TESIS DE LA UNIVERSIDAD DE ZARAGOZA

2013

55

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Algebraic and Topological Invariants of Curves and Surfaces with Quotient Singularities (Invariantes topológicos y algebraicos de curvas y superficies con singularidades cociente)

Departamento

Matemáticas

Director/es

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ISSN 2254-7606

Prensas de la Universidad
Universidad Zaragoza

Tesis Doctoral

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UNIVERSIDAD DE ZARAGOZA

Matemáticas

2013

Algebraic and Topological Invariants of Curves and Surfaces with Quotient Singularities

DOCTORAL THESIS

co-supervised by Professors
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by

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Zaragoza, May 6, 2013

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AGRADECIMIENTOS

Si alguien me hubiera dicho hace un año que a día de hoy estaría aquí sentado escribiendo estas líneas no le hubiese creído... Toda esta aventura, que ahora llega a su fin, comenzó en 2008. Han sido años muy duros pero plagados de experiencias inolvidables, buenos momentos y, sobre todo, personas que me han acompañado en este camino, allanándolo y empujándome hasta sacar lo mejor de mi. Es por eso que una parte de esta tesis es suya, queriendo aprovechar este momento para agradecerles todo el apoyo recibido.

Me gustaría dar las gracias en primer lugar a las instituciones que, con su soporte económico, han hecho posible el desarrollo de este trabajo. Al IUMA, por su beca destinada a los jóvenes investigadores al inicio del máster. Al Gobierno de Aragón por su beca de formación del personal investigador y por su contribución a través del grupo "E15 Grupo Consolidado Geometría". Al Ministerio de Educación, por la beca FPU recibida y su financiación por medio del proyecto "MTM2010-2010-21740-C02-02". A las universidades de Zaragoza y Pau junto con sus Departamentos de Matemáticas por acogerme hasta 2011. Finalmente, al Centro Universitario de la Defensa, mi casa desde 2011, por los medios que pone a nuestra disposición y la motivación personal que ha supuesto.

Este trabajo no habría sido posible sin la ayuda de distintos profesores que me han ido acompañando a lo largo de estos años. Javier Otal, que fue el que metió el gusanillo de la Geometría Algebraica en mi cuerpo. Luis Rández, por sus numerosos y buenos consejos. András Némethi, por su hospitalidad durante mi estancia de 3 meses en Budapest (muchas gracias por todo Elda). Agradecer a Vincent Florens y Jean Vallès por todo su apoyo, numerosas conversaciones de matemáticas, su corrección de mi francés y por su confianza en mí al pensar todos estos años que por "chapurrear" francés podían hablar a una velocidad normal y yo enterarme de todo. A Kike por la fuerza que siempre transmite, por todo su conocimiento que nunca deja de sorprenderte e intenta explicarte de forma “fácil”. Gracias por toda tu ayuda y apoyo durante estos años.

Mención especial merece José Ignacio, mi padre académico, por todo el mal que le he dado este tiempo (sobrevivir a dos hijos nuevos y un becario...). Este trabajo no habría sido posible sin toda su ayuda, su apoyo anímico en los malos momentos, su paciencia a la hora de explicarme su Memoirs (que aunque él diga que no... de fácil tiene poco) y su gran implicación en la fase final de esta tesis.

De estos años lo mejor que me llevo son las personas y amigos que han estado (y siguen estando) a mi lado y a las que quiero dar las gracias en estas líneas.

Echando la vista atrás, me acuerdo de cómo empezó esta andadura. Me viene a la cabeza el año del máster (2008-2009). Aquel mes de octubre donde comenzamos nuestro periplo en Bilbao, la primera lista de la compra con Julia en el supermercado de plaza Zabálburu (que fuimos incapaces de llevar a la resi solos...), nuestras “euskal” discusiones cariñosas con Nahikari y Arantzazu, las aventuras con el “Sunny” y los “Fluvis (de ambos sexos)” junto con Luis, la fabada Litoral con “nuggets” de pollo... y todos los maravillosos findes en Donosti que han ido aconteciendo durante estos años siguientes convirtiéndose en una tradición más (gracias especialmente a ti Nahikari por todo tu ánimo durante estos años).

Recordar a todo el grupo de “becarios aragoneses”, José Mari, Sergio, Raquel, Michel, Antonio Oller, Carmen, Ángeles, “Chavi” y “Chiquillo” por esos numerosos cafés que tanto ayudan, Antonio Otal (cuantas conversaciones y mutuo ánimo en nuestro anterior despacho), Pooo (formando, la ya famosa dualidad, “Pooo-Chatooo”), la nueva familia alternativa formada por Juan (gracias por tús ánimos y toda la ayuda burocrática francesa...), Adela (por su apoyo en la fase final) y Simón (nuestro pequeño “huevo kinder”) con su hija “la planta” moribunda. Los “becarios foráneos”, Guadalupe (con su juventud), Eva, Nacho, Irene y Ana; de los que tanto apoyo y cariño he recibido y que me han hecho sentirme como en casa cada vez que les visitaba.

Agradecer a mis compañeros de “Mates III” su apoyo en este último año de tesis, Marcos (con sus bromas de pique cariñoso), Tere y Etel por su ayuda, hacer fácil el trabajo diario y acompañarme dentro del grupo “afterhours” del CUD. Dar las gracias a todos mis compañeros de matemáticas por todos esos cafés y comidas: Javi, Eva, Martín (gracias por todas esas conversaciones de matemáticas), Álvaro (y su mascota el “perroala”), Silvia (que deja que la molestemos en el café), Rubén (con sus muelas en la pizarra), Verónica (con su misteriosa libreta para dominar el mundo. . .) y Mapi (por todo su cariño y sus visitas de ánimo). En especial, me gustaría devolver parte del cariño recibido por ese pasillo del primer año en el CUD, Fer y su mundo particular (menos mal que te vi aquel lunes. . .), Jorge Pastor (tú y yo sabemos cómo empezó nuestra relación. . .), Encarna (no te imaginas cómo te echamos de menos), Noemí (cuánto vales) y Jessica (nuestra “agitadora cultural”).

Los que han estado a mi lado estos años saben el mal que les he dado con la tesis durante este tiempo. Me gustaría dar las gracias a Carlos por nuestras “confesiones de cocina” (y ser mi pareja de juego perfecta contra las chicas). A Sandra y Jorge, con los que tan buenos momentos he pasado. Desde las clases de mates, hasta formar el “trío liberal alternativo” en nuestro fantástico periplo americano, pasando por esas comidas domingueras en las que Jorge nos da calabazas a veces.

Qué sería de una buena película sin grandes actores y un buen final. . . Hay cuatro amigos que merecen un papel protagonista en toda esta historia.

Dani, tú fuiste mi primer mentor y maestro en esto de la investigación. Nadie mejor que un becario para explicar a otro. Juntos cogimos un despacho-trastero y lo convertimos en un segundo hogar, con Klein como nuestro pequeñín, nuestro particular horario de trabajo (llegando para el café y volviendo a casa de madrugada). . . gracias por todo.

Marta, muchísimas gracias por estar siempre ahí, aconsejándome, ayudándome y animándome en todos esos momentos no tan buenos (when I get sad I stop being sad and be awesome instead, true story). “Nanopla” y “Qsi” van de la mano y en su camino han visto pasar piruletas, monitos, “Alfred Mc-Boobies”, hombres de negro, el pequeño “Hähnchenaugen” y muchas otras cosas. Muchísimas gracias por tantos buenos recuerdos que has dejado en esta tesis.

Alex, qué puedo decirte, juntos toda una vida, siempre que te he necesitado has estado. Tu casa ha sido mi segundo hogar (gracias a José Emilio y Carmen que así me lo han hecho sentir siempre). Podría escribir muchas líneas pero realmente no tengo palabras para agradecerte todo lo que has hecho por mí estos años.

Jorge, tú serías el cuarto director de esta tesis. Sin tu ayuda este trabajo no estaría acabado a día de hoy, gracias por todas tus explicaciones, tu apoyo y la tranquilidad que me has transmitido. Trabajar contigo hace que uno disfrute con lo que hace. Agradecer también a Patri por su continuo cariño y por aguantar que te llamara a horas intempestivas preguntándote dudas. Gracias de corazón.

Agradecer también a la madre de Goretti y mis cuñados, por su constante ánimo y apoyo estos años. Hay un invariante que me sostiene, y no es otro que mi familia. Sin ellos no sería lo que soy a día de hoy. Mis abuelos, Palmira y Joaquín, junto con mi tío Joaquín, de los que tanto he aprendido y tanto me han ayudado desde pequeño. Mi hermana Ester, que es un sol que siempre tiene un gesto de cariño para ti y a la que tanto quiero. Mis padres, Pili y José Antonio, no hay palabras que puedan agradecer todo lo que han hecho por mí siempre. Sin su apoyo, educación, valores, amor y comprensión no habría llegado a ser lo que soy.

Esta tesis está dedicada a las dos personas más importantes de mi vida, mi madre y Goretti. Gracias mamá por toda una vida. Goretti, esta tesis es tan tuya como mía. Tú has sufrido conmigo, siempre a mi lado, sintiendo mis alegrías y tristezas como tuyas propias. Sólo tú sabes cuánto ha costado llegar a este momento. Cualquier cosa que te diga se va a quedar corta. Doy gracias cada día por tenerte a mi lado. Te quiero.

Jorge Ortigas-Galindo Zaragoza, 24 de abril de 2013

INTRODUCTION

The main goal of this PhD thesis is the study of the cohomology ring of $\mathbb{P}^2_w \setminus \mathcal{R}$, being $\mathcal{R}$ a reduced algebraic curve in the complex weighted projective plane $\mathbb{P}^2_w$ whose irreducible components are all rational (possibly singular) curves. In particular, holomorphic (rational) representatives are found for the cohomology classes. In order to achieve our purpose one needs to develop an algebraic theory of curves on surfaces with quotient singularities and study techniques to compute some particularly useful invariants by means of embedded **Q**-resolutions.

The study of methods to compute different kinds of invariants from an embedded resolution constitutes a classical problem in Singularity Theory ([**Hir64**]). The main motivation for using embedded **Q**-resolutions rather than standard ones is that they provide almost the same information but their combinatorial and computational complexity is much lower than classical ones. The study of surface singularities using embedded resolutions is a classical procedure. The main ingredients appearing in these resolutions are projective planes and plane algebraic curves coming from the intersection of the strict transform with the exceptional divisor. In a natural way, using embedded **Q**-resolutions instead of classical ones, weighted projective planes and weighted projective curves appear. The study and understanding of algebraic and topological invariants of curves with quotient singularities in $\mathbb{P}^2_w$ gives a good insight into surface singularities in a simpler way.

To deal with these invariants we have to generalize, in an appropriate way, concepts such as: δ-invariant, Milnor fiber, genus formula, Noether's formula, concept of logarithmic forms on non-normal crossing $\mathbb{Q}$-divisors or an Adjunction-like Formula in $\mathbb{P}^2_w$ among others. The latter provides a very nice relation between a topological invariant, such as the genus of a generic (but not necessarily smooth) curve of quasi-homogeneous degree d, and the dimension of the space of polynomials of degree $d - \deg K$ (with K the canonical divisor in $\mathbb{P}^2_w$).

For technical reasons the main result in this thesis is presented for rational curves in $\mathbb{P}^2_w$. Following the work of Cogolludo-Agustín and Matei in [**CAM12**], the existence of genus makes the holomorphic classes not to be enough to generate the cohomology ring (anti-holomorphic forms are required).

The first approaches in the study of the cohomology algebra of the complement of hyperplane arrangements come from Arnold ([**Arn69**]), Brieskorn ([**Bri73**]) and Orlik-Solomon ([**OS80**]). In these works they prove that the cohomology algebra is combinatorial and every cohomology class has a holomorphic (rational) representative.

Theorem (Brieskorn Lemma)**.** *Let $\mathcal{L}$ be a line arrangement with components ℓ_0, ℓ_1, ..., ℓ_n, where each ℓ_i is the zero set of a linear form l_i. Then the logarithmic 1-forms*

$$\omega_i = \frac{1}{2\pi\sqrt{-1}}\ d\ \log\left(\frac{l_i}{l_0}\right) \quad i = 1, ..., n$$

generate the cohomology ring $H^\bullet(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{C})$. Moreover, $H^\bullet(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{C})$ is isomorphic to the subalgebra generated by the ω_i's in the algebra of meromorphic forms.

In [**Dim92**] the problem is presented in the case of complement of curves in $\mathbb{P}^2$. In this situation, generators of the cohomology ring are found by Lubicz in [**Lub00**]. The complete ring structure is solved for rational arrangements in [**CA02**] and for the general case in [**CAM12**], where the authors give a description which also depends on the combinatorial structure of the curves called *weak combinatorics* ([**CB10**]). The forms involved in all these previous results are of type

$$H\frac{\Omega^2}{C_i C_j C_k},$$

being $\mathcal{C} = \cup_i \mathcal{C}_i$ a curve in $\mathbb{P}^2$ with $\mathcal{C}_i = \{C_i = 0\}$, H a polynomial and $\Omega^2 := z dx \wedge dy + x dy \wedge dz + y dz \wedge dx$ the volume form.

To study the cohomology ring of the complement of a curve in $\mathbb{P}^2_w$ one has to start generalizing some local invariants on quotient singularities ([**CAMO13**]). Another generalizations can be found in [**ABFdBLMH10**], where the authors study Milnor fibers and Milnor numbers of germs on quotient singularities, or in [**BLSS02, uT77, STV05**]. In dimension 2, cyclic singularities coincide with toric singularities. Hence, the local study of singularities in this dimension can be done with techniques coming from toric geometry. However, our final purpose is the study of weighted projective curves which are not, in general, toric varieties. Therefore, the use of weighted blow-ups seems more appropriate to deal with these objects (see for instance [**AMO11a, AMO11b, AMO12, Dol82, Mar11, Ort09**]).

In this work three different techniques are involved: local theory on V-surface singularities, global theory of logarithmic forms and theory of lattice points and Dedekind sums. For the local part we have studied intersection theory, different local invariants previously exposed and embedded $\mathbf{Q}$-resolutions. We give an alternative definition of logarithmic forms, called here *log-resolution logarithmic forms* (see [**CAM12**]), which will be independent of the $\mathbf{Q}$-resolution. In general, the sheaf of such forms is smaller than the one of logarithmic forms on divisors with non-normal crossings ([**Sai80**]). A description of logarithmic sheaves is given in terms of valuations of $\mathbf{Q}$-resolution trees. To connect this local theory with the global one an Adjunction-like Formula is required. In order to show this formula, lattice points and Dedekind sum techniques are required ([**RG72, BR07**]). This shows the connection between geometry and combinatorics. Similar techniques can be found in [**Pom93, Lat95**].

As an application, the results obtained in this thesis are essential in the study of resonance varieties and formality. The cohomology ring provides the natural way to construct resonance varieties (see for instance [**CA02**]). The works of [**Bri73, OS80**] and [**CAM12**] allow one to prove that complements of hyperplane arrangements and curves are formal spaces. This present work could be used to study the formality of the complement of curves in weighted projective planes.

The rest of this introduction will be devoted to summarize the main results obtained divided into three different parts.

The first one (Chapter I) works as an *introductory part*, the basic concepts are presented and the necessary tools for the main results are stated. In the second one (Chapters II and III), *local invariants* of curves on spaces with quotient singularities are developed. In the last part (Chapters IV and V) all the previous results are used to obtain *global invariants*. In Chapter IV we provide a genus formula and an Adjunction-like Formula for curves on V-surfaces. Finally, in Chapter V, we join all these ingredients from Chapters I to IV and we focus on one of the most important invariants of the pair $(\mathbb{P}^2_w, \mathcal{R})$, namely, the cohomology ring of $\mathbb{P}^2_w \setminus \mathcal{R}$, where $\mathcal{R}$ is a reduced algebraic (possibly singular) curve in the complex weighted projective plane $\mathbb{P}^2_w$ whose irreducible components are all rational.

First part: basic tools

In Chapter I we start giving some basic definitions and properties of V-manifolds, weighted projective spaces, embedded **Q**-resolutions, and weighted blow-ups (for a detailed exposition see for instance [**AMO11a, AMO11b, AMO12, Dol82, Ort09, Mar11**]). The purpose of the first part of the chapter is to fix the notation and introduce several tools to calculate a special kind of embedded resolutions, called *embedded* **Q**-*resolutions* (see Definition (I.2.2)), for which the ambient space is allowed to contain abelian quotient singularities. To do this, we study weighted blow-ups at points. We will focus our attention on the case of V-surfaces.

Definition 1. A V-manifold of dimension n is a complex analytic space which admits an open covering $\{U_i\}$ such that U_i is analytically isomorphic to B_i/G_i where $B_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$.

V-manifolds were introduced in [**Sat56**] and have the same homological properties over $\mathbb{Q}$ as manifolds. For instance, they admit a Poincaré duality if they are compact and carry a pure Hodge structure if they are compact and Kähler (see [**Bai56**]). They have been classified locally by Prill ([**Pri67**]).

We are interested in V-surfaces where the quotient spaces B_i/G_i are given by (finite) abelian groups.

Let $\boldsymbol{G}_d$ be the cyclic group of d-th roots of unity. Consider a vector of weights $(a, b) \in \mathbb{Z}^2$ and the action

$$\begin{array}{ccc} \boldsymbol{G}_d \times \mathbb{C}^2 & \stackrel{\rho}{\longrightarrow} & \mathbb{C}^2, \\ (\xi_d, (x, y)) & \mapsto & (\xi_d^a\, x, \xi_d^b\, y). \end{array}$$

The set of all orbits $\mathbb{C}^2/\boldsymbol{G}_d$ is called a (*cyclic*) *quotient space of type* $(d; a, b)$ and it is denoted by $X(d; a, b)$.

The space $X(d; a, b)$ is written in a normalized form (see Definition (I.1.9)) if and only if $\gcd(d, a) = \gcd(d, b) = 1$. If this is not the case, one uses the isomorphism (assuming $\gcd(d, a, b) = 1$)

$$\begin{array}{ccc} X(d; a, b) & \longrightarrow & X\left(\frac{d}{(d,a)(d,b)}; \frac{a}{(d,a)}, \frac{b}{(d,b)}\right), \\ \big[(x, y)\big] & \mapsto & \big[(x^{(d,b)}, y^{(d,a)})\big] \end{array}$$

to normalize it.

One of the main examples of V-manifolds is the *weighted projective plane* (§I.4). Let $w := (w_0, w_1, w_2) \in \mathbb{N}^3$ be a weight vector, that is, a triple of pairwise coprime positive integers. There is a natural action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^3 \setminus \{0\}$ given by

$$(x_0, x_1, x_2) \longmapsto (t^{w_0}x_0, t^{w_1}x_1, t^{w_2}x_2).$$

The set of orbits $\frac{\mathbb{C}^3\setminus\{0\}}{\mathbb{C}^*}$ under this action is denoted by $\mathbb{P}^2_w$ and it is called the *weighted projective plane* of type w.

Let us recall the definition of one of the most important objects we are dealing with.

Definition 2. An *embedded* **Q**-*resolution* of $(H,0) \subset (M,0)$ is a proper analytic map $\pi : X \to (M,0)$ such that:

(1) X is a V-manifold with abelian quotient singularities,
(2) π is an isomorphism over $X \setminus \pi^{-1}(\mathrm{Sing}(H))$,
(3) $\pi^{-1}(H)$ is a **Q**-normal crossing hypersurface on X (see Definition (I.2.1)), and
(4) the strict transform $\hat{H} := \overline{\pi^{-1}(H \setminus \{0\})}$ is **Q**-smooth (see Definition (I.2.1)).

Embedded **Q**-resolutions are a natural generalization of the usual embedded resolutions, for which some of the invariants studied from Chapters II to V can be effectively calculated.

In Section I.3 we develop an intersection theory in order to study embedded **Q**-resolutions in dimension 2 (see [**AMO11b**] and [**Mar11**] for further details). One has to deal with two types of divisors on V-manifolds: Weil and Cartier divisors. Weil divisors are locally finite linear combinations with integral coefficients of irreducible subvarieties of codimension 1 and Cartier divisors are global sections of the quotient sheaf of meromorphic functions modulo non-vanishing holomorphic functions. The relationship between Cartier divisors and line bundles provides a useful way to define the intersection multiplicity of two divisors. In the smooth category, both notions coincide but this is not the case for singular varieties. Theorem (I.3.3) ([**AMO11a**]) allows one to develop a rational intersection theory on V-manifolds (see [**AMO11b**]).

Definition 3 (Local intersection number on $X(d;a,b)$, [**Ort09**])**.** Denote by X the cyclic quotient space $X(d;a,b)$ and consider two divisors $D_1 = \{f_1 = 0\}$ and $D_2 = \{f_2 = 0\}$ given by $f_1, f_2 \in \mathbb{C}\{x,y\}$ reduced and without common components. Assume that, $(d;a,b)$ is normalized.

Then as Cartier divisors $D_1 = \frac{1}{d}\{(X, f_1^d)\}$ and $D_2 = \frac{1}{d}\{(X, f_2^d)\}$. The local number $(D_1 \cdot D_2)_{[P]}$ at a point P of type $(d;a,b)$ is defined as

$$(D_1 \cdot D_2)_{[P]} = \frac{1}{d^2} \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\langle f_1^d, f_2^d \rangle}$$

where $\mathcal{O}_P = \mathbb{C}\{x,y\}^{\boldsymbol{G}_d}$ is the local ring of functions at P (recall §I.1–2).

This local intersection theory developed allows us, for instance, to compute the Weighted Bézout's Theorem for weighted projective planes (Proposition (I.4.7)), which will be of particular interest in some forthcoming results.

Proposition 1 ([**Ort09**])**.** *The intersection number of two* $\mathbb{Q}$*-divisors,* D_1 *and* D_2 *on* $\mathbb{P}^2_w$ *without common components is*

$$D_1 \cdot D_2 = \sum_{P \in D_1 \cap D_2} (D_1 \cdot D_2)_{[P]} = \frac{1}{\bar{w}} \deg_w(D_1) \deg_w(D_2) \in \mathbb{Q},$$

where $\bar{w} = w_0 w_1 w_2$ *and* $\deg_w(D_i) = \deg(\phi^*(D_i))$ *(see* (7)*).*

In §I.5 we develop a De Rham cohomology for projective varieties with quotient singularities. We shall recall some Hodge theoretical results for projective V-manifolds which will be of particular interest for us. All these results, with their respective proofs, can be found in the first chapter of [**Ste77**]. The definition of logarithmic forms and residues on non-normal crossing $\mathbb{Q}$-divisors on V-surfaces will be provided.

Let $\mathcal{D}$ be a $\mathbb{Q}$-divisor in $\mathbb{P}^2_w$. The complement of $\mathcal{D}$ will be denoted by $X_{\mathcal{D}}$. Let us fix $\pi : \overline{X}_{\mathcal{D}} \longrightarrow \mathbb{P}^2_w$ a **Q**-resolution of the singularities of $\mathcal{D}$ so that the reduced $\mathbb{Q}$-divisor $\overline{\mathcal{D}} = (\pi^*(\mathcal{D}))_{red}$ is a union of smooth $\mathbb{Q}$-divisors on $\overline{X}_{\mathcal{D}}$ with **Q**-normal crossings.

Definition 4. A C^∞ form φ on $X_{\mathcal{D}}$ shall be called *logarithmic (with respect to a divisor* $\mathcal{D}$ *and a* **Q***-resolution* π*)* if $\pi^*\varphi$ is logarithmic on $\overline{X}_{\mathcal{D}}$ with respect to the **Q**-normal crossing divisor $\overline{\mathcal{D}}$ (see Definition (I.5.6)). Therefore, one has the corresponding sheaf

$$\pi_* \Omega_{\overline{X}_{\mathcal{D}}}(\log\langle \overline{\mathcal{D}} \rangle).$$

Once $\mathcal{D}$ and π are fixed one can define the *residue map* $\operatorname{Res}^{[*]}_\pi(\varphi)$ *of a logarithmic form* φ as follows

$$\begin{array}{ccc} \pi_* \Omega^k_{\overline{X}_{\mathcal{D}}}(\log\langle \overline{\mathcal{D}} \rangle) & \stackrel{\operatorname{Res}^{[k]}_\pi}{\longrightarrow} & H^0(\overline{\mathcal{D}}^{[k]}; \mathbb{C}) \\ \varphi & \mapsto & \operatorname{Res}^{[k]}(\pi^*\varphi). \end{array}$$

This previous definition is independent from the **Q**-resolution. For instance, in the particular case of $X(d;a,b)$ and the $\operatorname{Res}^{[2]}$ one has the following.

Definition 5. Let h be an analytic germ on $X(d;a,b)$ written in normalized form (Definition (I.1.9)). Let $\varphi = h\dfrac{dx \wedge dy}{xy}$ be a logarithmic 2-form with poles at the origin. Then

$$\mathrm{Res}^{[2]}(\varphi) := \frac{1}{d} h(0,0).$$

Second part: local invariants

In Chapter II we extend the concept of Milnor fiber and Milnor number of a curve singularity allowing the ambient space to be a quotient surface singularity (§II.1). A generalization of the local δ-invariant is defined and described in terms of a **Q**-resolution of the curve singularity (§II.3). In particular, when applied to the classical case (the ambient space is a smooth surface) one obtains a formula for the classical δ-invariant in terms of a **Q**-resolution, which simplifies considerably effective computations.

All these tools will finally allow for an explicit description of the genus formula of a curve defined on a weighted projective plane in terms of its degree and the local type of its singularities in Chapter IV.

Definition 6 ([**CAMO13**]). Let $\mathcal{C} = \{f = 0\} \subset X(d; a, b)$ be a curve germ. The *Milnor fiber* F_t^w of $(\mathcal{C}, [0])$ is defined as follows,

$$F_t^w := \{F = t\}/\boldsymbol{G}_d.$$

The *Milnor number* μ^w of $(\mathcal{C}, P)$ is defined as follows,

$$\mu^w := 1 - \chi^{\mathrm{orb}}(F_t^w).$$

Note that alternative generalizations of Milnor numbers can be found, for instance, in [**ABFdBLMH10, BLSS02, uT77, STV05**]. The one proposed here seems more natural for quotient singularities (see Example (IV.1.18)), but more importantly, it allows for the existence of an explicit formula relating Milnor number, δ-invariant, and genus of a curve on a singular surface (Chapter IV).

In §II.3 we present a version of Noether's formula (see Theorem (II.2.1)) for curves on quotient singularities and **Q**-resolutions.

Theorem 2 (Noether's Formula, [**CAMO13**]). *Consider C and D two germs of $\mathbb{Q}$-divisors at $[0]$ without common components in a quotient surface singularity. Then the following formula holds:*

$$(C \cdot D)_{[0]} = \sum_{Q \prec [0]} \frac{\nu_{C,Q}\nu_{D,Q}}{pqd},$$

where Q runs over all the infinitely near points of $(CD, [0])$ and Q appears after a blow-up of type (p, q) of the origin in $X(d; a, b)$.

We define the local invariant δ^w for curve singularities on $X(d;a,b)$.

Definition 7 ([**CAMO13**])**.** Let C be a reduced curve germ at $[0] \in X(d;a,b)$, then we define δ^w as the number verifying

$$\chi^{\mathrm{orb}}(F_t^w) = r^w - 2\delta^w,$$

where r^w is the number of local branches of C at $[0]$, F_t^w denotes its Milnor fiber, and $\chi^{\mathrm{orb}}(F_t^w)$ denotes the orbifold Euler characteristic of F_t^w.

A recurrent formula for δ^w based on a **Q**-resolution of the singularity is provided in Theorem (II.2.5).

Theorem 3 ([**CAMO13**])**.** *Let $(C,[0])$ be a curve germ on an abelian quotient surface singularity. Then*

$$\delta^w = \frac{1}{2}\sum_{Q \prec [0]} \frac{\nu_Q}{dpq}\left(\nu_Q - p - q + e\right),$$

where Q runs over all the infinitely near points of a **Q***-resolution of $(C,[0])$, Q appears after a (p,q)-blow-up of the origin of $X(d;a,b)$, and $e := \gcd(d, aq-bp)$.*

In §II.3–1 an interpretation of the δ^w invariant as the dimension of a vector space is given. In the classical case this invariant can be interpreted as the dimension of a vector space. Since δ^w is in general a rational number, a similar result can only be expected in certain cases, namely, when associated with Cartier divisors (see Theorem (II.3.7)).

Theorem 4 ([**CAMO13**])**.** *Let $f : (X(d;a,b),P) \to (\mathbb{C},P)$ be a reduced analytic function germ. Assume $(d;a,b)$ is a normalized type. Consider $R = \frac{\mathcal{O}_P}{\langle f \rangle}$ the local ring associated with f and $\overline{R}$ its normalization ring. Then,*

$$\delta_P^w(f) = \dim_{\mathbb{C}}\left(\frac{\overline{R}}{R}\right) \in \mathbb{N}.$$

In §II.3–2 we will present, in some way, a generalization of this result. To do this some previous definitions are needed.

For a given $k \geq 0$, one has the module $\mathcal{O}_P(k)$ (for further details see §I.1–2),

$$\mathcal{O}_P(k) := \{h \in \mathbb{C}\{x,y\} \mid h(\xi_d^a x, \xi_d^b y) = \xi_d^k h(x,y)\}.$$

Let $\{f=0\}$ be a germ in $P \in X(d;a,b)$. Note that if $f \in \mathcal{O}_P(k)$, then one has the following $\mathcal{O}_P$-module, $\mathcal{O}_P(k-a-b)$ verifying

$$\mathcal{O}_P(k-a-b) = \{h \in \mathbb{C}\{x,y\} \mid h\frac{dx \wedge dy}{f} \text{ is } \boldsymbol{G}_d\text{-invariant}\}.$$

Definition 8. Let $\mathcal{D} = \{f = 0\}$ be a germ in $P \in X(d; a, b)$, where $f \in \mathcal{O}_P(k)$. Consider π a **Q**-resolution of $(\mathcal{D}, P)$.

(1) Let $\mathcal{M}^{log}_{\mathcal{D},\pi}$ denote the submodule of $\mathcal{O}_P$ consisting of all $h \in \mathcal{O}_P$ such that the 2-form
$$\omega = h\,\frac{dx \wedge dy}{f} \in \Omega^2_P(a + b - k)$$
is logarithmic at P, with respect to $\mathcal{D}$ and the embedded **Q**-resolution π (recall Definition 4).

(2) Let $\mathcal{M}^{nul}_{\mathcal{D},\pi}$ denote the submodule of $\mathcal{M}^{log}_{\mathcal{D},\pi}$ consisting of all $h \in \mathcal{M}^{log}_{\mathcal{D},\pi}$ such that the 2-form
$$\omega = h\,\frac{dx \wedge dy}{f}$$
admits a holomorphic extension outside the strict transform $\widehat{f}$.

This last module will play an important role in the construction of a presentation for the cohomology ring of $\mathbb{P}^2_w \setminus \mathcal{R}$ in Chapter V.

Definition 9. Let $\mathcal{D} = \{f = 0\}$ be a germ in $P \in X(d; a, b)$. Let us define the following dimension,
$$K_P(\mathcal{D}) = K_P(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\mathcal{M}^{nul}_{\mathcal{D},\pi}}.$$

The number $K_P(f)$ gives us the minimal number of conditions required to a generic germ $h \in \mathcal{O}_P(s)$ so that $h \in \mathcal{M}^{nul}_{\mathcal{D},\pi}(s)$.

Theorem 5. *Let $f, g \in \mathcal{O}(k)$, $k \in \mathbb{N}$, be two germs at $P \in X(d; a, b)$. Then,*
$$K_P(f) - K_P(g) = \delta^w_P(f) - \delta^w_P(g).$$

In Chapter III we continue defining some other logarithmic modules and sheaves associated with a $\mathbb{Q}$-divisor $\mathcal{D}$ and a **Q**-resolution π apart from the one in Definition 8. Their global sections will allow for a construction of logarithmic 2-forms on $\mathcal{D}$ in Chapter V. We will construct two kinds of trees associated with an analytic germ $\{f = 0\}$ at $P \in X(d; a, b)$, $\tilde{\mathcal{T}}^{nul}_P(f)$ (see §III.1) and $\tilde{\mathcal{T}}^{\delta_1\delta_2}_P(f)$ (see §III.2) being δ_1 and δ_2 two local branches of f at P. These trees will provide a useful description of the logarithmic modules previously defined.

Definition 10. Let $\mathcal{D} = \{f = 0\}$ be a germ in $P \in X(d; a, b)$, where $f \in \mathcal{O}_P(k)$. Consider π a **Q**-resolution of $(\mathcal{D}, P)$.

Let us define $\mathcal{M}^{\delta_i\delta_j}_{\mathcal{D},\pi}$ the submodule of $\mathcal{M}^{log}_{\mathcal{D},\pi}$ consisting of all $h \in \mathcal{M}^{log}_{\mathcal{D},\pi}$ such that the 2-form
$$\omega = h\,\frac{dx \wedge dy}{f}$$

has zero residues outside the edges of the path $\gamma(\delta_1, \delta_2)$.

As a consequence of the construction of the trees $\tilde{\mathcal{T}}_P^{nul}$ (§III.1) and $\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}$ (§III.2) and Definitions 8 and 10 one has the following characterization:

$$\mathcal{M}_{\mathcal{D},\pi}^{nul} = \{h \in \mathcal{O}_P \mid \tilde{\mathcal{T}}_P(\mathcal{D},\pi)|_h \ \geq \ \tilde{\mathcal{T}}_P^{nul}(\mathcal{D},\pi)\}.$$

$$\mathcal{M}_{\mathcal{D},\pi}^{\delta_i\delta_j} = \{h \in \mathcal{O}_P \mid \tilde{\mathcal{T}}_P(\mathcal{D},\pi)|_h \ \geq \ \tilde{\mathcal{T}}_P^{\delta_i\delta_j}(\mathcal{D},\pi)\}.$$

Consider now the following dimension,

$$K_P^{\delta_i\delta_j}(\mathcal{D}) = K_P^{\delta_i\delta_j}(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\mathcal{M}_{\mathcal{D},\pi}^{\delta_i\delta_j}}.$$

The number $K_P^{\delta_i\delta_j}(f)$ gives us the minimal number of conditions required to a generic germ $h \in \mathcal{O}_P(s)$ so that $h \in \mathcal{M}_{\mathcal{D},\pi}^{\delta_i\delta_j}(s)$.

