor F of G/B such that $F \cong B$. Since M/B is soluble, there exist normal subgroups N, K of G such that $M \leq N \leq K$ and $K/N \cong B$. In particular,

$$G/M = G / \left(G \cap (\text{Aut}(S))^\sharp \right) \cong G(\text{Aut}(S))^\sharp / (\text{Aut}(S))^\sharp,$$

which is a subgroup of $P_n \leq \text{Sym}(n)$. It follows that the order of G/M divides $n!$ and, in particular, $|K/N|$ divides $n!$. Since $K/N \cong S_1 \times \cdots \times S_n$ and all non-abelian simple groups have order divisible by 2, 2^n divides $|K/N|$. Consequently, 2^n divides $n!$. This contradicts Lemma 3.1. □

Now let us consider non-Frattini abelian chief factors. Recall that a Fermat prime is a prime of the form $2^{2^n} + 1$ for some $n \geq 0$. We apply the following consequence of a result of Giudici, Glasby, Li, and Verret [9, Theorem 1]. We thank one of the referees for drawing our attention to this beautiful result. Without it, we would not have been able to prove Theorem B in its current form and the proofs would have been longer.

Theorem 3.3 *If G is a primitive group with a unique minimal normal subgroup of order $q = p^d$, where p is a prime, then the number of composition factors of G of order p is at most $d + \frac{\varepsilon p^d - 1}{p - 1}$, where*

$$\varepsilon_p = \begin{cases} \frac{p}{p-1} & \text{if } p \text{ is a Fermat prime,} \\ 1 & \text{otherwise.} \end{cases}$$

Corollary 3.4 *Let G be a prime and let G be a primitive group in which $B = \text{Soc}(G)$ is an elementary abelian p -group of order p^d . Then G/B has no chief factors isomorphic to B .*

Proof By Theorem 3.3, the number of composition factors of order p of G is bounded by

$$d + \frac{\varepsilon_p d - 1}{p - 1} \leq \begin{cases} d + \frac{d-1}{2-1} = 2d - 1 < 2d, & \text{if } p = 2, \\ d + \frac{(3/2)d-1}{p-1} < 2d, & \text{if } p \geq 3. \end{cases}$$

Hence the number of composition factors of order p of G/B is less than d . If G/B had a chief factor of order p^d , then by refining the corresponding chief series to a composition series we would obtain that the number of composition factors of order p of G/B is at least d . This contradiction shows that G/B does not have chief factors isomorphic to B . \square

We are now in a position to prove Theorem A.

Proof of Theorem A We will construct a sequence

$$(C_0, R_0), (C_1, R_1), (C_2, R_2), \dots, (C_l, R_l)$$

of pairs of subgroups of G satisfying the following conditions:

1. C_i and R_i are normal subgroups of G with $R_i \leq C_i$, $1 \leq i \leq l$, and $C_0 = R_0 = G$.
2. $C_i/R_i = (N_{i,1}/R_i) \times \dots \times (N_{i,i}/R_i)$, where $N_{i,j}/R_i$ is a non-Frattini minimal normal subgroup of G/R_i isomorphic to A .
3. $C_i = \bigcap_{j=1}^i C_G^*(N_{i,j}/R_i)$.

This will be done by induction on i . For $i = 0$, we construct $C_0 = R_0 = G$. Assume that for some $i \geq 0$ we have constructed (C_i, R_i) satisfying the previous conditions. Consider a chief series of G passing through R_i and suppose that in this series there exists a non-Frattini chief factor H/K of G isomorphic to A such that $H \leq R_i$. Let C/R be a precrown associated to H/K . Let $R_{i+1} = R_i \cap R$. Note that $C = C_G^*(H/K) = C_G^*(C/R)$. Let $C_{i+1} = C_i \cap C$. The unique minimal normal subgroup of G/R is C/R . Since $HR = C$, H is not contained in R and so R_i is not contained in R . Since G/R is monolithic, we have that

$$R < C \leq R_i R = R_i C.$$

Therefore

$$\begin{aligned} (C \cap R_i)/(R \cap R_i) &= (C \cap R_i)/(C \cap R_i \cap R) \\ &\cong_G R(C \cap R_i)/R = (C \cap RR_i)/R = C/R. \end{aligned}$$

