Numerical Algorithms
https://doi.org/10.1007/s11075-021-01247-x

ORIGINAL PAPER

Inverse central ordering for the Newton interpolation formula

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Received: 21 October 2020 / Accepted: 16 December 2021

Abstract
An inverse central ordering of the nodes is proposed for the Newton interpolation formula. This ordering may improve the stability for certain distributions of nodes. For equidistant nodes, an upper bound of the conditioning is provided. This bound is close to the bound of the conditioning in the Lagrange interpolation formula, whose conditioning is the lowest. This ordering is related to a pivoting strategy of a matrix elimination procedure called Neville elimination. The results are illustrated with examples.

Keywords Newton interpolation formula · Conditioning · Inverse central ordering · Neville elimination

Mathematics Subject Classification (2010) 65D05 · 65F35 · 41A05 · 41A10

1 Introduction

The Lagrange interpolation operator associates to each function $f \in C[a, b]$ its Lagrange interpolating polynomial of degree less than or equal to n at $n + 1$ distinct nodes $x_0, \dots, x_n$ on the interval $[a, b]$. Using the Lagrange interpolation formula, the Lagrange interpolation operator L_n can be written in the form

$$L_n[f](x) = \sum_{k=0}^{n} f(x_k) l_k(x),$$

where

$$l_k(x) := \prod_{j \in \{0,\dots,n\} \setminus \{k\}} \frac{x - x_j}{x_k - x_j}, \quad k = 0, \dots, n,$$

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are the Lagrange fundamental polynomials. So, the Lagrange representation is $L_n[f] = \sum_{k=0}^{n}(\lambda_k f)l_k$, where $\lambda_k f = f(x_k)$, $k = 0, \ldots, n$.

In formula (2) of [5], a conditioning associated to a representation of the form $L_n[f] = \sum_{k=0}^{n}(\beta_k f)v_k$ is introduced,

$$\mathrm{cond}(x; \beta) := \sum_{k=0}^{n} ||\beta_k||_\infty |v_k(x)|, \tag{1}$$

where $\beta_0, \ldots, \beta_n$ are functionals belonging to the space generated by the evaluation functionals $\lambda_0, \ldots, \lambda_n$ and $(v_0, \ldots, v_n)$ is a basis of Π_n, the space of polynomials of degree not greater than n. By formula (4) and Theorem 4 of [5], we have

$$\lambda(x) := \sum_{k=0}^{n} |l_k(x)| = \mathrm{cond}(x; \lambda) \leq \mathrm{cond}(x; \beta), \tag{2}$$

that is, the conditioning of the Lagrange representation coincides with the Lebesgue function $\lambda(x)$ and it is lower than the conditioning of any other representation, and in particular, than the Newton representation. This representation is given by $\sum_{k=0}^{n} d_k f \omega_k(x)$, where d_k are the divided difference functionals

$$d_k f := [x_0, \ldots, x_k]f, \quad k = 0, \ldots, n,$$

and

$$\omega_0(x) := 1, \qquad \omega_k(x) := (x - x_0) \cdots (x - x_{k-1}), \quad k = 1, \ldots, n+1. \tag{3}$$

Since $d_k f$ is the coefficient of x^k in $L_n[f]$, we have

$$d_k f = \sum_{i=0}^{k} \frac{f(x_i)}{\omega'_{k+1}(x_i)}, \tag{4}$$

with $\omega'_{k+1}(x_i) = \prod_{j \in \{0,\ldots,k\} \setminus \{i\}}(x_i - x_j)$, $k = 0, \ldots, n$. (cf. formula (2.2) of [3]).

We are interested in the case of equidistant nodes $x_0, \ldots, x_n$ on $[a, b]$

$$x_i = a + ih, \quad i = 0, \ldots, n, \text{ with } h := \frac{b-a}{n}.$$

In order to compare the conditionings of the Lagrange and the Newton representations, we recall the following asymptotic formula for the Lebesgue constant shown by Schönhage in [10]

$$\max_{x \in [a,b]} \lambda(x) \sim \frac{2^{n+1}}{e\, n \log(n + \gamma)}, \tag{5}$$

where $\gamma \approx 0.5772156649$ is the Euler-Mascheroni constant. Newton's formula is sensitive to the ordering of the nodes in floating point arithmetic. An analysis of this fact can be found in [4, 5]. In [5] it is shown that

$$\max_{x \in [a,b]} \mathrm{cond}(x; d) = 3^n$$

for equidistant nodes in increasing order. In [6], a central ordering of the nodes, based on the distances of the nodes to a given center was considered. For equidistant nodes and the choice of the center $(a+b)/2$, it was proved that

$$\max_{x\in[a,b]} \mathrm{cond}(x;d) \le \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}.$$

In this paper, we propose the inverse central ordering, also based on the distance of the nodes to a center. The bound for the conditioning of the Newton representation with this ordering is smaller than the two previous bounds. More precisely, in Theorem 3, for a sequence of equidistant nodes in $[-l/2, l/2]$ of the form

$$x_{i,n} := \begin{cases} -\frac{n-i}{2}h, & \text{for } i \text{ even}, \\ \frac{n-i+1}{2}h, & \text{for } i \text{ odd}, \end{cases} \qquad i = 0, \dots, n, \tag{6}$$

with $h = l/n$, we show that

$$\max_{x\in[a,b]} \mathrm{cond}(x,d) \le 7\cdot 2^n.$$

In Section 2, we introduce the inverse central ordering. Bounds for the conditioning of the Newton representation are provided. The bounds for the inverse central ordering are close to the maximum conditioning of the Lagrange representation, which has the best conditioning. In this sense, the inverse central ordering for equidistant nodes is near optimal. Section 3 contains numerical experiments to illustrate the properties of the conditioning of the Newton formula with nodes following an inverse central ordering. In Section 4, Neville elimination, a matrix elimination procedure alternative to Gaussian elimination, is considered. Neville elimination is especially useful when dealing with some structured matrices (see [2, 7, 8]) including Vandermonde matrices. We show that the inverse central ordering corresponds to partial pivoting for Neville elimination in Vandermonde matrices.

2 Inverse central ordering and conditioning

The inverse central order consists in arranging a set of nodes, starting with the furthest node to a given center and finishing with the closest one.

Definition 1 (Inverse central order) A sequence of nodes $x_0, \dots, x_n$ follows a inverse central ordering with respect to a center c if the nodes satisfy

$$|x_0 - c| \ge |x_1 - c| \ge \cdots \ge |x_n - c|.$$

We now consider equidistant nodes and, for the sake of simplicity, we take symmetric intervals and the center $c = 0$. Therefore, the interval has the form $[-nh/2, nh/2]$, where n is the degree and h is the *distance between neighboring nodes*. This ordering may not be unique. So, we propose the following choice: in the case where two nodes lie at the same distance from the center, we take first the least of both nodes, leading to the ordering defined in (6). With formula (6), we can compute the difference between two nodes,

$$\begin{aligned} x_{2j,n} - x_{2i,n} &= (j-i)h, \quad i,j = 0,\ldots,\left\lfloor \frac{n}{2} \right\rfloor, \\ x_{2j+1,n} - x_{2i+1,n} &= (i-j)h, \quad i,j = 0,\ldots,\left\lfloor \frac{n-1}{2} \right\rfloor, \\ x_{2j+1,n} - x_{2i,n} &= (n-j-i)h, \quad i = 0,\ldots,\left\lfloor \frac{n}{2} \right\rfloor, \quad j = 0,\ldots,\left\lfloor \frac{n-1}{2} \right\rfloor. \end{aligned} \tag{7}$$

We note

$$d_k^n f := [x_{0,n},\ldots,x_{k,n}]f, \quad k = 0,\ldots,n, \tag{8}$$

where $x_{0,n},\ldots,x_{k,n}$ are given by (6).