Definition 11. The *degree* of a weighted tree $\mathcal{T}$ will be defined as follows

$$\deg(\mathcal{T}) := \sum_{Q\in|\mathcal{T}|} \frac{w(\mathcal{T},Q)}{2dpq}\left(w(\mathcal{T},Q) + p + q - e\right),$$

where $w(\mathcal{T},Q)$ denotes the weight of $\mathcal{T}$ at Q, the vertex Q runs over all the infinitely near points of a **Q**-resolution of V_f, Q appears after a (p,q)-blow-up of the origin of $X(d;a,b)$, and $e := \gcd(d, aq - bp)$.

One has the following result (Lemma (III.4.3)) for weighted plane curves in $\mathbb{P}^2_w$ which extends Lemma 2.35 in [**CA02**] for curves in $\mathbb{P}^2$ and classical resolutions.

Lemma 6. *The following result holds,*

$$\deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}(f)) = \deg(\tilde{\mathcal{T}}_P^{nul}(f)) - 1.$$

Note that in the case of germs at P on $\mathbb{C}^2$ and classical blow-ups, the degree of a tree $\mathcal{T}$ is related with the number of conditions imposed to a germ g so that $\mathcal{T}|_g \geq \mathcal{T}$. In this situation $K_P(f) = \deg \mathcal{T}_P^{nul}(f) = \delta_P(f)$ (see [**CA02**]). In our case, $\deg \mathcal{T}_P^{nul}(f) = \delta_P^w(f)$, independent from the **Q**-resolution, which is in general a rational number. Therefore, $K_P(f) = \deg \mathcal{T}_P^{nul}(f)$ can only be expected when f defines a function on $X(d;a,b)$.

The previous Lemma 6, together with the following Proposition 7 (see Proposition (III.5.6)), will be useful in Chapter V to allow for the proof of Theorem 13.

Proposition 7. *Let* $\{f = 0\}$ *be an analytic germ of curve singularity at* $[0]$ *on* $X(d;a,b)$. *Consider* δ_1, δ_2 *any two different local branches of* f *at* $[0]$, *then*

$$K_P^{\delta_1\delta_2}(f) = K_P(f) - 1.$$

Third part: global invariants

In §IV.1 a genus formula for weighted projective curves is provided by means of the δ^w-invariant (Definition 7). For a given $d \in \mathbb{N}$ and a normalized weight list $w \in \mathbb{N}^3$ the *virtual genus* associated with d and w is defined as

$$g_{d,w} := \frac{d(d-|w|)}{2\bar{w}} + 1,$$

having the following result (see Theorem (IV.1.12)).

Theorem 8 (**[CAMO13]**). *Let $\mathcal{C} \subset \mathbb{P}^2_w$ be an irreducible curve of degree $d > 0$, then*

$$g(\mathcal{C}) = g_{d,w} - \sum_{P \in \mathrm{Sing}(\mathcal{C})} \delta^w_P.$$

We also show some practical examples in which the genus of different curves in different weighted projective planes is computed.

During the rest of this Chapter IV we focus our efforts on obtaining an Adjunction-like Formula relating the genus of a generic curve of quasi-homogeneous degree d, and the dimension of the space of polynomials of degree $d + \deg K$ (note that $\deg K = -|w| = -(w_0 + w_1 + w_2)$), with K the canonical divisor in $\mathbb{P}^2_w$ (this dimension will be denoted by $D_{d-|w|,w}$).

One has the following results (see Theorem (IV.4.3) and Corollary (IV.4.4)) which will play an important role in Chapter V.

Theorem 9. *Let w_0, w_1, w_2 be pairwise coprime integers, $d \in \mathbb{N}$ and denote by $\bar{w} = w_0w_1w_2$, $|w| = w_0+w_1+w_2$ where $w = (w_0, w_1, w_2)$. Let us consider positive integers $p_i = w_i$, $q_i = -w_j^{-1}w_k \mod w_i \in \mathbb{N}$ with $j < k$ (recall that $X(w_i; w_j, w_k) = X(p_i; -1, q_i)$), $r_i = w_k^{-1}d \mod w_i \in \mathbb{N}$. Consider*

$$\begin{aligned} D_{d-|w|,w} &= \#\left\{(x,y,z) \in \mathbb{N}^3 \mid w_0x + w_1y + w_2z = d - |w|\right\}, \\ A^{(p_i,q_i)}_{r_i} &= \#\left\{(x,y) \in \mathbb{N}^2 \mid p_ix + q_iy \leq q_ir_i,\ x, y \geq 1\right\}, \end{aligned}$$

$$\delta^{(p_i,q_i)}_{r_i} = \frac{r_i(p_ir_i - p_i - q_i + 1)}{2p_i}.$$

Then

$$D_{d-|w|,w} = g_{d,w} + \sum_{i=0}^{2} \left(\delta^{(p_i,q_i)}_{r_i} - A^{(p_i,q_i)}_{r_i}\right).$$

Let $\mathcal{C} \subset \mathbb{P}^2_w$ be a reduced curve of degree d, the *number of global conditions of $\mathcal{C}$* is defined as follows

$$K(\mathcal{C}) := \sum_{P \in \mathrm{Sing}(\mathcal{C})} K_P(f).$$

Corollary 10 (Adjunction-like Formula)**.** *Let $\mathcal{C} \subset \mathbb{P}^2_w$ be a reduced curve of degree d, then*

$$h^0(\mathbb{P}^2_w; \mathcal{O}(d-|w|)) = D_{d-|w|,w} = g_{d,w} - \sum_{P \in \mathrm{Sing}(\mathcal{C})} \delta^w_P + K(\mathcal{C}).$$

From now on, we will denote by $X_{\mathcal{C}}$ the complement of $\mathcal{C}$ in the complex projective plane $\mathbb{P}^2_w$.

With all the necessary ingredients previously developed, we will finally focus, in Chapter V, on one of the most important invariants of the pair $(\mathbb{P}^2_w, \mathcal{R})$, namely, the cohomology ring of $X_{\mathcal{R}}$, where $\mathcal{R}$ is a reduced algebraic (possibly singular) curve in the complex projective plane $\mathbb{P}^2_w$ whose irreducible components $\mathcal{R}_i$ are all rational ($g(\mathcal{R}_i) = 0$). Such curves will be called *rational arrangements*. The aim of this chapter is to find a presentation for the cohomology ring of $X_{\mathcal{R}}$.

Let $\mathcal{D}$ be a reduced $\mathbb{Q}$-divisor in $\mathbb{P}^2_w$. In §V.2, a basis for $H^1(\mathbb{P}^2_w \setminus \mathcal{D}; \mathbb{C})$ is given and, in §V.4, a holomorphic presentation for $H^2(\mathbb{P}^2_w \setminus \mathcal{R}; \mathbb{C})$ is provided.

Consider a system of coordinates $[X : Y : Z]$ in $\mathbb{P}^2_w$. If one writes $\mathcal{D} := \{D = 0\}$, D can be expressed as a product $C_0 \cdot C_1 \cdot \ldots \cdot C_n$ where $\mathcal{C}_i := \{C_i = 0\}$, C_i are irreducible components of D.

One can consider the following differential forms

$$\sigma_{ij} := d\left(\log \frac{C_i^{d_j}}{C_j^{d_i}}\right) = d_j \; d(\log C_i) - d_i \; d(\log C_j).$$

where $i, j = 0, ..., n$, $d_i := \deg_w(\mathcal{C}_i)$.

Take π a **Q**-resolution of $\mathcal{D}$ then, the pull-back $\pi^*\sigma_{ij}$ defines a logarithmic 1-form on $\overline{X}_{\mathcal{D}}$. The following result holds.

Theorem 11. *The cohomology classes of*

$$\mathcal{B}_1(\mathcal{D}) := \{\sigma_{ik}\}_{i=0}^n$$

$i \neq k$, constitute a basis for $H^1(X_{\mathcal{D}}; \mathbb{C})$.

It is easy to check that, in general, Brieskorn's Theorem does not hold, that is, $\wedge^2 H^1(X_{\mathcal{R}}; \mathbb{C})$ does not generate $H^2(X_{\mathcal{R}}; \mathbb{C})$.

In §V.3 some examples of the computation of the ring structure of $H^2(X_{\mathcal{D}}; \mathbb{C})$ are provided. Finally, in §V.4, a holomorphic presentation for $H^2(X_{\mathcal{R}}; \mathbb{C})$, for a rational arrangement $\mathcal{R}$, is given. Let us sketch this last result.

Let $\mathcal{C}_i,\ \mathcal{C}_j,\ \mathcal{C}_k$ be three curves in $\mathbb{P}^2_w$ (not necessarily different). We will denote by $\mathcal{C}_{ijk}$ the union $\mathcal{C}_i \cup \mathcal{C}_j \cup \mathcal{C}_k$ and consider C_{ijk} a reduced equation for $\mathcal{C}_{ijk}$. We also use $d_{ijk} := \deg_w \mathcal{C}_{ijk}$.

For instance, if $i = j = k$, $\mathcal{C}_{ijk} = \mathcal{C}_i$, $C_{ijk} = C_i$ and $d_{ijk} = \deg_w(\mathcal{C}_i)$.

Using the modules described in terms of logarithmic trees seen in Chapters II and III one can construct the following sheaf $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$.

Definition 12. Let $\mathcal{R} = \bigcup_i \mathcal{R}_i$ be a rational arrangement and π a **Q**-resolution of singularities for $\mathcal{R}$. For every triple $(\mathcal{R}_i, \mathcal{R}_j, \mathcal{R}_k)$, not necessarily $i \neq j \neq k$, let us take three points $P_1 \in \operatorname{Sing}(\mathcal{R}_i \cap \mathcal{R}_j)$, $P_2 \in \operatorname{Sing}(\mathcal{R}_j \cap \mathcal{R}_k)$ and $P_3 \in \operatorname{Sing}(\mathcal{R}_i \cap \mathcal{R}_k)$. For every P_l choose two local branches, $\delta_l^{i_l}$ of $\mathcal{R}_i$ and $\delta_l^{j_l}$ of $\mathcal{R}_j$. Consider

$$\Delta := \left[(P_1, \delta_1^{i_1}, \delta_1^{j_1}),\ (P_2, \delta_2^{j_2}, \delta_2^{k_2}),\ (P_3, \delta_3^{k_3}, \delta_3^{i_3})\right].$$

Let us construct a sheaf $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ associated with Δ. Let $Q \in \mathcal{R}_{ijk}$, one has the following module

$$(\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi})_Q := \left\{ \begin{array}{ll} \mathcal{O}_Q & \text{if } Q \notin \operatorname{Sing}(\mathcal{R}_{ijk}) \\ (\mathcal{M}^{nul}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } P_l \neq Q \in \operatorname{Sing}(\mathcal{R}_{ijk}) \\ (\mathcal{M}^{\delta_l^i,\delta_l^j}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } Q = P_l \text{ with } \delta_l^i \neq \delta_l^j \\ (\mathcal{M}^{nul}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } Q = P_l \text{ with } \delta_l^i = \delta_l^j \end{array} \right\}.$$

This module lead us to the corresponding sheaf $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ which will be called *sheaf of* Δ*-logarithmic forms along* $\mathcal{R}_{ijk}$ *w.r.t.* π.

The previous sheaf $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ does not depend on the choice of the resolution π. For a given $\mathcal{R}$ we will simply write $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}$ if no ambiguity seems no likely to arise.

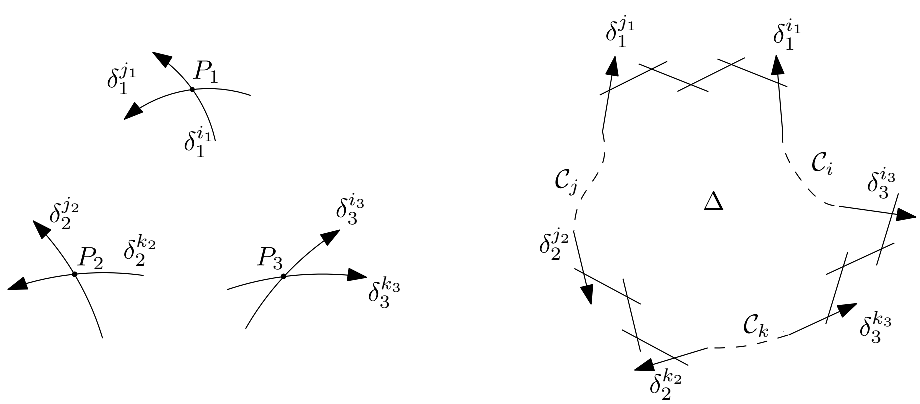


FIGURE 1. Δ in $H^1(\bar{\mathcal{R}}_{ijk}; \mathbb{C})$.

With the previous definition, using the Adjunction-like Formula (Corollary 10), Lemma 6 and Proposition 7 one has the following results.

Proposition 12. *Let $\mathcal{R}$ be a rational arrangement in $\mathbb{P}^2_w$ as in Definition 12, then*

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk} - |w|)) > 0.$$

Theorem 13. *Let $\mathcal{R} = \bigcup_i \mathcal{R}_i$ be a rational arrangement in $\mathbb{P}^2_w$ and π a $\mathbf{Q}$-resolution of singularities for $\mathcal{R}$. Let H be a polynomial of quasi-homogeneous degree $d_{ijk} - |w|$, such that*

$$H \in H^0(\mathbb{P}^2_w, \mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk} - |w|)).$$

The well-defined global 2-forms $\omega = H\dfrac{\Omega^2}{R_{ijk}}$ form a holomorphic presentation of $H^2(\mathbb{P}^2_w \setminus \mathcal{R}, \mathbb{C})$.

The proof in Theorem 13 will provide a method to find the relations among the generators in $H^2(\mathbb{P}^2_w \setminus \mathcal{R}; \mathbb{C})$ by means of the residue operator (Definitions 4 and 5).

Most of the results seen from Chapters I to V are illustrated in the particular case of $\mathcal{D} = V(xyz(xyz + (x^3 - y^2)^2)) \subset \mathbb{P}^2_w$ and $w = (2, 3, 7)$. In Chapter I we study a $\mathbf{Q}$-resolution of its singularities (Example (I.2.8)). In Chapter III we construct different logarithmic trees associated with them (Examples (III.3.2) and (III.3.5)). The local concepts studied in Chapters I and II give us the tools to compute its genus in Chapter IV (Example (IV.1.18)). See also Example (IV.4.5) for an illustrative example of the Adjunction-like Formula. Finally, all these results allow us in Chapter V to study its cohomology ring $H^\bullet(\mathbb{P}^2_w \setminus \mathcal{D}; \mathbb{C})$ in §V.3–2 and Example (V.4.9).

RESUMEN (Spanish)

El objetivo principal de esta tesis doctoral es el estudio del anillo de cohomología de $\mathbb{P}^2_w \setminus \mathcal{R}$, siendo $\mathcal{R}$ una curva algebraica reducida en el plano proyectivo ponderado complejo $\mathbb{P}^2_w$ cuyas componentes son curvas racionales irreducibles (con o sin puntos singulares). En particular, encontramos representantes holomorfos (racionales) para las clases de cohomología. Para lograr nuestro objetivo es necesario desarrollar una teoría algebraica de curvas en superficies con singularidades cociente y estudiar técnicas para calcular algunos invariantes, particularmente útiles, por medio de $\mathbf{Q}$-resoluciones encajadas.

El estudio de métodos para calcular diferentes tipos de invariantes a partir de una resolución encajada constituye un problema clásico en Teoría de Singularidades ([**Hir64**]). La motivación principal de usar $\mathbf{Q}$-resoluciones encajadas en lugar de las resoluciones clásicas es que éstas ofrecen casi la misma información siendo su complejidad combinatoria y computacional mucho menor. El estudio de singularidades de superficie utilizando resoluciones encajadas es un procedimiento estándar. Los principales ingredientes que aparecen en estas resoluciones son los planos proyectivos y las curvas algebraicas planas provenientes de la intersección de la transformada estricta con el divisor excepcional. De forma natural, al utilizar $\mathbf{Q}$-resoluciones encajadas en lugar de las usuales, aparecen planos proyectivos ponderados y curvas proyectivas ponderadas. El estudio y la comprensión de invariantes algebraicos y topológicos de curvas con singularidades cociente en $\mathbb{P}^2_w$ ofrece una buena perspectiva de las singularidades de superficie de una forma simple y más rápida que con los métodos usuales.

Para estudiar ciertos invariantes hemos tenido que generalizar, de manera adecuada, los siguientes conceptos: invariante δ, fibra de Milnor, fórmula del género, fórmula de Noether, concepto de forma logarítmica en $\mathbb{Q}$-divisores con cruces no normales o una Fórmula de tipo Adjunción en $\mathbb{P}^2_w$ entre otros.

Esta última fórmula proporciona una relación muy interesante entre un invariante topológico, como es el género de una curva genérica (no necesariamente lisa) de grado cuasi-homogéneo d, y la dimensión del espacio de polinomios de grado $d - \deg K$ (siendo K el divisor canónico en $\mathbb{P}^2_w$).

Por razones técnicas, el resultado principal de esta tesis se presenta para curvas racionales en $\mathbb{P}^2_w$. Siguiendo el trabajo de Cogolludo-Agustín y Matei en [**CAM12**], la existencia de género hace que las clases holomorfas no sean suficientes para generar el anillo de cohomología (siendo necesario trabajar con formas anti-holomorfas).

Las primeras aproximaciones al estudio del álgebra de cohomología del complementario de configuraciones de hiperplanos provienen de los trabajos de Arnold ([**Arn69**]), Brieskorn ([**Bri73**]) y Orlik-Solomon ([**OS80**]). En éstos, los autores demuestran que el álgebra de cohomología es combinatoria y que cada clase de cohomología tiene un representante holomorfo (racional).

Teorema (Lema de Brieskorn)**.** *Sea $\mathcal{L}$ una configuración de rectas con componentes ℓ_0, ℓ_1 , ..., ℓ_n, donde cada ℓ_i es el lugar de ceros de una forma lineal l_i. Entonces las* 1*-formas logarítmicas*

$$\omega_i = \frac{1}{2\pi\sqrt{-1}}\ d\ \log\left(\frac{l_i}{l_0}\right) \quad i = 1, ..., n$$

generan el anillo de cohomología $H^\bullet(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{C})$. Es más, $H^\bullet(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{C})$ es isomorfo a la subálgebra generada por los ω_i en el álgebra de formas meromorfas.

En [**Dim92**] el problema se presenta para el caso del complementario de curvas en $\mathbb{P}^2$. En esta situación, Lubicz encuentra los generadores del anillo de cohomología en [**Lub00**]. La estructura de anillo completa se resuelve para configuraciones racionales en [**CA02**] y para el caso general en [**CAM12**]. En este último artículo los autores dan una descripción que también depende de la estructura combinatoria de las curvas, llamada *combinatoria débil* ([**CB10**]). Las formas que participan en todos los resultados anteriores son de la forma

$$H\frac{\Omega^2}{C_iC_jC_k},$$

donde $\mathcal{C} = \cup_i \mathcal{C}_i$ es una curva en $\mathbb{P}^2$ con $\mathcal{C}_i = \{C_i = 0\}$, H un polinomio y $\Omega^2 := zdx \wedge dy + xdy \wedge dz + ydz \wedge dx$ la forma de volumen.

Para estudiar el anillo de cohomología del complementario de una curva en $\mathbb{P}^2_w$ uno tiene que comenzar generalizando algunos invariantes locales de curvas en espacios con singularidades cociente ([**CAMO13**]). Otras generalizaciones se pueden encontrar en [**ABFdBLMH10**], donde los autores estudian la fibra de Milnor y el número de Milnor para gérmenes en superficies con singularidades cociente, o en [**BLSS02, uT77, STV05**]. En dimensión 2, las singularidades cíclicas coinciden con las tóricas. De ahí que el estudio local de singularidades en esta dimensión se pueda hacer con técnicas provenientes de la geometría tórica. Sin embargo, nuestro propósito final es el estudio de curvas proyectivas ponderadas, que no son, en general, variedades tóricas. Por tanto, el uso de explosiones ponderadas parece más adecuado para tratar este tipo de objetos (véase, por ejemplo [**AMO11a, AMO11b, AMO12, Dol82, Ort09, Mar11**]).

En este trabajo se usan principalmente tres técnicas diferentes: teoría local de singularidades en V-superficies, teoría global de formas logarítmicas y teoría de puntos en retículos junto con sumas de Dedekind. Desde el punto de vista local se ha estudiado teoría de intersección, diferentes invariantes locales anteriormente expuestos y $\mathbf{Q}$-resoluciones encajadas. En la tesis damos una definición alternativa de formas logarítmicas, las llamadas *formas logarítmicas de log-resolución* (ver [**CAM12**]), que son independientes de la $\mathbf{Q}$-resolución escogida. En general, el haz de tales formas es más pequeño que el de las formas logarítmicas sobre divisores con cruces no normales ([**Sai80**]). Mostramos una descripción de los haces logarítmicos en términos de valoraciones de árboles de $\mathbf{Q}$-resolución. Para conectar la teoría local con la global entra en juego la Fórmula de tipo Adjunción. Para demostrar dicha fórmula es necesario el estudio de métodos de conteo de puntos y sumas de Dedekind ([**RG72, BR07**]). Todo ello muestra la conexión entre la geometría y la combinatoria. Técnicas similares pueden encontrarse en [**Pom93, Lat95**].

Como aplicación, los resultados obtenidos en esta tesis se pueden aplicar al estudio de las variedades de resonancia y la formalidad. El anillo de cohomología proporciona una manera natural para el cálculo de las variedades de resonancia (véase, por ejemplo [**CA02**]). Los trabajos de [**Bri73, OS80**] y [**CAM12**] permiten demostrar que los complementarios de configuraciones de hiperplanos y curvas son espacios formales. El presente trabajo podría ser usado para estudiar la formalidad del complementario de curvas en planos proyectivos ponderados.

El resto de esta introducción está dedicada a resumir los principales resultados obtenidos divididos en tres partes.

La primera (Capítulo I) funciona como una *parte introductoria*, en ella se presentan los conceptos básicos y las herramientas necesarias para que los resultados principales puedan ser desarrollados. En la segunda (Capítulos II y III), se estudian *invariantes locales* de curvas en espacios con singularidades cociente. En la última parte (Capítulos IV y V) todos los resultados anteriores se utilizan para obtener *invariantes globales*. En el Capítulo IV damos una fórmula para el género y una Fórmula de tipo Adjunción para curvas en V-superficies. Por último, en el Capítulo V, unimos todos los ingredientes vistos a lo largo de los Capítulos I a IV para centrarnos en uno de los invariantes más importantes del par $(\mathbb{P}^2_w, \mathcal{R})$, es decir, el anillo de cohomología de $\mathbb{P}^2_w \setminus \mathcal{R}$, donde $\mathcal{R}$ es una curva algebraica reducida (con o sin puntos singulares) en el plano proyectivo ponderado complejo $\mathbb{P}^2_w$ cuyas componentes irreducibles son todas racionales.

Primera parte: herramientas básicas

En el Capítulo I empezamos dando algunas definiciones básicas y propiedades de V-variedades, espacios proyectivos ponderados, $\mathbf{Q}$-resoluciones encajadas y explosiones ponderadas (para una exposición detallada, véase, por ejemplo, [**AMO11a, AMO11b, AMO12, Dol82, Mar11, Ort09**]). El objetivo de la primera parte del capítulo es fijar la notación e introducir varias herramientas para calcular un tipo especial de resoluciones encajadas, las llamadas $\mathbf{Q}$-*resoluciones encajadas* (ver Definición (I.2.2)), para las cuales, el espacio ambiente puede contener singularidades cociente de tipo abeliano. Para ello, estudiamos explosiones ponderadas de puntos. Centraremos nuestra atención en el caso de V-superficies.

Definición 1. Una V-variedad de dimensión n es un espacio analítico complejo que admite un recubrimiento por abiertos $\{U_i\}$ tales que los U_i son analíticamente isomorfos a B_i/G_i donde $B_i \subset \mathbb{C}^n$ es una bola abierta y G_i es un subgrupo finito de $\mathrm{GL}(n, \mathbb{C})$.

Las V-variedades fueron introducidas en [**Sat56**] y tienen las mismas propiedades homológicas sobre $\mathbb{Q}$ que las variedades. Por ejemplo, admiten una dualidad de Poincaré si son compactas y tienen una estructura de Hodge pura si son compactas y Kähler (véase [**Bai56**]). Han sido clasificadas a nivel local por Prill ([**Pri67**]).

Estamos interesados en V-variedades donde los espacios cociente B_i/G_i vienen dados por grupos abelianos finitos.

Sea $\boldsymbol{G}_d$ el grupo cíclico de las d-ésimas raíces de la unidad. Consideremos un vector de pesos $(a, b) \in \mathbb{Z}^2$ y la acción

$$\begin{array}{rcl} \boldsymbol{G}_d \times \mathbb{C}^2 & \stackrel{\rho}{\longrightarrow} & \mathbb{C}^2, \\ (\xi_d, (x, y)) & \mapsto & (\xi_d^a\, x, \xi_d^b\, y). \end{array}$$

El conjunto de todas las órbitas $\mathbb{C}^2/\boldsymbol{G}_d$ se llama *espacio (cíclico) cociente the tipo* $(d; a, b)$ y se denota por $X(d; a, b)$.

Diremos que el espacio $X(d; a, b)$ está escrito en forma normalizada (Definición (I.1.9)) si y sólo si $\gcd(d, a) = \gcd(d, b) = 1$. Si no es el caso, usaremos el siguiente isomorfismo (asumiendo $\gcd(d, a, b) = 1$)

$$\begin{array}{ccc} X(d; a, b) & \longrightarrow & X\left(\frac{d}{(d,a)(d,b)}; \frac{a}{(d,a)}, \frac{b}{(d,b)}\right), \\ \big[(x, y)\big] & \mapsto & \big[(x^{(d,b)}, y^{(d,a)})\big] \end{array}$$

para normalizarlo.

Uno de los principales ejemplos de V-variedades es el *plano proyectivo ponderado* (§I.4). Sea $w := (w_0, w_1, w_2) \in \mathbb{N}^3$ un vector de pesos, es decir, una terna de enteros positivos primos dos a dos. Hay una acción natural del grupo multiplicativo $\mathbb{C}^*$ sobre $\mathbb{C}^3 \setminus \{0\}$ dada por

$$(x_0, x_1, x_2) \longmapsto (t^{w_0} x_0, t^{w_1} x_1, t^{w_2} x_2).$$

El conjunto de órbitas $\frac{\mathbb{C}^3 \setminus \{0\}}{\mathbb{C}^*}$ bajo esta acción se denota $\mathbb{P}^2_w$ y es llamado *plano proyectivo ponderado* de tipo w.

Recordemos la definición de uno de los objetos más importantes que vamos a tratar.

Definición 2. Una **Q**-*resolución encajada* de $(H, 0) \subset (M, 0)$ es una aplicación analítica propia $\pi : X \to (M, 0)$ tal que:

(1) X es una V-variedad con singularidades cociente abelianas.
(2) π es un isomorfismo sobre $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
(3) $\pi^{-1}(H)$ es una hipersuperficie con **Q**-cruces normales en X (véase Definición (I.2.1)).
(4) La transformada estricta $\hat{H} := \overline{\pi^{-1}(H \setminus \{0\})}$ es **Q**-lisa (ver Definición (I.2.1)).

Las **Q**-resoluciones encajadas son una generalización natural de las resoluciones estándar. El uso de **Q**-resoluciones permite calcular con eficacia algunos de los invariantes estudiados a lo largo de los Capítulos II a V.

En la Sección I.3 se desarrolla una teoría de intersección con el fin de estudiar **Q**-resoluciones encajadas en dimensión 2 (ver [**AMO11b**] y [**Mar11**] para más detalles). Uno tiene que lidiar con dos tipos de divisores en V-variedades: divisores de Weil y de Cartier. Los divisores de Weil son combinaciones locales lineales finitas con coeficientes enteros de subvariedades irreducibles de codimensión 1 y los divisores de Cartier son secciones globales del haz cociente de funciones meromorfas modulo funciones holomorfas no nulas. La relación entre los divisores Cartier y los fibrados en línea proporciona una manera útil de definir la multiplicidad de intersección de dos divisores. En la categoría lisa, ambas nociones coinciden pero este no es el caso para las variedades singulares. El Teorema (I.3.3) ([**AMO11a**]) permite desarrollar una teoría de intersección en V-variedades (ver [**AMO11b**]).

Definición 3 (Multiplicidad de intersección local en $X(d;a,b)$, [**Ort09**])**.** Denotemos por X el espacio cíclico cociente $X(d;a,b)$ y consideremos dos divisores $D_1 = \{f_1 = 0\}$ y $D_2 = \{f_2 = 0\}$ dados por $f_1, f_2 \in \mathbb{C}\{x,y\}$ reducidos sin componentes comunes. Asumamos que, $(d;a,b)$ está normalizado.

Entonces como divisores de Cartier $D_1 = \frac{1}{d}\{(X, f_1^d)\}$ y $D_2 = \frac{1}{d}\{(X, f_2^d)\}$. La multiplicidad de intersección $(D_1 \cdot D_2)_{[P]}$ en el punto P de tipo $(d;a,b)$ se define como

$$(D_1 \cdot D_2)_{[P]} = \frac{1}{d^2} \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\langle f_1^d, f_2^d \rangle}$$

donde $\mathbb{C}\{x,y\}^{\boldsymbol{G}_d}$ ($\equiv \mathcal{O}_P$) es el anillo local de las funciones en P (véase §I.1–2).

Esta teoría de intersección local desarrollada nos permite, por ejemplo, calcular el Teorema de Bézout Ponderado en planos proyectivos ponderados (Proposición (I.4.7)), que será particularmente interesante en algunos de los resultados futuros.

Proposición 1 ([**Ort09**])**.** *La multiplicidad de intersección de dos $\mathbb{Q}$-divisores sin componentes comunes, D_1 y D_2 en $\mathbb{P}^2_w$ es*

$$D_1 \cdot D_2 = \sum_{P \in D_1 \cap D_2} (D_1 \cdot D_2)_{[P]} = \frac{1}{\bar{w}} \deg_w(D_1) \deg_w(D_2) \in \mathbb{Q},$$

donde $\bar{w} = w_0 w_1 w_2$ y $\deg_w(D_i) = \deg(\phi^*(D_i))$ (ver (7)).

En §I.5 desarrollamos una cohomología de De Rham para variedades proyectivas con singularidades cociente. Recordaremos algunos resultados teóricos de Teoría de Hodge en V-variedades proyectivas que serán de especial interés para nosotros. Todos estos resultados, con sus respectivas demostraciones, se pueden encontrar en el primer capítulo de [**Ste77**]. Finalmente proporcionaremos una definición de formas logarítmicas y residuos sobre $\mathbb{Q}$-divisores con cruces no normales en V-superficies.

Sea $\mathcal{D}$ un $\mathbb{Q}$-divisor en $\mathbb{P}^2_w$. El complementario de $\mathcal{D}$ se denotará $X_{\mathcal{D}}$. Fijemos $\pi : \overline{X}_{\mathcal{D}} \longrightarrow \mathbb{P}^2_w$ una $\mathbf{Q}$-resolución de las singularidades de $\mathcal{D}$ tal que el $\mathbb{Q}$-divisor reducido $\overline{\mathcal{D}} = (\pi^*(\mathcal{D}))_{red}$ es una unión de $\mathbb{Q}$-divisores lisos en $\overline{X}_{\mathcal{D}}$ con $\mathbf{Q}$-cruces normales.

Definición 4. Una forma C^∞, φ en $X_{\mathcal{D}}$ se llamará *logarítmica (con respecto al divisor $\mathcal{D}$ y a la* **Q***-resolución π)* si $\pi^*\varphi$ es logarítmica en $\overline{X}_{\mathcal{D}}$ con respecto al divisor con **Q**-cruces normales $\overline{\mathcal{D}}$ (véase Definición (I.5.6)). Entonces, tendremos el correspondiente haz

$$\pi_*\Omega_{\overline{X}_{\mathcal{D}}}(\log\langle\overline{\mathcal{D}}\rangle).$$

Una vez que $\mathcal{D}$ y π están fijados, podemos definir la *aplicación residuo* $\operatorname{Res}_\pi^{[*]}(\varphi)$ *de una forma logarítmica* φ de la siguiente manera

$$\begin{array}{ccc} \pi_*\Omega^k_{\overline{X}_{\mathcal{D}}}(\log\langle\overline{\mathcal{D}}\rangle) & \overset{\operatorname{Res}_\pi^{[k]}}{\longrightarrow} & H^0(\overline{\mathcal{D}}^{[k]};\mathbb{C}) \\ \varphi & \mapsto & \operatorname{Res}^{[k]}(\pi^*\varphi). \end{array}$$

La definición anterior es independiente de la **Q**-resolución. Por ejemplo, en el caso particular de $X(d;a,b)$, para el $\operatorname{Res}^{[2]}$ tenemos lo siguiente.

Definición 5. Sea h un germen analítico en $X(d;a,b)$ escrito en forma normalizada (Definición (I.1.9)). Sea $\varphi = h\dfrac{dx\wedge dy}{xy}$ una 2-forma logarítmica con polos en el origen. Entonces

$$\operatorname{Res}^{[2]}(\varphi) := \frac{1}{d}h(0,0).$$

Segunda parte: invariantes locales

En el Capítulo II se amplía el concepto de fibra de Milnor y número de Milnor de una singularidad de curva permitiendo que el espacio ambiente tenga singularidades cociente (§II.1). Definimos una generalización del invariante δ y damos una descripción de éste en términos de una **Q**-resolución de las singularidades de la curva (§II.3). En particular, aplicando lo visto al caso clásico (el espacio de ambiente es una superficie lisa) uno obtiene una fórmula para el invariante δ clásico en términos de una **Q**-resolución, lo que simplifica considerablemente los cálculos efectivos.