Consequently $(C \cap R_i)/R_{i+1}$ is a normal subgroup of G/R_{i+1} that is G -isomorphic to C/R . Consider now $1 \leq j \leq i$, then

$$N_{i,j}R/RR_i = N_{i,j}RR_i/RR_i \cong_G N_{i,j}/(N_{i,j} \cap RR_i).$$

By the minimality of $N_{i,j}/R_i$, we have that either $N_{i,j} = N_{i,j} \cap RR_i$ or $N_{i,j} \cap RR_i = R_i$. Assume that the second case holds for a given j with $1 \leq j \leq i$. Hence $(N_{i,j}/R_i) \cap (RR_i/R_i) = 1$ and so $R < C \leq RR_i \leq C_G(N_{i,j}/R_i) \leq C_G^*(N_{i,j}/R_i)$. But now in a chief series of $G/C_G^*(N_{i,j}/R_i)$ there are no chief factors isomorphic to A . Moreover, $C/R \cong A$ and $RR_i < RN_{i,j} \leq C_G^*(N_{i,j}/R_i)$ with $RN_{i,j}/RR_i \cong A$. It follows that there are at least two chief factors isomorphic to A in a chief series of the primitive group associated with C/R , which is isomorphic to the primitive group associated with H/K . This contradicts

Theorem B. Consequently, for all j with $1 \leq j \leq i$, the condition $N_{i,j} = N_{i,j} \cap RR_i$ holds, that is, $N_{i,j} \leq RR_i$. Therefore, $C_i \leq RR_i$. Now

$$(C_i \cap R)/(R_i \cap R) = (C_i \cap R)/(R_1 \cap C_i \cap R) \cong_G R_1(C_i \cap R)/R_1 = (C_i \cap RR_i)/R_i = C_i/R_i.$$

Moreover, $(C_i \cap R) \cap (C \cap R_i) = R \cap R_i = R_{i+1}$. It follows that G/R_{i+1} has a normal subgroup $N_{i+1,i+1}/R_{i+1} = (C \cap R_i)/R_{i+1}$ G -isomorphic to C/R , which is in turn G -isomorphic to H/K , and another normal subgroup $(C_i \cap R)/R_{i+1}$ which is G -isomorphic to C_i/R_i . Hence $(C_i \cap R)/R_{i+1}$ is isomorphic to a direct product of minimal normal subgroups $N_{i+1,j}/R_{i+1}$ of G/R_{i+1} G -isomorphic, respectively, to $N_{i,j}/R_i$, $1 \leq j \leq i$. By construction,

$$C_{i+1} = C_i \cap C = \bigcap_{j=1}^i C_G^*(N_{i,j}/R_i) \cap C_G^*(H/K) = \bigcap_{j=1}^{i+1} C_G^*(N_{i+1,j}/R_{i+1}).$$

This construction can be done until we reach an $i = l$ such that there are no non-Frattini chief factors isomorphic to A below R_l . Note that, in this case, $l = k$, because all primitive groups associated to non-Frattini chief factors isomorphic to A have no chief factors isomorphic to A and, hence, $G/C_G^*(H/K)$ has no chief factor isomorphic to A , consequently, there are no chief factor of G isomorphic to A that could appear above C_i . □

Finally, we prove Corollary 1.4.

Proof of Corollary 1.4 The normal section C/R obtained in Theorem A is the product of k minimal normal subgroups of G/R isomorphic to A , hence $k \leq r_A(G)$. Since obviously $r_A(G) \leq k$, we obtain the result. □

Example 3.5 Let S be a non-abelian simple group, then the group $H = \text{Inn}(S) \cong S$ of inner automorphisms of S acts on S and we can consider the corresponding semidirect product $G = [S]H$. As a consequence of Theorem A we can obtain the well-known fact that G is isomorphic to the direct product $S \times S$ of two copies of S .

Example 3.6 Consider the group

$$G = \langle (1, 2, 3), (1, 4, 5), (1, 2), (6, 7, 8), (6, 9, 10), (6, 7) \rangle$$

isomorphic to the direct product of two copies of the symmetric group of degree 5 acting naturally on the sets $\{1, 2, 3, 4, 5\}$ and $\{6, 7, 8, 9, 10\}$. Then G has two crowns corresponding to chief factors isomorphic to the alternating group $\text{Alt}(5)$ of degree 5, namely C_1/R_1 with

$$C_1 = \langle (1, 2, 3), (1, 4, 5), (6, 7, 8), (6, 9, 10), (6, 7) \rangle$$

and

$$R_1 = \langle (6, 7, 8), (6, 9, 10), (6, 7) \rangle$$

and C_2/R_2 with

$$C_2 = \langle (1, 2, 3), (1, 4, 5), (1, 2), (6, 7, 8), (6, 9, 10) \rangle$$

and

$$R_2 = \langle (1, 2, 3), (1, 4, 5), (1, 2) \rangle.$$

We note that the chief factors C_1/R_1 and C_2/R_2 correspond to different crowns because their innerisers, C_1 and C_2 , respectively, are different. In this case, Theorem A gives the normal

section C/R with $C = 1$ and $R = \langle (1, 2, 3), (1, 4, 5), (6, 7, 8), (6, 9, 10) \rangle$ isomorphic to the direct product of two copies of the alternating group of degree 5.

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