Let us introduce the falling factorial symbol and the generalized binomial coefficient

$$x^{(k)} := x(x-1)\cdots(x-k+1), \quad \binom{x}{k} := \frac{x^{(k)}}{k!}.$$

Remark 1 The polynomial $\binom{x}{k}$ is an increasing function of x for $x \geq k-1$. In particular, we have that

$$\binom{x}{k} \geq 1, \quad x \geq k, \tag{9}$$

and $\binom{x}{k} \geq k+1$, for $x \geq k+1$.

In order to obtain an explicit expression for the norm of the divided difference functionals, we derive recurrence formulae.

Proposition 1 *Let $x_{0,n},\ldots,x_{n,n}$ be nodes given by* (6) *and d_k^n, $k = 0,\ldots,n$, the divided difference functionals* (8). *We have*

$$||d_{2k}^n||_\infty = \frac{1}{kh}||d_{2k-1}^{n-1}||_\infty, \quad k = 1,\ldots,\left\lfloor \frac{n}{2} \right\rfloor,$$

and

$$||d_{2k+1}^n||_\infty = \frac{2}{nh}||d_{2k}^{n-1}||_\infty, \quad k = 0,\ldots,\left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof Using formula (4) and Proposition 2 of [5], we have

$$||d_k^n||_\infty = \sum_{i=0}^{k} \frac{1}{|\omega'_{k+1}(x_{i,n})|}.$$

Using (7), we compute

$$
\begin{aligned}
|\omega'_{2k}(x_{2i,n})| &= \prod_{\substack{j=0\\ j\neq i}}^{k-1} |x_{2i,n} - x_{2j,n}| \prod_{j=0}^{k-1} |x_{2i,n} - x_{2j+1,n}| \\
&= h^{2k-1} \prod_{\substack{j=0\\ j\neq i}}^{k-1} |i-j| \prod_{j=0}^{k-1} (n-j-i) = h^{2k-1} i!(k-1-i)!(n-i)^{(k)}, \\
|\omega'_{2k}(x_{2i+1,n})| &= \prod_{j=0}^{k-1} |x_{2i+1,n} - x_{2j,n}| \prod_{\substack{j=0\\ j\neq i}}^{k-1} |x_{2i+1,n} - x_{2j+1,n}| \\
&= h^{2k-1} \prod_{j=0}^{k-1} (n-j-i) \prod_{\substack{j=0\\ j\neq i}}^{k-1} |i-j| = h^{2k-1} i!(k-1-i)!(n-i)^{(k)}.
\end{aligned}
$$

Similarly, for the rest of the cases

$$
\begin{aligned}
|\omega'_{2k+1}(x_{2i,n})| &= h^{2k} i!(k-i)!(n-i)^{(k)}, \\
|\omega'_{2k+1}(x_{2i+1,n})| &= h^{2k} i!(k-1-i)!(n-i)^{(k+1)}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
||d_{2k}^n||_\infty &= \sum_{i=0}^{2k} \frac{1}{|\omega'_{2k+1}(x_{i,n})|} = \sum_{i=0}^{k} \frac{1}{|\omega'_{2k+1}(x_{2i,n})|} + \sum_{i=0}^{k-1} \frac{1}{|\omega'_{2k+1}(x_{2i+1,n})|} \\
&= \frac{1}{h^{2k}} \left(\sum_{i=0}^{k} \frac{1}{i!(k-i)!(n-i)^{(k)}} + \sum_{i=0}^{k-1} \frac{1}{i!(k-i-1)!(n-i)^{(k+1)}} \right). \quad (10)
\end{aligned}
$$

Analogously, we have that

$$
||d_{2k+1}^n||_\infty = \sum_{i=0}^{2k+1} \frac{1}{|\omega'_{2k+2}(x_{i,n})|} = \frac{2}{h^{2k+1}} \sum_{i=0}^{k} \frac{1}{i!(k-i)!(n-i)^{(k+1)}}. \quad (11)
$$

By formula (10), we have

$$
k!h^{2k}||d_{2k}^n||_\infty = \sum_{i=0}^{k} \binom{k}{i} \frac{1}{(n-i)^{(k)}} + \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{k}{(n-i)^{(k+1)}}.
$$

Using Pascal's identity,

$$
\binom{k}{i} = \binom{k-1}{i} + \binom{k-1}{i-1}, \quad (12)
$$

and formula (11) we obtain

$$
\begin{aligned}
k!h^{2k}||d_{2k}^n||_\infty &= \sum_{i=0}^{k-1}\binom{k-1}{i}\frac{1}{(n-i)^{(k)}}+\sum_{i=1}^{k}\binom{k-1}{i-1}\frac{1}{(n-i)^{(k)}}\\
&\quad+\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{k}{(n-i)^{(k+1)}}=\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{n-i-k+n-i+k}{(n-i)^{(k+1)}}\\
&=2\sum_{i=0}^{k-1}\frac{(k-1)!}{i!(k-1-i)!(n-i-1)^{(k)}}=(k-1)!h^{2k-1}||d_{2k-1}^{n-1}||_\infty .
\end{aligned}
$$

We proceed in a similar way for the divided difference functionals of odd order. By formulae (11) and (10), we have

$$
\begin{aligned}
&n\frac{h^{2k+1}}{2}||d_{2k+1}^n||_\infty=\sum_{i=0}^{k}\frac{n}{i!(k-i)!(n-i)^{(k+1)}}=\sum_{i=0}^{k}\frac{n-i+i}{i!(k-i)!(n-i)^{(k+1)}}\\
&=\sum_{i=0}^{k}\frac{1}{i!(k-i)!(n-i-1)^{(k)}}+\sum_{i=1}^{k}\frac{1}{(i-1)!(k-i)!(n-i)^{(k+1)}}=h^{2k}||d_{2k}^{n-1}||_\infty ,
\end{aligned}
$$

and the result follows. □

We obtain in the following result explicit formulae for the ∞-norm of the divided difference functionals.

Theorem 1 *Let $x_{0,n},\ldots,x_{n,n}$ be the nodes given by (6) and d_k^n, $k=0,\ldots,n$, the divided difference functionals (8). We have*

$$
||d_{2k}^n||_\infty=\frac{1}{h^{2k}k!\left(\frac{n-1}{2}\right)^{(k)}},\quad k=0,\ldots,\left\lfloor\frac{n}{2}\right\rfloor, \tag{13}
$$

and

$$
||d_{2k+1}^n||_\infty=\frac{1}{h^{2k+1}k!\left(\frac{n}{2}\right)^{(k+1)}},\quad k=0,\ldots,\left\lfloor\frac{n-1}{2}\right\rfloor. \tag{14}
$$

Proof Let us prove the result by induction on k. Since $d_0^n(f)=f(x_{0,n})$, we have that $\|d_0^n\|_\infty=1$ and (13) follows for $k=0$. For d_1^n, we have

$$
||d_1^n||_\infty=\frac{1}{|\omega_2'(x_{0,n})|}+\frac{1}{|\omega_2'(x_{1,n})|}=\frac{2}{|x_{0,n}-x_{1,n}|}=\frac{2}{nh},
$$

and (14) follows for $k = 0$. Assuming that (13) and (14) hold for k, let us prove them for $k+1$. By Proposition 1 and the induction hypothesis, we have

$$||d^n_{2(k+1)}||_\infty = \frac{1}{(k+1)h}||d^{n-1}_{2k+1}||_\infty = \frac{1}{(k+1)h}\frac{1}{h^{2k+1}k!\left(\frac{n-1}{2}\right)^{(k+1)}}$$

$$= \frac{1}{h^{2(k+1)}(k+1)!\left(\frac{n-1}{2}\right)^{(k+1)}},$$

and (13) follows. Let us show (14) for $k+1$. By Proposition 1 and the induction hypothesis, we have

$$||d^n_{2(k+1)+1}||_\infty = \frac{2}{nh}||d^{n-1}_{2(k+1)}||_\infty = \frac{2}{nh}\frac{1}{h^{2(k+1)}(k+1)!\left(\frac{n-2}{2}\right)^{(k+1)}}$$

$$= \frac{1}{h^{2(k+1)+1}(k+1)!\left(\frac{n}{2}\right)^{(k+2)}},$$

and the result follows. □

Next we derive some formulae for the conditioning of the Newton formula with equidistant nodes and inverse central ordering.