Finalmente, todas estas herramientas nos permitirán dar, en el capítulo IV, una descripción explícita de la fórmula de género para una curva definida en un plano proyectivo ponderado en términos de su grado y el tipo de sus singularidades locales.

Definición 6 ([**CAMO13**])**.** Sea $\mathcal{C} = \{f=0\} \subset X(d;a,b)$ un germen de curva. La *fibra de Milnor* F_t^w de $(\mathcal{C},[0])$ se define como,

$$F_t^w := \{F=t\}/\boldsymbol{G}_d.$$

El *número de Milnor* μ^w de $(\mathcal{C},P)$ se define como,

$$\mu^w := 1-\chi^{\mathrm{orb}}(F_t^w).$$

Nótese que generalizaciones alternativas del número de Milnor se pueden encontrar, por ejemplo, en [**ABFdBLMH10, BLSS02, uT77, STV05**]. La que aquí se propone parece más natural para el caso de singularidades cociente (véase el Ejemplo (IV.1.18)), pero más importante aún, permite la existencia de una fórmula explícita que relaciona el número de Milnor, el invariante δ y el género de una curva en una superficie singular (Capítulo IV).

En §II.3 presentamos una versión de la fórmula de Noether para curvas en espacios con singularidades cociente usando **Q**-resoluciones (véase Teorema (II.2.1)).

Teorema 2 (Fórmula de Noether, [**CAMO13**])**.** *Consideremos C y D dos gérmenes de $\mathbb{Q}$-divisor en $[0]$ sin componentes comunes en una superficie con singularidades cociente. Se tiene la siguiente fórmula:*

$$(C \cdot D)_{[0]} = \sum_{Q \prec [0]} \frac{\nu_{C,Q}\nu_{D,Q}}{pqd},$$

donde Q recorre todos los puntos infinitamente próximos a $(CD, [0])$ y Q aparece tras una explosión ponderada de tipo (p, q) del origen de $X(d; a, b)$.

A continuación definimos el invariante δ^w local para singularidades de curvas en $X(d; a, b)$.

Definición 7 ([**CAMO13**])**.** Sea C un germen reducido en $[0] \in X(d; a, b)$, definimos δ^w como el número que verifica la siguiente ecuación

$$\chi^{\text{orb}}(F_t^w) = r^w - 2\delta^w,$$

donde r^w es el número de ramas locales de C en $[0]$, F_t^w denota su fibra de Milnor y $\chi^{\text{orb}}(F_t^w)$ denota la característica de Euler orbifold de F_t^w.

En el Teorema (II.2.5) proporcionamos una fórmula recursiva para δ^w basada en una **Q**-resolución de la singularidad.

Teorema 3 ([**CAMO13**])**.** *Sea $(C, [0])$ un germen de curva en una superficie con singularidades cociente abelianas. Entonces*

$$\delta^w = \frac{1}{2} \sum_{Q \prec [0]} \frac{\nu_Q}{dpq} \left(\nu_Q - p - q + e\right),$$

*donde Q recorre todos los puntos infinitamente próximos de una **Q**-resolución de $(C, [0])$, Q aparece tras una explosión ponderada de tipo (p, q) del origen de $X(d; a, b)$ y $e := \gcd(d, aq - bp)$.*

En §II.3–1 damos una interpretación del invariante δ^w como dimensión de un espacio vectorial. En el caso clásico este invariante se puede interpretar como la dimensión de un espacio vectorial, sin embargo, dado que δ^w es en general, un número racional, un resultado similar sólo se puede esperar en ciertos casos, más concretamente, cuando se asocie a divisores de Cartier (ver Teorema (II.3.7)).

Teorema 4 ([**CAMO13**]). *Sea* $f : (X(d;a,b),P) \to (\mathbb{C},P)$ *un germen reducido de función analítica. Asumamos que* $(d;a,b)$ *es de tipo normalizado. Consideremos* $R = \frac{\mathcal{O}_P}{\langle f \rangle}$ *el anillo local asociado a* f *y* $\overline{R}$ *su anillo normalizado. Entonces,*

$$\delta_P^w(f) = \dim_{\mathbb{C}} \left(\frac{\overline{R}}{R} \right) \in \mathbb{N}.$$

En §II.3–2 presentaremos una generalización de este resultado. Para ello, necesitaremos algunas de las definiciones anteriormente vistas.

Dado $k \geq 0$, tenemos el módulo $\mathcal{O}_P(k)$ (para más detalles, véase §I.1–2),

$$\mathcal{O}_P(k) := \{h \in \mathbb{C}\{x,y\} |\ h(\xi_d^a x, \xi_d^b y) = \xi_d^k h(x,y)\}.$$

Sea $\{f = 0\}$ un germen en $P \in X(d;a,b)$. Notar que si $f \in \mathcal{O}_P(k)$, entonces el siguiente $\mathcal{O}_P$-módulo, $\mathcal{O}_P(k-a-b)$, verifica

$$\mathcal{O}_P(k-a-b) = \{h \in \mathbb{C}\{x,y\} |\ h\frac{dx \wedge dy}{f} \text{ es } \boldsymbol{G}_d\text{-invariante}\}.$$

Definición 8. Sea $\mathcal{D} = \{f = 0\}$ un germen en $P \in X(d;a,b)$ con $f \in \mathcal{O}_P(k)$. Considerar π una $\mathbf{Q}$-resolución de $(\mathcal{D},P)$.

(1) Denotemos $\mathcal{M}_{\mathcal{D},\pi}^{log}$ el submódulo de $\mathcal{O}_P$ consistente en los $h \in \mathcal{O}_P$ tales que la 2-forma

$$\omega = h\ \frac{dx \wedge dy}{f} \in \Omega_P^2(a+b-k)$$

es logarítmica en P, con respecto a $\mathcal{D}$ y la $\mathbf{Q}$-resolución π (recordar Definición 4).

(2) Sea $(\mathcal{M}_{\mathcal{D},\pi}^{nul})_P$ el submódulo de $\mathcal{M}_{\mathcal{D},\pi}^{log}$ consistente en todos los $h \in \mathcal{M}_{\mathcal{D},\pi}^{log}$ tales que la 2-forma

$$\omega = h\ \frac{dx \wedge dy}{f}$$

admite una extensión holomorfa fuera de la transformada estricta $\widehat{f}$.

Este último módulo jugará un papel importante a la hora de construir una presentación para el anillo de cohomología de $\mathbb{P}_w^2 \setminus \mathcal{R}$ en el Capítulo V.

Definición 9. Sea $\mathcal{D} = \{f = 0\}$ un germen en $P \in X(d;a,b)$, definimos la siguiente dimensión,

$$K_P(\mathcal{D}) = K_P(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\mathcal{M}_{\mathcal{D},\pi}^{nul}}.$$

El número $K_P(f)$ nos da el mínimo número de condiciones que tenemos que pedirle a un germen genérico $h \in \mathcal{O}_P(s)$ para que $h \in \mathcal{M}^{nul}_{\mathcal{D},\pi}(s)$.

En el caso particular de que $f \in \mathcal{O}_P$, es decir, $(f,[0])$ sea un germen de función en $X(d;a,b)$, entonces (véase Corolario (II.3.14))

$$K_P(f) = \delta_P(f).$$

Teorema 5. *Sean $f, g \in \mathcal{O}(k)$, $k \in \mathbb{N}$, dos gérmenes en $P \in X(d;a,b)$. Entonces,*

$$K_P(f) - K_P(g) = \delta^w_P(f) - \delta^w_P(g).$$

En el Capítulo III continuamos definiendo otros módulos y haces logarítmicos asociados a un $\mathbb{Q}$-divisor $\mathcal{D}$ y a una $\mathbf{Q}$-resolución π. Sus secciones globales nos permitirán, en el capítulo V, construir 2-formas logarítmicas sobre $\mathcal{D}$.

Construiremos dos tipos de árboles asociados a un germen analítico $\{f = 0\}$ en $P \in X(d;a,b)$, $\tilde{\mathcal{T}}^{nul}_P(f)$ (véase §III.1) y $\tilde{\mathcal{T}}^{\delta_1\delta_2}_P(f)$ (véase §III.2) siendo δ_1 y δ_2 dos ramas locales de f en P. Estos árboles nos permitirán dar una descripción útil de los módulos logarítmicos previamente definidos.

Definición 10. Sea $\mathcal{D} = \{f = 0\}$ un germen en $P \in X(d;a,b)$ con $f \in \mathcal{O}_P(k)$. Considerar π una $\mathbf{Q}$-resolución de $(\mathcal{D}, P)$. Definimos $\mathcal{M}^{\delta_i\delta_j}_{\mathcal{D},\pi}$ el submódulo de $\mathcal{M}^{log}_{\mathcal{D},\pi}$ consistente en todos los $h \in \mathcal{M}^{log}_{\mathcal{D},\pi}$ tales que la 2-forma

$$\omega = h\ \frac{dx \wedge dy}{f}$$

tiene residuo nulo fuera de los bordes del camino $\gamma(\delta_1, \delta_2)$.

Como consecuencia de la construcción de los árboles $\tilde{\mathcal{T}}^{nul}_P$ (§III.1) y $\tilde{\mathcal{T}}^{\delta_1,\delta_2}_P$ (§III.2) y las Definiciones 8 y 10, tenemos la siguiente caracterización:

$$\mathcal{M}^{nul}_{\mathcal{D},\pi} = \{h \in \mathcal{O}_P \mid \tilde{\mathcal{T}}_P(\mathcal{D},\pi)|_h \ \geq\ \tilde{\mathcal{T}}^{nul}_P(\mathcal{D},\pi)\}.$$

$$\mathcal{M}^{\delta_i\delta_j}_{\mathcal{D},\pi} = \{h \in \mathcal{O}_P \mid \tilde{\mathcal{T}}_P(\mathcal{D},\pi)|_h \ \geq\ \tilde{\mathcal{T}}^{\delta_i\delta_j}_P(\mathcal{D},\pi)\}.$$

Consideremos la siguiente dimensión,

$$K^{\delta_i\delta_j}_P(\mathcal{D}) = K^{\delta_i\delta_j}_P(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\mathcal{M}^{\delta_i\delta_j}_{\mathcal{D},\pi}}.$$

El número $K^{\delta_i\delta_j}_P(f)$ nos da el menor número de condiciones que tenemos que imponerle a un germen genérico $h \in \mathcal{O}_P(s)$ para que $h \in \mathcal{M}^{\delta_i\delta_j}_{\mathcal{D},\pi}(s)$.

Definición 11. Definimos el *grado* de un árbol con pesos, $\mathcal{T}$, de la siguiente manera

$$\deg(\mathcal{T}) := \sum_{Q \in |\mathcal{T}|} \frac{w(\mathcal{T},Q)}{2dpq} \left(w(\mathcal{T},Q) + p + q - e \right),$$

donde $w(\mathcal{T}, Q)$ denota el peso de $\mathcal{T}$ en Q, el vértice Q recorre todos los puntos infinitamente próximos de una **Q**-resolución de V_f, Q aparece tras una explosión de tipo (p, q) del origen de $X(d; a, b)$ y $e := \gcd(d, aq - bp)$.

Obtenemos el siguiente resultado (Lema (III.4.3)) para curvas planas ponderadas en $\mathbb{P}^2_w$ mediante **Q**-resoluciones que generaliza el Lema 2.35 en [**CA02**] para curvas en $\mathbb{P}^2$ usando resoluciones clásicas.

Lema 6. *Tenemos que,*

$$\deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}(f)) = \deg(\tilde{\mathcal{T}}_P^{nul}(f)) - 1.$$

Nótese que en el caso de gérmenes en un punto P de $\mathbb{C}^2$ y explosiones clásicas, el grado de un árbol $\mathcal{T}$ está relacionado con el número de condiciones que hay que imponer a un germen g para que $\mathcal{T}|_g \geq \mathcal{T}$. En esta situación $K_P(f) = \deg \mathcal{T}_P^{nul}(f) = \delta_P(f)$ (véase [**CA02**]). En nuestro caso, $\deg \mathcal{T}_P^{nul}(f) = \delta_P^w(f)$, independiente de la **Q**-resolución, siendo este grado un número racional. Por tanto, $K_P(f) = \deg \mathcal{T}_P^{nul}(f)$ sólo se puede esperar cuando f sea una función en $X(d; a, b)$.

El Lema 6, junto con la Proposición 7 que veremos a continuación (ver Proposición (III.5.6)), serán útiles en el Capítulo V a la hora de probar el Teorema 13.

Proposición 7. *Sea $\{f = 0\}$ un germen analítico de una singularidad de curva en el punto P de $X(d; a, b)$. Denotemos por δ_1, δ_2, dos ramas locales cualesquiera de f en P, entonces*

$$K_P^{\delta_1\delta_2}(f) = K_P(f) - 1.$$

Tercera parte: invariantes globales

En §IV.1 damos una fórmula para calcular el género de curvas en el plano proyectivo ponderado por medio del invariante δ^w (Definición 7). Dado $d \in \mathbb{N}$ y una lista de pesos normalizados $w \in \mathbb{N}^3$, definimos el *género virtual* asociado a d y w como

$$g_{d,w} := \frac{d(d - |w|)}{2\bar{w}} + 1,$$

dando lugar al siguiente resultado (véase Teorema (IV.1.12)).

Teorema 8 ([**CAMO13**])**.** *Sea $\mathcal{C} \subset \mathbb{P}^2_w$ una curva irreducible de grado $d > 0$, entonces*

$$g(\mathcal{C}) = g_{d,w} - \sum_{P \in \operatorname{Sing}(\mathcal{C})} \delta_P^w.$$

En el capítulo también se muestran algunos ejemplos prácticos en los que se calcula el género de diferentes curvas en distintos $\mathbb{P}^2_w$.

Durante el resto del Capítulo IV centraremos nuestros esfuerzos en obtener una Fórmula de tipo Adjunción que relacione el género de una curva genérica de grado cuasihomogéneo d y la dimensión del espacio de polinomios de grado $d + \deg K$ (notar que $\deg K = -|w| = -(w_0 + w_1 + w_2)$), siendo K el divisor canónico de $\mathbb{P}^2_w$ (esta dimensión se denotará $D_{d-|w|,w}$).

Obtenemos los siguientes resultados (véanse Teorema (IV.4.3) y Corolario (IV.4.4)) que jugaran un papel importante en el Capítulo V.

Teorema 9. *Sean w_0, w_1, w_2 enteros primos dos a dos, $d \in \mathbb{N}$ y denotemos por $\bar{w} = w_0 w_1 w_2$, $|w| = w_0 + w_1 + w_2$ donde $w = (w_0, w_1, w_2)$. Consideremos los siguientes enteros positivos $p_i = w_i$, $q_i = -w_j^{-1} w_k \mod w_i \in \mathbb{N}$ con $j < k$ (notar que $X(w_i; w_j, w_k) = X(p_i; -1, q_i)$), $r_i = w_k^{-1} d \mod w_i \in \mathbb{N}$. Consideremos*

$$\begin{aligned} D_{d-|w|,w} &= \# \left\{ (x,y,z) \in \mathbb{N}^3 \mid w_0 x + w_1 y + w_2 z = d - |w| \right\}, \\ A_{r_i}^{(p_i,q_i)} &= \# \left\{ (x,y) \in \mathbb{N}^2 \mid p_i x + q_i y \leq q_i r_i,\ x, y \geq 1 \right\}, \end{aligned}$$

$$\delta_{r_i}^{(p_i,q_i)} = \frac{r_i(p_i r_i - p_i - q_i + 1)}{2 p_i}.$$

Entonces

$$D_{d-|w|,w} = g_{d,w} + \sum_{i=0}^{2} \left(\delta_{r_i}^{(p_i,q_i)} - A_{r_i}^{(p_i,q_i)} \right).$$

Sea $\mathcal{C} \subset \mathbb{P}^2_w$ una curva reducida de grado d, definimos el *número de condiciones globales para* $\mathcal{C}$ de la siguiente manera

$$K(\mathcal{C}) := \sum_{P \in \operatorname{Sing}(\mathcal{C})} K_P(f).$$

Corolario 10 (Fórmula de tipo Adjunción)**.** *Sea $\mathcal{C} \subset \mathbb{P}^2_w$ una curva reducida de grado d, entonces*

$$h^0(\mathbb{P}^2_w; \mathcal{O}(d - |w|)) = D_{d-|w|,w} = g_{d,w} - \sum_{P \in \operatorname{Sing}(\mathcal{C})} \delta_P^w + K(\mathcal{C}).$$

A partir de ahora, denotaremos por $X_{\mathcal{C}}$ al complementario de $\mathcal{C}$ en el plano proyectivo ponderado $\mathbb{P}^2_w$.

Con todos los ingredientes previamente vistos, nos centraremos, a lo largo del Capítulo V, en uno de los invariantes más importantes del par $(\mathbb{P}^2_w, \mathcal{R})$, el anillo de cohomolgía de $X_{\mathcal{R}}$, donde $\mathcal{R}$ es una curva algebraica plana reducida (con o sin puntos singulares) en el plano proyectivo complejo ponderado

$\mathbb{P}^2_w$ cuyas componentes irreducibles $\mathcal{R}_i$ son todas racionales ($g(\mathcal{R}_i)=0$). Tales curvas serán llamadas *configuraciones racionales*. El objetivo del capítulo será encontrar una presentación para el anillo de cohomología de $X_{\mathcal{R}}$.

Sea $\mathcal{D}$ un $\mathbb{Q}$-divisor reducido en $\mathbb{P}^2_w$. En §V.2, se proporciona una base para $H^1(\mathbb{P}^2_w \setminus \mathcal{D};\mathbb{C})$ y, en §V.4, damos una presentación holomorfa para $H^2(\mathbb{P}^2_w \setminus \mathcal{R};\mathbb{C})$.

Tomemos un sistema de coordenadas $[X:Y:Z]$ en $\mathbb{P}^2_w$. Si escribimos $\mathcal{D}:=\{D=0\}$, D se puede expresar como producto de $C_0\cdot C_1\cdot\ldots\cdot C_n$ donde $\mathcal{C}_i:=\{C_i=0\}$, siendo C_i las componentes irreducibles de D.

Consideremos las siguientes formas diferenciales

$$\sigma_{ij} := d\left(\log \frac{C_i^{d_j}}{C_j^{d_i}}\right) = d_j\ d(\log C_i) - d_i\ d(\log C_j).$$

donde $i,j=0,...,n$, $d_i:=\deg_w(\mathcal{C}_i)$.

Tomemos π una **Q**-resolución de $\mathcal{D}$ entonces, el pull-back $\pi^*\sigma_{ij}$ define una 1-forma logarítmica en $\overline{X}_{\mathcal{D}}$. Tenemos el siguiente resultado.

Teorema 11. *Las clases de cohomología de*

$$\mathcal{B}_1(\mathcal{D}) := \{\sigma_{ik}\}_{i=0}^n$$

$i\neq k$*, constituyen una base para* $H^1(X_{\mathcal{D}};\mathbb{C})$.

Es fácil comprobar que, en general, no podemos esperar el Teorema de Brieskorn, es decir, $\wedge^2 H^1(X_{\mathcal{R}};\mathbb{C})$ no genera $H^2(X_{\mathcal{R}};\mathbb{C})$.

En §V.3 presentamos algunos ejemplos del cálculo de la estructura de anillo de $H^2(X_{\mathcal{D}};\mathbb{C})$. Finalmente, en §V.4, damos una presentación holomorfa para $H^2(X_{\mathcal{R}};\mathbb{C})$, con $\mathcal{R}$ una configuración racional. Veamos en detalle este último resultado.

Sean $\mathcal{C}_i$, $\mathcal{C}_j$, $\mathcal{C}_k$ tres curvas en $\mathbb{P}^2_w$ (no necesariamente distintas). Denotaremos por $\mathcal{C}_{ijk}$ a la unión $\mathcal{C}_i\cup\mathcal{C}_j\cup\mathcal{C}_k$. Consideremos C_{ijk} una ecuación reducida para $\mathcal{C}_{ijk}$. También usaremos $d_{ijk}:=\deg_w \mathcal{C}_{ijk}$.

Por ejemplo, en el caso $i=j=k$, tendremos $\mathcal{C}_{ijk}=\mathcal{C}_i$, $C_{ijk}=C_i$ y $d_{ijk}=\deg_w(\mathcal{C}_i)$.

Usando los módulos descritos en términos de árboles logarítmicos en los Capítulos II y III podemos construir el siguiente haz $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$.

Definición 12. Sea $\mathcal{R}=\bigcup_i \mathcal{R}_i$ una configuración racional y π una **Q**-resolución de singularidades para $\mathcal{R}$. Para cada triple $(\mathcal{R}_i,\mathcal{R}_j,\mathcal{R}_k)$, no necesariamente $i\neq j\neq k$, tomemos tres puntos $P_1\in\operatorname{Sing}(\mathcal{R}_i\cap\mathcal{R}_j)$, $P_2\in$

$\operatorname{Sing}(\mathcal{R}_j \cap \mathcal{R}_k)$ y $P_3 \in \operatorname{Sing}(\mathcal{R}_i \cap \mathcal{R}_k)$. Para cada P_l elegimos dos ramas, $\delta_l^{i_l}$ de $\mathcal{R}_i$ y $\delta_l^{j_l}$ de $\mathcal{R}_j$. Consideremos

$$\Delta := \left[(P_1, \delta_1^{i_1}, \delta_1^{j_1}),\ (P_2, \delta_2^{j_2}, \delta_2^{k_2}),\ (P_3, \delta_3^{k_3}, \delta_3^{i_3})\right].$$

Vamos a construir un haz $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ asociado a Δ. Sea $Q \in \mathcal{R}_{ijk}$, tenemos el siguiente módulo

$$(\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi})_Q := \left\{ \begin{array}{ll} \mathcal{O}_Q & \text{if } Q \notin \operatorname{Sing}(\mathcal{R}_{ijk}) \\ (\mathcal{M}^{nul}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } P_l \neq Q \in \operatorname{Sing}(\mathcal{R}_{ijk}) \\ (\mathcal{M}^{\delta_l^i,\delta_l^j}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } Q = P_l \text{ with } \delta_l^i \neq \delta_l^j \\ (\mathcal{M}^{nul}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } Q = P_l \text{ with } \delta_l^i = \delta_l^j \end{array} \right\}.$$

Este módulo nos lleva a la definición del haz $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ que llamaremos *haz de formas Δ-logarítmicas sobre $\mathcal{R}_{ijk}$ con respecto a π.*

Este haz $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ no depende de la elección de la resolución π. Si no hay ambigüedad, dado $\mathcal{R}$, simplemente escribiremos $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}$.

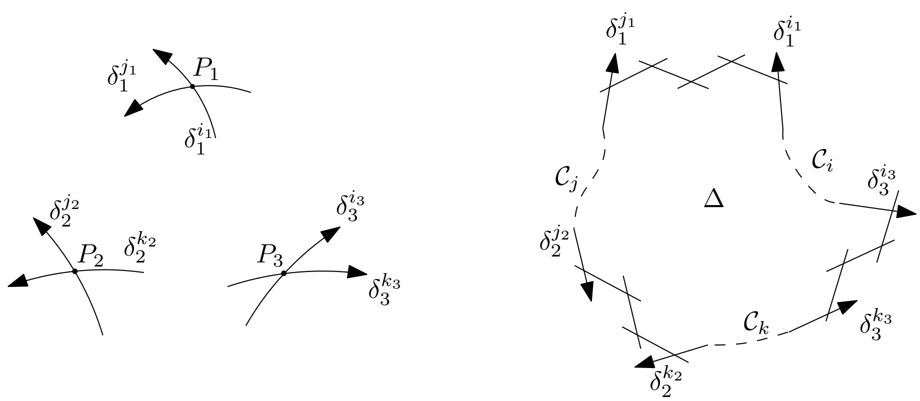


FIGURA 2. Δ en $H^1(\bar{\mathcal{R}}_{ijk};\mathbb{C})$.

Con la definición previa, usando la Fórmula de tipo Adjunción (Corolario 10), el Lema 6 y la Proposición 7 obtenemos los siguientes resultados.

Proposición 12. *Sea $\mathcal{R}$ una configuración racional en $\mathbb{P}^2_w$ como en la Definición 12, entonces*

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk} - |w|)) > 0.$$

Teorema 13. *Sea $\mathcal{R} = \bigcup_i \mathcal{R}_i$ una configuración racional en $\mathbb{P}^2_w$ y π una* **Q***-resolución de singularidades para $\mathcal{R}$. Sea H un polinomio de grado cuasi-homogéneo $d_{ijk} - |w|$, tal que*

$$H \in H^0(\mathbb{P}^2_w, \mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk} - |w|)).$$

Las 2-formas $\omega = H\dfrac{\Omega^2}{R_{ijk}}$ *forman una presentación holomorfa para* $H^2(\mathbb{P}^2_w \setminus \mathcal{R}, \mathbb{C})$.

La demostración del Teorema 13 proporciona un método para encontrar las relaciones entre los generadores de $H^2(\mathbb{P}^2_w \setminus \mathcal{R}; \mathbb{C})$ por medio de las relaciones en $H^1(\bar{\mathcal{R}}^{[1]}; \mathbb{C})$ y la inyectividad del operador residuo (Definiciones 4 y 5).

La mayor parte de los resultados vistos a lo largo de los Capítulos I a V están ilustrados en el caso particular de $\mathcal{D} = V(xyz(xyz + (x^3 - y^2)^2)) \subset \mathbb{P}^2_w$ con $w = (2, 3, 7)$. En el Capítulo I estudiamos una **Q**-resolución de sus singularidades (Ejemplo (I.2.8)). En el Capítulo III construimos diferentes árboles logarítmicos asociados a $\mathcal{D}$ (Ejemplos (III.3.2) y (III.3.5)). Los conceptos locales estudiados en los Capítulos I y II nos darán las herramientas necesarias para calcular el género de $\mathcal{D}$ en el Capítulo IV (Ejemplo (IV.1.18)). Véase también (IV.4.5) para un ejemplo ilustrativo de la Fórmula de tipo Adjunción. Finalmente, todos estos resultados nos permitirán, en el Capítulo V, estudiar el anillo de cohomología $H^\bullet(\mathbb{P}^2_w \setminus \mathcal{D}; \mathbb{C})$ en §V.3–2 y el Ejemplo (V.4.9).

RÉSUMÉ (French)

Le but principal de cette thèse de doctorat est l'étude de l'anneau de cohomologie de $\mathbb{P}^2_w \setminus \mathcal{R}$, $\mathcal{R}$ étant une courbe algébrique réduite dans le plan projectif pondéré complexe $\mathbb{P}^2_w$, dont les composantes irréductibles sont des courbes rationnelles (avec ou sans points singuliers). En particulier, des représentants holomorphes (rationnels) sont obtenus pour les classes de cohomologie. Pour atteindre notre objectif, il est nécessaire de développer une théorie algébrique des courbes sur des surfaces avec des singularités quotient et d'étudier des techniques pour calculer certains invariants particulièrement utiles à travers des **Q**-résolutions plongées.

L'étude des méthodes de calcul de différents types d'invariants à partir d'une résolution plongée est un problème classique en Théorie des Singularités ([**Hir64**]). La motivation principale pour utiliser des **Q**-résolutions plongées, au lieu des résolutions classiques, provient du fait qu'elles offrent essentiellement la même information, avec une structure combinatoire plus simple et un procédé de calcul moins coûteux. La plupart des invariants des singularités de surface (dans une variété de dimension 3) s'obtiennent à partir des résolutions plongées.

Pour obtenir la résolution plongée d'une surface il faut éclater des points ou éclater le long de courbes lisses. Quand on éclate un point, le diviseur exceptionnel est un plan projectif et son intersection avec la transformée stricte de la surface est une courbe algébrique. Lorsque l'on utilise des **Q**-résolutions plongées, les éclatements classiques sont remplacés par des éclatements pondérés de sorte que des plans projectifs pondérés et des courbes projectives pondérées apparaissent de façon naturelle. L'étude et la compréhension des invariants algébriques et topologiques des courbes avec des singularités quotient dans $\mathbb{P}^2_w$ permettent une approche alternative pour l'étude des singularités de surface, qui est d'habitude plus simple du point de vue combinatoire.

Pour étudier les invariants à partir de ce point de vue, on a dû généraliser des objets classiques : δ-invariant, fibre de Milnor, formule du genre, formule de Noether, forme logarithmique ayant des pôles le long de $\mathbb{Q}$-diviseurs à croisements non-normaux, Formule de type Adjonction dans $\mathbb{P}^2_w$. Cette dernière formule donne une relation très intéressante entre un invariant topologique, le genre d'une courbe générique (pas nécessairement lisse) de degré quasi-homogène d, et la dimension de l'espace des polynômes de degré $d - \deg K$ (K étant le diviseur canonique dans $\mathbb{P}^2_w$).

Pour des raisons techniques, le résultat principal de cette thèse est présenté pour les courbes rationnelles dans $\mathbb{P}^2_w$. Déjà dans le cas classique, s'il y a des courbes de genre positif, les classes holomorphes sont insuffisantes pour engendrer l'anneau de cohomologie (il est nécessaire de travailler avec des formes anti-holomorphes), d'après le travail de Cogolludo-Agustín et Matei [**CAM12**].

Les premières études de l'algèbre de cohomologie du complémentaire des arrangements des hyperplans proviennent des travaux d'Arnol'd ([**Arn69**]), Brieskorn ([**Bri73**]) et Orlik-Solomon ([**OS80**]). Dans ces travaux, les auteurs montrent que l'algèbre de cohomologie est combinatoire et que chaque classe de cohomologie a un représentant holomorphe (rationnel).

Théorème (Lemme de Brieskorn)**.** *Soit $\mathcal{L}$ un arrangement de droites avec des composantes ℓ_0, ℓ_1 , ..., ℓ_n, dont ℓ_i est le lieu de zéros d'une forme linéaire l_i. Alors, les 1-formes logarithmiques*

$$\omega_i = \frac{1}{2\pi\sqrt{-1}}\ d\ \log\left(\frac{l_i}{l_0}\right) \quad i = 1, ..., n$$

engendrent l'anneau de cohomologie $H^\bullet(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{C})$. De plus, $H^\bullet(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{C})$ est isomorphe à la sous-algèbre engendrée par ω_i dans l'algèbre des formes méromorphes.

Dans [**Dim92**] le problème se pose dans le cas des compléments de courbes dans $\mathbb{P}^2$. Dans cette situation, Lubicz trouve des générateurs de l'anneau de cohomologie dans [**Lub00**]. La structure complète de l'anneau est décrite pour les arrangements rationnels dans [**CA02**] et pour le cas général dans [**CAM12**]. Dans ce dernier article, les auteurs donnent une description qui dépend aussi de la structure combinatoire des courbes appelée *combinatoire faible* ([**CB10**]). Les formes impliquées dans l'ensemble des résultats ci-dessus sont de la forme

$$H\frac{\Omega^2}{C_iC_jC_k},$$

où $\mathcal{C} = \bigcup_i \mathcal{C}_i$ est une courbe dans $\mathbb{P}^2$ avec $\mathcal{C}_i = \{C_i = 0\}$, H un polynôme et $\Omega^2 := zdx \wedge dy + xdy \wedge dz + ydz \wedge dx$ la forme volume.

Pour étudier l'anneau de cohomologie du complémentaire d'une courbe dans $\mathbb{P}^2_w$ il est nécessaire de généraliser les invariants locaux des courbes au cas où l'espace ambiant a des singularités quotient ([**CAMO13**]). D'autres généralisations peuvent être trouvées dans [**ABFdBLMH10**], où les auteurs étudient la fibre de Milnor et le nombre de Milnor pour germes dans des surfaces avec des singularités quotient, ou dans [**BLSS02, uT77, STV05**]. En dimension 2, les singularités cycliques coïncident avec les toriques. C'est pourquoi l'étude locale des singularités dans cette dimension peut être faite avec des techniques de géométrie torique. Cependant, notre objectif ultime est l'étude des courbes projectives pondérées, qui ne sont généralement pas des variétés toriques. Par conséquent, l'utilisation des éclatements pondérés semble aussi approprié pour traiter ces objets (voir par exemple [**AMO11a, AMO11b, AMO12, Dol82, Ort09, Mar11**]).

Dans cet travail, on utilise principalement trois techniques : théorie locale des singularités dans des V-surfaces, théorie globale des formes logarithmiques et théorie des réseaux avec des sommes de Dedekind. Du point de vue local, on a étudié la théorie d'intersection de différents invariants locaux et des $\mathbf{Q}$-résolutions plongées. Dans cette thèse, on donne une définition alternative des formes logarithmiques, appelées *formes logarithmiques de log-résolution* (voir [**CAM12**]), qui sont indépendantes de la $\mathbf{Q}$-résolution choisie. En général, le faisceau de ces formes est plus petit que celui des formes logarithmiques sur des diviseurs avec croisements non-normaux ([**Sai80**]). On montre une description des faisceaux logarithmiques en termes de valorisations d'arbres de $\mathbf{Q}$-résolution. Pour passer la théorie locale avec la théorie globale il faut la Formule de type Adjonction. Cette formule, dont la preuve a besoin des techniques de théorie des réseaux et des sommes de Dedekind ([**RG72, BR07**]), fait le lien entre la géométrie et la combinatoire. Des techniques similaires peuvent être trouvées dans [**Pom93, Lat95**].

Comme application, les résultats obtenus dans cette thèse s'appliquent à l'étude des variétés de résonance et de la formalité. L'anneau de cohomologie fournit un moyen naturel de construire des variétés de résonance (voir, par exemple [**CA02**]). Les travaux de [**Bri73, OS80**] et [**CAM12**] permettent de montrer que les complémentaires d'arrangements d'hyperplans (ou des courbes) sont des espaces formels. Ce travail pourrait être utilisé pour étudier la formalité des complémentaires de courbes dans des plans projectifs pondérés.