Lemma 1 *For any* $0 \le i \le \lfloor \frac{n}{2} \rfloor$, *we have*

$$\mathrm{cond}(x; d) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\left|\binom{i+t}{k}\right|\left|\binom{n-i-t}{k}\right|}{\binom{(n-1)/2}{k}} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\left|\binom{i+t}{k+1}\right|\left|\binom{n-i-t}{k}\right|}{\binom{n/2}{k+1}}, \quad \textit{where } t = \frac{x - x_{2i,n}}{h},$$

and for any i *with* $0 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$,

$$\mathrm{cond}(x; d) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\left|\binom{i+t}{k}\right|\left|\binom{n-i-t}{k}\right|}{\binom{(n-1)/2}{k}} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\left|\binom{i+t}{k}\right|\left|\binom{n-i-t}{k+1}\right|}{\binom{n/2}{k+1}}, \quad \textit{where } t = \frac{x_{2i+1,n} - x}{h}.$$

Proof In order to compute

$$\mathrm{cond}(x; d) = \sum_{k=0}^{n} \|d^n_k\|_\infty |\omega_k(x)|,$$

let us derive some formulae for

$$|\omega_{2k}(x)| = \prod_{j=0}^{k-1} |x - x_{2j,n}| \prod_{j=0}^{k-1} |x_{2j+1,n} - x|, \quad k = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$$

and

$$|\omega_{2k+1}(x)| = \prod_{j=0}^{k} |x - x_{2j,n}| \prod_{j=0}^{k-1} |x_{2j+1,n} - x|, \quad k = 0, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor.$$

First consider the case $x = x_{2i,n} + th$, for some $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$. Substituting $x - x_{2j,n} = h(i + t - j)$, $x_{2j+1,n} - x = h(n - j - i - t)$ in $|\omega_{2k}(x)|$, we can write

$$|\omega_{2k}(x_{2i,n} + th)| = h^{2k} \prod_{j=0}^{k-1} |i + t - j| \prod_{j=0}^{k-1} |n - j - i - t| = h^{2k} |(i + t)^{(k)}| \, (n - i - t)^{(k)}$$

and

$$|\omega_{2k}(x_{2i,n} + th)| \|d_{2k}^{n}\|_\infty = \frac{|(i + t)^{(k)}| (n - i - t)^{(k)}|}{k! \left(\frac{n-1}{2}\right)^{(k)}} = \frac{\left|\binom{i + t}{k}\right| \binom{n - i - t}{k}}{\binom{(n - 1)/2}{k}}.$$

Similarly, we have

$$|\omega_{2k+1}(x_{2i,n} + th)| = h^{2k+1} \prod_{j=0}^{k} |i + t - j| \prod_{j=0}^{k-1} |n - j - i - t| = h^{2k+1} |(i + t)^{(k+1)}| \, (n - i - t)^{(k)}$$

and

$$|\omega_{2k+1}(x_{2i,n} + th)| \|d_{2k+1}^{n}\|_\infty = \frac{(i + t)^{(k+1)} (n - i - t)^{(k)}}{k! \left(\frac{n}{2}\right)^{(k+1)}} = \frac{\left|\binom{i + t}{k + 1}\right| \binom{n - i - t}{k}}{\binom{n/2}{k + 1}}.$$

Hence the first formula in the statement of Lemma 1 holds.

Now assume that $x = x_{2i+1,n} - th$, for some $i = 0, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor$. Substituting $x - x_{2j,n} = h(n - i - j - t)$, $x_{2j+1,n} - x = h(i + t - j)$, we can write

$$|\omega_{2k}(x_{2i+1,n} - th)| = h^{2k} \prod_{j=0}^{k-1} (n - j - i - t) \prod_{j=0}^{k-1} |i + t - j| = h^{2k} |(i + t)^{(k)}| \, (n - i - t)^{(k)}$$

and

$$|\omega_{2k}(x_{2i+1,n} - th)| \|d_{2k+1}^{n}\|_\infty = \frac{(i + t)^{(k)} (n - i - t)^{(k)}}{k! \left(\frac{n-1}{2}\right)^{(k)}} = \frac{\left|\binom{i + t}{k}\right| \binom{n - i - t}{k}}{\binom{(n - 1)/2}{k}}.$$

Similarly

$$|\omega_{2k+1}(x_{2i+1,n} + th)| = h^{2k+1} \prod_{j=0}^{k} (n - j - i - t) \prod_{j=0}^{k-1} |i + t - j| = h^{2k+1} |(i + t)^{(k)}| \, (n - i - t)^{(k+1)}$$

and

$$|\omega_{2k+1}(x_{2i,n}+th)|\|d^n_{2k+1}\|_\infty = \frac{\left|\binom{i+t}{k}\right|\binom{n-i-t}{k+1}}{\binom{n/2}{k+1}}.$$

□

Taking into account that $\frac{\binom{k}{i}}{\binom{k}{i-1}} = \frac{k-i+1}{i}$, we deduce that $\binom{k}{i} \geq \binom{k}{i-1}$ if and only if $i \leq \frac{k+1}{2}$ and we have the following auxiliary result.

Lemma 2 *For* $0 < i \leq \frac{k+1}{2}$, *we have* $\binom{k}{i} \geq \binom{k}{i-1}$ *and for* $\frac{k+1}{2} \leq i \leq k$, $\binom{k}{i-1} \geq \binom{k}{i}$.

In order to obtain a bound for $\text{cond}(x; d) = \sum_{k=0}^{n} \|d^n_k\|_\infty |\omega_k(x)|$, we will use the well-known Vandermonde identity

$$\binom{r+q}{i} = \sum_{k=0}^{i} \binom{r}{k}\binom{q}{i-k}, \quad r, q, i \in \mathbb{N} \cup \{0\}. \tag{15}$$

We will also use the following relation between generalized binomial coefficients:

$$\binom{k-1/2}{k} = 2^{-2k}\binom{2k}{k}. \tag{16}$$

Let us obtain an upper bound for $\text{cond}(x; d)$ at the interpolation nodes.

Theorem 2 *Let* $x_{0,n}, \ldots, x_{n,n}$ *be the nodes given by* (6). *Then, we have*

$$\text{cond}(x_{i,n}; d) \leq \binom{n+1}{\lfloor (i+1)/2 \rfloor} + n, \quad i = 0, \ldots, n,$$

and so we have

$$\max_{i=0,\ldots,n} \text{cond}(x_{i,n}; d) \leq \binom{n+1}{\left\lfloor \frac{n+1}{2} \right\rfloor} + n.$$

Proof Let us start by bounding $\text{cond}(x_{2i}; d)$, $i = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$. By Lemma 1 we have that

$$\text{cond}(x_{2i,n}; d) = \sum_{k=0}^{i} \frac{\binom{i}{k}\binom{n-i}{k}}{\binom{(n-1)/2}{k}} + \sum_{k=0}^{i-1} \frac{\binom{i}{k+1}\binom{n-i}{k}}{\binom{n/2}{k+1}}$$

because $\binom{i}{k} = 0$ for any integer $i < k$.