La suite de cette introduction est consacrée au résumé des principaux résultats divisé en trois parties.

La première (Chapitre I) fonctionne comme une *introduction*, les concepts de base et les outils nécessaires sont présentés pour que les principaux résultats puissent être développées. Dans la seconde (Chapitres II et III), on étudie les *invariants locaux* de courbes dans des espaces avec des singularités quotient. Dans la dernière partie (Chapitres IV et V) tous les résultats ci-dessus sont utilisés pour obtenir des *invariants globaux*. Dans le Chapitre IV on donne une formule pour le genre et une Formule de type Adjonction pour des courbes dans des V-surfaces. Finalement, dans le Chapitre V, le contenu des Chapitres I à IV sert à l'étude de l'un des invariants les plus importants de la paire $(\mathbb{P}^2_w, \mathcal{R})$, c'est-à-dire, l'anneau de cohomologie de $\mathbb{P}^2_w \setminus \mathcal{R}$, où $\mathcal{R}$ est une courbe algébrique réduite (avec ou sans points singuliers) dans le plan projectif pondéré complexe $\mathbb{P}^2_w$, dont les composantes irréductibles sont toutes rationnelles.

Première partie : outils de base

Dans le Chapitre I on commence à donner quelques définitions et propriétés de base des V-variétés, des espaces projectifs pondérés, des $\mathbf{Q}$-résolutions plongées et des éclatements pondérés (pour une discussion détaillée, voir, par exemple, [**AMO11a, AMO11b, AMO12, Dol82, Mar11**] et [**Ort09**]). L'objectif de la première partie de ce chapitre est de fixer les notations et de mettre en place plusieurs outils pour calculer une variante de résolutions plongées, les $\mathbf{Q}$-*résolutions plongées* (voir la Définition (I.2.2)), pour lesquelles l'espace ambiant peut contenir des singularités quotient de type abélien. Pour cela, on étudié des éclatements pondérés des points. On se concentre sur le cas de V-surfaces.

Définition 1. Une V-variété de dimension n est un espace analytique complexe qui admet un recouvrement ouvert $\{U_i\}$ tel que chaque ouvert U_i est analytiquement isomorphe à B_i/G_i où $B_i \subset \mathbb{C}^n$ est une boule ouverte et G_i est un sous-groupe fini de $\mathrm{GL}(n, \mathbb{C})$.

Les V-variétés ont été introduites dans [**Sat56**] et ont les mêmes propriétés homologiques sur $\mathbb{Q}$ que les variétés. Par exemple, elles admettent une dualité de Poincaré si elles sont compactes ; dans le cas compact-Kähler elles possèdent une structure de Hodge pure (voir [**Bai56**]). La classification locale est due à Prill ([**Pri67**]).

On est intéressé par des V-variétés où les espaces quotients B_i/G_i sont donnés par des groupes abéliens finis.

Soit $\boldsymbol{G}_d$ le groupe cyclique des d-ièmes racines de l'unité. Considérons un vecteur de poids $(a,b) \in \mathbb{Z}^2$ et l'action

$$\begin{array}{ccc} \boldsymbol{G}_d \times \mathbb{C}^2 & \stackrel{\rho}{\longrightarrow} & \mathbb{C}^2, \\ (\xi_d, (x,y)) & \mapsto & (\xi_d^a\, x, \xi_d^b\, y). \end{array}$$

L'ensemble des orbites $\mathbb{C}^2/\boldsymbol{G}_d$ s'appelle *espace (cyclique) quotient du type* $(d;a,b)$ et il est désigné par $X(d;a,b)$.

On dit que l'espace $X(d;a,b)$ est écrit sous forme standard (Définition (I.1.9)) si et seulement si $\gcd(d,a) = \gcd(d,b) = 1$. Si ce n'est pas le cas, on utilise l'isomorphisme suivant (supposant $\gcd(d,a,b) = 1$)

$$\begin{array}{ccc} X(d;a,b) & \longrightarrow & X\left(\frac{d}{(d,a)(d,b)}; \frac{a}{(d,a)}, \frac{b}{(d,b)}\right), \\ \big[(x,y)\big] & \mapsto & \big[(x^{(d,b)}, y^{(d,a)})\big] \end{array}$$

pour normaliser la singularité.

Un des premiers exemples de V-variétés est le *plan projectif pondéré*(§I.4). Soit $w := (w_0, w_1, w_2) \in \mathbb{N}^3$ un vecteur de poids, c'est-à-dire , une liste de trois entiers positifs deux à deux premiers entre eux. Il y a une action naturelle du groupe multiplicatif $\mathbb{C}^*$ sur $\mathbb{C}^3 \setminus \{0\}$ donnée par

$$(x_0, x_1, x_2) \longmapsto (t^{w_0} x_0, t^{w_1} x_1, t^{w_2} x_2).$$

L'ensemble des orbites $\frac{\mathbb{C}^3\setminus\{0\}}{\mathbb{C}^*}$ sous cette action est notée $\mathbb{P}^2_w$ et s'appelle *plan projectif pondéré* de type w.

On rappelle la définition de l'un des objets les plus importants de ce travail.

Définition 2. Une **Q**-*résolution plongée* de $(H,0) \subset (M,0)$ est une application analytique propre $\pi : X \to (M,0)$ de telle sorte que :

1. X est une V-variété abélienne avec des singularités quotient.
2. π est un isomorphisme sur $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
3. $\pi^{-1}(H)$ est une hypersurface avec **Q**-croisements normaux en X (voir Définition (I.2.1)).
4. La transformée stricte $\hat{H} := \overline{\pi^{-1}(H \setminus \{0\})}$ est **Q**-lisse (voir Définition (I.2.1)).

Les **Q**-résolutions plongées sont une généralisation naturelle des résolutions standards. L'utilisation des **Q**-résolutions permet le calcul efficace de certains invariants étudiés dans les Chapitres II à V.

Dans la Section I.3 on développe une théorie d'intersection avec le but d'étudier les **Q**-résolutions plongées en dimension 2 (voir [**AMO11b**] et [**Mar11**] pour plus de détails). On doit faire face à deux types de diviseurs dans les V-variétés : les diviseurs de Weil et de Cartier. Les diviseurs de Weil sont des combinaisons linéaires, avec coefficients entiers et localement finies, de sous-variétés irréductibles de codimension 1 ; les diviseurs de Cartier sont des sections globales du faisceau quotient de fonctions méromorphes modulo les fonctions holomorphes dans $\mathbb{C}^*$. La correspondance entre les diviseurs de Cartier et le fibré en droites offre un moyen utile pour définir la multiplicité d'intersection de deux diviseurs. Dans la catégorie lisse, les deux notions de diviseurs coïncident, mais ce n'est pas le cas pour les variétés singulières. Le Théorème (I.3.3) ([**AMO11a**]) permet de développer une théorie d'intersection dans les V-variétés (voir [**AMO11b**]).

Définition 3 (Multiplicité d'intersection locale en $X(d;a,b)$, [**Ort09**])**.** On note X l'espace quotient cyclique $X(d;a,b)$ et on considère deux diviseurs $D_1 = \{f_1 = 0\}$ et $D_2 = \{f_2 = 0\}$ donnés par $f_1, f_2 \in \mathbb{C}\{x,y\}$ réduits sans composants communs. On suppose $(d;a,b)$ normalisé.

Alors, en tant que diviseurs de Cartier on exprime $D_1 = \frac{1}{d}\{(X, f_1^d)\}$ et $D_2 = \frac{1}{d}\{(X, f_2^d)\}$. La multiplicité d'intersection $(D_1 \cdot D_2)_{[P]}$ dans le point P de type $(d;a,b)$ est définie comme

$$(D_1 \cdot D_2)_{[P]} = \frac{1}{d^2} \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\langle f_1^d, f_2^d \rangle}$$

où $\mathbb{C}\{x,y\}^{\boldsymbol{G}_d}$ ($\equiv \mathcal{O}_P$) est l'anneau local des fonctions en P (voir §I.1–2).

Cette théorie d'intersection locale permet, par exemple, de calculer le Théorème de Bézout Pondéré pour les plans projectifs pondérés (Proposition (I.4.7)), qui sera particulièrement intéressant dans certains des résultats futurs.

Proposition 1 ([**Ort09**])**.** *La multiplicité d'intersection de deux $\mathbb{Q}$-diviseurs sans composante commune, D_1 et D_2 dans $\mathbb{P}^2_w$, est*

$$D_1 \cdot D_2 = \sum_{P \in D_1 \cap D_2} (D_1 \cdot D_2)_{[P]} = \frac{1}{\bar{w}} \deg_w(D_1) \deg_w(D_2) \in \mathbb{Q},$$

où $\bar{w} = w_0 w_1 w_2$ et $\deg_w(D_i) = \deg(\phi^*(D_i))$ (voir (7)).

Dans §I.5 on étudie la cohomologie de de Rham pour les variétés projectives avec singularités quotient. On va rappeler certains résultats théoriques de Théorie de Hodge dans des V-variétés projectives qui nous intéressent. Tous ces résultats, avec leurs preuves, peuvent être trouvés dans le premier chapitre de [**Ste77**]. Finalement, on va donner une définition des formes logarithmiques et de résidus sur $\mathbb{Q}$-diviseurs à croisements non-normaux dans des V-surfaces.

Soit $\mathcal{D}$ un $\mathbb{Q}$-diviseur dans $\mathbb{P}^2_w$. Le complémentaire de $\mathcal{D}$ est noté $X_{\mathcal{D}}$. On fixe $\pi : \overline{X}_{\mathcal{D}} \longrightarrow \mathbb{P}^2_w$ une **Q**-résolution des singularités de $\mathcal{D}$ de telle sorte que le $\mathbb{Q}$-diviseur réduit $\overline{\mathcal{D}} = (\pi^*(\mathcal{D}))_{red}$ est une réunion de $\mathbb{Q}$-diviseurs lisses dans $\overline{X}_{\mathcal{D}}$ à **Q**-croisements normaux.

Définition 4. Une forme C^∞ φ dans $X_{\mathcal{D}}$ est dite *logarithmique (le long du diviseur $\mathcal{D}$ par rapport à la* **Q***-résolution* π*)* si $\pi^*\varphi$ est logarithmique en $\overline{X}_{\mathcal{D}}$ par rapport au diviseur à **Q**-croisements normaux $\overline{\mathcal{D}}$ (voir Définition (I.5.6)). Alors, on a le faisceau correspondant

$$\pi_*\Omega_{\overline{X}_{\mathcal{D}}}(\log\langle\overline{\mathcal{D}}\rangle).$$

Une fois que $\mathcal{D}$ et π sont fixés, on peut définir l'*application résidu* $\mathrm{Res}^{[*]}_{\pi}(\varphi)$ *d'une forme logarithmique* φ comme suit

$$\begin{array}{ccc} \pi_*\Omega^k_{\overline{X}_{\mathcal{D}}}(\log\langle\overline{\mathcal{D}}\rangle) & \overset{\mathrm{Res}^{[k]}_{\pi}}{\longrightarrow} & H^0(\overline{\mathcal{D}}^{[k]};\mathbb{C}) \\ \varphi & \mapsto & \mathrm{Res}^{[k]}(\pi^*\varphi). \end{array}$$

La définition ci-dessus est indépendante de la **Q**-résolution. Par exemple, dans le cas particulier de $X(d;a,b)$, $\mathrm{Res}^{[2]}$ s'exprime comme suit.

Définition 5. Soit h un germe analytique dans $X(d;a,b)$ écrit sous forme standard (Définition (I.1.9)). Soit $\varphi = h\dfrac{dx\wedge dy}{xy}$ une 2-forme logarithmique avec des pôles à l'origine. Alors

$$\mathrm{Res}^{[2]}(\varphi) := \frac{1}{d}h(0,0).$$

Deuxième partie : invariants locaux

Dans le Chapitre II on étend le concept de fibre de Milnor et de nombre de Milnor d'une singularité de courbe dans un espace ambiant ayant des singularités quotient (§II.1). On définit une généralisation du δ-invariant et on donne une description de celui-ci en termes d'une **Q**-résolution des singularités de la courbe (§II.3). En particulier, quand on applique ce qu'on a vu dans le cas classique (l'espace ambiant est une surface lisse), on obtient une formule pour l'invariant δ classique en fonction d'une **Q**-résolution, ce qui simplifie considérablement les calculs.

Enfin, tous ces outils nous permettent de donner, dans le Chapitre IV,une description explicite de la formule du genre d'une courbe définie dans un plan projectif pondéré en fonction de son degré et du type de singularités locales.

Définition 6 ([**CAMO13**])**.** Soit $\mathcal{C} = \{f = 0\} \subset X(d; a, b)$ un germe de courbe. La *fibre de Milnor* F_t^w de $(\mathcal{C}, [0])$ est définie comme

$$F_t^w := \{F = t\}/\boldsymbol{G}_d.$$

Le *nombre de Milnor* μ^w de $(\mathcal{C}, P)$ est défini comme

$$\mu^w := 1 - \chi^{\text{orb}}(F_t^w).$$

Notez que des généralisations alternatives pour le nombre de Milnor peuvent être trouvées, par exemple, dans [**ABFdBLMH10, BLSS02, uT77, STV05**]. La généralisation proposée ici semble plus naturelle dans le cas des singularités quotient (voir l'Exemple (IV.1.18)) ; en particulier, elle permet de généraliser la formule qui rélie le nombre de Milnor, le δ-invariant et le genre d'une courbe sur une surface singulière (Chapitre IV).

Dans §II.3 on présente une version de la formule de Noether pour des courbes dans des espaces avec des singularités quotient à l'aide des **Q**-résolutions (voir Théorème (II.2.1)).

Théorème 2 (Formule de Noether, [**CAMO13**])**.** *On considère deux germes* C, D *de* $\mathbb{Q}$*-diviseurs en* $[0]$*, sans composantes communes, dans un surface avec des singularités quotient et une suite d'éclatements pondérés qui séparent les branches de* C *et* D*. Ces germes satisfont la formule suivante :*

$$(C \cdot D)_{[0]} = \sum_{Q \prec [0]} \frac{\nu_{C,Q}\nu_{D,Q}}{pqd},$$

où Q *parcourt les points infiniment voisins de* $[0]$ *par la suite d'éclatements pondérés.*

Ensuite, on définit l'invariant local δ^w pour les singularités de courbes dans $X(d; a, b)$.

Définition 7 ([**CAMO13**])**.** Soit C un germe réduit en $[0] \in X(d; a, b)$, on définit δ^w comme le nombre déterminé par l'égalité suivante :

$$\chi^{\text{orb}}(F_t^w) = r^w - 2\delta^w,$$

où r^w est le nombre de branches locales C en $[0]$, F_t^w désigne la fibre de Milnor, et $\chi^{\text{orb}}(F_t^w)$ représente la caractéristique d'Euler de l'orbifold F_t^w.

Cette définition suit [**Mil68**, Theorem 10.5]. Dans le Théorème (II.2.5) on énonce une formule récursive pour δ^w pour les **Q**-résolutions de la singularité de surface qui généralise la formule classique.

Théorème 3 ([**CAMO13**])**.** *Soit* $(C, [0])$ *un germe de courbe dans une surface avec des singularités quotient abéliennes. Alors*

$$\delta^w = \frac{1}{2} \sum_{Q \prec [0]} \frac{\nu_Q}{dpq} \left(\nu_Q - p - q + e\right),$$

où Q qui apparaît lors d'un éclatement pondéré (p,q) d'un point de type $X(d;a,b)$ (normalisé), parcourt les points infiniment proches d'une $\mathbf{Q}$-résolution du $(C,[0])$ et $e := \gcd(d, aq-bp)$.

Dans §II.3–1 on donne une interprétation de l'invariant δ^w comme la dimension d'un espace vectoriel. Dans le cas classique cet invariant est aussi défini comme la codimension de l'anneau du germe dans sa normalisation. δ^w est en général un nombre rationnel ; ce résultat est encore vrai dans certains cas, par exemple, lorsqu'il est associé à des diviseurs de Cartier (voir Théorème (II.3.7)).

Théorème 4 ([**CAMO13**]). *Soit $f : (X(d;a,b),P) \to (\mathbb{C},P)$ un germe réduit de fonction analytique. On suppose que $(d;a,b)$ est de type normalisé. On considère $R = \frac{\mathcal{O}_P}{\langle f\rangle}$ l'anneau local associé à f et $\overline{R}$ la normalisation de R. Alors,*

$$\delta^w_P(f) = \dim_{\mathbb{C}} \left(\frac{\overline{R}}{R} \right) \in \mathbb{N}.$$

Dans §II.3–2 on présente une généralisation de ce résultat. Pour ce faire, on a besoin de quelques définitions.

On fixe $k \geq 0$, et l'on considère le module $\mathcal{O}_P(k)$ (pour plus de détails, voir §I.1–2),

$$\mathcal{O}_P(k) := \{h \in \mathbb{C}\{x,y\} |\ h(\xi_d^a x, \xi_d^b y) = \xi_d^k h(x,y)\}.$$

Soit $\{f = 0\}$ un germe dans $P \in X(d;a,b)$. Si $f \in \mathcal{O}_P(k)$, alors le $\mathcal{O}_P$-module $\mathcal{O}_P(k-a-b)$ vérifie :

$$\mathcal{O}_P(k-a-b) = \left\{ h \in \mathbb{C}\{x,y\} \;\middle|\; h\frac{dx \wedge dy}{f} \text{ est } \boldsymbol{G}_d\text{-invariant} \right\}.$$

Définition 8. Soit $\mathcal{D} = \{f = 0\}$ un germe dans $P \in X(d;a,b)$ avec $f \in \mathcal{O}_P(k)$. On considère π une $\mathbf{Q}$-resolution de $(\mathcal{D},P)$.

(1) Soit $\mathcal{M}^{log}_{\mathcal{D},\pi}$ le sous-module de $\mathcal{O}_P$ des $h \in \mathcal{O}_P$ tels que la 2-forme

$$\omega = h\ \frac{dx \wedge dy}{f} \in \Omega^2_P(a+b-k)$$

est logarithmique en P, par rapport à $\mathcal{D}$ et π (Définition 4).

(2) Soit $\mathcal{M}^{nul}_{\mathcal{D},\pi}$ le sous-module de $\mathcal{M}^{log}_{\mathcal{D},\pi}$ des $h \in \mathcal{M}^{log}_{\mathcal{D},\pi}$ tels que la 2-forme

$$\omega = h\ \frac{dx \wedge dy}{f}$$

admet une extension holomorphe en dehors de la transformée stricte $\widehat{f}$.

Ce dermier module joue un rôle important dans la construction de la présentation de l'anneau de cohomologie $\mathbb{P}^2_w \setminus \mathcal{R}$ du Chapitre V.

Définition 9. Soit $\mathcal{D} = \{f = 0\}$ un germe dans $P \in X(d; a, b)$. On pose :

$$K_P(\mathcal{D}) = K_P(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\mathcal{M}^{nul}_{\mathcal{D},\pi}}.$$

Le nombre $K_P(f)$ nous donne le nombre minimum de conditions qu'on doit demander à un germe générique $h \in \mathcal{O}_P(s)$ pour que $h \in \mathcal{M}^{nul}_{\mathcal{D},\pi}(s)$.

Dans le cas particulier où $f \in \mathcal{O}_P$, c'est à dire, quand $(f, [0])$ est un germe de fonction dans $X(d; a, b)$, on a (voir Corollaire (II.3.14))

$$K_P(f) = \delta_P(f).$$

Théorème 5. *Soient $f, g \in \mathcal{O}(k)$, $k \in \mathbb{N}$, deux germes en $P \in X(d; a, b)$. Alors,*

$$K_P(f) - K_P(g) = \delta^w_P(f) - \delta^w_P(g).$$

Dans le Chapitre III on définit d'autres modules et faisceaux logarithmiques associés à un $\mathbb{Q}$-diviseur $\mathcal{D}$ et à une **Q**-résolution π. Leurs sections globales nous permettent, dans le Chapitre V, de construire des 2-formes logarithmiques sur $\mathcal{D}$.

On va construire deux types d'arbres associés à un germe analytique $\{f = 0\}$ en $P \in X(d; a, b)$, $\tilde{\mathcal{T}}^{nul}_P(f)$ (voir §III.1) et $\tilde{\mathcal{T}}^{\delta_1\delta_2}_P(f)$ (voir §III.2) où δ_1 et δ_2 sont deux branches locales f au point P. Ces arbres nous permettent de donner une description utile des modules logarithmiques définis précédemment.

Définition 10. Soit $\mathcal{D} = \{f = 0\}$ un germe dans $P \in X(d; a, b)$ avec $f \in \mathcal{O}_P(k)$ et soit π une **Q**-resolution de $(\mathcal{D}, P)$. On définit $\mathcal{M}^{\delta_i\delta_j}_{\mathcal{D},\pi}$ comme le sous-module de $\mathcal{M}^{log}_{\mathcal{D},\pi}$ des $h \in \mathcal{M}^{log}_{\mathcal{D},\pi}$ tels que la 2-forme

$$\omega = h\,\frac{dx \wedge dy}{f}$$

a des résidus nuls le long du chemin $\gamma(\delta_1, \delta_2)$.

À la suite de la construction des arbres $\tilde{\mathcal{T}}^{nul}_P$ (§III.1) et $\tilde{\mathcal{T}}^{\delta_1,\delta_2}_P$ (§III.2) et des Définitions 8 et 10, on a les caractérisations suivantes :

$$\mathcal{M}^{nul}_{\mathcal{D},\pi} = \{h \in \mathcal{O}_P \mid \tilde{\mathcal{T}}_P(\mathcal{D},\pi)|_h \;\geq\; \tilde{\mathcal{T}}^{nul}_P(\mathcal{D},\pi)\}.$$

$$\mathcal{M}^{\delta_i\delta_j}_{\mathcal{D},\pi} = \{h \in \mathcal{O}_P \mid \tilde{\mathcal{T}}_P(\mathcal{D},\pi)|_h \;\geq\; \tilde{\mathcal{T}}^{\delta_i\delta_j}_P(\mathcal{D},\pi)\}.$$

On considère la dimension suivante,

$$K^{\delta_i\delta_j}_P(\mathcal{D}) = K^{\delta_i\delta_j}_P(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\mathcal{M}^{\delta_i\delta_j}_{\mathcal{D},\pi}}.$$

Le nombre $K^{\delta_i\delta_j}_P(f)$ donne le plus petit nombre de conditions à imposer un germe générique $h \in \mathcal{O}_P(s)$ pour que $h \in \mathcal{M}^{\delta_i\delta_j}_{\mathcal{D},\pi}(s)$.

Définition 11. On définit le *degré* d'un arbre $\mathcal{T}$ avec des poids comme suit :

$$\deg(\mathcal{T}) := \sum_{Q\in|\mathcal{T}|} \frac{w(\mathcal{T},Q)}{2dpq}\left(w(\mathcal{T},Q)+p+q-e\right),$$

où $w(\mathcal{T},Q)$ désigne le poids de $\mathcal{T}$ en Q, le sommet Q passe par chaque point infiniment proche d'une **Q**-résolution de V_f, Q apparaît lors d'un éclatement de type (p,q) de $(X(d;a,b),[0])$ et $e := \gcd(d, aq-bp)$.

On obtient le résultat suivant (Lemme (III.4.3)) pour des courbes dans $\mathbb{P}^2_w$ en utilisant des **Q**-résolutions ; ce résultat généralise le Lemme 2.35 dans [**CA02**] pour des courbes dans $\mathbb{P}^2$ et des résolutions classiques.

Lemme 6. $\deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}_P(f)) = \deg(\tilde{\mathcal{T}}^{nul}_P(f)) - 1$.

On remarque que dans le cas des germes ($P \in \mathbb{C}^2$ et des éclatements classiques), le degré d'un arbre $\mathcal{T}$ est lié au nombre de conditions que doit satisfaire un germe g pour que $\mathcal{T}|_g \geq \mathcal{T}$. Dans cette situation, $K_P(f) = \deg \mathcal{T}^{nul}_P(f) = \delta_P(f)$ (voir [**CA02**]). Dans notre cas, le nombre $\deg \mathcal{T}^{nul}_P(f) = \delta^w_P(f) \in \mathbb{Q}$ est indépendant de la **Q**-résolution. Ainsi, l'egalité $K_P(f) = \deg \mathcal{T}^{nul}_P(f)$ est vraie seulement si f est une fonction à $X(d;a,b)$.

Le Lemme 6, avec la Proposition 7 (voir Proposition (III.5.6)), seront utiles dans le Chapitre V pour démontrer le Théorème 13.

Proposition 7. *Soit* $\{f=0\}$ *un germe analytique d'une singularité de courbe en* P *de* $X(d;a,b)$*. On dénote* δ_1,δ_2*, deux branches locales de* f *en* P*. Alors,*

$$K^{\delta_1\delta_2}_P(f) = K_P(f) - 1.$$

Troisième partie : invariants globaux

Dans §IV.1 on donne une formule pour le genre de courbes dans le plan projectif pondéré qui utilise l'invariant δ^w (Définition 7). Soit $d \in \mathbb{N}$ et soit $w \in \mathbb{N}^3$ une liste de poids normalisés, on définie le *genre virtuel* associé à d et w comme

$$g_{d,w} := \frac{d(d-|w|)}{2\bar{w}} + 1.$$

Il donne lieu au résultat suivant (voir Théorème (IV.1.12)).

Théorème 8 ([**CAMO13**])**.** *Soit* $\mathcal{C} \subset \mathbb{P}^2_w$ *une courbe irréductible de degré* $d > 0$*, alors*

$$g(\mathcal{C}) = g_{d,w} - \sum_{P\in \mathrm{Sing}(\mathcal{C})} \delta^w_P.$$

Dans ce chapitre on montre également quelques exemples pratiques où les genres de différentes courbes dans $\mathbb{P}^2_w$ sont calculés.

Pour le reste du Chapitre IV on concentre nos efforts dans l'obtention d'une Formule de type Adjonction concernant le genre d'une courbe générique de degré quasi-homogène d et la dimension de l'espace des polynômes de degré $d + \deg K$ (noter que $\deg K = -|w| = -(w_0 + w_1 + w_2)$), où K le diviseur canonique de $\mathbb{P}^2_w$ (cette dimension sera notée $D_{d-|w|,w}$).

Les résultats suivants (voir Théorème (IV.4.3) et Corollaire (IV.4.4)) vont jouer un rôle clé dans le Chapitre V.

Théorème 9. *Soient w_0, w_1, w_2 des entiers deux à deux premiers entre eux, soit $d \in \mathbb{N}$ et on dénote $\bar{w} = w_0w_1w_2$, $|w| = w_0 + w_1 + w_2$ où $w = (w_0, w_1, w_2)$. On considère les entiers positifs suivants $p_i = w_i$, $q_i = -w_j^{-1}w_k \mod w_i \in \mathbb{N}$, $j < k$ (on note que $X(w_i; w_j, w_k) = X(p_i; -1, q_i)$), $r_i = w_k^{-1}d \mod w_i \in \mathbb{N}$). On considère*

$$\begin{array}{rcl} D_{d-|w|,w} & = & \# \left\{ (x,y,z) \in \mathbb{N}^3 \mid w_0x + w_1y + w_2z = d - |w| \right\}, \\ A_{r_i}^{(p_i,q_i)} & = & \# \left\{ (x,y) \in \mathbb{N}^2 \mid p_ix + q_iy \leq q_ir_i,\ x, y \geq 1 \right\}, \end{array}$$

$$\delta_{r_i}^{(p_i,q_i)} = \frac{r_i(p_ir_i - p_i - q_i + 1)}{2p_i}.$$

Alors

$$D_{d-|w|,w} = g_{d,w} + \sum_{i=0}^{2} \left(\delta_{r_i}^{(p_i,q_i)} - A_{r_i}^{(p_i,q_i)} \right).$$

Soit $\mathcal{C} \subset \mathbb{P}^2_w$ une courbe réduite de degré d, on définit le *nombre de conditions globales pour* $\mathcal{C}$ comme suit :

$$K(\mathcal{C}) := \sum_{P \in \operatorname{Sing}(\mathcal{C})} K_P(f).$$

Corollaire 10 (Formule de type Adjonction)**.** *Soit $\mathcal{C} \subset \mathbb{P}^2_w$ une courbe réduite de degré d, alors*

$$h^0(\mathbb{P}^2_w; \mathcal{O}(d - |w|)) = D_{d-|w|,w} = g_{d,w} - \sum_{P \in \operatorname{Sing}(\mathcal{C})} \delta_P^w + K(\mathcal{C}).$$

Dès maintenant, on note $X_{\mathcal{C}}$ le complémentaire de $\mathcal{C}$ dans le plan projectif pondéré $\mathbb{P}^2_w$.

Les invariants calculés précédemment vont nous servir, dans le Chapitre V, pour calculer l'un des invariants les plus importants de la paire $(\mathbb{P}^2_w, \mathcal{R})$, l'anneau de cohomologie $X_{\mathcal{R}}$, où $\mathcal{R}$ est une courbe algébrique plane réduite (avec ou sans point singulier) dans le plan projectif pondéré $\mathbb{P}^2_w$, dont les composantes irréductibles $\mathcal{R}_i$ sont toutes rationnelles ($g(\mathcal{R}_i) = 0$). Ces courbes sont appelées des *arrangements rationnels*. Le but du chapitre est de trouver une présentation de l'anneau de cohomologie de $X_{\mathcal{R}}$.

Soi $\mathcal{D}$ un $\mathbb{Q}$-diviseur réduit dans $\mathbb{P}^2_w$. Dans §V.2, on fournit une base pour $H^1(\mathbb{P}^2_w \setminus \mathcal{D}; \mathbb{C})$ et, dans §V.4, on donne une présentation *holomorphe* de $H^2(\mathbb{P}^2_w \setminus \mathcal{R}; \mathbb{C})$.

On prend un système de coordonnées $[X : Y : Z]$ en $\mathbb{P}^2_w$. Si l'on écrit $\mathcal{D} := \{D = 0\}$, la fonction D peut s'exprimer comme le produit de $C_0 \cdot C_1 \cdot \ldots \cdot C_n$, avec $\mathcal{C}_i := \{C_i = 0\}$, où $\mathcal{C}_i$ sont les composantes irréductibles de $\mathcal{D}$.

On considère les formes différentielles suivantes

$$\sigma_{ij} := d\left(\log \frac{C_i^{d_j}}{C_j^{d_i}}\right) = d_j \; d(\log C_i) - d_i \; d(\log C_j).$$

avec $i, j = 0, ..., n$, $d_i := \deg_w(\mathcal{C}_i)$.

On prend π une **Q**-résolution de $\mathcal{D}$ alors, le pull-back $\pi^* \sigma_{ij}$ définit une 1-forme logarithmique dans $\overline{X}_{\mathcal{D}}$. On a le résultat suivant.

Théorème 11. *Les classes de cohomologie de*

$$\mathcal{B}_1(\mathcal{D}) := \{\sigma_{ik}\}_{i=0}^n, \quad i \neq k,$$

fournissent une base pour $H^1(X_{\mathcal{D}}; \mathbb{C})$.

Il est facile de voir que, en général, on ne peut pas espérer récuperer le Théorème de Brieskorn, c'est à dire, $\bigwedge^2 H^1(X_{\mathcal{R}}; \mathbb{C})$ n'engendre pas $H^2(X_{\mathcal{R}}; \mathbb{C})$.

Dans §V.3 on présente quelques exemples de calcul de la structure d'anneau de $H^2(X_{\mathcal{D}}; \mathbb{C})$. Finalement, dans §V.4, on donne une présentation *holomorphe* de $H^2(X_{\mathcal{R}}; \mathbb{C})$, où $\mathcal{R}$ est un arrangement rationnel. Voyons en détail ce dernier résultat.

Soient $\mathcal{C}_i$, $\mathcal{C}_j$, $\mathcal{C}_k$ trois courbes dans $\mathbb{P}^2_w$ (pas nécessairement differentes). On note $\mathcal{C}_{ijk}$ l'union $\mathcal{C}_i \cup \mathcal{C}_j \cup \mathcal{C}_k$ et on considère C_{ijk} une équation réduite pour $\mathcal{C}_{ijk}$. On utilise également $d_{ijk} := \deg_w \mathcal{C}_{ijk}$.

Par exemple, dans le cas $i = j = k$, on a $\mathcal{C}_{ijk} = \mathcal{C}_i$, $C_{ijk} = C_i$ et $d_{ijk} = \deg_w(\mathcal{C}_i)$.

En utilisant les modules décrits en termes d'arbres logarithmiques dans les Chapitres II et III on peut construire le faisceau $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$.