By (9), all denominators are greater than or equal to 1 in the above formula, except in the case where n is even and $k = i = n/2$. We use (9) again to show that

$$\binom{(n-1)/2}{n/2} = \frac{1}{n}\binom{(n-1)/2}{(n-2)/2} \geq \frac{1}{n}.$$

So taking $i = n/2$, we obtain

$$\mathrm{cond}(x_{n,n}; d) \leq \sum_{k=0}^{n/2-1} \binom{n/2}{k}\binom{n/2}{k} + \sum_{k=0}^{n/2-1} \binom{n/2}{k+1}\binom{n/2}{k} + n.$$

Using Pascal's identity (12) and Vandermonde identity (15), we deduce that

$$\mathrm{cond}(x_{n,n}; d) \leq \sum_{k=0}^{n/2-1} \binom{n/2}{k}\binom{n/2+1}{k+1} + n \leq \binom{n+1}{n/2} - 1 + n.$$

For the rest of the cases, $i < (n-1)/2$, all the binomial coefficients in the denominators are larger than or equal to 1 by (9). Applying Pascal's identity (12) and (15), we have

$$\begin{aligned}\mathrm{cond}(x_{2i,n}; d) &\leq \sum_{k=0}^{i} \binom{n-i}{k}\binom{i}{k} + \sum_{k=0}^{i} \binom{n-i}{k}\binom{i}{k+1} \\ &= \sum_{k=0}^{i} \binom{n-i}{k}\binom{i+1}{k+1} = \sum_{k=0}^{i} \binom{n-i}{k}\binom{i+1}{i-k} = \binom{n+1}{i}.\end{aligned}$$

Analogously, for nodes of odd indices

$$\mathrm{cond}(x_{2i+1,n}; d) = \sum_{k=0}^{i} \frac{\binom{n-i}{k}\binom{i}{k}}{\binom{(n-1)/2}{k}} + \sum_{k=0}^{i} \frac{\binom{n-i}{k+1}\binom{i}{k}}{\binom{n/2}{k+1}}.$$

If $i < (n-1)/2$, we can apply (9), Pascal's identity (12) and formula (15) to deduce that

$$\begin{aligned}\mathrm{cond}(x_{2i+1,n}; d) &\leq \sum_{k=0}^{i} \binom{n-i}{k}\binom{i}{k} + \sum_{k=0}^{i} \binom{n-i}{k+1}\binom{i}{k} = \sum_{k=1}^{i+1} \binom{n-i+1}{k}\binom{i}{k-1} \\ &= \sum_{k=1}^{i+1} \binom{n-i+1}{k}\binom{i}{i+1-k} = \binom{n+1}{i+1}.\end{aligned}$$

If n is odd and $i = (n-1)/2$, there is a term with denominator

$$\binom{n/2}{(n+1)/2} = \frac{1}{n+1}\binom{n/2}{(n-1)/2} \geq \frac{1}{n+1}.$$

Then we deduce from (12) and (15) that

$$\begin{aligned}\operatorname{cond}(x_{n,n};d) &\le \sum_{k=0}^{(n-1)/2}\binom{(n+1)/2}{k}\binom{(n-1)/2}{k}+\sum_{k=0}^{(n-3)/2}\binom{(n+1)/2}{k+1}\binom{(n-1)/2}{k}\\ &+1+n=\sum_{k=0}^{(n-1)/2}\binom{(n+3)/2}{k+1}\binom{(n-1)/2}{k}+n\\ &=\sum_{k=0}^{(n+1)/2}\binom{(n+3)/2}{(n+1)/2-k}\binom{(n-1)/2}{k}+n=\binom{n+1}{(n+1)/2}+n.\end{aligned}$$

So, the first formula follows. The second one follows from the fact that the sequence $\binom{n+1}{\lfloor (i+1)/2\rfloor}$, $i=0,\dots,n$, is nondecreasing by Lemma 2. □

Remark 2 Recall that the Lagrange representation is optimal in the sense that it provides the least conditioning (see (2)). Corollary 6 of [5] shows that the ratio between the conditioning of the Newton representation and the conditioning of the Lagrange representation is attained at a node. Using this result and Theorem 2, we obtain the following inequality for equidistant nodes arranged according to the inverse central ordering (6):

$$\max_{x\in[a,b]}\frac{\operatorname{cond}(x;d)}{\lambda(x)}=\max_{i=0,\dots,n}\operatorname{cond}(x_{i,n};d)\le\binom{n+1}{\left\lfloor\frac{n+1}{2}\right\rfloor}+n.$$

We observe that the maximum bound for the condition number at the nodes is attained for the nodes closest to the origin.

In the following theorem we bound $\operatorname{cond}(x;d)$.

Theorem 3 *Let $x_{0,n},\dots,x_{n,n}$ be nodes given by* (6). *Then*

$$\operatorname{cond}(x;d)\le 7\cdot 2^n.$$

Proof First assume that $x\in[x_{2i,n},x_{2i,n}+h]$, $i=0,\dots\lfloor n/2\rfloor$, and let us define $t:=h^{-1}(x-x_{2i,n})\in[0,1]$. Using Lemma 1, we can write

$$\operatorname{cond}(x;d)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{\left|\binom{i+t}{k}\right|\binom{n-i-t}{k}}{\binom{(n-1)/2}{k}}+\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{\left|\binom{i+t}{k+1}\right|\binom{n-i-t}{k}}{\binom{n/2}{k+1}}. \tag{17}$$

If $k\ge i+1$, we deduce, by Remark 1, that

$$\left|\binom{i+t}{k}\right|=\frac{\binom{i+t}{i+1}\left|\binom{t-1}{k-i-1}\right|}{\binom{k}{i+1}}=\frac{\binom{i+t}{i+1}\binom{k-i-1-t}{k-i-1}}{\binom{k}{i+1}}\le\frac{1}{\binom{k}{i+1}}. \tag{18}$$

Observe that, by (9), the binomial coefficients $\binom{(n-1)/2}{k}$ are always larger than or equal to 1, except if $k = \lfloor n/2 \rfloor$ and n is even. Using (16), the corresponding term can be bounded from above by

$$\frac{\left|\binom{i+t}{n/2}\right|\binom{n-i-t}{n/2}}{\binom{(n-1)/2}{n/2}} \leq 2^n \frac{\left|\binom{i+t}{n/2}\right|\binom{n-i-t}{n/2}}{\binom{n}{n/2}}.$$

If $i = n/2$, we can deduce that

$$\binom{n/2+t}{n/2}\binom{n/2-t}{n/2} = \prod_{k=0}^{n/2-1} \frac{(n/2-k+t)(n/2-k-t)}{(n/2-k)^2} = \prod_{k=0}^{n/2-1} \frac{(n/2-k)^2-t^2}{(n/2-k)^2} \leq 1$$

and so

$$2^n \frac{\binom{n/2+t}{n/2}\binom{n/2-t}{n/2}}{\binom{n}{n/2}} \leq \frac{2^n}{\binom{n}{n/2}}.$$

If $i \leq n/2-1$, we deduce from Remark 1 that $\binom{n-i-t}{n/2} \leq \binom{n}{n/2}$ and, by (18),

$$2^n \frac{\left|\binom{i+t}{n/2}\right|\binom{n-i-t}{n/2}}{\binom{n}{n/2}} \leq 2^n \left|\binom{i+t}{n/2}\right| \leq \frac{2^n}{\binom{n/2}{i+1}}.$$

Analogously, the binomial coefficients $\binom{n/2}{k+1}$ are always larger than 1, except if $k = (n-1)/2$ and n is odd. Using (16), the corresponding term can be written as

$$\frac{\left|\binom{i+t}{(n+1)/2}\right|\binom{n-i-t}{(n-1)/2}}{\binom{n/2}{(n+1)/2}} \leq 2^{n+1} \frac{\left|\binom{i+t}{(n+1)/2}\right|\binom{n-i-t}{(n-1)/2}}{\binom{n+1}{(n+1)/2}} = 2^n \frac{\left|\binom{i+t}{(n+1)/2}\right|\binom{n-i-t}{(n-1)/2}}{\binom{n}{(n-1)/2}}.$$

If $i = (n-1)/2$, we have

$$\begin{aligned} &\binom{t+(n-1)/2}{(n+1)/2}\binom{(n+1)/2-t}{(n-1)/2} \\ &= \frac{(n-1)/2+t}{(n+1)/2} \prod_{k=0}^{(n-1)/2-1} \frac{((n-1)/2-k+(1-t))((n-1)/2-k-(1-t))}{((n-1)/2-k)^2} \\ &= \frac{(n-1)/2+t}{(n+1)/2} \prod_{k=0}^{(n-1)/2-1} \frac{((n-1)/2-k)^2-(1-t)^2}{((n-1)/2-k)^2} \leq 1 \end{aligned}$$

and so

$$2^n \frac{\binom{t+(n-1)/2}{(n+1)/2}\binom{(n+1)/2-t}{(n-1)/2}}{\binom{n}{(n-1)/2}} \leq \frac{2^n}{\binom{n}{(n-1)/2}}.$$