Définition 12. Soit $\mathcal{R} = \bigcup_i \mathcal{R}_i$ un arrangement rationnel et π une **Q**-résolution des singularités pour $\mathcal{R}$. Pour chaque triplet $(\mathcal{R}_i, \mathcal{R}_j, \mathcal{R}_k)$ (les indices ne sont pas nécessairement distincts), on prend trois points $P_1 \in \operatorname{Sing}(\mathcal{R}_i \cap \mathcal{R}_j)$, $P_2 \in \operatorname{Sing}(\mathcal{R}_j \cap \mathcal{R}_k)$ et $P_3 \in \operatorname{Sing}(\mathcal{R}_i \cap \mathcal{R}_k)$. Pour chaque P_l

on choisi deux branches, $\delta_l^{i_l}$ de $\mathcal{R}_i$ et $\delta_l^{j_l}$ de $\mathcal{R}_j$. On considère

$$\Delta := \left[(P_1, \delta_1^{i_1}, \delta_1^{j_1}),\ (P_2, \delta_2^{j_2}, \delta_2^{k_2}),\ (P_3, \delta_3^{k_3}, \delta_3^{i_3})\right].$$

On va construire un faisceau $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ associé à Δ. Soit $Q \in \mathcal{R}_{ijk}$; on considère le module suivant

$$(\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi})_Q := \left\{ \begin{array}{ll} \mathcal{O}_Q & \text{if } Q \notin \operatorname{Sing}(\mathcal{R}_{ijk}) \\ (\mathcal{M}^{nul}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } P_l \neq Q \in \operatorname{Sing}(\mathcal{R}_{ijk}) \\ (\mathcal{M}^{\delta_l^i,\delta_l^j}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } Q = P_l \text{ with } \delta_l^i \neq \delta_l^j \\ (\mathcal{M}^{nul}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } Q = P_l \text{ with } \delta_l^i = \delta_l^j \end{array} \right\}.$$

Ce module conduit à la définition du faisceau $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ qu'on appelle le *faisceau des formes Δ-logarithmiques sur $\mathcal{R}_{ijk}$ par rapport à π.*

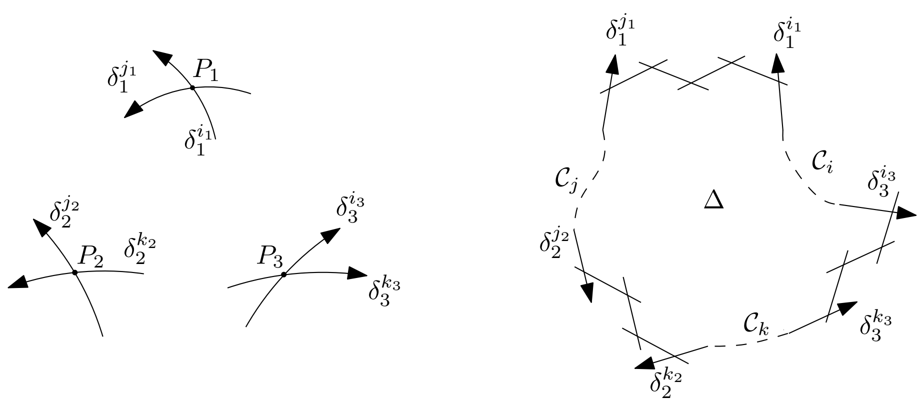


FIGURE 3. Δ en $H^1(\bar{\mathcal{R}}_{ijk}; \mathbb{C})$.

Avec la définition précédente, en utilisant la Formule de type Adjonction (Corollaire 10), le Lemme 6 et la Proposition 7 on a les résultats suivants.

Proposition 12. *Soit $\mathcal{R}$ un arrangement rationnel dans $\mathbb{P}^2_w$ comme dans le Définition 12. Alors,*

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk} - |w|)) > 0.$$

Théorème 13. *Soit $\mathcal{R} = \bigcup_i \mathcal{R}_i$ un arrangement rationnel dans $\mathbb{P}^2_w$ et π une $\mathbf{Q}$-résolution des singularités pour $\mathcal{R}$. Soit H un polynôme de degré quasi-homogène $d_{ijk} - |w|$, tel que*

$$H \in H^0(\mathbb{P}^2_w, \mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk} - |w|)).$$

Les 2-formes $\omega = H\dfrac{\Omega^2}{R_{ijk}}$ forment une présentation holomorphe *pour l'espace $H^2(\mathbb{P}^2_w \setminus \mathcal{R}, \mathbb{C})$.*

La démonstration du Théorème 13 fournit une méthode pour trouver la lien entre les générateurs $H^2(\mathbb{P}^2_w \setminus \mathcal{R}; \mathbb{C})$ grâce aux relations dans $H^1(\bar{\mathcal{R}}^{[1]}; \mathbb{C})$ et à l'injectivité de l'opérateur résidu (Définitions 4 et 5).

La plupart des résultats vus dans les Chapitres I à V sont illustrés dans le cas particulier de $\mathcal{D} = V(xyz(xyz+(x^3-y^2)^2)) \subset \mathbb{P}^2_w$ avec $w = (2,3,7)$. Dans le Chapitre I on étude une **Q**-résolution des singularités (Exemple (I.2.8)). Dans le Chapitre III on construit différents arbres logarithmiques associés à $\mathcal{D}$ (Exemples (III.3.2) et (III.3.5)). Les concepts locaux étudiés dans les Chapitres I et II nous donnent les outils nécessaires pour calculer le genre de $\mathcal{D}$ dans le Chapitre IV (Exemple (IV.1.18)). Voir aussi (IV.4.5) pour un exemple illustratif de la Formule de type Adjonction. Finalement, ces résultats nous permettent, dans le Chapitre V, d'étudier l'anneau de cohomologie $H^\bullet(\mathbb{P}^2_w \setminus \mathcal{D}; \mathbb{C})$ dans §V.3–2 et l'Exemple (V.4.9).

I

V- manifolds: Quotient Singularities, Embedded $\mathbb{Q}$-Resolutions, Intersection Numbers and Logarithmic Complex

Let us start with some basic definitions and properties of V-manifolds, weighted projective spaces, embedded $\mathbf{Q}$-resolutions, weighted blow-ups, intersection theory on V-surfaces and logarithmic forms. See for example [**AMO11a, AMO11b, AMO12, Dol82, Mar11, Ort09, Ste77**] for a more detailed exposition of the previous concepts.

One of our purposes in this chapter is to fix the notation and provide several tools to calculate *embedded* $\mathbf{Q}$*-resolutions* (see Definition (I.2.2)). To do this, weighted blow-ups will be studied. We will center our attention in the case of V-surfaces and weighted blow-ups at points. All these techniques will be frequently used along the successive chapters.

One of the main examples of V-manifolds are the weighted projective spaces (§I.4) and closely related with them we have the weighted blow-ups. As opposed to standard ones, these blow-ups do not produce smooth varieties, but the result may only have abelian quotient singularities. They can be used to obtain the mentioned $\mathbf{Q}$-resolutions of singularities, where the usual conditions are weakened: we allow the total space to have abelian quotient singularities and the condition of normal crossing divisors is replaced by $\mathbf{Q}$-normal crossing divisors. One of the main interests of $\mathbf{Q}$-resolutions of singularities is the following: their combinatorial complexity is extremely lower than the complexity of smooth resolutions, but they provide essentially the same information for the properties of the singularity.

In Section I.3, we will develop an intersection theory in the context of surfaces with abelian quotient singularities, see [**AMO11b**] for a detailed exposition. This theory was first introduced by Mumford over normal surfaces, see [**Mum61**]. The tools presented here will allow us to compute the self-intersection numbers of the exceptional divisors of weighted blow-ups in dimension two, see Proposition (I.3.7).

In the last section we extend the notions of C^∞ log complex of quasi-projective algebraic varieties to the case of V-manifolds. A De Rham cohomology for projective varieties with quotient singularities is developed. We shall recall some results in the Hodge Theory of projective V-manifolds. We will also focus on logarithmic forms on non-normal crossing $\mathbb{Q}$-divisors (called here *log-resolution logarithmic forms*) and residues on $\mathbb{P}^2_w$. These results will be of particular interest in Chapter V.

Section § I.1
V-manifolds and Quotient Singularities

Definition (I.1.1). A V-manifold of dimension n is a complex analytic space which admits an open covering $\{U_i\}$ such that U_i is analytically isomorphic to B_i/G_i where $B_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$.

V-manifolds were introduced in [**Sat56**] and have the same homological properties over $\mathbb{Q}$ as manifolds. For instance, they admit a Poincaré duality if they are compact and carry a pure Hodge structure if they are compact and Kähler (see [**Bai56**]). They have been classified locally by Prill ([**Pri67**]). To state this local result we need the following.

Definition (I.1.2). A finite subgroup G of $\mathrm{GL}(n, \mathbb{C})$ is called *small* if no element of G has 1 as an eigenvalue of multiplicity precisely $n-1$, that is, G does not contain rotations around hyperplanes other than the identity.

(I.1.3). For every finite subgroup G of $\mathrm{GL}(n, \mathbb{C})$ denote by G_{big} the normal subgroup of G generated by all rotations around hyperplanes. Then, the G_{big}-invariant polynomials form a polynomial algebra and hence $\mathbb{C}^n/G_{\text{big}}$ is isomorphic to $\mathbb{C}^n$.

The group G/G_{big} maps isomorphically to a small subgroup of $\mathrm{GL}(n, \mathbb{C})$, once a basis of invariant polynomials has been chosen. Hence the local classification of V-manifolds reduces to the classification of actions of small subgroups of $\mathrm{GL}(n, \mathbb{C})$.

Theorem (I.1.4) ([**Pri67**]). *Let G_1 and G_2 be small subgroups of $\mathrm{GL}(n, \mathbb{C})$. Then $\mathbb{C}^n/G_1$ is isomorphic to $\mathbb{C}^n/G_2$ if and only if G_1 and G_2 are conjugate subgroups.*

I.1–1. The abelian case: normalized types

We are interested in V-manifolds where the quotient spaces B_i/G_i are given by (finite) abelian groups. In this case the following notation is used.

(I.1.5). Let $\boldsymbol{G}_{\mathbf{d}} := \boldsymbol{G}_{d_1} \times \cdots \times \boldsymbol{G}_{d_r}$ be an arbitrary finite abelian group written as a product of finite cyclic groups, that is, $\boldsymbol{G}_{d_i}$ is the cyclic group of d_i-th roots of unity. Consider a matrix of weight vectors

$$A := (a_{ij})_{i,j} = [\ a_1 \mid \cdots \mid\ a_n] \in \mathrm{Mat}(r \times n, \mathbb{Z})$$

and the action $(\boldsymbol{G}_{d_1} \times \cdots \times \boldsymbol{G}_{d_r}) \times \mathbb{C}^n \xrightarrow{\rho} \mathbb{C}^n$, defined as

$$(1) \qquad (\boldsymbol{\xi}_{\mathbf{d}}, \mathbf{x}) \mapsto (\xi_{d_1}^{a_{11}} \cdot \ldots \cdot \xi_{d_r}^{a_{r1}}\, x_1, \ldots, \xi_{d_1}^{a_{1n}} \cdot \ldots \cdot \xi_{d_r}^{a_{rn}}\, x_n)$$

with $\boldsymbol{\xi}_{\mathbf{d}} = (\xi_{d_1}, \ldots, \xi_{d_r})$ and $\mathbf{x} = (x_1, \ldots, x_n)$.

Note that the i-th row of the matrix A can be considered modulo d_i. The set of all orbits $\mathbb{C}^n/\boldsymbol{G}_{\mathbf{d}}$ is called (*cyclic*) *quotient space of type* $(\mathbf{d}; A)$ and it is denoted by

$$X(\mathbf{d}; A) := X \left(\begin{array}{c|ccc} d_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ d_r & a_{r1} & \cdots & a_{rn} \end{array} \right).$$

The orbit of an element $\mathbf{x}$ under this action is denoted by $[(x_1, \ldots, x_n)]_{(\mathbf{d};A)}$ and the subindex is omitted if no ambiguity seems likely to arise. Sometimes it is convenient to use multi-index notation

$$\mathbf{d} = (d_1, \ldots, d_r), \qquad a_j = (a_{1j}, \ldots, a_{rj}),$$

$$\boldsymbol{\xi}_{\mathbf{d}} = (\xi_{d_1}, \ldots, \xi_{d_r}), \qquad x = (x_1, \ldots, x_n), \qquad \boldsymbol{G}_{\mathbf{d}} = \boldsymbol{G}_{d_1} \times \cdots \times \boldsymbol{G}_{d_r},$$

so that the action (1) takes the simple form

$$\begin{array}{rcl} \boldsymbol{G}_{\mathbf{d}} \times \mathbb{C}^n & \xrightarrow{\rho} & \mathbb{C}^n, \\ (\boldsymbol{\xi}_{\mathbf{d}}, \mathbf{x}) & \mapsto & \boldsymbol{\xi}_{\mathbf{d}} \cdot \mathbf{x} := (\boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_1}\, x_1, \ldots, \boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_n}\, x_n). \end{array}$$

Example (I.1.6) (Dimension 2: $X(d; a, b)$)**.** Let $\boldsymbol{G}_d$ be the cyclic group of d-th roots of unity. Consider a vector of weights $(a, b) \in \mathbb{Z}^2$ and the action seen before (1)

$$\begin{array}{rcl} \boldsymbol{G}_d \times \mathbb{C}^2 & \xrightarrow{\rho} & \mathbb{C}^2, \\ (\xi_d, (x, y)) & \mapsto & (\xi_d^a\, x, \xi_d^b\, y). \end{array}$$

The set of all orbits $\mathbb{C}^2/\boldsymbol{G}_d$ is called a (*cyclic*) *quotient space of type* $(d; a, b)$ and it is denoted by $X(d; a, b)$.

The following result shows that the family of varieties which can locally be written as $X(\mathbf{d}; A)$ is exactly the same as the family of V-manifolds with abelian quotient singularities.

Lemma (I.1.7) ([**AMO11b, Mar11**])**.** *Let G be a finite abelian subgroup of* $\mathrm{GL}(n,\mathbb{C})$*. Then,* $\mathbb{C}^n/G$ *is isomorphic to some quotient space of type* $(\mathbf{d};A)$*.*

(I.1.8). The action shown in (1) is free on $(\mathbb{C}^*)^n$, that is,

$$\left[\boldsymbol{\xi}_{\mathbf{d}}\cdot x = x,\ \forall x\in(\mathbb{C}^*)^n\right] \Longrightarrow \boldsymbol{\xi}_{\mathbf{d}} = 1,$$

if and only if the group homomorphism $\boldsymbol{G}_{\mathbf{d}} \to \mathrm{GL}(n,\mathbb{C})$ given by

$$\boldsymbol{\xi}_{\mathbf{d}} = (\xi_{d_1},\ldots,\xi_{d_r}) \longmapsto \begin{pmatrix} \boldsymbol{\xi}_{\mathbf{d}}^{a_1} & & \\ & \ddots & \\ & & \boldsymbol{\xi}_{\mathbf{d}}^{a_r} \end{pmatrix} \tag{2}$$

is injective. Otherwise, let H be the kernel of this group homomorphism. Then $\mathbb{C}^n/H \equiv \mathbb{C}^n$ and the group $\boldsymbol{G}_{\mathbf{d}}/H$ acts freely on $(\mathbb{C}^*)^n$ under the previous identification.

Thus one can always assume that the free (as well as the small) condition is satisfied. This motivates the following definition.

Definition (I.1.9). The type $(\mathbf{d};A)$ is said to be *normalized* if the following two conditions hold.

(1) The action is free on $(\mathbb{C}^*)^n$.
(2) The group $\boldsymbol{G}_{\mathbf{d}}$ is identified with a small subgroup of $\mathrm{GL}(n,\mathbb{C})$ under the group homomorphism given in (2).

By abuse of language we often say the space $X(\mathbf{d};A)$ is written in a normalized form when we actually mean the type $(\mathbf{d};A)$ is normalized.

Proposition (I.1.10) ([**AMO11b, Mar11**])**.** *The space $X(\mathbf{d};A)$ is written in a normalized form if and only if the stabilizer subgroup of P is trivial for all $P\in\mathbb{C}^n$ with exactly $n-1$ coordinates different from zero.*

In the cyclic case the stabilizer of a point as above (with exactly $n-1$ coordinates different from zero) has order $\gcd(d,a_1,\ldots,\widehat{a}_i,\ldots,a_n)$*.*

The procedures described in (I.1.8) and (I.1.3) can be used to convert general types $(\mathbf{d};A)$ into their normalized form. Theorem (I.1.4) allows one to decide whether two quotient spaces are isomorphic. In particular, one can use this result to check if the space $X(\mathbf{d};A)$ is or not singular.

Example (I.1.11) (Dimension 1, [**AMO11b, Mar11**])**.** When $n=1$ all spaces $X(\mathbf{d};A)$ are isomorphic to $\mathbb{C}$. Note that $X((d_1,\ldots,d_r);(a_{11},\ldots,a_{r1})^t)$ is the same space as $X((d'_1,\ldots,d'_r);(a'_{11},\ldots,a'_{r1})^t)$ where $d'_i=\frac{d_i}{\gcd(d_i,a_{i1})}$ and $a'_{i1}=\frac{a_{i1}}{\gcd(d_i,a_{i1})}$. Therefore we can assume that $\gcd(d_i,a_{i1})=1$.

The map $x\mapsto x^{d_1}$ gives an isomorphism between $X(d_1;a_{11})$ and $\mathbb{C}$. For $r=2$ one has that (we write the symbol "=" when the isomorphism is induced by the identity map)

$$\begin{array}{ccccccc}\dfrac{\mathbb{C}}{\boldsymbol{G}_{d_1}\times\boldsymbol{G}_{d_2}} = \dfrac{\mathbb{C}/\boldsymbol{G}_{d_1}}{\boldsymbol{G}_{d_2}} & \stackrel{\cong}{\longrightarrow} & \mathbb{C}/\boldsymbol{G}_{d_2} \stackrel{(*)}{=} X(d_2; a_{21}d_1) & \stackrel{\cong}{\longrightarrow} & \mathbb{C}, \\ x & \mapsto & x^{d_1}, \qquad\qquad x & \mapsto & x^{\frac{d_2}{\gcd(d_1,d_2)}}.\end{array}$$

To see the equality $(*)$ observe that

$$\xi_{d_2}\cdot x^{d_1} \equiv \xi_{d_2}\cdot x = (\xi_{d_2}^{a_{21}}x) \equiv \xi_{d_2}^{a_{21}d_1}x^{d_1}.$$

It follows that the corresponding quotient space is isomorphic to $\mathbb{C}$ under the map $x \mapsto x^{\operatorname{lcm}(d_1,d_2)}$.

Example (I.1.12) (Dimension 2,[**AMO11b, Mar11**])**.** All quotient singularities of surfaces are cyclic. Recall that in dimension 2, any abelian quotient singularity can be written as $X(\mathbf{d}; A)$ where $A \in \operatorname{Mat}(r\times 2)$. If $r=1$ the space $X(d;a,b)$ is written in a normalized form if and only if $\gcd(d,a)=\gcd(d,b)=1$. Otherwise, one uses the isomorphism (assuming $\gcd(d,a,b)=1$)

$$\begin{array}{ccc} X(d;a,b) & \longrightarrow & X\left(\frac{d}{(d,a)(d,b)}; \frac{a}{(d,a)}, \frac{b}{(d,b)}\right), \\ \big[(x,y)\big] & \mapsto & \big[(x^{(d,b)}, y^{(d,a)})\big] \end{array}$$

to convert it into a normalized one.

If $r>1$, then $X(d;a,b)$ is written in a normalized form if and only if $\gcd(d_1,\ldots,d_r)=1$, in which case it can reduced to $r=1$.

Example (I.1.13) (Cyclic case, [**AMO11b, Mar11**])**.** In the cyclic case the order of the stabilizer subgroup is specially easy to compute and hence the normalized form can be described explicitly. In fact, $X(d;a_1,\ldots,a_n)$ is written in a normalized form if and only if $\gcd(d,a_1,\ldots,\widehat{a}_i,\ldots,a_n)=1$, $\forall i=1,\ldots,n$. Here we summarize how to convert types $(d;a_1,\ldots,a_n)$ into their normalized form.

(1) $X(d;a_1,\ldots,a_n) \simeq X(d;\, a_{\sigma(1)},\ldots,a_{\sigma(n)})$, $\forall\sigma\in\Sigma_n$.
(2) $X(d;0,a_2,\ldots,a_n) = \mathbb{C}\times X(d;\, a_2,\ldots,a_n)$.
(3) $X(d;a_1,\ldots,a_n) = X(\frac{d}{k};\, \frac{a_1}{k},\ldots,\frac{a_n}{k})$ if k divides d and all a_i's.
(4) $X(d;a_1,\ldots,a_n) = X(d;ka_1,\ldots,ka_n)$ if $\gcd(d,k)=1$.
(5) $X(d;a_1,\ldots,a_n) \simeq X(\frac{d}{k};a_1,\frac{a_2}{k},\ldots,\frac{a_n}{k})$, the isomorphism is given by $[(x_1,x_2,\ldots,x_n)] \mapsto [(x_1^k,x_2,\ldots,x_n)]$.

I.1–2. Orbisheaves

The content of this section can be found in detail in [**BG08**, §4] and [**Dol82**]. Here we will focus on the case of dimension 2. We present different properties of some important sheaves associated to a V-surface.

Proposition (I.1.14) ([**BG08**]). *Let $\mathcal{O}_X$ be the structure sheaf of a V-surface X then,*

- *If P is not a singular point of X then $\mathcal{O}_P$ is isomorphic to the ring of convergent power series $\mathbb{C}\{x, y\}$.*
- *If P is a singular point of X then $\mathcal{O}_P$ is isomorphic to the ring of $\boldsymbol{G}_d$ invariant convergent power series $\mathbb{C}\{x, y\}^{\boldsymbol{G}_d}$ ($\mathcal{O}_P$ is not an UFD).*
- *For any $P \in X$ the local ring $\mathcal{O}_P$ is integrally closed.*

Definition (I.1.15). Let $\boldsymbol{G}_d$ be an arbitrary finite cyclic group, a vector of weights $(a, b) \in \mathbb{Z}^2$ and the action seen in Example (I.1.6). Associated with $X(d; a, b)$ one has the following $\mathcal{O}_{X,P}$-modules:

$$\mathcal{O}_{X,P}(k) := \{h \in \mathbb{C}\{x, y\} |\ h(\xi_d^a x, \xi_d^b y) = \xi_d^k h(x, y)\}.$$

Remark **(I.1.16).** Note that

$$\mathbb{C}\{x, y\} = \bigoplus_{k=0}^{d-1} \mathcal{O}_{X,P}(k) \tag{3}$$

Remark **(I.1.17)** ([**BG08**]). Let $l, k \in \mathbb{N}$. Using the notation above one clearly has the following properties:

- $\mathcal{O}_{X,P}(k) = \mathcal{O}_P(d + k)$,
- $\mathcal{O}_{X,P}(l) \otimes \mathcal{O}_{X,P}(k) = \mathcal{O}_{X,P}(l + k)$.

These modules produce the corresponding sheaves $\mathcal{O}_X(k)$ on a V-surface X also called *orbisheaves*.

Section § I.2
Weighted Blow-ups and Embedded Q-Resolutions

Classically an embedded resolution of $\{f = 0\} \subset \mathbb{C}^n$ is a proper map $\pi : X \to (\mathbb{C}^n, 0)$ from a smooth variety X satisfying, among other conditions, that $\pi^{-1}(\{f = 0\})$ is a normal crossing divisor. In order to generalize this concept in the context of V-manifolds one needs to extend the concept of *normal crossing*. This notion of normal crossing divisor on V-manifolds was first introduced by Steenbrink in [**Ste77**].

We recall that in the class of V-manifolds, the abelian groups of Cartier and Weil divisors are not isomorphic. However the isomorphism can be achieved after tensoring by $\mathbb{Q}$. Such divisors will be referred to as $\mathbb{Q}$-*divisors* (Section I.3).

Definition (I.2.1). Let X be a V-manifold with abelian quotient singularities. A hypersurface D on X is said to be $\mathbf{Q}$-*normal crossing* if it is locally isomorphic to the quotient of a normal crossing divisor under a group action of type $(\mathbf{d}; A)$.

That is, for any $x \in X$, there is an isomorphism of germs $(X, x) \simeq (X(\mathbf{d}; A), [0])$ such that $(D, x) \subset (X, x)$ is identified under this morphism with a germ of the form

$$\big(\{[x] \in X(\mathbf{d}; A) \mid x_1^{m_1} \cdots x_k^{m_k} = 0\}, [0]\big). \tag{4}$$

Whenever (D, x) is $\mathbf{Q}$-normal crossing with $k = 1$ in (4) we say x is a $\mathbf{Q}$-*smooth point* of D. A $\mathbf{Q}$-divisor with only $\mathbf{Q}$-smooth points will be referred to as a $\mathbf{Q}$-*smooth divisor*.

Let $M = \mathbb{C}^{n+1}/\boldsymbol{G}$ be an abelian quotient space not necessarily cyclic or written in normalized form. Consider $H \subset M$ an analytic subvariety of codimension one.

Definition (I.2.2). An *embedded* $\mathbf{Q}$-*resolution* of $(H, 0) \subset (M, 0)$ is a proper analytic map $\pi : X \to (M, 0)$ such that:

(1) X is a V-manifold with abelian quotient singularities,
(2) π is an isomorphism over $X \setminus \pi^{-1}(\mathrm{Sing}(H))$,
(3) $\pi^{-1}(H)$ is a $\mathbf{Q}$-normal crossing hypersurface on X, and
(4) the strict transform $\hat{H} := \overline{\pi^{-1}(H \setminus \{0\})}$ is $\mathbf{Q}$-smooth.

Remark **(I.2.3)** ([**AMO11b, Mar11**]). Let $f : (M, 0) \to (\mathbb{C}, 0)$ be a non-constant analytic function germ. Consider $(H, 0)$ the hypersurface defined by f. Let $\pi : X \to (M, 0)$ be an embedded $\mathbf{Q}$-resolution of $(H, 0) \subset (M, 0)$. Then $\pi^{-1}(H) = (f \circ \pi)^{-1}(0)$ is locally given by a function of the form $x_1^{m_1} \cdots x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C}$.

Remark **(I.2.4)** ([**CAMO13**]). In some cases, one needs to consider a stronger condition on $\mathbf{Q}$-resolutions, namely, the strict transform of H does not contain any singular points of X. This can always be achieved by blowing up eventually once more the strict preimage of the hypersurface. Such an embedded resolution will be referred to as a *strong* $\mathbf{Q}$-*resolution*.

In what follows we will use weighted blow-ups of points as a tool for finding embedded $\mathbf{Q}$-resolutions.

Let X be an analytic surface with abelian quotient singularities. Consider $\pi : \widehat{X} \to X$ the weighted blow-up at a point $P \in X$ with respect to $w = (p, q)$, which will be assumed to be coprime. We distinguish three cases.

(i) The point P is smooth. In this case $X = \mathbb{C}^2$ and $\pi = \pi_w : \widehat{\mathbb{C}}^2_w \to \mathbb{C}^2$ is the weighted blow-up at the origin with respect to $w = (p, q)$. The new ambient space is covered as

$$\widehat{\mathbb{C}}^2_w = U_1 \cup U_2 = X(p; -1, q) \cup X(q;\, p, -1)$$

and the charts are given by

$$\text{First chart} \left| \begin{array}{rcl} X(p; -1, q) & \longrightarrow & U_1, \\ [(x, y)] & \mapsto & ((x^p, x^q y), [1 : y]_w). \end{array} \right.$$

$$\text{Second chart} \left| \begin{array}{rcl} X(q;\, p, -1) & \longrightarrow & U_2, \\ [(x, y)] & \mapsto & ((x y^p, y^q), [x : 1]_w). \end{array} \right.$$

The exceptional divisor $E = \pi_w^{-1}(0)$ is isomorphic to $\mathbb{P}^1_w$ which is in turn isomorphic to $\mathbb{P}^1$ under the map

$$[x : y]_w \longmapsto [x^q : y^p].$$

The singular points of $\widehat{\mathbb{C}}^2_w$ are cyclic quotient singularities located at the exceptional divisor. They actually coincide with the origins of the two charts and they are written in a normalized form.

Example (I.2.5) (**[AMO11b, Mar11]**)**.** Let $f : \mathbb{C}^2 \to \mathbb{C}$ be the function given by $f = x^p + y^q$ with $\gcd(p, q) = 1$. Consider $\pi_{(q,p)} : \widehat{\mathbb{C}}^2_{(q,p)} \to \mathbb{C}^2$ the (q, p)-weighted blow-up at the origin. In U_1 the total transform is given by the function

$$x^{pq}(1 + y^q) \,:\, X(q; -1, p) \longrightarrow \mathbb{C}.$$

The equation $y^q = -1$ has just one solution in U_1 and the local equation of the total transform at this point is of the form $x^{pq} y = 0$.

Hence the proper map $\pi_{(q,p)}$ is an embedded **Q**-resolution of the plane curve $C = \{f = 0\} \subset \mathbb{C}^2$ where all spaces are written in a normalized form.

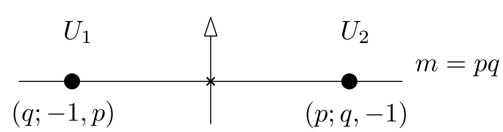


FIGURE I.1. Embedded **Q**-resolution of $\{x^p + y^q = 0\} \subset \mathbb{C}^2$.

(ii) The point P is of type $(d; p, q)$. Assume $X = X(d; p, q)$ is written in a normalized form, i.e. $\gcd(d, p) = \gcd(d, q) = 1$. Without loss of generality,

p and q can assumed to be coprime. We will describe $\pi = \pi_{w,d} : \widehat{\mathbb{C}}^2_{w,d} \to X(d;p,q)$ the weighted blow-up at the origin with respect to $w = (p,q)$. The new ambient space is covered as

$$\widehat{\mathbb{C}}^2_{w,d} = U_1 \cup U_2 = X(p;-d,q) \cup X(q;p,-d)$$

and the charts are given by

$$\text{First chart} \left|\begin{array}{rcl} X(p;-d,q) & \longrightarrow & U_1, \\ \big[(x^d,y)\big] & \mapsto & \big([(x^p,x^q y)]_d,[1:y]_w\big). \end{array}\right.$$

$$\text{Second chart} \left|\begin{array}{rcl} X(q;\,p,-d) & \longrightarrow & U_2, \\ \big[(x,y^d)\big] & \mapsto & \big([(xy^p,y^q)]_d,[x:1]_w\big). \end{array}\right.$$

As above, the exceptional divisor $E = \pi_w^{-1}(0)$ is identified with $\mathbb{P}^1_w$ which is isomorphic to $\mathbb{P}^1$ under the map

$$[x:y]_w \longmapsto [x^q : y^p].$$

The singular points of $\widehat{\mathbb{C}}^2_{w,d}$ are cyclic quotient singularities at the origin of each chart and they are written in a normalized form.

Example (I.2.6) ([**AMO11b, Mar11**]). Assume $\gcd(p,q) = 1$ and $p < q$. Let $f = (x^p + y^q)(x^q + y^p)$ and consider $C_1 = \{x^p + y^q = 0\}$ and $C_2 = \{x^q + y^p = 0\}$ the two irreducible components of $\{f = 0\}$.

Let $\pi_1 : \widehat{\mathbb{C}}^2_{(q,p)} \to \mathbb{C}^2$ be the (q,p)-weighted blow-up at the origin. The new space has two singular points of type $(q;-1,p)$ and $(p;q,-1)$ located on the exceptional divisor E_1. The local equation of the total transform in the first chart is given by the function

$$x^{p(p+q)}(1+y^q)(x^{q^2-p^2}+y^p) : \; X(q;-1,p) \longrightarrow \mathbb{C}.$$

Here $x = 0$ is the equation of the exceptional divisor and the other factors correspond to the strict transform of C_1 and C_2 (denoted again by the same symbol).

Hence E_1 has multiplicity $p(p+q)$; it intersects transversely C_1 at a smooth point while it intersects C_2 at a singular point (the origin of the first chart) without **Q**-normal crossings.

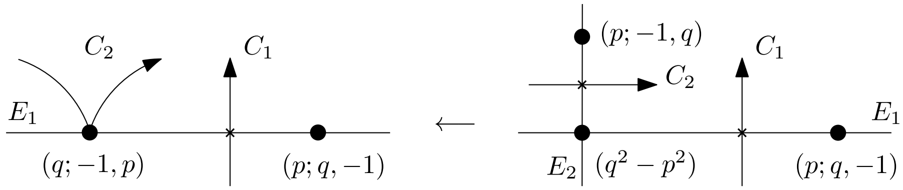


Figure I.2. Embedded **Q**-resolution of $f = (x^p + y^q)(x^q + y^p)$.

Let us consider π_2 the $w = (p, q^2 - p^2)$-weighted blow-up at the origin of $X(q; -1, p)$,

$$\pi_2 : \widehat{\mathbb{C}}^2_{w,q} \longrightarrow X(q; p, q^2 - p^2) = X(q; -1, p).$$

The new space has two singular points of type $(p; -q, q^2-p^2) = (p; -1, q)$ and $(q^2 - p^2; p, -q)$. In the first chart, the local equation of the total transform of $x^{p(p+q)}(x^{q^2-p^2} + y^p)$ is given by the function

$$x^{p(p+q)}(1 + y^p) : X(p; -1, q) \longrightarrow \mathbb{C}.$$

Thus the new exceptional divisor E_2 has multiplicity $p(p+q)$ and intersects transversely the strict transform of C_2 at a smooth point. Hence the composition $\pi = \pi_2 \circ \pi_1$ is an embedded **Q**-resolution of $\{f = 0\} \subset \mathbb{C}^2$. Figure I.2 illustrates the whole process.