If $i \leq (n-3)/2$, we deduce from Remark 1 that $\binom{n-i-t}{(n-1)/2} \leq \binom{n}{(n-1)/2}$ and, by (18),

$$2^n \frac{\left|\binom{i+t}{(n+1)/2}\right|\binom{n-i-t}{(n-1)/2}}{\binom{n}{(n-1)/2}} \leq 2^n \left|\binom{i+t}{(n+1)/2}\right| \leq \frac{2^n}{\binom{(n+1)/2}{i+1}}.$$

Using Remark 1 and (18) we have that

$$\left|\binom{i+t}{k}\right| \leq \begin{cases} \binom{i+1}{k}, & k \leq i+1, \\ \frac{1}{\binom{k}{i+1}} \leq \frac{1}{i+2}, & k \geq i+2, \end{cases} \quad \binom{n-i-t}{k} \leq \binom{n-i}{k}, \quad k \leq n-i,$$

and we deduce the following inequality from (17)

$$\begin{aligned} \operatorname{cond}(x;d) \leq & \sum_{k=0,k\neq n/2}^{i+1} \frac{\binom{i+1}{k}\binom{n-i}{k}}{\binom{(n-1)/2}{k}} + \frac{1}{i+2} \sum_{k=i+2,k\neq n/2}^{\lfloor n/2 \rfloor} \frac{\binom{n-i}{k}}{\binom{(n-1)/2}{k}} \\ & + \sum_{k=0,k\neq(n-1)/2}^{i} \frac{\binom{i+1}{k+1}\binom{n-i}{k}}{\binom{n/2}{k+1}} + \frac{1}{i+2} \sum_{k=i+1,k\neq(n-1)/2}^{\lfloor (n-1)/2 \rfloor} \frac{\binom{n-i}{k}}{\binom{n/2}{k+1}} + 2^n. \end{aligned}$$

By (9), all binomial coefficients in the denominators are larger than or equal to 1 and we derive the following inequality

$$\begin{aligned} \operatorname{cond}(x;d) \leq & \sum_{k=0}^{i+1} \binom{i+1}{k}\binom{n-i}{k} + \frac{1}{i+2} \sum_{k=i+2}^{\lfloor (n-1)/2 \rfloor} \binom{n-i}{k} \\ & + \sum_{k=0}^{i} \binom{i+1}{k+1}\binom{n-i}{k} + \frac{1}{i+2} \sum_{k=i+1}^{\lfloor (n-2)/2 \rfloor} \binom{n-i}{k} + 2^n \\ \leq & \sum_{k=0}^{i+1} \binom{i+2}{k+1}\binom{n-i}{k} + \frac{2^{n-i+1}}{i+2} + 2^n. \end{aligned}$$

Using Vandermonde's identity (15) we have

$$\sum_{k=0}^{i+1} \binom{i+2}{k+1} \binom{n-i}{k} = \sum_{k=0}^{i+1} \binom{i+2}{i+1-k} \binom{n-i}{k} = \binom{n+2}{i+1}$$

and we obtain

$$\operatorname{cond}(x, d) \le \binom{n+2}{i+1} + \frac{2^{n-i+1}}{i+2} + 2^n.$$

Now, we take $x \in [x_{2i+1} - h, x_{2i+1}]$, $i = 0, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$, and we proceed in an analogous way. We take $t := (x_{2i+1,n} - x)/h \in [0, 1]$. By Lemma 1, we have

$$\operatorname{cond}(x; d) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\left|\binom{i+t}{k}\right| \binom{n-i-t}{k}}{\binom{(n-1)/2}{k}} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\left|\binom{i+t}{k}\right| \binom{n-i-t}{k+1}}{\binom{n/2}{k+1}}. \tag{19}$$

The terms corresponding to binomial coefficients in the denominators smaller than 1 in the first sum can be bounded again by 2^n. For the second sum, we first consider the case where n is odd and $k = (n-1)/2$, $i < (n-1)/2$. By (16) and (18), we bound the term in the form

$$\frac{\left|\binom{i+t}{(n-1)/2}\right| \binom{n-i-t}{(n+1)/2}}{\binom{n/2}{(n+1)/2}} = 2^{n+1} \frac{\left|\binom{i+t}{(n-1)/2}\right| \binom{n-i-t}{(n+1)/2}}{\binom{n+1}{(n+1)/2}}$$

$$\le 2^{n+1} \left|\binom{i+t}{(n-1)/2}\right| \le \frac{2^{n+1}}{\binom{(n-1)/2}{i+1}}.$$

If n is odd, $k = i = (n-1)/2$, we have that

$$\binom{(n-1)/2+t}{(n-1)/2} \binom{(n+1)/2-t}{(n+1)/2}$$

$$= t \frac{(n+1)/2 - t}{(n+1)/2} \prod_{k=0}^{(n-1)/2-1} \frac{((n-1)/2 - k + t)((n-1)/2 - k - t)}{((n-1)/2 - k)^2} \le 1$$

and then

$$\frac{\left|\binom{(n-1)/2+t}{(n-1)/2}\right| \binom{(n+1)/2-t}{(n+1)/2}}{\binom{n/2}{(n+1)/2}} = 2^{n+1} \frac{\binom{(n-1)/2+t}{(n-1)/2} \binom{(n+1)/2-t}{(n+1)/2}}{\binom{n+1}{(n+1)/2}} \le \frac{2^{n+1}}{\binom{n+1}{(n+1)/2}}.$$

Similarly to the previous case, we can obtain, by (19), an upper bound for $\mathrm{cond}(x; d)$ in terms of binomial coefficients:

$$\mathrm{cond}(x; d) \le \sum_{k=0, k\neq n/2}^{i+1} \frac{\binom{i+1}{k}\binom{n-i}{k}}{\binom{(n-1)/2}{k}} + \frac{1}{i+2} \sum_{k=i+2, k\neq n/2}^{\lfloor n/2 \rfloor} \frac{\binom{n-i}{k}}{\binom{(n-1)/2}{k}} + \sum_{k=0, k\neq (n-1)/2}^{i+1} \frac{\binom{i+1}{k}\binom{n-i}{k+1}}{\binom{n/2}{k+1}} + \frac{1}{i+2} \sum_{k=i+2, k\neq (n-1)/2}^{\lfloor (n-1)/2 \rfloor} \frac{\binom{n-i}{k+1}}{\binom{n/2}{k+1}} + 2^{n+1}.$$

Since, by (9), all binomial coefficients in the denominators are larger than or equal to 1, we deduce that

$$\mathrm{cond}(x; d) \le \sum_{k=0}^{i+1} \binom{n-i}{k}\binom{i+1}{k} + \frac{1}{i+2} \sum_{k=i+2}^{\lfloor n/2 \rfloor} \binom{n-i}{k} + \sum_{k=0}^{i+1} \binom{n-i}{k+1}\binom{i+1}{k} + \frac{1}{i+2} \sum_{k=i+2}^{\lfloor (n-1)/2 \rfloor} \binom{n-i}{k+1} + 2^{n+1}.$$

Applying Pascal's identity (12) and Vandermonde identity (15), we obtain the bound

$$\sum_{k=0}^{i+1} \binom{n-i+1}{k+1}\binom{i+1}{k} + \frac{2^{n+1-i}}{i+2} + 2^n = \sum_{k=0}^{i+1} \binom{n-i+1}{k+1}\binom{i+1}{i+1-k} + \frac{2^{n+1-i}}{i+2} + 2^n = \binom{n+2}{i+2} + 2^{n+1} + \frac{2^{n+1-i}}{i+2}.$$

□

Observe that the bound obtained in Theorem 3 is close to the formula (5) for the Lebesgue constant. Therefore, the inverse central ordering is a convenient strategy to order the nodes in the Newton formula, in addition to easy implementation.

3 Numerical experiments

We have computed the conditioning of the Newton formula for the inverse central ordering. Figures 1 and 2 show a comparison between the Lebesgue function and $\mathrm{cond}(x; d)$ at equidistant nodes in the inverse central ordering. We see that $\mathrm{cond}(x; d)$ is close to the Lebesgue function at the extremities of the intervals but the difference between both functions grows in a neighborhood of the origin. We can also see that the maximum of the conditioning is attained in a neighborhood of the center for $n = 10$ and that the largest value is attained in a neighborhood of the extremities of the interval for $n = 19$.

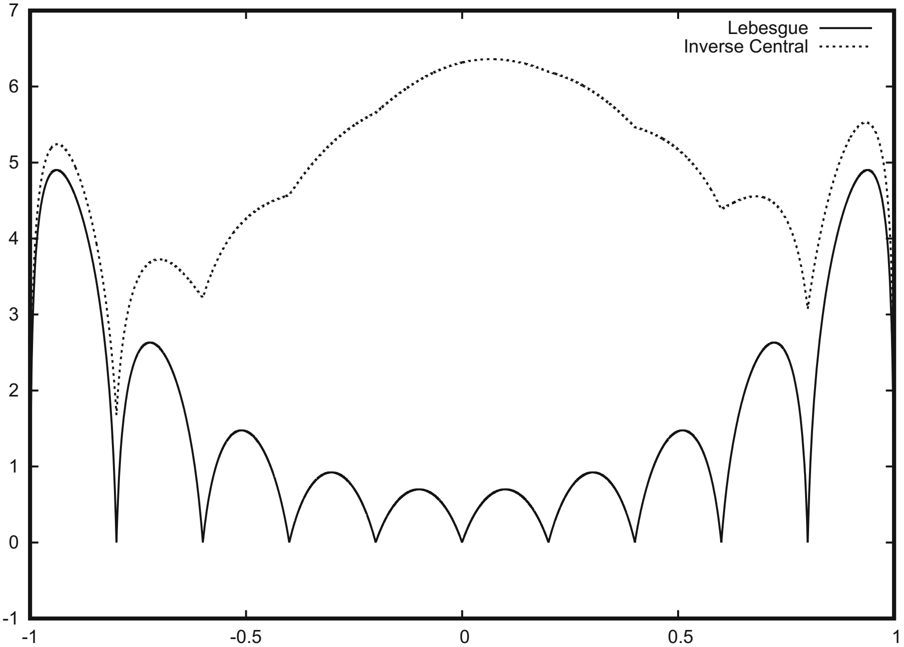

Fig. 1 $\log_2(\lambda(x))$ and $\log_2(\text{cond}(x; d))$ at equidistant nodes following a inverse central ordering for $n = 10$

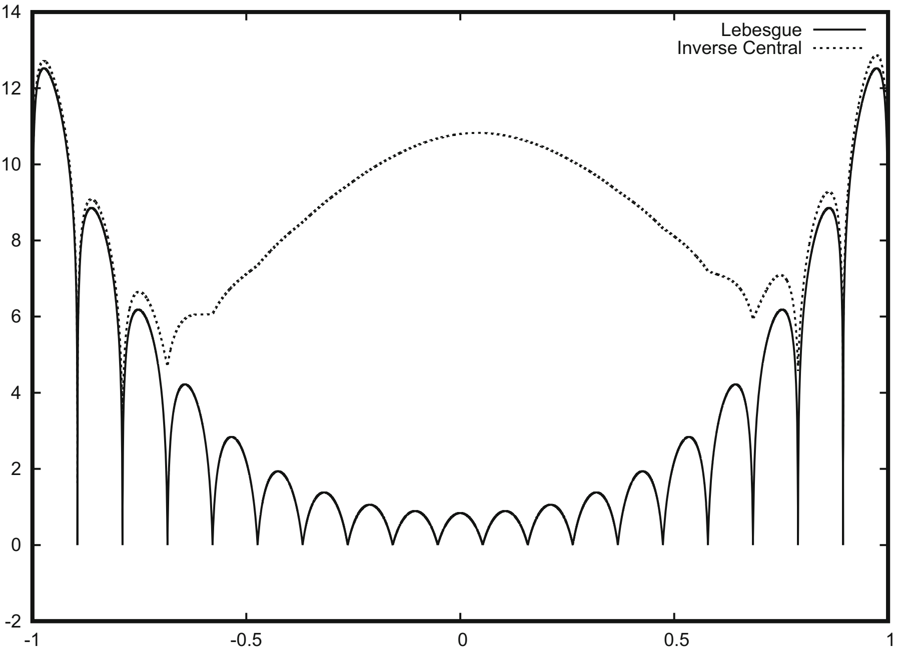

Fig. 2 $\log_2(\lambda(x))$ and $\log_2(\text{cond}(x; d))$ at equidistant nodes following an inverse central ordering for $n = 19$

Let us compare the condition of the Newton formula for equidistant nodes for the increasing ordering and an inverse central ordering. Figure 3 (left) shows the condition for nodes in increasing order for degree $n = 70$. Figure 3 (right) shows the condition for nodes following an inverse central ordering for the same degree. We see that the condition is much lower for the inverse central ordering. In each case, we have also computed the numerical error $e(x)$ for interpolating the smooth function $f(x) = u^{-1}\sin(x/2)$, where $u = 2^{-24}$ is the unit roundoff in single precision. The numerical error $e(x)$ has been computed subtracting the value of $f(x)$ and the computed value of $p(x)$ by Newton's formula in single precision arithmetic. We see in both cases that the error is lower than the bound and even imitates its shape. We also see that the highest numerical error is found for the increasing order in a neighborhood of the right extremity of the interval.

We have also checked other distributions of nodes. Generally the conditioning of the Newton formula near the extremities of the interval is lower with the inverse central ordering that with other orderings. In Fig. 4, we have compared the conditioning of the Newton formula for Chebyshev nodes using different orderings. The worst condition corresponds to the nodes in increasing order. The inverse central ordering leads to a relatively worse conditioning in the center of the interval as compared with the Lebesgue function. In contrast, the Leja order remains closer to the Lebesgue function.

4 Inverse central ordering and Neville elimination

In this section, we interpret the inverse central ordering in terms of a matrix elimination with a pivoting strategy. Specifically, we show a connection between the inverse central ordering for equidistant nodes and Neville elimination with partial pivoting. Neville elimination (NE) is an elimination algorithm alternative to Gaussian elimination where one subtracts to each row a multiple of the previous one (see [7]). Let

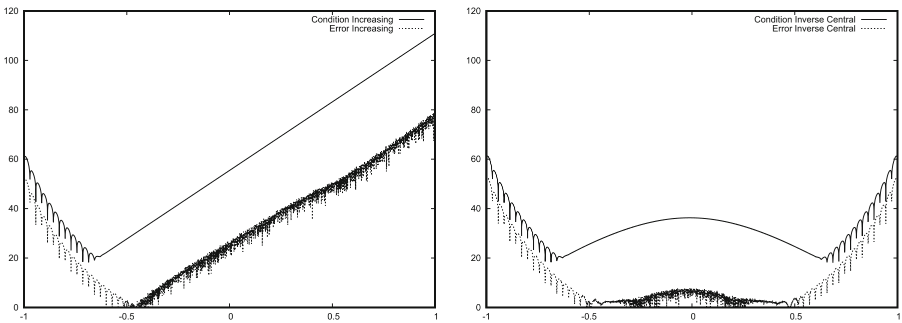

Fig. 3 $\log_2(\text{cond}(x; d))$ and $\log_2(e(x))$ at equidistant nodes for $n = 70$ in increasing order (left) or following an inverse central ordering (right)