(iii) The point P is of type $(d; a, b)$. As above, assume that $X = X(d; a, b)$ and the map

$$\pi = \pi_{(d;a,b),w} : \widehat{X(d; a, b)}_w \longrightarrow X(d; a, b) \tag{5}$$

is the weighted blow-up at the origin of $X(d; a, b)$ with respect to $w = (p, q)$. The new space is covered as

$$\widehat{U}_1 \cup \widehat{U}_2 = X\left(\begin{array}{c|cc} p & -1 & q \\ pd & a & pb - qa \end{array} \right) \cup X\left(\begin{array}{c|cc} q & p & -1 \\ qd & qa - pb & b \end{array} \right).$$

Or equivalently

$$\widehat{X(d; a, b)}_w = \widehat{U}_1 \cup \widehat{U}_2 = X\left(\frac{pd}{e}; 1, \frac{-q + a'pb}{e}\right) \cup X\left(\frac{qd}{e}; \frac{-p + b'qa}{e}, 1\right) \tag{6}$$

with $a'a = b'b \equiv 1 \mod (d)$ and $e = \gcd(d, pb - qa)$. The charts are given by

First chart
$$X\left(\begin{array}{c|cc} p & -1 & q \\ pd & a & pb - qa \end{array} \right) \longrightarrow \widehat{U}_1,$$
$$\big[(x, y)\big] \mapsto \big[((x^p, x^q y), [1 : y]_w)\big]_{(d;a,b)}.$$

Second chart
$$X\left(\begin{array}{c|cc} q & p & -1 \\ qd & qa - pb & b \end{array} \right) \longrightarrow \widehat{U}_2,$$
$$\big[(x, y)\big] \mapsto \big[((xy^p, y^q), [x : 1]_w)\big]_{(d;a,b)}.$$

Equivalently, see [**Mar11**, Remark I.3.14],

$$\text{First chart}\quad \left|\ \begin{array}{c} X\left(\dfrac{pd}{e};1,\dfrac{-q+a'pb}{e}\right) \longrightarrow \widehat{U}_1,, \\ \big[(x^e,y)\big] \mapsto \big[((x^p,x^q y),[1:y]_w)\big]_{(d;a,b)}. \end{array}\right.$$

$$\text{Second chart}\quad \left|\ \begin{array}{c} X\left(\dfrac{qd}{e};\dfrac{-p+b'qa}{e},1\right) \longrightarrow \widehat{U}_2, \\ \big[(x,y^e)\big] \mapsto \big[((xy^p,y^q),[x:1]_w)\big]_{(d;a,b)}. \end{array}\right.$$

The exceptional divisor $E = \pi^{-1}_{(d;a,b),w}(0)$ is identified with the quotient space $\mathbb{P}^1_w(d;a,b) := \mathbb{P}^1_w/\boldsymbol{G}_d$ which is isomorphic to $\mathbb{P}^1$ under the map

$$\begin{array}{ccc} \mathbb{P}^1_w(d;a,b) & \longrightarrow & \mathbb{P}^1 \\ [x:y]_w & \mapsto & [x^{dq/e}:y^{dp/e}], \end{array}$$

where $e = \gcd(dp, dq, pb - qa)$. Again the singular points are cyclic and correspond to the origins. They may be not written in normalized form even if $\gcd(p,q) = 1$ and $(d;a,b)$ is normalized.

Example (I.2.7) ([**AMO11b, Mar11**])**.** Assume $\gcd(p,q) = \gcd(r,s) = 1$ and $\frac{p}{q} < \frac{r}{s}$. Let $f = (x^p + y^q)(x^r + y^s)$ and consider

$$C_1 = \{x^p + y^q = 0\}, \quad C_2 = \{x^r + y^s = 0\}$$

the two irreducible components of f.

Working as in Example (I.2.6), one obtains the following picture ((I.2.7)) representing an embedded **Q**-resolution of $\{f = 0\} \subset \mathbb{C}^2$.

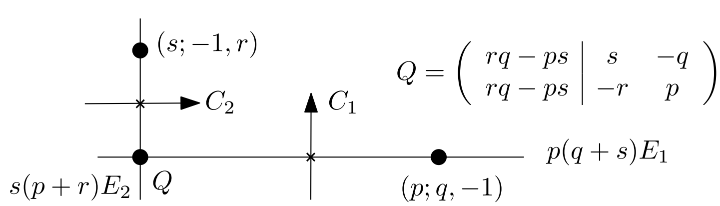


FIGURE I.3. Embedded **Q**-resolution of $f = (x^p + y^q)(x^r + y^s)$.

After writing the quotient spaces in their normalized form one checks that this resolution coincides with the one given in Example (I.2.6) assuming $r = q$ and $s = p$.

Example (I.2.8). Let $\{f = xy(xy + (x^3 - y^2)^2) = 0\}$ be a $\mathbb{Q}$-divisor on $X(7;2,3)$. Let us compute a **Q**-resolution of $\{f = 0\} \in X(7;2,3)$. Consider

$w = (1,5)$ and let $\pi_{(7;2,3),w} : \widehat{X(7;2,3)}_w \longrightarrow X(7;3,2)$ be the $(1,5)$-weighted blow-up at the origin.

$$\widehat{X(7;2,3)}_w = \widehat{U}_1 \cup \widehat{U}_2 = \mathbb{C}^2 \cup X\,(5;2,1)$$

$$\text{First chart} \quad \left| \begin{array}{c} \mathbb{C}^2 \longrightarrow \widehat{U}_1,, \\ \\ \big[(x^7, y)\big] \mapsto \big[((x, x^5 y), [1:y]_w)\big]_{(7;2,3)}. \end{array} \right.$$

$$\text{Second chart} \quad \left| \begin{array}{c} X\,(5;2,1) \longrightarrow \widehat{U}_2, \\ \\ \big[(x, y^7)\big] \mapsto \big[((xy, y^5), [x:1]_w)\big]_{(7;2,3)}. \end{array} \right.$$

The local equation of the total transform in the first chart is given by the function

$$\underbrace{x^{12}}_{E_1} \cdot \underbrace{y}_{C_2} \cdot \underbrace{y + (1 - x^7 y^2)^2}_{C_4},$$

on $\mathbb{C}^2$. The local equation of the total transform in the second chart is given by the equation

$$\underbrace{y^{12}}_{E_1} \cdot \underbrace{x}_{C_1} \cdot \underbrace{x + (x^3 - y^7)^2}_{C_3},$$

on $X(5;2,1)$. After a second weighted blow-up with weight $w = (2,1)$ of $[(0,0)] \in X(5;2,1)$ one finally has

$$\widehat{X(5;2,1)}_w = \widehat{U}_1 \cup \widehat{U}_2 = X\,(2;1,1) \cup \mathbb{C}^2$$

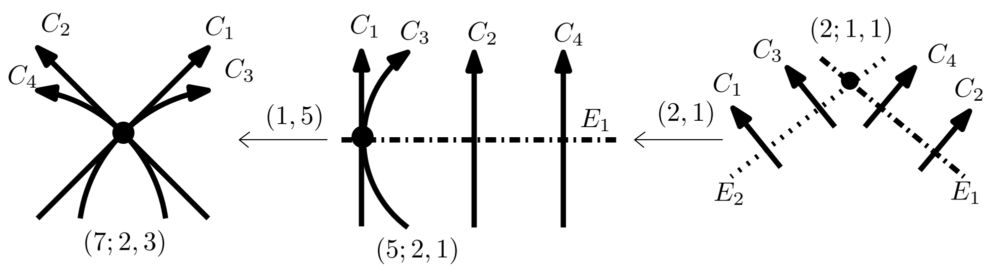


FIGURE I.4. Embedded $\mathbf{Q}$-resolution of $f = xy(xy + (x^3 - y^2)^2)$.

First chart
$$X\,(2;1,1) \longrightarrow \widehat{U}_1,,$$
$$\big[(x^5,y)\big] \mapsto \big[((x^2,xy),[1:y]_w)\big]_{(5;2,1)}.$$

Second chart
$$\mathbb{C}^2 \longrightarrow \widehat{U}_2,$$
$$\big[(x,y^5)\big] \mapsto \big[((xy^2,y),[x:1]_w)\big]_{(5;2,1)}.$$

Section § **I.3**

Intersection Theory on Abelian-Quotient V-Surfaces

Intersection theory is a powerful tool in complex algebraic (and analytic) geometry, see [**Ful98**] for a wonderful exposition. It will be frequently used in the successive chapters.

The main objects involved in intersection theory on surfaces are divisors, which have two main incarnations, Weil and Cartier. These coincide in the smooth case, but not in general. In the singular case the two concepts are different and a geometric interpretation of intersection theory is yet to be developed. A general definition for normal surfaces was given by Mumford [**Mum61**] and it was applied by Sakai to study Weil divisors on normal surfaces [**Sak84**].

I.3–1. Cartier and Weil $\mathbb{Q}$-Divisors on V-Manifolds

Let us recall the definitions of Cartier and Weil divisors. The content of this section can be found in detail in [**AMO11a**]. Let X be an irreducible normal complex analytic variety. Denote by $\mathcal{O}_X$ the structure sheaf of X and $\mathcal{K}_X$ the sheaf of total quotient rings of $\mathcal{O}_X$. Denote by $\mathcal{K}_X^*$ the (multiplicative) sheaf of invertible elements in $\mathcal{K}_X$. Similarly $\mathcal{O}_X^*$ is the sheaf of invertible elements in $\mathcal{O}_X$. Note that an irreducible subvariety V corresponds to a prime ideal in the ring of sections of any local complex model space meeting V.

Definition (I.3.1). A *Cartier divisor* on X is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$ and it can be represented by giving an open covering $\{U_i\}_{i\in I}$ of X and, for all $i \in I$, an element $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$ such that
$$\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*), \quad \forall i,j \in I.$$

Two systems $\{(U_i, f_i)\}_{i\in I}$, $\{(V_j, g_j)\}_{j\in J}$ represent the same Cartier divisor if and only if on $U_i \cap V_j$, f_i and g_j differ by a multiplicative factor in $\mathcal{O}_X(U_i \cap V_j)^*$. The abelian group of Cartier divisors on X is denoted by $\mathrm{CaDiv}(X)$. If $D := \{(U_i, f_i)\}_{i\in I}$ and $E := \{(V_j, g_j)\}_{j\in J}$ then $D + E = \{(U_i \cap V_j, f_i g_j)\}_{i\in I, j\in J}$.

The functions f_i above are called *local equations* of the divisor on U_i. A Cartier divisor on X is *effective* if it can be represented by $\{(U_i, f_i)\}_i$ with all local equations $f_i \in \Gamma(U_i, \mathcal{O}_X)$. Any global section $f \in \Gamma(X, \mathcal{K}_X^*)$ determines a *principal* Cartier divisor $(f)_X := \{(X, f)\}$ by taking all local equations equal to f.

Definition (I.3.2). A *Weil divisor* on X is a locally finite linear combination with integral coefficients of irreducible subvarieties of codimension one. The abelian group of Weil divisors on X is denoted by $\mathrm{WeDiv}(X)$. If all coefficients appearing in the sum are non-negative, the Weil divisor is called *effective.*

The following theorem allows us to identify both notions on V-manifolds after tensorizing by $\mathbb{Q}$.

Theorem (I.3.3) ([**AMO11a, Mar11**])**.** *Let X be a V-manifold. Then the notion of Cartier and Weil divisor coincide over $\mathbb{Q}$. More precisely, the linear map*

$$T_X \otimes 1 : \mathrm{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism of $\mathbb{Q}$-vector spaces. In particular, for a given Weil divisor D on X there always exists $k \in \mathbb{Z}$ such that $kD \in \mathrm{CaDiv}(X)$.

Definition (I.3.4). Let X be a V-manifold. The vector space of $\mathbb{Q}$-Cartier divisors is identified under T_X with the vector space of $\mathbb{Q}$-Weil divisors. A $\mathbb{Q}$-*divisor* on X is an element in $\mathrm{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The set of all $\mathbb{Q}$-divisors on X is denoted by $\mathbb{Q}\text{-}\mathrm{Div}(X)$.

In [**AMO11a**], we give a way to construct the inverse of $T_X \otimes 1$. Here we summarize how to write a Weil divisor as a $\mathbb{Q}$-Cartier divisor where X is an algebraic V-manifold.

(1) Write $D = \sum_{i\in I} a_i [V_i] \in \mathrm{WeDiv}(X)$, where $a_i \in \mathbb{Z}$ and $V_i \subset X$ irreducible. Also choose $\{U_j\}_{j\in J}$ an open covering of X such that $U_j = B_j / G_j$ where $B_j \subset \mathbb{C}^n$ is an open ball and G_j is a small finite subgroup of $\mathrm{GL}(n, \mathbb{C})$.

(2) For each $(i, j) \in I \times J$ choose a reduced polynomial $f_{i,j} : U_j \to \mathbb{C}$ such that $V_i \cap U_j = \{f_{i,j} = 0\}$, then

$$[V_i|_{U_j}] = \frac{1}{|G_j|}\{(U_j, f_{i,j}^{|G_j|})\}.$$

(3) Identifying $\{(U_j, f_{i,j}^{|G_j|})\}$ with its image $\mathrm{CaDiv}(U_j) \hookrightarrow \mathrm{CaDiv}(X)$, one finally writes D as a sum of locally principal Cartier divisors over $\mathbb{Q}$,

$$D = \sum_{(i,j)\in I\times J} \frac{a_i}{|G_j|}\{(U_j, f_{i,j}^{|G_j|})\}.$$

See [**AMO11a**] for a detailed explanation.

I.3–2. Rational Intersection Number and Weighted Blow-ups

Now we have all the necessary ingredients to develop a rational intersection theory on surfaces with quotient singularities. The content of this section can be found in detail in [**AMO11b**].

Let X be an algebraic V-manifold of dimension 2. Consider D_1 and D_2 two effective $\mathbb{Q}$-divisors on X, and $P \in X$ a point. The divisor D_i is locally given in a neighborhood of P by a reduced polynomial f_i, $i = 1, 2$. On the other hand the point P can be assumed to be a normalized type of the form $(d; a, b)$. Hence the computation of $(D_1 \cdot D_2)_P$ is reduced to the following particular case.

Definition (I.3.5) (Local intersection number on $X(d; a, b)$, [**AMO11b, Mar11, Ort09**])**.** Denote by X the cyclic quotient space $X(d; a, b)$ and consider two divisors $D_1 = \{f_1 = 0\}$ and $D_2 = \{f_2 = 0\}$ given by $f_1, f_2 \in \mathbb{C}\{x, y\}$ reduced and without common components. Assume that, $(d; a, b)$ is normalized.

Then as Cartier divisors $D_1 = \frac{1}{d}\{(X, f_1^d)\}$ and $D_2 = \frac{1}{d}\{(X, f_2^d)\}$. The local number $(D_1 \cdot D_2)_{[P]}$ at a point P of type $(d; a, b)$ is defined as

$$(D_1 \cdot D_2)_{[P]} = \frac{1}{d^2} \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\langle f_1^d, f_2^d\rangle}$$

where $\mathcal{O}_P = \mathbb{C}\{x, y\}^{\boldsymbol{G}_d}$ is the local ring of functions at P (recall §I.1–2).

Example (I.3.6) ([**AMO11b, Mar11, Ort09**])**.** Let x_1 and x_2 be the local coordinates of the axes in $M := X(d; a, b)$. Consider $X_i := \{(M, x_i)\}$ the $\mathbb{Q}$-divisors associated with them. Then

$$(X_1 \cdot X_2)_0 = \frac{1}{d}.$$

See [**AMO11b**] for further details and [**Ort09**] for a more direct approach.

In the preceeding sections, weighted blow-ups were introduced as a tool for computing embedded **Q**-resolutions. Here we calculate self-intersection numbers of exceptional divisors of weighted blow-ups on analytic surfaces with abelian quotient singularities.

Proposition (I.3.7) ([**AMO11b, Mar11**]). *Let X be an analytic surface with abelian quotient singularities and let $\pi : \widehat{X} \to X$ be the (p,q)-weighted blow-up at a point $P \in X$ of type $(d;a,b)$. Assume* $\gcd(p,q) = 1$ *and $(d;a,b)$ is a normalized type, i.e.* $\gcd(d,a) = \gcd(d,b) = 1$. *Also write* $e = \gcd(d, pb - qa)$.

Consider two $\mathbb{Q}$-divisors C and D on X. As usual, denote by E the exceptional divisor of π, and by $\widehat{C}$ (resp. $\widehat{D}$) the strict transform of C (resp. D). Let ν and μ be the (p,q)-multiplicities of C and D at P, i.e. x (resp. y) has (p,q)-multiplicity p (resp. q). Then the following equalities hold:

(1) $\pi^*(C) = \widehat{C} + \frac{\nu}{e} E.$

(2) $E \cdot \widehat{C} = \frac{e\nu}{dpq}.$

(3) $E^2 = -\frac{e^2}{dpq}.$

(4) $\widehat{C} \cdot \widehat{D} = C \cdot D - \frac{\nu\mu}{dpq}.$

In addition, if D has compact support then $\widehat{D}^2 = D^2 - \frac{\mu^2}{dpq}$.

Example (I.3.8) ([**AMO11b**]). Let us compute now the self-intersection of the divisors in Example (I.2.6). After the first blow-up (of type (q,p) over a smooth point) the divisor E_1 has self-intersection $\frac{-1}{pq}$. Let us consider the second blow-up, of type $(p, q^2 - p^2)$ at a point of type $(q; p, q^2 - p^2)$; the exceptional divisor is E_2 and its self-intersection is $-\frac{q}{p(q^2-p^2)}$. The strict transform of E_1 has multiplicity p and hence its self-intersection is $\frac{-1}{pq} - \frac{p}{q(q^2-p^2)} = -\frac{q}{p(q^2-p^2)}$.

For a detailed example of this intersection theory see [**AMO12**, §6].

Section § I.4
Weighted projective plane

The main reference that has been used in this section is [**Dol82**], see also [**Ort09**] for a more detailed and down to earth exposition. Here we concentrate our attention on describing the analytic structure and singularities.

Let $w := (w_0, w_1, w_2)$ be a weight vector, that is, a triple of pairwise coprime positive integers. There is a natural action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^3 \setminus \{0\}$ given by

$$(x_0, x_1, x_2) \longmapsto (t^{w_0} x_0, t^{w_1} x_1, t^{w_2} x_2).$$

The set of orbits $\frac{\mathbb{C}^3\setminus\{0\}}{\mathbb{C}^*}$ under this action is denoted by $\mathbb{P}^2_w$ and it is called the *weighted projective plane* of type w. The class of a nonzero element $(X_0, X_1, X_2) \in \mathbb{C}^3$ is denoted by $[X_0 : X_1 : X_2]_w$ and the weight vector is omitted if no ambiguity seems likely to arise. When $(w_0, w_1, w_2) = (1, 1, 1)$ one obtains the usual projective space and the weight vector is always omitted. For $\mathbf{x} \in \mathbb{C}^3 \setminus \{0\}$, the closure of $[\mathbf{x}]_w$ in $\mathbb{C}^3$ is obtained by adding the origin and it is an algebraic curve.

Consider the decomposition $\mathbb{P}^2_w = U_0 \cup U_1 \cup U_2$, where U_i is the open set consisting of all elements $[X_0 : X_1 : X_2]_w$ with $X_i \neq 0$. The map

$$\widetilde{\psi}_0 : \mathbb{C}^2 \longrightarrow U_0, \quad \widetilde{\psi}_0(x_1, x_2) := [1 : x_1 : x_2]_w$$

defines an isomorphism ψ_0 if we replace $\mathbb{C}^2$ by $X(w_0;\, w_1, w_2)$. Analogously, $X(w_1;\, w_0, w_2) \cong U_1$ and $X(w_2;\, w_0, w_1) \cong U_2$ under the obvious analytic map.

Remark **(I.4.1).** (Another way to describe $\mathbb{P}^2_w$). Let $\mathbb{P}^2$ be the classical projective space and $\boldsymbol{G}_w = \boldsymbol{G}_{w_0} \times \boldsymbol{G}_{w_1} \times \boldsymbol{G}_{w_2}$ the product of cyclic groups.

Consider the group action

$$\begin{array}{ccc} \boldsymbol{G}_w \times \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2, \\ \big((\xi_{w_0}, \xi_{w_1}, \xi_{w_2}), [X_0 : X_1 : X_2]\big) & \mapsto & [\xi_{w_0} X_0 : \xi_{w_1} X_1 : \xi_{w_2} X_2]. \end{array}$$

Then the set of all orbits $\mathbb{P}^2/\boldsymbol{G}_w$ is isomorphic to the weighted projective plane of type w and the isomorphism is induced by the branched covering

$$\mathbb{P}^2 \ni [X_0 : X_1 : X_2] \overset{\phi}{\longmapsto} [X_0^{w_0} : X_1^{w_1} : X_2^{w_2}]_w \in \mathbb{P}^2_w. \tag{7}$$

Note that this branched covering is unramified over

$$\mathbb{P}^2_w \setminus \{[X_0, X_1, X_2]_w \mid X_0 \cdot X_1 \cdot X_2 = 0\}$$

and has $\bar{w} = w_0 \cdot w_1 \cdot w_2$ sheets. Moreover, the covering respects the coordinate axes.

Example (I.4.2) ([**AMO11b, Mar11**])**.** Let $\phi : \mathbb{P}^2 \to \mathbb{P}^2_w$ be the branched covering defined above with weights $w = (1, 2, 3)$. For instance, the preimage of $[1 : 1 : 1]_w$ consists of 6 points, namely the set $\{[1 : \xi_2 : \xi_3] \in \mathbb{P}^2 \mid \xi_2 \in \boldsymbol{G}_2,\ \xi_3 \in \boldsymbol{G}_3\}$.

The following result is well known.

Proposition (I.4.3) ([**AMO11b, Mar11**])**.** *Let* $d_0 := \gcd(w_1, w_2)$*,* $d_1 := \gcd(w_0, w_2)$ *,* $d_2 := \gcd(w_0, w_1)$*,* $e_0 := d_1 \cdot d_2$*,* $e_1 := d_0 \cdot d_2$*,* $e_2 := d_0 \cdot d_1$ *and* $p_i := \frac{w_i}{e_i}$*. The following map is an isomorphism:*

$$\begin{array}{ccc} \mathbb{P}^2\big(w_0, w_1, w_2\big) & \longrightarrow & \mathbb{P}^2(p_0, p_1, p_2), \\ [X_0 : X_1 : X_2] & \mapsto & \big[\, X_0^{d_0} : X_1^{d_1} : X_2^{d_2} \,\big]. \end{array}$$

Remark **(I.4.4).** Note that, due to the preceding proposition, one can always assume the weight vector satisfies that (w_0, w_1, w_2) are pairwise relatively prime numbers. Note that following a similar argument, $\mathbb{P}^1_{(w_0,w_1)} \cong \mathbb{P}^1$.

The space $\mathbb{P}^2_w$ is a normal irreducible projective algebraic variety of dimension 2. Its Euler characteristic is $\chi(\mathbb{P}^2_w) = \chi(\mathbb{P}^2) = 3$.

Definition (I.4.5). Let be $H_i = \{X_i = 0\}$, the *canonical divisor* in $\mathbb{P}^2_w$ is given by

$$K_{\mathbb{P}^2_w} := -\sum_i H_i.$$

Note that $\deg K_{\mathbb{P}^2_w} = -(w_0 + w_1 + w_2)$.

Remark **(I.4.6).** Notice that Definition (I.4.5) of the canonical divisor is the one obtained from the canonical divisor in $\mathbb{P}^2$ after using the covering (7) and applying the Riemann-Hurwitz formula. Another approach can be found in [**CLS11**, Theorem 8.2.3].

I.4–1. Weighted Bézout's Theorem for Weighted Projective Planes

For a proof of this classical theorem a wonderful approach can be found in [**Mar11**]. Here we are going to state the theorem for weighted projective planes.

Proposition (I.4.7) (Weighted Bézout's Theorem on $\mathbb{P}^2_w$, [**Ort09**])**.** *The intersection number of two $\mathbb{Q}$-divisors, D_1 and D_2 on $\mathbb{P}^2_w$ without common components is*

$$D_1 \cdot D_2 = \sum_{P \in D_1 \cap D_2} (D_1 \cdot D_2)_{[P]} = \frac{1}{\bar{w}} \deg_w(D_1) \deg_w(D_2) \in \mathbb{Q},$$

where $\bar{w} = w_0 w_1 w_2$ and $\deg_w(D_i) = \deg(\phi^*(D_i))$ (7).

Section § I.5
Logarithmic Complex and residues on V-manifolds

In this section we develop a De Rham cohomology for projective varieties with quotient singularities. We shall sketch some results in the Hodge Theory of projective V-manifolds which will be of particular interest. All these results, with their respective proofs, can be found in the first chapter of [**Ste77**]. Remark that the reader can find all the necessary preliminaries about C^∞ log complex of quasi projective algebraic varieties and residues in [**CA02**, §1.3].

We are interested in the Hodge theory of projective V-manifolds. The following proposition shows that we can expect an analogous situation as in the smooth projective case.

Proposition (I.5.1) ([**Ste77**])**.** *Every V-manifold is a rational homology manifold.*

As a result we get the following.

Corollary (I.5.2) ([**Ste77**])**.** *If X is a complete algebraic V-manifold, then the canonical Hodge structure on $H^k(X)$ is purely of weight k for all $k \geq 0$. If $\pi : \hat{X} \to X$ is a resolution of singularities for X, then the map $\pi^* : H^k(X) \to H^k(\hat{X})$ is injective for all $k \geq 0$.*

Definition (I.5.3). Let Ω_Y be the sheaf of analytic forms on a smooth algebraic variety Y. Let X be a V-manifold and denote by Σ its singular locus. Denote by $j : X \setminus \Sigma \to X$ the inclusion map. Then we define $\hat{\Omega}^\bullet_X = j_* \Omega^\bullet_{X\setminus\Sigma}$.

Lemma (I.5.4) ([**Ste77**])**.** *Let B an open ball with center 0 in $\mathbb{C}^n$. Let G be a small subgroup of $\mathrm{GL}(n, \mathbb{C})$ which leaves B invariant and let $U = B/G$. Denote by $\rho : B \to U$ the quotient map. Then for all $p \geq 0$ one has*

$$\hat{\Omega}^\bullet_U \cong (\rho^* \Omega^\bullet_B)^G.$$

With the previous Definition one has the following result.

Lemma (I.5.5) ([**Ste77**])**.** *Let $\pi : \hat{X} \to X$ be a resolution of singularities for the V-manifold X. Then $\hat{\Omega}^\bullet_X \cong \pi_* \Omega^\bullet_{\hat{X}}$.*

Definition (I.5.6) ([**Ste77**])**.** Let X be a V-manifold and Y a $\mathbb{Q}$-divisor with $\mathbf{Q}$-normal crossings. Define the complex $\hat{\Omega}^\bullet_X(\log Y)$ on X by

$$\hat{\Omega}^\bullet_X(\log Y) = j_* \Omega^\bullet_{X\setminus\Sigma}(\log Y \setminus \Sigma),$$

where $\Sigma = \mathrm{Sing}(X)$ and $j : X \setminus \Sigma \to$ is the inclusion map.

Consider $\rho : B \subseteq \mathbb{C}^n \to X$ the one in 1. If $(X, Y) = (B, \mathcal{D})/\boldsymbol{G}_{\mathbf{d}}$, $\mathcal{D} \subset B$ a $\boldsymbol{G}_{\mathbf{d}}$-invariant divisor with normal crossings, then

$$\hat{\Omega}^\bullet_X(\log Y) = (\rho_* \Omega^\bullet_B(\log \mathcal{D}))^{\boldsymbol{G}_{\mathbf{d}}}.$$

If $\pi : \hat{X} \to X$ is a resolution of singularities for X such that the total transform $\hat{Y}$ of Y is a divisor with normal crossings on $\hat{X}$, then

$$\hat{\Omega}^\bullet_X(\log Y) = \pi_* \Omega^\bullet_X(\log \hat{Y}).$$

Definition (I.5.7) ([**Ste77**]). Let $k \in \mathbb{Z}$, the weight filtration $\mathcal{W}$ on $\hat{\Omega}^{\bullet}_X(\log Y)$ is defined by

$$\mathcal{W}_k \hat{\Omega}^p_X(\log Y) = \hat{\Omega}^k_X(\log Y) \wedge \hat{\Omega}^{p-k}_X.$$

Assume that Y is a union of irreducible components $Y_1, \ldots, Y_m$ without self-intersection. Denote by $\hat{Y}^{[p]}$ the disjoint union of all p-fold intersections $Y_{i_1} \cap \cdots \cap Y_{i_p}$ for $1 \leq i_1 < \cdots < i_p \leq m$. Denote by $a_p : \hat{Y}^{[p]} \to X$ the natural map. Analogous to the smooth case (see [**CA02**, §1.3]) one has a residue map

$$R^{[k]} : W_k \hat{\Omega}^p_X(\log Y) \to (a_k)_* \hat{\Omega}^{p-k}_{Y^{[k]}},$$

with $p, k \geq 0$. Remark that $\hat{Y}^{[k]}$ is a V-manifold for every $k \geq 0$.

I.5–1. Logarithmic forms and residues on $\mathbb{P}^2_w$

Let us focus the previous results on the case of non-normal crossing $\mathbb{Q}$-divisors in weighted projective planes. Let $\mathcal{D}$ be a $\mathbb{Q}$-divisor in $\mathbb{P}^2_w$. The complement of $\mathcal{D}$ will be denoted by $X_{\mathcal{D}}$. Let us fix $\pi : \overline{X}_{\mathcal{D}} \longrightarrow \mathbb{P}^2_w$ a **Q**-resolution of the singularities of $\mathcal{D}$ so that the reduced $\mathbb{Q}$-divisor $\overline{\mathcal{D}} = (\pi^*(\mathcal{D}))_{red}$ is a union of smooth $\mathbb{Q}$-divisors on $\overline{X}_{\mathcal{D}}$ with **Q**-normal crossings.

The purpose of the forthcoming Chapter V will be to construct global logarithmic forms on $\mathcal{D}$ that provide a basis for the spaces $H^k(X_{\mathcal{D}}; \mathbb{C})$.

Let us rewrite with the previous notation Definitions (I.5.6) and (I.5.7) for a given non-normal crossing $\mathbb{Q}$-divisor in $\mathbb{P}^2_w$.

Definition (I.5.8) (*log-resolution logarithmic forms*).

(1) A C^∞ form φ on $X_{\mathcal{D}}$ shall be called *logarithmic (with respect to a divisor $\mathcal{D}$ and a **Q**-resolution π)* if $\pi^*\varphi$ is logarithmic on $\overline{X}_{\mathcal{D}}$ with respect to the **Q**-normal crossing divisor $\overline{\mathcal{D}}$ (see Definition (I.5.6)). Therefore, one has the corresponding sheaf

$$\pi_* \Omega_{\overline{X}_{\mathcal{D}}}(\log \langle \overline{\mathcal{D}} \rangle).$$

In the sequel, a logarithmic form with respect to a $\mathbb{Q}$-divisor $\mathcal{D}$ and a **Q**-resolution π will be referred to as simply a logarithmic form if $\mathcal{D}$ and π are known and no ambiguity is likely to arise.

(2) Once $\mathcal{D}$ and π are fixed one can define the *residue map* $\mathrm{Res}^{[*]}_\pi(\varphi)$ *of a logarithmic form* φ as follows

$$\begin{array}{ccc} \pi_* \Omega^k_{\overline{X}_{\mathcal{D}}}(\log \langle \overline{\mathcal{D}} \rangle) & \stackrel{\mathrm{Res}^{[k]}_\pi}{\longrightarrow} & H^0(\overline{\mathcal{D}}^{[k]}; \mathbb{C}) \\ \varphi & \mapsto & \mathrm{Res}^{[k]}(\pi^*\varphi). \end{array}$$

Remark **(I.5.9).** The Poincaré residue operator can be generalized to all the log sheaves relative to $\bar{\mathcal{D}}$. We omit these results because are analogues to the same results for $\mathbb{C}^\infty$ log complex of quasi-projective algebraic varieties in [**CA02**, §1.3].

Remark **(I.5.10).** Let h be an analytic germ on $X(d;a,b)$ written in normalized form. It is important to notice that in the case of logarithmic 2-forms $\omega = h\dfrac{dx\wedge dy}{xy}$ we do not have to impose any condition to h to obtain a logarithmic form w with poles along xy. However, logarithmic 2-forms $\tilde\omega = h\dfrac{dx\wedge dy}{x}$ are not logarithmic for an arbitrary h because, in general, they are not invariant under $\boldsymbol{G}_d$.

Lemma (I.5.11). *Let π and π' be two* **Q**-*resolutions such that π' dominates π, that is, if one has the following commutative diagram of birational morphisms*

$$\begin{array}{ccc} \overline{X}'_{\mathcal{D}} & \stackrel{\rho}{\to} & \overline{X}_{\mathcal{D}} \\ \pi' & \searrow & \downarrow \pi \\ & & \mathbb{P}^2_w \end{array}$$

then

$$\pi_*\Omega_{\overline{X}_{\mathcal{D}}}(\log\langle\overline{D}\rangle) \subset \pi'_*\Omega_{\overline{X}'_{\mathcal{D}}}(\log\langle\overline{D}'\rangle).$$

Proof. The result can be checked locally. We will do it in detail for 2-forms. For 1-forms or functions the proof is essentially the same. Let ψ be a logarithmic 2-form at $P\in\mathbb{P}^2_w$ with respect to $\mathcal{D}$ and π. The form ψ can be written locally as

$$\varphi_Q\,\frac{dx\wedge dy}{xy}$$

at any point $Q\in\pi^{-1}(P)$. If the map ρ doesn't blow-up Q, then its inverse image is automatically logarithmic on $\overline{X}'_{\mathcal{D}}$. Otherwise, Q is the center of a weighted (p,q)-weighted blow-up ε. Then, on one chart, its inverse image becomes (recall (6)).

$$\varphi_Q\,\frac{dx\wedge dy}{xy} \quad \xleftarrow{\substack{x=\bar u^p,\ u=\bar u^e \\ y=\bar u^q v}} \quad \frac{p}{e}\tilde\varphi_Q\,\frac{\bar u^{p-1+q-e+1}d\bar u^e\wedge dv}{\bar u^{p+q}v} = \frac{p}{e}\tilde\varphi_Q\,\frac{du\wedge dv}{uv}.$$

which is also logarithmic at 0. Therefore, since ρ is a finite sequence of weighted blow-ups the result follows.

□

Corollary (I.5.12). *The concept of being logarithmic does not depend on the* **Q**-*resolution.*

Proof. Suppose we have two **Q**-resolutions π and π' and a form φ which is logarithmic on $X_{\mathcal{D}}$ with respect π but not π'. There exists another **Q**-resolution π'' dominating π and π' and the commutativity of the pull-back would lead to contradiction. □

Remark **(I.5.13).** Let ψ be a logarithmic 2-form at a point P, and assume $\mathcal{D}$ is a **Q**-normal crossing divisor at P. As a consequence of Lemma (I.5.11), if one makes a weighted blow-up of the point P, then the pull-back of ψ is a logarithmic 2-form on the pull-back of $\mathcal{D}$ and has poles along the strict transform of $\mathcal{D}$ and the exceptional $\mathbb{Q}$-divisor.

In the particular case of $X(d;a,b)$ and the $\mathrm{Res}^{[2]}$ one has the following.