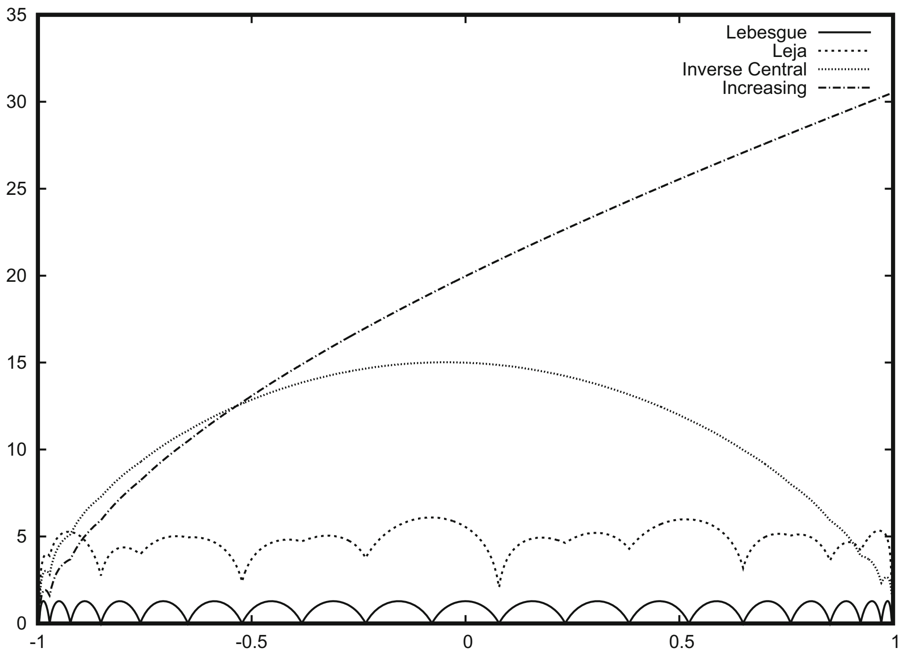

Fig. 4 $\log_2(\text{cond}(x; d))$ for different orderings and $\log_2(\lambda(x))$ corresponding to Chebyshev nodes for degree $n = 19$

$A = (a_{ij})_{0\le i,j\le n}$ be a nonsingular matrix. The NE of A consists of n steps that give rise to a sequence of matrices

$$A = \tilde{A}^{(0)} \to A^{(0)} \to \tilde{A}^{(1)} \to A^{(1)} \to \cdots \to \tilde{A}^{(n)} = A^{(n)} = U,$$

where U is an upper triangular matrix. For each k, $1 \le k \le n$, the matrix $A^{(k)} = (a_{ij}^{(k)})_{0\le i,j\le n}$ has zeros under the main diagonal in the first k columns, that is, $a_{ij}^{(k)} = 0$, $i > j$, $0 \le j \le k-1$. For $k = 0, \ldots, n-1$, the matrix $A^{(k)}$ is obtained from $\tilde{A}^{(k)}$ by reordering the rows with indices $i = k, k+1, \ldots, n$ of $\tilde{A}^{(k)}$ so that the zero entries of the k-th column are placed at the end. At this stage pivoting strategies can be applied. To compute $\tilde{A}^{(k+1)}$ from $A^{(k)}$ one subtracts a multiple of the i-th row from the $(i+1)$-th one, $i = n-1, \ldots, k+1, k$, in order to produce zeros in the k-th column under the main diagonal. The (i, j) pivot of the NE of A is defined by

$$p_{ij} := a_{ij}^{(j)}, \quad 0 \le j < i \le n.$$

Let us denote by

$$A[i, \ldots, i+k | j, \ldots, j+k] := (a_{i+p,j+q})_{p,q=0,\ldots,k}$$

the $(k+1)\times(k+1)$ submatrix of A corresponding to rows with indices $i, i+1, \ldots, i+k$ and columns with indices $j, j+1, \ldots, j+k$. If all the pivots are nonzero, then NE does not require row interchanges and, by Lemma 2.6 of [7], we have

$$p_{i0} = a_{i0}, \quad i = 0, \ldots, n,$$
$$p_{ij} = \frac{\det A[i-j, \ldots, i|0, \ldots, j]}{\det A[i-j, \ldots, i-1|0, \ldots, j-1]}, \quad 0 < j \le i \le n. \tag{20}$$

The number

$$m_{ij} := \begin{cases} \frac{a_{ij}^{(j)}}{a_{i-1,j}^{(j)}} = \frac{p_{ij}}{p_{i-1,j}}, & \text{if } a_{i-1,j}^{(j)} \neq 0, \\ 0, & \text{if } a_{i-1,j}^{(j)} = 0, \end{cases} \tag{21}$$

is the (i, j)-multiplier of the NE of A, for $0 \le j < i \le n$.

The NE with partial pivoting is a strategy (see [1]) in which the pivots corresponding to the k-th column of $A^{(k)}$ form a nonincreasing sequence:

$$|p_{kk}| \ge |p_{k+1,k}| \ge \cdots \ge |p_{nk}|, \quad k = 0, \ldots, n-1,$$

or equivalently,

$$|m_{ij}| \le 1, \quad j = 0, \ldots, n-1, \quad i = j+1, \ldots, n. \tag{22}$$

This condition is analogous to the property obeyed by the multipliers of Gaussian elimination with partial pivoting.

Recall that the Vandermonde matrix is the collocation matrix of the monomial basis at the nodes $x_0, \ldots, x_n$,

$$V(x_0, \ldots, x_n) := \begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix} = (x_i^j)_{0\le i,j\le n}. \tag{23}$$

Let us calculate the pivots and the multipliers of NE for $V(x_0, \ldots, x_n)$ with distinct nodes. By formula (20), $p_{i0} = v_{i0} = 1$ for $i = 0, \ldots, n$, and, using the well-known formula for the Vandermonde determinant, we have

$$p_{ij} = \frac{\det V(x_{i-j}, \ldots, x_i)}{\det V(x_{i-j}, \ldots, x_{i-1})} = \frac{\prod_{i-j\le k<l\le i}(x_l - x_k)}{\prod_{i-j\le k<l\le i-1}(x_l - x_k)}. \tag{24}$$

We have that NE does not require row interchanges of $V(x_0, \ldots, x_n)$ because the nodes are distinct. Then, by formula (21), we have $m_{i0} = 1$, $i = 1, \ldots, n$, and

$$m_{ij} = \frac{p_{ij}}{p_{i-1,j}} = \prod_{k=1}^{j} \frac{x_i - x_{i-k}}{x_{i-1} - x_{i-1-k}}, \tag{25}$$

for $0 \le j < i \le n$. Observe that by (25) the multipliers (21) are invariant under any affine transformation of the nodes.

In the following we show that, for certain ordering of the nodes, NE with partial pivoting does not require row interchanges of the associated Vandermonde matrix. The Leja order satisfies a similar property for the Gaussian elimination with partial pivoting. In fact, Gaussian elimination with partial pivoting leads to an ordering of rows corresponding essentially to the Leja order (see Section 22.3.3 of [9]).

Theorem 4 *NE with partial pivoting of a Vandermonde matrix $V(x_0, \ldots, x_n)$ with distinct nodes $x_0 < \cdots < x_n$ does not require row interchanges if and only if $d_i := x_{i+1} - x_i$, $i = 0, \ldots, n-1$, form a nonincreasing sequence.*

Proof Let us assume that NE with partial pivoting does not require row interchanges. Then, by (25),

$$|m_{i1}| = \frac{d_{i-1}}{d_{i-2}} \leq 1, \quad i = 2, \ldots, n.$$

Conversely, we compute

$$x_i - x_{i-k} = \sum_{l=1}^{k} x_{i+1-l} - x_{i-l} = \sum_{l=1}^{k} d_{i-l}. \tag{26}$$

Taking into account that the nodes form an increasing sequence, as well as (25) and (26), we have

$$|m_{ij}| = \prod_{k=1}^{j} \frac{x_i - x_{i-k}}{x_{i-1} - x_{i-1-k}} = \prod_{k=1}^{j} \frac{\sum_{l=1}^{k} d_{i-l}}{\sum_{l=1}^{k} d_{i-1-l}}.$$

Because the sequence $\{d_i\}_{i=0,\ldots,n-1}$ is nonincreasing, $\sum_{l=1}^{k} d_{i-l} \leq \sum_{l=1}^{k} d_{i-1-l}$, and $|m_{ij}| \leq 1$. □

If the nodes are equidistant, $d_i := x_{i+1} - x_i = h$, $i = 0, \ldots, n-1$, and then Theorem 4 implies that NE of $V(x_0, \ldots, x_n)$ with partial pivoting does not require row interchanges. In the next result, we do not impose the condition that the nodes form an increasing sequence.

Theorem 5 *Let $V(x_0, \ldots, x_n)$ be the Vandermonde matrix (23) with nodes $x_0, \ldots, x_n$ such that there is an $s \in \{-1, 1\}$ with*

$$d_i := |x_{i+1} - x_i| = s(-1)^i (x_{i+1} - x_i) > 0, \quad i = 0, \ldots, n-1.$$

If $\{d_i\}_{i=0,\ldots,n-1}$ and $\{d_i - d_{i+1}\}_{i=0,\ldots,n-1}$ are nonincreasing sequences, then NE with partial pivoting does not require row interchanges of $V(x_0, \ldots, x_n)$.

Proof Observe that

$$\begin{aligned} x_i - x_{i-k} &= \sum_{l=0}^{k-1} (x_{i-l} - x_{i-l-1}) = s \sum_{l=0}^{k-1} (-1)^{i-l-1} d_{i-l-1} \\ &= s(-1)^{i-1} \sum_{l=0}^{k-1} (-1)^l d_{i-l-1}. \end{aligned}$$

Taking into account that the distances d_i form a nonincreasing sequence, we have

$$|x_i - x_{i-k}| = \sum_{l=0}^{k-1} (-1)^l d_{i-l-1} > 0. \tag{27}$$

We note $\Delta d_i := d_{i+1} - d_i$, $i = 0, \ldots, n-1$. Since $|\Delta d_i| = d_i - d_{i+1}$, we have that $\{|\Delta d_i|\}_{i=0,\ldots,n-1}$ is a nonincreasing sequence. Using (27), we obtain

$$|x_i - x_{i-k}| = \begin{cases} d_{i-1} + \sum_{l=1}^{\lfloor k/2 \rfloor} |\Delta d_{i-1-2l}|, & k \text{ odd}, \\ \sum_{l=1}^{\lfloor k/2 \rfloor} |\Delta d_{i-2l}|, & k \text{ even}. \end{cases} \tag{28}$$

Now, by (28), the sequence $\{|x_i - x_{i-k}|\}_{i=k,\ldots,n}$ is nonincreasing in i for each fixed k. So, $\prod_{k=1}^{j} |x_i - x_{i-k}| \le \prod_{k=1}^{j} |x_{i-1} - x_{i-1-k}|$. Therefore, by (25), we deduce that $|m_{ij}| \le 1$. □

If the distances between adjacent nodes are greater in the center than close to the extremities, as in the case of Chebyshev nodes, then the hypotheses of Theorem 5 do not hold. This fact suggests that the inverse central ordering is not a good ordering for dealing with Chebyshev nodes, in agreement with the observations on the conditioning illustrated by Fig. 4. If the distances between adjacent nodes decrease as we approach the center of the interval, then the hypotheses of Theorem 5 hold, which implies more stable computations.

In the following result we consider nodes following the inverse central ordering (6). We show that NE with partial pivoting does not require row interchanges. Furthermore, we prove that the double sequence $(|m_{ij}|)_{0 \le j < i \le n}$ is bimonotonically nonincreasing or, equivalently,

$$|m_{j+1,j}| \ge |m_{j+2,j}| \ge \cdots \ge |m_{n,j}|, \quad j = 0, \ldots, n-1, \tag{29}$$

and

$$|m_{i,0}| \ge |m_{i,1}| \ge \cdots \ge |m_{i,i-1}|, \quad i = 1, \ldots, n. \tag{30}$$

Recall that a double sequence $(a_{ij})_{i,j \in I}$, with $I \subset \mathbb{Z}^2$, is bimonotonically nonincreasing if $a_{ij} \le a_{i',j'}$ for all pairs of indices $(i, j), (i', j') \in I$ such that $i' \le i$, $j' \le j$.

Theorem 6 *Let $V(x_{0,n}, \ldots, x_{n,n})$ be the Vandermonde matrix (23) at equidistant nodes $x_{0,n}, \ldots, x_{n,n}$ following the inverse central ordering given by (6). Then NE with partial pivoting of $V(x_{0,n}, \ldots, x_{n,n})$ does not require row interchanges and, furthermore, the double sequence $(|m_{ij}|)_{0 \le j < i \le n}$ is bimonotonically nonincreasing.*

Proof We note $x_i := x_{i,n}$, $i = 0, \ldots, n$. Using (6), we have that the

$$d_i = (-1)^i (x_{i+1} - x_i) = (n - i)h, \quad i = 0, \ldots, n-1,$$

form a nonincreasing sequence. With Δd_i defined in the proof of Theorem 5, we also have that $|\Delta d_i| = h$, $i = 0, \ldots, n-1$. Hence, by Theorem 5, NE with partial pivoting does not require row interchanges of $V(x_0, \ldots, x_n)$.

Let us analyze the distance between two nodes. If the indices have the same parity, by (6), we have

$$|x_i - x_k| = \frac{|i-k|}{2}h, \quad i \equiv k \pmod 2, \tag{31}$$

and, otherwise,

$$|x_i - x_k| = \left(n - \frac{k+i-1}{2}\right)h, \quad i \not\equiv k \pmod 2. \tag{32}$$

By (31) and (32), the multipliers are in this case

$$\prod_{k=i-j}^{i-1} |x_i - x_k| = \prod_{k=1}^{\lfloor j/2 \rfloor} |x_i - x_{i-2k}| \prod_{k=1}^{\lfloor (j+1)/2 \rfloor} |x_i - x_{i+1-2k}|$$
$$= h^j \prod_{k=1}^{\lfloor j/2 \rfloor} k \prod_{k=1}^{\lfloor (j+1)/2 \rfloor} (n-i+k) = h^j \frac{\lfloor j/2 \rfloor!(n-i+\lfloor (j+1)/2 \rfloor)!}{(n-i)!}. \tag{33}$$

Hence, by (33), we can explicitly compute the absolute value of the multipliers as

$$|m_{ij}| = \frac{\prod_{k=i-j}^{i-1} |x_i - x_k|}{\prod_{k=i-1-j}^{i-2} |x_{i-1} - x_k|} = \frac{(n-i+\lfloor (j+1)/2 \rfloor)!(n-i+1)!}{(n-i)!(n-i+1+\lfloor (j+1)/2 \rfloor)!}$$
$$= \frac{n-i+1}{n-i+1+\lfloor (j+1)/2 \rfloor} \leq 1, \tag{34}$$

which shows, again, that NE with partial pivoting does not require row interchanges.

Observe that, by (34),

$$|m_{ij}|^{-1} = 1 + \frac{\lfloor (j+1)/2 \rfloor}{n-i+1}, \quad 0 \leq j < i \leq n.$$

Clearly, the $|m_{ij}|^{-1}$ form a nondecreasing sequence in each of the indices i and j and, so, $(|m_{ij}|)_{0 \leq j < i \leq n}$ is a bimonotonically nonincreasing sequence. □

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. This work has been partially supported by the PGC2018-096321-B-I00 Spanish Research Grant, by Gobierno de Aragón E41_17R and Feder 2014-2020 "Construyendo Europa desde Aragón".

Declarations

Conflict of interest The authors declare no competing interests.

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