Definition (I.5.14). Let h be an analytic germ on $X(d;a,b)$ written in normalized form (Definition (I.1.9)). Let $\varphi = h\dfrac{dx\wedge dy}{xy}$ be a logarithmic 2-form with poles at the origin. Then

$$\mathrm{Res}^{[2]}(\varphi) := \frac{1}{d}h(0,0).$$

Note that this is invariant under (p,q) blow-ups

$$\frac{dx\wedge dy}{xy} \overset{\substack{x=\bar{u}^p,\ u=\bar{u}^e\\ y=\bar{u}^q v}}{\longleftarrow} \frac{p}{e}\,\frac{\bar{u}^{p-1+q-e+1}d\bar{u}^e\wedge dv}{\bar{u}^{p+q}v} = \frac{p}{e}\,\frac{du\wedge dv}{uv}. \tag{8}$$

$$\mathrm{Res}^{[2]}(\frac{dx\wedge dy}{xy}) = \frac{1}{d} = \mathrm{Res}^{[2]}(\frac{p}{e}\,\frac{du\wedge dv}{uv}) = \frac{p}{e}\frac{e}{pd}.$$

Recall (6), $\widehat{X(d;a,b)}_w = X\left(\dfrac{pd}{e};1,\dfrac{-q+a'pb}{e}\right) \cup X\left(\dfrac{qd}{e};\dfrac{-p+b'qa}{e},1\right)$.

Proposition (I.5.15). *The residue map does not depend on the resolution.*

Proof. For dimension 2 see (8), an analogous proof holds for dimension one o higher. □

Let us see some useful examples of how to compute the residues in different situations.

Example (I.5.16). Consider the form $\omega = \frac{w_0\ xdy - w_1\ ydx}{xy}$ in $\mathbb{P}^1_{(w_0,w_1)}$. Denote by $P_1 = [(0:1)] \in X(w_1;w_0)$ and $P_2 = [(1:0)] \in X(w_0;w_1)$ Let us compute the residues of ω at the vertices of the weighted projective line.

One has

$$\mathrm{Res}^{[1]}_{P_1}(\varphi) = \mathrm{Res}^{[1]} - w_1\frac{dx}{x} = -(-1)^{\sigma(2,1)} = -(-1)^1 = 1,$$

$$\mathrm{Res}^{[1]}_{P_2}(\varphi) = \mathrm{Res}^{[1]} w_0\frac{dy}{y} = (-1)^{\sigma(1,2)} = (-1)^0 = 1.$$

As in the classical smooth case (see [**CA02**, Definition 1.21]), σ denotes the signature of the permutation realized in the variables.

Example (I.5.17). Consider now the 2-form

$$\tau := \frac{\Omega^2}{xyz} = w_2\ z\frac{dx \wedge dy}{xyz} + w_0\ x\frac{dy \wedge dz}{xyz} + w_1\ y\frac{dz \wedge dx}{xyz}$$

in $\mathbb{P}^2_{(w_0,w_1,w_2)}$ with

$$\Omega^2 := w_2\ zdx \wedge dy + w_0\ xdy \wedge dz + w_1\ ydz \wedge dx. \tag{9}$$

This form (9) will be called from now on the *weighted volume form.* Denote by $P_0 := [1 : 0 : 0]_w$, $P_1 := [0 : 1 : 0]_w$ and $P_2 := [0 : 0 : 1]_w$ the three vertices of $\mathbb{P}^2_w$. Let us compute the residues of τ at the three origins of the weighted projective space. One has

$$\begin{aligned}
\mathrm{Res}^{[2]}_{P_0}(\tau) &= \mathrm{Res}^{[2]} w_2\ \frac{dx \wedge dy}{xy} = (-1)^{\sigma(3,1,2)} = (-1)^2 = 1,\\
\mathrm{Res}^{[2]}_{P_1}(\tau) &= \mathrm{Res}^{[2]} w_1\ \frac{dz \wedge dx}{zx} = (-1)^{\sigma(2,3,1)} = (-1)^2 = 1,\\
\mathrm{Res}^{[2]}_{P_2}(\tau) &= \mathrm{Res}^{[2]} w_0\ \frac{dy \wedge dz}{yz} = (-1)^{\sigma(1,2,3)} = (-1)^0 = 1.
\end{aligned}$$

Definition (I.5.18). In general we define the *weighted volume form* on $\mathbb{P}^n_w$ with $w = (w_0, \ldots, w_n)$ as

$$\Omega^n := \sum_{i=0}^{n} (-1)^i w_i x_i dx_{\check{i}}, \tag{10}$$

where $dx_{\check{i}} = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$.

II

Local invariants on quotient singularities

In this chapter we define and investigate some of the properties of local invariants of curve germs on quotient singular surfaces. In particular, we define a Milnor fiber, Milnor number, and a δ-invariant which generalize their analogues over smooth surfaces. In §II.1 and §II.3 we show that such invariants can be effectively computed in terms of embedded **Q**-resolutions. These results will allow us in Chapter IV to obtain a formula for the genus of a curve in the weighted projective plane in terms of its degree and the previous invariants. Part of the results of this chapter can be found in [**CAMO13**].

Let X be a surface quotient singularity and $\mathcal{C} = \{f = 0\}$ a $\mathbb{Q}$-divisor on X. By means of the cyclic action, one can canonically obtain a function F on X and thus define the Milnor fiber and Milnor number μ^w of F in a standard way. Note that alternative generalizations of Milnor numbers can be found, for instance, in [**ABFdBLMH10, BLSS02, uT77, STV05**].

The approach proposed in the present work seems more natural for quotient singularities (see Example (IV.1.18)), but more importantly, it allows for the existence of an explicit formula relating Milnor number, δ-invariant, and genus of a curve on a singular surface. The new local invariant δ^w_0 can be given in terms of the Milnor fiber of $(\mathcal{C}, 0)$ by means of the formula $\mu^w = 2\delta^w - r^w + 1$, where μ^w is the Milnor number of $(\mathcal{C}, 0)$ and r^w is the number of local branches of $\mathcal{C}$ at 0. In the classical case ($X = \mathbb{C}^2$) the invariant δ can be obtained from a resolution of the local singularity $(\mathcal{C}, 0)$ in $(\mathbb{C}^2, 0)$ as

$$\delta = \sum_{Q \prec 0} \frac{\nu_Q(\nu_Q - 1)}{2}, \tag{11}$$

where Q runs over all infinitely near points of 0 of the σ-process in a resolution of $(\mathcal{C},0)$ and ν_Q denotes the multiplicity of the strict transform of $\mathcal{C}$ at Q. A two-fold generalization of this result is shown here: first by allowing the resolution to be an embedded **Q**-resolution and second by allowing X to be a quotient surface singularity (see Theorem (II.2.5)).

In the classical case, the δ-invariant can be interpreted as the dimension of a vector space. Since, in general, δ^w is a rational number, a similar result can only be expected in certain cases, namely, when associated with Cartier divisors. However, we can associate a natural number $\widetilde{\delta^w}$ to δ^w which can also be understood as a difference of two dimensions of vector spaces. Some of the techniques used in the study of $\widetilde{\delta^w}$ will be used in Chapter II to obtain a numerical version of the Adjunction Formula to compute the degree of the canonical divisor of (non-smooth) curves in $\mathbb{P}^2_w$.

Section § II.1
Milnor fibers on quotient singularities

Our purpose in this section is to provide a definition for the Milnor fiber of the germ of an isolated curve singularity $(f,[0])$ defined on an abelian V-surface $X(d;a,b)$. For the sake of completeness we include this chapter which can be found in [**CAMO13**].

In the classical case, if $(\mathcal{C}=\{f=0\},0)\subset(\mathbb{C}^2,0)$ defines a local singularity, the Milnor fiber is defined as $F_t=\{f=t\}$ and it satisfies $\chi(F_t)=r-2\delta$, where r is the number of local branches of $(\mathcal{C},0)$. Note that this cannot be extended directly to the case $\mathcal{C}\subset X(d;a,b)$ because, in general, the germ $(f,[0])$ does not define a function on $X(d;a,b)$. However, $F:=\prod_{g\in\boldsymbol{G}} f^g = uf^d$ (u a unit) is a well-defined function on $X(d;a,b)$ and hence the set $\{F=t\}$ is also well defined and invariant under the action of the cyclic group $\boldsymbol{G}_d$. One can offer the following alternative definition for the Milnor fiber of $\mathcal{C}$.

Definition (II.1.1) ([**CAMO13**])**.** Let $\mathcal{C}=\{f=0\}\subset X(d;a,b)$ be a curve germ. The *Milnor fiber* F_t^w of $(\mathcal{C},[0])$ is defined as follows,

$$F_t^w := \{F=t\}/\boldsymbol{G}_d.$$

The *Milnor number* μ^w of $(\mathcal{C},P)$ is defined as follows,

$$\mu^w := 1-\chi^{\text{orb}}(F_t^w).$$

The symbol $\chi^{\text{orb}}(M)$ denotes the orbifold Euler characteristic of $M\subset X=\mathbb{C}^2/\boldsymbol{G}_d$ as a subvariety of a quotient space which carries an orbifold structure. Note that one can also consider $\mathcal{C}$ as a germ in $(\mathbb{C}^2,0)$ in which case, $\chi^{\text{orb}}(F_t^w) = \frac{1}{d}\chi(F_t)$, where $F_t = \{F = t\} \subset \mathbb{C}^2$. Therefore, $1 - \mu^w = \frac{1}{d} - \frac{\mu}{d}$, which implies

$$\mu^w = \frac{d-1}{d} + \frac{\mu}{d}. \tag{12}$$

Note the difference between this definition and [**ABFdBLMH10**, Definition 2.10].

Also note that $\mu^w(x) = \mu^w(y) = \frac{d-1}{d}$, which extends immediately to all local curves that can become an axis after an action-preserving change of coordinates. This motivates the following.

Definition (II.1.2) ([**CAMO13**])**.** The curve $\{f = 0\} \subset X(d; a, b)$ is called a $\mathbf{Q}$-*smooth* curve if there exists $g \in \mathbb{C}\{x, y\}$ such that $\mathbb{C}\{f, g\}^{\boldsymbol{G}} = \mathbb{C}\{x, y\}^{\boldsymbol{G}}$.

Corollary (II.1.3) ([**CAMO13**])**.**

$$\{f = 0\} \text{ is a } \mathbf{Q}\text{-smooth curve} \iff \mu^w(f) = \frac{d-1}{d}.$$

Section § II.2
Local invariants on quotient singularities

II.2–1. Noether's Formula

In this section we present a version of Noether's formula for curves on quotient singularities with respect to $\mathbf{Q}$-resolutions. During this section we will use the notation introduced in Chapter I. As an immediate consequence of Proposition (I.3.7)(4) one has the following formula,

$$(C \cdot D)_{[0]} = \frac{\nu_C \nu_D}{pqd} + \sum_{Q \prec [0]} (\widehat{C} \cdot \widehat{D})_Q, \tag{13}$$

where Q runs over all infinitely near points to $[0] \in X(d; a, b)$ after a weighted (p, q)-blow-up.

By induction, using formula (13), one can prove the following generalization of Noether's formula for $\mathbb{Q}$-divisors on quotient surface singularities.

Theorem (II.2.1) (Noether's Formula, [**CAMO13**])**.** *Consider C and D two germs of $\mathbb{Q}$-divisors at $[0]$ without common components in a quotient surface singularity. Then the following formula holds:*

$$(C \cdot D)_{[0]} = \sum_{Q \prec [0]} \frac{\nu_{C,Q} \nu_{D,Q}}{pqd},$$

where Q runs over all the infinitely near points of $(C \cdot D, [0])$ and Q appears after a blow-up of type (p, q) of the origin in $X(d; a, b)$.

Remark **(II.2.2)** ([**CAMO13**])**.** Note that p, q, d, a, b in Theorem (II.2.1) depend on Q and its predecessor.

II.2–2. Definition of the δ-invariant on quotient singularities

In this section the local invariant δ^w for curve singularities on $X(d; a, b)$ is defined.

Definition (II.2.3) ([**CAMO13**])**.** Let C be a reduced curve germ at $[0] \in X(d; a, b)$. We define δ^w as the number verifying

$$\chi^{\mathrm{orb}}(F_t^w) = r^w - 2\delta^w, \tag{14}$$

where r^w is the number of local branches of C at $[0]$, F_t^w denotes its Milnor fiber, and $\chi^{\mathrm{orb}}(F_t^w)$ denotes the orbifold Euler characteristic of F_t^w.

Using the same argument as in (12), one can check that

$$\delta^w = \frac{1}{d}\delta + \frac{1}{2}\left(r^w - \frac{r}{d}\right) \tag{15}$$

where δ denotes the classical δ-invariant of C as a germ in $(\mathbb{C}^2, 0)$ and r is the number of local branches of C in $(\mathbb{C}^2, 0)$.

Remark **(II.2.4)** ([**CAMO13**])**.** At this point it is worth mentioning that r^w and r do not necessarily coincide. For instance, $x^2 - y^4$ defines an irreducible curve germ in $X(2; 1, 1)$ (hence $r^w = 1$), but it is not irreducible in $\mathbb{C}^2$ (where $r = 2$). One can check that $\delta^w = 1$, $\delta = 2$ (see Example (II.2.7)), which verify (15).

The purpose of this section is to give a recurrent formula for δ^w based on a **Q**-resolution of the singularity. For technical reasons it seems more natural to use strong **Q**-resolutions (see Remark (I.2.4)) for the statement, but this is not a restriction (see Remark (II.2.8)).

Before we state the result, let us introduce some notation. Assume $(f, 0) \subset X(d; a, b)$ and consider a (p, q) blow-up π at the origin. Denote by $\nu_0(f)$ the (p, q)-multiplicity of f at 0. We will use the following notation:

$$\delta^w_{0,\pi}(f) := \frac{\nu_0(f)}{2dpq}\left(\nu_0(f) - p - q + e\right), \tag{16}$$

The result now can be stated as.

Theorem (II.2.5) ([**CAMO13**])**.** *Let $(C, [0])$ be a curve germ on an abelian quotient surface singularity. Then*

$$\delta^w(C) = \sum_{Q \prec [0]} \delta^w_{Q,\pi_{(p,q)}}(f) \tag{17}$$

where Q runs over all the infinitely near points of a strong **Q***-resolution of* $(C,[0])$*,* $\pi_{(p,q)}$ *is a* (p,q)*-blow-up of* Q*, the origin of* $X(d;a,b)$*, and* $e := \gcd(d, aq-bp)$*.*

Proof. Since we want to proceed by induction, let us assume $[0] \in X(d;a,b)$. After a (p,q)-blow-up of $[0]$ there are three types of infinitely near points to $[0]$, namely, P_1 (resp. P_2) a point on the surface of local type $X(\frac{pd}{e}; 1, \frac{-q+a'pb}{e})$ (resp. $X(\frac{qd}{e}; \frac{-p+b'qa}{e}, 1)$) and $P_3, \ldots, P_n$ smooth points on the surface (see (6)). Outside the neighborhoods $\mathbb{B}_i$ of the points P_i, the preimage of the Milnor fiber is a covering of degree $\frac{\nu}{e}$ over E.

Therefore

$$\chi^{\mathrm{orb}}(F_t^w) = \chi^{\mathrm{orb}}(E \setminus \{P_1, \ldots, P_n\})\frac{\nu}{e} + \sum_i \chi^{\mathrm{orb}}(\mathbb{B}_i \cap \{x^{\frac{d\nu}{e}} \tilde{F}_i = t\}),$$

since the intersection is glued over puncture disks, whose Euler characteristic is zero. Here $\tilde{F}_i$ denotes the strict preimage of F at P_i.

In order to compute $\chi^{\mathrm{orb}}(\mathbb{B}_i \cap \{x^{\frac{d\nu}{e}} \tilde{F}_i = t\})$ we have to distinguish three cases: P_1, P_2, and P_i $(i = 3, \ldots, n)$.

Assume first that $P = P_i$, $i = 3, \ldots, n$, then one can push $\tilde{F}_i$ into $\tilde{F}_i'$ in a direction transversal to E as in Figure II.1, then $\chi^{\mathrm{orb}}(\mathbb{B}_i \cap \{x^{\frac{d\nu}{e}} \tilde{F}_i = t\}) = \chi((\tilde{F}_i')_t) - (E, \tilde{F}_i)_{P_i}$.

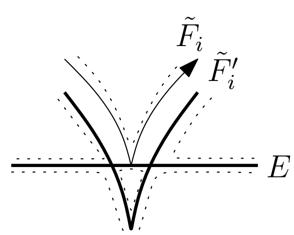


Figure II.1. Pushing $\tilde{F}_i$

In case $P = P_1 \in X(\frac{pd}{e}; 1, \frac{-q+a'pb}{e})$, one has $\chi^{\mathrm{orb}}(\mathbb{B}_1 \cap \{x^{\frac{d\nu}{e}} \tilde{F}_1 = t\}) = \frac{e}{pd}\chi(\mathbb{B}_1 \cap \{x^{\frac{d\nu}{e}\frac{1}{d}} \tilde{f}_1 = t\})$, where $\tilde{f}_1$ is the preimage of $\tilde{F}_1$ in $\mathbb{C}^2$. Therefore, after applying the pushing strategy, $\chi(\mathbb{B}_1 \cap \{x^{\frac{\nu}{e}} \tilde{f}_1 = t\}) = \frac{\nu}{e}(1 - (E, f_1)_{P_1})$, and hence

$$\chi^{\mathrm{orb}}(\mathbb{B}_1 \cap \{x^{\frac{d\nu}{e}} \tilde{F}_1 = t\}) = \frac{\nu}{e}\left(\frac{e}{pd} - (E, \tilde{F}_1)_{P_1}\right).$$

One obtains an analogous formula for P_2. Adding up all the terms and applying Proposition (I.3.7)(1) one obtains:

$$\begin{aligned}\chi^{\text{orb}}(F_t^w) &= \frac{\nu}{dp}+\frac{\nu}{dq}-\frac{e\nu}{dpq}(1+\frac{\nu}{e})+\sum_i \chi^{\text{orb}}(\tilde{F}_i)\\ &= -\frac{\nu}{dpq}(\nu-p-q+e)+\sum_i \chi^{\text{orb}}(\tilde{F}_i).\end{aligned}$$

The formula follows by induction since, after a strong **Q**-resolution, $\sum_i \chi^{\text{orb}}(\tilde{F}_i)=r^w$ and hence

$$\chi^{\text{orb}}(F_t^w)=r^w-\sum \frac{\nu}{dpq}(\nu-p-q+e)=r^w-2\delta^w.$$

□

As an immediate consequence of Theorem (II.2.5), one can obtain a recursive formula for $\delta_0^w(C)$ in terms of $\delta_Q^w(\hat{C})$ for $Q \prec 0$ which can be useful for our purposes.

Corollary (II.2.6). *Let C be a curve singularity at $0 \in X(d;a,b)$, π a weighted (p,q) blow-up and $Q_1,\dots,Q_r$ infinitely near points of C after π. Then*

$$\delta_0^w(C)=\delta_{0,\pi}^w(C)+\sum_{j=1}^{r}\delta_{Q_j}^w(\hat{C}),$$

with the notation used above.

A similar comment to Remark (II.2.2) applies to Theorem (II.2.5).

Example (II.2.7) ([**CAMO13**])**.** Assume $x^p-y^q=0$, for p, q coprime, defines a curve on a surface singularity of type $X(d;a,b)$. Note that a simple (q,p)-blow-up will be a (strong) **Q**-resolution of the singular point. Therefore, using Theorem (II.2.5) one obtains

$$\delta^w=\frac{\nu}{2dqp}(\nu-q-p+e).$$

Note that $\nu=pq$. Also, since $x^p-y^q=0$ defines a set of zeros in $X(d;a,b)$ by hypothesis, this implies $ap\equiv bq \mod d$ and hence $e:=\gcd(d,ap-bq)=d$. Thus

$$\delta^w=\frac{(pq-p-q+d)}{2d}. \tag{18}$$

Note that this provides a direct proof, for the classical case ($d=1$), that

$$\delta=\frac{(p-1)(q-1)}{2}$$

for a singularity of type x^p-y^q in $(\mathbb{C}^2,0)$.

Another direct consequence of (18) is that

$$\delta^w(x) = \delta^w(y) = \frac{d-1}{2d} \tag{19}$$

and the same formula holds for any **Q**-smooth curve (see Definition (II.1.2)).

Remark **(II.2.8).** By (19), a **Q**-resolution is enough to obtain δ^w. Note that a **Q**-resolution ends when the branches are separated and each strict transform is a **Q**-smooth curve. Therefore if (17) is applied to a (not necessarily strong) **Q**-resolution, one needs to add $\frac{d_i-1}{2d_i}$ for each local branch $\gamma_i \subset X(d_i; a_i, b_i)$, $i = 1, \ldots, r^w$.

Corollary (II.2.9) (**[CAMO13]**). *Let C and D be two reduced $\mathbb{Q}$-divisors at $[0] \in X(d; a, b)$ without common components. Then*

$$\delta^w(C \cdot D) = \delta^w(C) + \delta^w(D) + (C \cdot D)_{[0]}.$$

Proof. One has,

$$\begin{aligned} \nu_{C\cdot D}(\nu_{C\cdot D} - p - q + e) &= (\nu_C + \nu_D)(\nu_C + \nu_D - p - q + e) \\ &= \nu_C(\nu_C - p - q + e) + \nu_D(\nu_D - p - q + e) + 2\nu_C\nu_D. \end{aligned} \tag{20}$$

Dividing (20) by $2dpq$ and adding over all the infinitely near points to P one obtains,

$$\delta^w(C \cdot D) = \delta^w(C) + \delta^w(D) + \sum_{Q \prec [0]} \frac{\nu_{C,Q}\nu_{D,Q}}{pqd} = \delta^w(C) + \delta^w(D) + (C \cdot D)_{[0]}.$$

□

As an inmediate consequence of Corollary (II.2.9) after applying induction one has the following result.

Lemma (II.2.10). *Let C_i, $i = 1 \ldots, n \in \mathbb{N}$ be reduced $\mathbb{Q}$-divisors at $[0] \in X(d; a, b)$ without common components. Then*

$$\delta^w(C_1 \cdot \ldots \cdot C_n) = \sum_{i=1}^{n} \delta^w(C_i) + \sum_{i \neq j} (C_i \cdot C_j)_{[0]}.$$

Example (II.2.11). Let $x = 0$ and $y = 0$ on $X(d; a, b)$, by (19) and Corollary (II.2.9) one has

$$\delta^w(xy) = \delta^w(x) + \delta^w(y) + (x \cdot y)_{[0]} = 2\frac{d-1}{2d} + \frac{1}{d} = 1.$$

Definition (II.2.12). Let E be a reduced $\mathbb{Q}$-divisor. We define

$$\delta^w_E := \sum_{P \in E} \delta^w_P(E).$$

Note that the sum is finite since $\delta^w_P(E) = 0$ for non-singular points.

Example (II.2.13). Recall that after a weighted blow-up at the origin of $X(d;a,b)$ with respect to $w=(p,q)$ (see 6) one has

$$\widehat{X(d;a,b)}_w = X\left(\frac{pd}{e};1,\frac{-q+a'pb}{e}\right) \cup X\left(\frac{qd}{e};\frac{-p+b'qa}{e},1\right). \tag{21}$$

Let us compute δ^w_E, where $E=\pi^{-1}_{(d;a,b),w}(0)$ is the exceptional divisor after a weighted blow-up of type w at the origin of $X(d;a,b)$. Using (19), (21), and Definition (II.2.12), one has

$$\delta^w_E = \frac{\frac{qd}{e}-1}{\frac{2qd}{e}} + \frac{\frac{pd}{e}-1}{\frac{2pd}{e}} = 1 - \frac{(p+q)e}{2pqd}.$$

Using Corollary (II.2.6), note that one can obtain δ^w from a sequence of weighted blow-ups such that the strict preimage of C is **Q**-smooth, which is weaker than asking for a (strong) **Q**-resolution as follows:

$$\delta^w = \frac{1}{2}\sum_{Q\prec[0]} \frac{\nu_Q}{dpq}\left(\nu_Q - p - q + e\right) + \delta^w_{\hat{C}}, \tag{22}$$

where $\delta^w_{\hat{C}}$ is a finite sum of terms of the form $\frac{d_i-1}{2d_i}$, for $Q_i \in X(d_i;a_i,b_i)$.

Section § II.3
The δ^w-invariant

In the classical case this invariant can be interpreted as the dimension of a vector space. Since, in general, δ^w is a rational number, a similar result cannot be expected for any germ. The following section is devoted to proving that when the germ is defined by a function, the number δ^w is a non-negative integer and in fact it can also be interpreted as the dimension of a vector space. After that, in the second part we present a discussion on the δ^w-invariant for general germs.

II.3–1. The δ^w-invariant for function germs

Let us start with the following constructive result which allows one to see any singularity on the quotient $X(d;a,b)$ as the strict transform of some $\{g=0\}\subset\mathbb{C}^2$ after performing a certain weighted blow-up.

Remark **(II.3.1).** The Weierstrass division theorem states that given $f,g\in\mathbb{C}\{x,y\}$ with f y-general of order k, there exist $q\in\mathbb{C}\{x,y\}$ and $r\in\mathbb{C}\{x\}[y]$

of degree in y less than or equal to $k-1$, both uniquely determined by f and g, such that $g = qf + r$. The uniqueness and the linearity of the action ensure that the division can be performed equivariantly for the action of $\boldsymbol{G}_d$ on $\mathbb{C}\{x,y\}$ (see (3)), i.e. if $f, g \in \mathcal{O}(l)$, then so are q and r. In other words, the Weierstrass preparation theorem still holds for zero sets in $\mathbb{C}\{x,y\}^{\boldsymbol{G}_d}$.

Let $\{f = 0\} \subset (X(d;a,b),0)$ be a reduced analytic germ. Assume $(d;a,b)$ is a normalized type. After a suitable change of coordinates of the form $X(d;a,b) \to X(d;a,b)$, $[(x,y)] \mapsto [(x+\lambda y^k, y)]$ where $bk \equiv a \mod d$, one can assume $x \nmid f$. Moreover, by Remark (II.3.1), f can be written in the form

$$f(x,y) = y^r + \sum_{i>0,\ j<r} a_{ij} x^i y^j \ \in\ \mathbb{C}\{x\}[y]. \tag{23}$$

For technical reasons, in the following results the space $X(d;a,b)$ will be considered to be of type $X(p;-1,q)$. Note that this is always possible by changing the action of the variables, which also preserves the condition $x \nmid f$.

Lemma (II.3.2). *Let $f \in \mathcal{O}(k)$ define an analytic germ on $X(p;-1,q)$, $\gcd(p,q) = 1$, such that $x \nmid f$. Then there exist $g \in \mathbb{C}\{x,y\}$ with $x \nmid g$ such that $g(x^p, x^q y) = x^{qr} f(x,y)$. Moreover, f is reduced (resp. irreducible) if and only if g is.*

Proof. By the discussion after Remark (II.3.1) one can assume $f \in \mathbb{C}\{x\}[y]$ as in (23). We have $-i + qj \equiv qr \equiv k \mod p$ for all i,j so $p | (i + q(r-j))$ and $i + q(r-j) > 0$. Consider

$$g(x,y) = y^r + \sum_{i>0,\ j<r} a_{ij} x^{\frac{i+q(r-j)}{p}} y^j \ \in\ \mathbb{C}\{x\}[y].$$

$$g(x^p, x^q y) = x^{qr} y^r + \sum_{i>0,\ j<r} a_{ij} x^{i+qr} y^j = x^{qr} \left(y^r + \sum_{i>0,\ j<r} a_{ij} x^i y^j \right).$$

Note that the strict transform passes only through the origin of the first chart. □

As an immediate result one has.

Corollary (II.3.3) ([CAMO13]). *Let $f \in \mathbb{C}\{x\}[y]$ defining an analytic function germ on $X(p;-1,q)$, $\gcd(p,q) = 1$, such that $x \nmid f$. Then there exist $g \in \mathbb{C}\{x\}[y]$ with $x \nmid g$ and $\nu \in \mathbb{N}$ multiple of p such that $g(x^p, x^q y) = x^\nu f(x,y)$.*

Proof. In this case one can write $f = \sum_{i,j} a_{ij} x^i y^j$. Since every monomial of f is $\boldsymbol{G}_p$-invariant, p divides $-i + qj$ for all i, j. Consider

$$g_1(x,y) = \sum_{i,j} a_{ij} x^{\frac{i-qj}{p}} y^j \ \in \ \mathbb{C}\{x^{\pm 1}\}[y]$$

and take the minimal non-negative integer ℓ such that $g(x,y) := x^\ell g_1(x,y)$ is an element of $\mathbb{C}\{x\}[y]$. Note that ℓ exists because f is a polynomial in y.

Then $g(x^p, x^q y) = x^{p\ell} f(x,y)$ and the minimality of ℓ ensures that $x \nmid g$. ☐

Let $I_{p,q} := \overline{\langle x^q, y^p \rangle} \subset \mathbb{C}\{x,y\}$ be the integral closure of $\langle x^q, y^p\rangle$ in $\mathbb{C}\{x,y\}$ with $\gcd(p,q) = 1$. Note that the integral closure of a monomial ideal is well understood. In fact, the exponent set of the integral closure of a monomial ideal equals all the integer lattice points in the convex hull of the exponent set of such an ideal, (see for instance [**HS06**, Proposition 1.4.6]). In our case, $I_{p,q}^n = \overline{\langle x^q, y^p\rangle^n}$ and the following property holds for any $h \in \mathbb{C}\{x,y\}$,

$$h \in I_{p,q}^n \iff \nu_{p,q}(h) \geq pqn. \tag{24}$$

Lemma (II.3.4) ([**CAMO13**]). *Assume $f \in \mathbb{C}\{x,y\}$ has (p,q)-order ν a multiple of pq. Then, for all $n \geq \frac{\nu}{pq}$, one has*

$$\dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{x,y\}}{I_{p,q}^n + \langle f \rangle} \right) = n\nu - \frac{\nu(\nu - p - q + 1)}{2pq}.$$

Proof. Since $pq \,|\, \nu$, by (24), the multiplication by f gives rise to the short exact sequence for all $n \geq \frac{\nu}{pq}$,

$$0 \longrightarrow \frac{\mathbb{C}\{x,y\}}{I_{p,q}^{n-\frac{\nu}{pq}}} \xrightarrow{\cdot f} \frac{\mathbb{C}\{x,y\}}{I_{p,q}^n} \longrightarrow \frac{\mathbb{C}\{x,y\}}{I_{p,q}^n + \langle f\rangle} \longrightarrow 0.$$

Note that, by virtue of Pick's Theorem and (24), the dimension of the middle term in the short exact sequence above is given by the formula

$$F(n) := \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{x,y\}}{I_{p,q}^n} \right) = pq\, \frac{n(n+1)}{2} - \frac{(p-1)(q-1)}{2}\, n.$$

Finally, the required dimension is $F(n) - F(n - \frac{\nu}{pq})$ and the claims follow. ☐

Although the following results can be stated in a more general setting, considering a direct sum of vector spaces in the ring R^1 (see below), we proceed in this way for the sake of simplicity.

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an analytic function germ and $C := \{f = 0\}$. Consider the weighted blow-up at the origin with $\gcd(p,q) = 1$. Assume the exceptional divisor E and the strict transform $\widehat{C}$ intersect just at the origin of the first chart $X(p; -1, q)$ and the latter divisor is given by a well-defined function $\widehat{f}$ on the quotient space. Also denote by $R = \frac{\mathbb{C}\{x,y\}}{\langle f\rangle}$ and $R^1 = \frac{\mathcal{O}_X}{\langle \widehat{f}\rangle}$ their corresponding local rings.

Assume f is a Weierstrass polynomial in y of degree b with $\nu := \nu_{p,q}(f) = qb$. Then its strict transform $\widehat{f}$ is again a Weierstrass polynomial in y of the same degree. Classical arguments using the Weierstrass division theorem, see for instance [**CA00**, Theorem 1.8.8], provide the following isomorphisms $R \simeq \frac{\mathbb{C}\{x\}[y]}{\langle f\rangle}$ and $R^1 \simeq \frac{(\mathbb{C}\{x\}[y])^{\boldsymbol{G}_p}}{\langle \widehat{f}\rangle}$, which allow one to prove that the pull-back morphism

$$\begin{array}{rccc} \varphi: & R & \longrightarrow & R^1 \\ & h(x,y) & \mapsto & h(x^p, x^q y) \end{array}$$

is in fact injective. Hereafter R is identified with a subring of R^1 and thus one simply writes $R \subset R^1$.

Lemma (II.3.5) ([**CAMO13**])**.** *For all $n \gg 0$,*

$$R\, I_{p,q}^n = R^1 x^{pqn}.$$

Proof. By (24), the ideal $I_{p,q}^n$ is generated by all monomials $x^i y^j$ with $pi + qj \geq pqn$. Each monomial $x^i y^j$ is converted under φ in $x^{pi+qj} y^j = x^{pi+qj-pqn} y^j \cdot x^{pqn}$ which belongs to $R^1 x^{pqn}$.

For the other inclusion, given $g = \sum_{i,j} a_{ij} x^i y^j \cdot x^{pqn} \in (\mathbb{C}\{x\}[y])^{\boldsymbol{G}_p} x^{pqn}$, consider as in the proof of Corollary (II.3.3),

$$h(x,y) = \sum_{i,j} a_{ij} x^{\frac{pqn+i-qj}{p}} y^j \;\in\; \mathbb{C}\{x^{\pm 1}\}[y]$$

which is an element of $\mathbb{C}\{x\}[y]$ for all $n \gg 0$. The (p,q)-order of each monomial of h is greater than or equal to pqn hence they are in $I_{p,q}^n$ by (24). Finally, it is clear that $\varphi(h + \langle f\rangle) = g + \langle \widehat{f}\,\rangle$ which concludes the proof. $\square$

Proposition (II.3.6) ([**CAMO13**])**.** *Using the previous conventions and assumptions, the order $\nu := \nu_{p,q}(f)$ is a multiple of pq and*

$$\dim_{\mathbb{C}} \left(\frac{R^1}{R} \right) = \frac{\nu(\nu - p - q + 1)}{2pq} \in \mathbb{N}.$$

Proof. Since E and $\widehat{C}$ only intersect at the origin of the first chart, the (p,q)-initial part of f is of the form $f_\nu = \lambda y^b$, $\lambda \in \mathbb{C}^*$; thus $q \,|\, \nu$. On the other hand, $f(x^p, x^q y) = x^\nu \widehat{f}(x,y)$ and $\widehat{f}(x,y)$ define functions on $X(p;-1,q)$ and hence x^ν is $\boldsymbol{G}_p$-invariant. Consequently, $p \,|\, \nu$ and also $pq \,|\, \nu$ because p and q are coprime.

For the second part of the statement, by Lemma (II.3.5), there is a short exact sequence, for all $n \gg 0$,

$$0 \longrightarrow \frac{R}{RI_{p,q}^{n}} \longrightarrow \frac{R^1}{R^1 x^{pqn}} \longrightarrow \frac{R^1}{R} \longrightarrow 0.$$

The dimension of the first vector space is calculated in Lemma (II.3.4). The second one is a consequence of $E \cdot \widehat{C} = \frac{\nu}{pq}$, see Proposition (I.3.7)(2),

$$\dim_{\mathbb{C}} \left(\frac{R^1}{R^1 x^{pqn}} \right) = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{x,y\}^{\boldsymbol{G}_p}}{\langle \widehat{f}, x^{pqn} \rangle} \right) = pqn\, E \cdot \widehat{C} = n\nu. \qquad \square$$

Now we are ready to state the main result of this section which allows one to interpret the invariant δ^w as the dimension of a vector space given by the normalization of a singularity when defined by a function.

Theorem (II.3.7) ([**CAMO13**]). *Let $f : (X(d;a,b),0) \to (\mathbb{C},0)$ be a reduced analytic function germ. Assume $(d;a,b)$ is a normalized type. Consider $R = \frac{\mathcal{O}_{X,0}}{\langle f \rangle}$ the local ring associated with f and $\overline{R}$ its normalization ring. Then,*

$$\delta_0^w(f) = \dim_{\mathbb{C}} \left(\frac{\overline{R}}{R} \right) \in \mathbb{N}.$$

Proof. By Remark (II.3.1) and the discussion after it, we can assume that

$$f(x,y) = y^r + \sum_{i>0\leq j<r} a_{ij} x^i y^j \ \in \ \mathbb{C}\{x\}[y]. \tag{25}$$

in $X(p;-1,q)$ ($p = d, q \equiv -ba^{-1} \mod d$).

Consider $g \in \mathbb{C}\{x,y\}$ the reduced germ obtained after applying Lemma (II.3.3) to f. Denote by $R^{-1} = \frac{\mathbb{C}\{x,y\}}{\langle g \rangle}$ its corresponding local ring and by $\pi_{(p,q)}$ the blowing-up at the origin of $(\mathbb{C}^2,0)$.

By Corollary (II.2.6) one has the following:

$$\delta_0(g) = \delta^w_{\pi_{(p,q)}}(g) + \delta_0^w(f).$$

Since $\overline{R}$ is also the normalization of g because $R^{-1} \subset R \subset \overline{R}$, by the classical case

$$\delta_0(g) = \dim_{\mathbb{C}} \left(\frac{\overline{R}}{R^{-1}} \right) = \dim_{\mathbb{C}} \left(\frac{\overline{R}}{R} \right) + \dim_{\mathbb{C}} \left(\frac{R}{R^{-1}} \right).$$

Since f defines a function, hypotheses of Proposition (II.3.6) are satisfied. Hence $\delta^w_{\pi_{(p,q)}}(g) = \dim_{\mathbb{C}} \frac{R}{R^{-1}}$ and the proof is completed. $\square$

II.3–2. The δ^w-invariant: the general case of local germs

We have already studied in §II.3–1 that the δ^w-invariant can be interpreted as the dimension of a vector space in certain cases, namely, when defined by a function ([**CAMO13**]). In general, δ^w is a rational number.

In this section we will give, in some way, a generalization of this result. Let us start with some basic definitions.

Let $\{f = 0\}$ be a germ in $P \in X(d; a, b)$, note that if $f \in \mathcal{O}_P(k)$, then one has the following $\mathcal{O}_P$-module, $\mathcal{O}_P(k-a-b)$, verifying

$$\mathcal{O}_P(k-a-b) = \{h \in \mathbb{C}\{x,y\} \mid h\frac{dx \wedge dy}{f} \text{ is } \boldsymbol{G}_d\text{-invariant}\}.$$

Definition (II.3.8). Let $\mathcal{D} = \{f = 0\}$ be a germ in $P \in X(d; a, b)$, where $f \in \mathcal{O}_P(k)$. Consider π a $\mathbf{Q}$-resolution of $(\mathcal{D}, P)$.

(1) Let $\mathcal{M}^{log}_{\mathcal{D},\pi}$ denote the submodule of $\mathcal{O}_P$ consisting of all $h \in \mathcal{O}_P$ such that the 2-form
$$\omega = h\ \frac{dx \wedge dy}{f} \in \Omega^2_P(a+b-k)$$
is logarithmic at P, with respect to $\mathcal{D}$ and the embedded $\mathbf{Q}$-resolution π (recall Definition (I.5.8)).

(2) Let $\mathcal{M}^{nul}_{\mathcal{D},\pi}$ denote the submodule of $\mathcal{M}^{log}_{\mathcal{D},\pi}$ consisting of all $h \in \mathcal{M}^{log}_{\mathcal{D},\pi}$ such that the 2-form
$$\omega = h\ \frac{dx \wedge dy}{f}$$
admits a holomorphic extension outside the strict transform $\widehat{f}$.

(3) Any module $\mathcal{M} \subseteq \mathcal{M}^{log}_{\mathcal{D},\pi}$ wil be called *logarithmic module.*

Definition (II.3.9). Let $\mathcal{D} = \{f = 0\}$ be a germ in $P \in X(d; a, b)$. Let us define the following dimension,

$$K_P(\mathcal{D}) = K_P(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\mathcal{M}^{nul}_{\mathcal{D},\pi}}.$$

Remark **(II.3.10).** Note that $K_P(f)$ turns out to be a finite number (see Remark (III.5.2)) independent on the chosen $\mathbf{Q}$-resolution. Intitutively, the number $K_P(f)$ provides the minimal number of conditions required for a generic germ $h \in \mathcal{O}_P(s)$ so that $h \in \mathcal{M}^{nul}_{\mathcal{D},\pi}(s)$.

Remark **(II.3.11).** It is known (see [**CA02**, Chapter 2]) that if f is a holomorphic germ in $(\mathbb{C}^2, 0)$, then

$$K_0(f) = \delta_0(f).$$

The following Proposition (II.3.12) will be useful to give a generalization of Remark (II.3.11).

Before we state the result we need some notation. Given $r, p, q \in \mathbb{Z}_{>0}$ we define the following the combinatorial number which generalizes $\binom{d}{2}$:

$$\delta_r^{(p,q)} := \frac{r(qr - p - q + 1)}{2p}. \tag{26}$$

Note that $\binom{d}{2} = \delta_d^{(1,1)}$.

Proposition (II.3.12). *Let be $p, q, a, r \in \mathbb{N} \setminus \{0\}$ with $\gcd(p, q) = 1$ and $r_1 = r + pa$. Consider the following cardinal,*

$$A_r^{(p,q)} := \#\{(i, j) \in \mathbb{N}^2 \mid pi + qj \leq qr;\ i, j \geq 1\}.$$

Then,

1) If $r = pa$, one has $\delta_r^{(p,q)} = A_r^{(p,q)}$.
2) The following equalities hold:

$$\delta_{r_1}^{(p,q)} - \delta_r^{(p,q)} = \delta_{r_1-r}^{(p,q)} + aqr, \tag{27}$$

$$A_{r_1}^{(p,q)} - A_r^{(p,q)} = A_{r_1-r}^{(p,q)} + aqr. \tag{28}$$

PROOF.

1) To prove this fact it is enough to apply Pick's Theorem (see for example [**BR07**, §2.6]) noticing that the number of points on the diagonal without counting the ones in the axes is $a - 1$ (see Figure II.2).

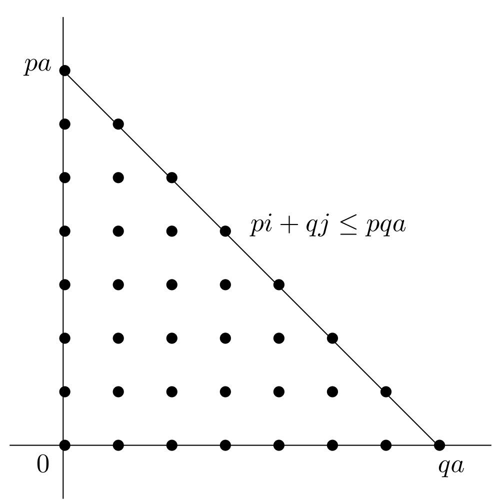


FIGURE II.2.

Finally one gets,

$$A_{pa}^{(p,q)} = \frac{a(pqa - p - q + 1)}{2} = \delta_{pa}^{(p,q)}.$$

2) Proving equation (27) is simple and direct computation. To prove equation (28), let us describe $A_{r_1}^{(p,q)}$, $A_r^{(p,q)}$ and $A_{r_1-r}^{(p,q)}$:

$$A_{r_1}^{(p,q)} = \#\{(i,j) \in \mathbb{N}^2 |\ pi + qj \leq qr + pqa;\ i,j \geq 1\}.$$

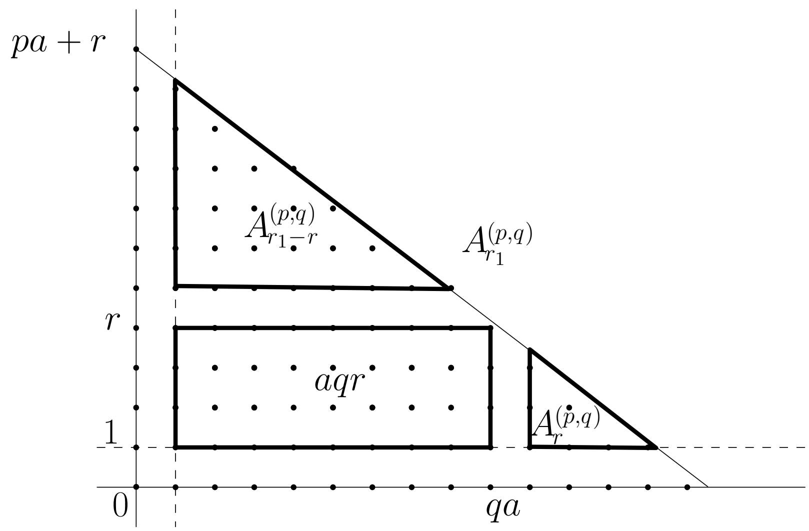


Figure II.3.

$$\begin{aligned} A_r^{(p,q)} &= \#\{(i,j) \in \mathbb{N}^2 |\ pi + qj \leq qr;\ i,j \geq 1\} \\ &= \#\{(i,j) \in \mathbb{N}^2 |\ pi + qj \leq qr + apq - apq;\ i,j \geq 1\} \\ &= \#\{(i,j) \in \mathbb{N}^2 |\ p(i + aq) + qj \leq qr_1;\ i \geq 1,\ j \geq 1\} \\ &= \#\{(i,j) \in \mathbb{N}^2 |\ pi + qj \leq qr_1;\ i \geq aq + 1,\ j \geq 1\}. \end{aligned}$$

$$\begin{aligned} A_{r_1-r}^{(p,q)} &= \#\{(i,j) \in \mathbb{N}^2 |\ pi + qj \leq qr_1 - qr;\ i,j \geq 1\} \\ &= \#\{(i,j) \in \mathbb{N}^2 |\ pi + q(j + r) \leq qr_1;\ i,j \geq 1\} \\ &= \#\{(i,j) \in \mathbb{N}^2 |\ p + qj \leq qr_1;\ i \geq 1,\ j \geq r + 1\}. \end{aligned}$$

To conclude the proof it is enough to apply Pick's Theorem again and look at Figure II.3 which represent the situation one has.

□

As a result of this Proposition (II.3.12) one has the following result.

Theorem (II.3.13). *Let $f_1, f_2 \in \mathcal{O}(k)$ be two germs at $[0] \in X(d;a,b)$. Then,*

$$K_0(f_1) - K_0(f_2) = \delta_0^w(f_1) - \delta_0^w(f_2).$$

Proof. By Remark (II.3.1) and the discussion after it, we can assume that

$$f_\ell(x,y) = y^{r_\ell} + \sum_{i>0\leq j<r_\ell} a_{ij}x^i y^j \ \in\ \mathbb{C}\{x\}[y].$$

in $X(p;-1,q)$ ($p = d, q \equiv -ba^{-1} \mod d$). Consider $g_1 \in \mathbb{C}\{x,y\}$ the reduced germ obtained after applying Lemma (II.3.2) to f_1. Denote by $\pi_{(p,q)}$ the blowing-up at the origin. Note that $\nu_{p,q}(g_1) = qr_1$, and thus $\delta_{r_1}^{(p,q)} = \delta^w_{\pi_{(p,q)}}(g_1)$ (see (26) and (16)).

Consider the form $\omega := \phi\frac{dx\wedge dy}{g_1}$, $\phi \in \mathbb{C}\{x,y\}$ and let us calculate the local equations for the pull-back of ω after blowing-up the origin on $\mathbb{C}^2$,

$$\phi\frac{dx\wedge dy}{g_1} \stackrel{\pi_{(p,q)}}{\longleftarrow} x^{\nu_\phi+p+q-1-qr_1}h\frac{dx\wedge dy}{f_1}. \tag{29}$$

Using the definitions of $\mathcal{M}^{nul}_{g_1}$ and $\mathcal{M}^{nul}_{f_1}$ (see Definition (II.3.8)) this implies that

$$\phi \in \mathcal{M}^{nul}_{g_1} \Leftrightarrow h \in \mathcal{M}^{nul}_{f_1}(\nu_\phi), \text{ and } \nu_\phi + p + q - 1 - qr_1 \geq 0.$$

Therefore $\phi(x,y) \mapsto \phi(x^p, x^q y)$ induces an isomorphism

$$\mathcal{M}^{nul}_{g_1} \cong \mathcal{A}^{(p,q)}_{r_1} \cap \mathcal{M}^{nul}_{f_1},$$

where $\mathcal{A}^{(p,q)}_{r_1} := \{h \in \mathbb{C}\{x,y\} \mid \operatorname{ord}_h + p + q - 1 - qr_1 \geq 0\}$ and ord_h is the order of h. Since $\dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\mathcal{A}^{(p,q)}_{r_1}} = A^{(p,q)}_{r_1}$, one obtains

$$K_0(g_1) = A^{(p,q)}_{r_1} + K_0(f_1) \tag{30}$$

On the other hand (see Remark (II.3.11) and Corollary (II.2.6)),

$$K_0(g_1) = \delta_0(g_1) = \delta^w_{\pi_{(p,q)}}(f) + \delta_0^w(f_1) = \delta^{(p,q)}_{r_1} + \delta^w_0(f_1). \tag{31}$$

Therefore, from (30) and (31),

$$K_0(f_1) = \delta^w_0(f_1) + \delta^{(p,q)}_{r_1} - A^{(p,q)}_{r_1}. \tag{32}$$

Following a similar procedure we get,

$$K_0(f_2) = \delta^w_0(f_2) + \delta^{(p,q)}_{r_2} - A^{(p,q)}_{r_2}. \tag{33}$$

Notice that $k \equiv qr_1 \equiv qr_2 \mod p$, which implies $r_1 \equiv r_2 \mod p$ since p and q are coprime. Therefore by Proposition (II.3.12)

$$A^{(p,q)}_{r_1} - \delta^{(p,q)}_{r_1} = A^{(p,q)}_{r_2} - \delta^{(p,q)}_{r_2},$$

and finally from (32) and (33) it can be concluded that

$$K_0(f_1) - K_0(f_2) = \delta_0^w(f_1) - \delta_0^w(f_2).$$

□

From Proposition (II.3.12)(1) and (32) one has the following result which generalizes Remark (II.3.11).

Corollary (II.3.14). *If $(f, [0])$ is a function germ on $X(d; a, b)$, then*

$$K_0(f) = \delta_0^w(f).$$

The result in Theorem (II.3.13) motivates the following definition.

Definition (II.3.15). Let $\{f = 0\}$ be a germ in $0 \in X(d; a, b)$, where $f \in \mathcal{O}_0(k)$. Consider $g \in \mathcal{O}_0(k)$ generic. The Δ_0^w-*invariant* is defined as follows

$$\Delta_0^w(f) := \delta_0^w(f) - \delta_0^w(g).$$

Remark **(II.3.16).** Notice that $\Delta_P^w(f)$ is always a well-defined integer. Note that if g is a generic holomorphic germ in $(\mathbb{C}^2, P)$ (and thus smooth), then $\delta_P^w(g) = 0$. Hence, if f is a holomorphic germ in $(\mathbb{C}^2, P)$ then

$$\Delta_P^w(f) = \delta_P(f).$$

III

Logarithmic Trees

Some basic definitions of how to construct *logarithmic modules* (Definition (II.3.8)) and *trees* (Definition (III.1.10)) associated with a **Q**-resolution will be given in the following sections. They extend those given in [**CA02**, §2.5 and §2.6]. In this chapter we also define some new logarithmic modules associated with a germ $f \in (X(d;a,b),P)$ and a **Q**-resolution π, recall the ones already defined in Chapter II (Definition (II.3.8)). A useful description of these logarithmic modules will result from the use of multiplicity trees. Their global sections will allow for the construction of logarithmic 2-forms on $\mathcal{D} \subset \mathbb{P}^2_w$ with respect to π. All these results will allow us in Chapter V to provide a (rational) presentation for the cohomology ring of $\mathbb{P}^2_w \setminus \mathcal{R}$, where $\mathcal{R}$ is a reduced algebraic curve (possibly singular) in the complex projective plane $\mathbb{P}^2_w$ whose irreducible components are all rational.

Section § III.1
Construction of logarithmic trees: $\mathcal{T}_P^{nul}$

Let us start this section with some technical definitions about the construction of multiplicity trees. We will end this section describing a basic example of a logarithmic tree.

Remark **(III.1.1).** For the sake of simplicity and if no ambiguity seems no likely to arise, when refering to a (p_P, q_P)-weighted blow-up of a point P, the subindex P will be omitted. Analogously, the type of surface singularity appearing after weighted blow-ups $X(d;a,b)$ will be omitted when possible.

Notation (III.1.2). Let f be an analytic germ at $P \in X_0 = X(d;a,b)$ whose set of zeros is a germ of curve $V_f \subset X_0$. Consider the sequence of weighted blow-ups

$$X_0 \overset{\varepsilon_1}{\longleftarrow} X_1 \overset{\varepsilon_2}{\longleftarrow} X_2 \overset{\varepsilon_3}{\longleftarrow} \dots \overset{\varepsilon_m}{\longleftarrow} X_m = \hat{X}$$

in a **Q**-resolution of X_0 at P.

- Denote by π_k the composition of the first k weighted blow-ups $\pi_k = \varepsilon_k \circ \dots \circ \varepsilon_1$.
- The germ of curve $\hat{V}_{f,k} = \pi_k^{-1}(V_f \setminus \{P\})$ shall be called the *strict transform of* V_f *in* X_k and its equation denoted by $\hat{f}_k$.
- The reduced divisor $\pi_k^*(V_f)_{red}$ shall be denoted by $\overline{V}_{f,k}$ and called the *total transform of* V_f *in* X_k. For simplicity let us write $\hat{V}_f := \hat{V}_{f,m}$ and $\overline{V}_f := \overline{V}_{f,m}$.
- The exceptional divisor in X_k resulting from the weighted blow-up of a point in X_{k-1} shall be denoted by E_k and the points $P_k^1, \dots, P_k^{N_k}$ in $E_k \cap \hat{V}_{f,k}$ will be called the *infinitely near points to* P *in* E_k. For convenience, the point P is also considered to be infinitely near to itself.
- For the sake of simplicity, if a (p,q)-weighted blow-up occurs at a point Q we shall denote by $\nu_Q(f)$ its (p,q)-weighted multiplicity, omitting the weights if no ambiguity seems no likely to arise.

Definition (III.1.3) (Multiplicity tree of π at P)**.** Let π be a **Q**-resolution of singularities. Let us construct the *multiplicity tree of* π *at* P which will be denoted by $\mathcal{T}_P(\pi, f)$, or simply by $\mathcal{T}_P(f)$ if the **Q**-resolution π of X_0 is fixed. $\mathcal{T}_P(f)$ is a labeled tree with triples (w, ℓ_1, ℓ_2) at each vertex and is defined as follows.

T_1. The vertices of $\mathcal{T}_P(f)$ are in bijection with the infinitely near points to P (for simplicity we shall denote the vertices of the tree by their corresponding infinitely near points).

T_2. Two vertices of $\mathcal{T}_P(f)$, say Q and Q', are joined by an edge if and only if: one of the points, say Q', belongs to X_k; the other point Q belongs to X_{k-1} and $Q' \in \varepsilon_k^{-1}(Q) = E_k$.

T_3. For convenience, this tree is considered to simply be a vertex if X_0 is smooth and P is not a singular point of f. If $f(P) \neq 0$, then $\mathcal{T}_P(f) := \emptyset$.

T_4. The weight w of the label at a vertex Q will be defined as $\nu_Q(f)$ and denoted by $w(\mathcal{T}_P(f), Q) = \nu_Q(f)$. If a (p,q)-weighted blow-up occurs at a point Q of type $X(d; a, b)$, its labels will be $\ell_1 = (p,q)$, $\ell_2 = (d; a, b)$. For technical reasons, if no blow-up occurs at a point Q of type $X(d; a, b)$, then Q must be a **Q**-smooth point and its label will be $(\nu_Q(f), (1,1), (d; 1,1))$ or simply $(\nu_Q(f), (1,1), (d))$.

Notice that both $\operatorname{mult}_Q(\hat{V}_{f,k}, E_k) = \dfrac{1}{d}$ and $\delta_Q(E_k) = \delta_Q(\hat{V}_{f,k}) = \dfrac{d-1}{2d}$ do not depend on (p,q) and (a,b).

T_5. Depending on the context, for the sake of simplicity and if no ambiguity seems likely to arise, we might only write in each vertex Q of the tree the weight $w(\mathcal{T}_P(f), Q)$.

Let us define an extension of these multiplicity trees.

Definition (III.1.4) (Extended multiplicity tree at P)**.** The *extended multiplicity tree at* P will be denoted by $\tilde{\mathcal{T}}_P(f)$. It contains the multiplicity tree $\mathcal{T}_P(f)$ as a subtree and it can be constructed as follows.

$\tilde{T}_1$. The vertices of $\tilde{\mathcal{T}}_P(f)$ correspond to the points of $\operatorname{Sing}(\hat{X})$. Note that each extra vertex can be of two types:
 (a) Those corresponding to a non-infinitely near point in $\operatorname{Sing}(\hat{X})$ in the intersection of two exceptional divisors, say E_i and E_j, which shall be denoted by e_{ij} with the convention $i < j$.
 (b) Those corresponding to a non-infinitely near point in $\operatorname{Sing}(\hat{X})$ which only belong to one exceptional divisor, say E_i which shall be denoted by e_{i_n}, where $n \in \{1, 2\}$. Note that there are at most two of these points on each exceptional divisor E_i. The corresponding vertices will be shown in the tree as $\circ$ instead of the usual solid vertices $\bullet$.

Note that the singular points of type (b) are necessarily **Q**-smooth in $\bar{V}_f$.

$\tilde{T}_2$. A vertex of type (a), say e_{ij}, will be joined to a vertex Q of $\mathcal{T}_P(f)$ if Q belongs to X_j and $e_{ij} \in \varepsilon_j^{-1}(Q)$. A vertex of type (b), say e_{i_n} is joined to a vertex Q of $\mathcal{T}_P(f)$ if Q belongs to X_i and $e_{i_n} \in \varepsilon_i^{-1}(Q)$. These new edges will be shown on the extended tree as dashed segments.

$\tilde{T}_3$. The corresponding label at each e_{ij} or e_{i_j} shall be $(0, (1,1), (d; 1, 1))$ or simply $(0, (1,1), (d))$. The triples at the remaining vertices coincide with those as vertices of $\mathcal{T}_P(f)$.

Remark **(III.1.5).** Notice that existence of points of type $\tilde{T}_1$.(b) in Definition (III.1.4) constitutes a remarkable difference with the classical construction in [**CA02**, §2.5].

Example (III.1.6). Assume $\gcd(p,q) = 1$ and $p < q$. Let $f = (x^p + y^q)(x^q + y^p)$ be a germ of curve singularity at $0 \in \mathbb{C}^2$. Recall Examples (I.2.6) and (I.3.8). Figure III.1 is a possible graph for $\tilde{\mathcal{T}}_0(f)$.

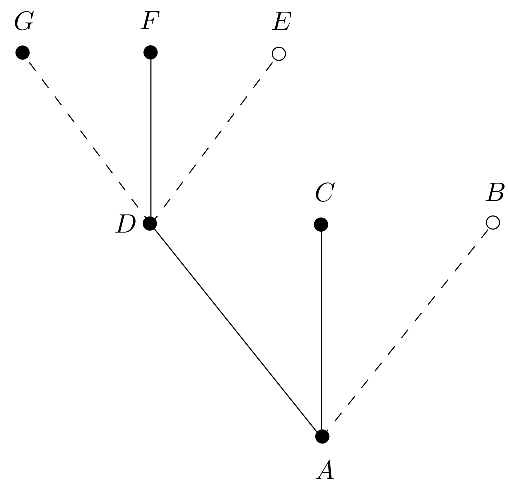


FIGURE III.1. Extended multiplicity tree $\tilde{\mathcal{T}}_0(f)$.

$$\begin{aligned}
A &= (p(p+q),(q,p),(1;1,1)).\\
B &= (0,(1,1),(p;1,1)).\\
C &= (1,(1,1),(1;1,1)).\\
D &= (p(p+q),(p,q^2-p^2),(q;-1,p)).\\
E &= (0,(1,1),(p;1,1)).\\
F &= (1,(1,1),(1;1,1)).\\
G &= (0,(1,1),(q^2-p^2;1,1)).
\end{aligned}$$

Let us see now some basic definitions coming from graph theory.

Definition (III.1.7). The set of vertices $|\tilde{\mathcal{T}}_P(f)|$ of an extended multiplicity tree $\tilde{\mathcal{T}}_P(f)$ is endowed with a partial order as follows. Consider the preferred point (*root*), P and direct the edges of the tree towards P. In this directed tree, a point Q is said to be greater than Q' (denoted $Q \geq Q'$) if there is a directed path from Q to Q'. That means that all arrows are pointing in the same direction. In graph theory this situation is commonly described by calling Q an *ancestor* of Q', or Q' a *descendant* of Q. Given a set of points $\{P_1, ..., P_n\} \subset \tilde{\mathcal{T}}_P(f)$ one can define

$$\mathrm{Asc}(P_1, ..., P_n) = \{Q \in \mathcal{T}_P(f) \mid Q \geq P_i \quad i = 1, ..., n\},$$

$$\mathrm{Desc}(P_1, ..., P_n) = \{Q \in \mathcal{T}_P(f) \mid Q \leq P_i \quad i = 1, ..., n\}.$$

Remark **(III.1.8).** Multiplicity trees are *quasi-strongly connected trees*, which means that the set of common descendants $\mathrm{Desc}(P_1, ..., P_n)$ is non vacuous and inherits a linear order from $\mathcal{T}_P(f)$. The maximal element in

$\operatorname{Desc}(P_1, ..., P_n)$ is called the *greatest common descendant* and it is denoted by $\gcd(P_1, ..., P_n)$, for a reference see [**Ber70**].

Notation (III.1.9). In order to simplify, we shall write $\mathcal{T} \cong \mathcal{T}'$ for two weighted trees that are isomorphic as trees and $\ell_i = \ell'_i$, $i = 1, 2$. We will say $\mathcal{T} = \mathcal{T}'$ (resp. $\geq, \leq, <$ or $>$) if $\mathcal{T} \cong \mathcal{T}'$ and $w(\mathcal{T}, Q) = w(\mathcal{T}', Q)$ (resp. $\geq$, $\leq$, $<$ or $>$) for any $Q \in |\mathcal{T}| = |\mathcal{T}'|$ (note that we are using the isomorphism of trees to identify the vertices).

Sometimes it will be necessary to compare empty trees. In this case, the conditions $=, \leq, \geq$ are vacuous and hence always satisfied.

In what follows we are going to define some logarithmic trees and tree invariants which will be of particular interest.

Definition (III.1.10). Let (f, P) be a germ and $\mathcal{T}_P(f)$ its multiplicity tree. A labeled tree $\mathcal{T}$ is said to be a *logarithmic tree* for (f, P) if it satisfies the following properties:

- $\mathcal{T} \cong \mathcal{T}_P(f)$, and
- the $\mathcal{O}_P$-module $\mathcal{M}_{\mathcal{T}} := \{h \in \mathcal{O}_P \mid \mathcal{T}_P(f)|_h \geq \mathcal{T}\}$ is logarithmic (recall Definition (II.3.8)).

Remark **(III.1.11).** Note that if $\mathcal{T}_1 \geq \mathcal{T}_2$, then $\mathcal{M}_{\mathcal{T}_2} \subset \mathcal{M}_{\mathcal{T}_1}$.

Definition (III.1.12). Consider a labeled tree $\mathcal{T}$ whose labels are triples of type $(w, (p, q), (d; a, b))$ as above. The *degree* of $\mathcal{T}$ shall be defined as

$$\deg(\mathcal{T}) := \sum_{Q \in |\mathcal{T}|} \frac{w(\mathcal{T}, Q)}{2dpq} \left(w(\mathcal{T}, Q) + p + q - e \right), \tag{34}$$

where $w(\mathcal{T}, Q)$ denotes the weight of $\mathcal{T}$ at Q and $e := \gcd(d, aq - bp)$.

Also, one can define the *null tree associated with* $\mathcal{T}$ (denoted by $\mathcal{T}^{nul}$) as a labeled tree isomorphic to $\mathcal{T}$ whose labels are triples of type $(w', (p, q), (d; a, b))$ where

$$w(\mathcal{T}^{nul}, Q) = w(\mathcal{T}, Q) - p - q + e. \tag{35}$$

Remark **(III.1.13).** Note that $\mathcal{T}_P^{nul}(f)$ is a logarithmic tree for $(f, P) \in (X(d; a, b), P)$.

Definition (III.1.14). Let $f, g \in \mathcal{O}_P(k)$ be germs at. One can define the *restriction of* g *to* $\tilde{\mathcal{T}}_P(f)$ or $\tilde{\mathcal{T}}_P(f)|_g$ as a labeled tree isomorphic to $\tilde{\mathcal{T}}_P(f)$ where $w(\tilde{\mathcal{T}}_P(f)|_g, Q) = \nu_Q(g)$ is the (p, q)-multiplicity of g at Q a point of label $(w, (p, q), (d; a, b))$.

Example (III.1.15). Recall Example (III.1.6). Assume $\gcd(p, q) = 1$ and $p < q$. Let $g = (x^q - 8y^p)$ be a germ of curve singularity at $0 \in \mathbb{C}^2$. The restriction $\tilde{\mathcal{T}}_P(f)|_g$ is given by the tree in Figure III.1 with the following labels:

$$\begin{array}{rcl} A &=& (p^2,(q,p),(1;1,1)).\\ B &=& (0,(1,1),(p;1,1)).\\ C &=& (0,(1,1),(1;1,1)).\\ D &=& (p(p+q),(p,q^2-p^2),(q;-1,p)).\\ E &=& (0,(1,1),(p;1,1)).\\ F &=& (0,(1,1),(1;1,1)).\\ G &=& (0,(1,1),(q^2-p^2;1,1)). \end{array}$$

Assume $\gcd(p,q)=1$ and $p<q$. Let $f=(x^p+y^q)(x^q+y^p)$ be a germ of curve singularity at $0\in\mathbb{C}^2$. Recall Examples (I.2.6) and (I.3.8). Figure III.1 is a possible graph for $\tilde{\mathcal{T}}_0(f)$.

Remark **(III.1.16).** Notice that from (34), Theorem (II.2.5) and Corollary (II.2.6) one has

$$\deg(\tilde{\mathcal{T}}_P^{nul}(f)) = \deg(\mathcal{T}_P^{nul}(f)) = \delta_P^w(f).$$

Remark **(III.1.17).** Note that according to (III.1.4), in the case of germs in $(\mathbb{C}^2,P)$ the degree of a tree $\mathcal{T}$ is related with the number of conditions imposed to a germ g so that $\mathcal{T}|_g \geq \mathcal{T}$. In this situation $K_P(f)=\deg\mathcal{T}_P^{nul}(f)=\delta_P(f)$ (see also [**CA02**, Lemma 2.35]). In our case, $\deg\mathcal{T}_P^{nul}(f)=\delta_P^w(f)$, which is in general a rational number. However, when f defines a function on $X(d;a,b)$ one still obtains the equality

$$K_P(f) = \deg\mathcal{T}_P^{nul}(f) = \delta_P^w(f)$$

(see Corollary (II.3.14)).

Let us see a lemma which will be of particular interest through this chapter. It gives local conditions under which meromorphic forms are logarithmic. Such conditions can be expressed in terms of multiplicity trees.

Lemma (III.1.18). *Let f and h be analytic germs where x and y are local equations of P on a surface $X(d;a,b)$. Let*

$$\psi := h\,\frac{dx\wedge dy}{f}$$

*be a invariant 2-form on $X(d;a,b)$. Let π be a **Q**-resolution of $V_f\subset X_0$ such that*

$$\tilde{\mathcal{T}}_P(f)|_h \;\geq\; \tilde{\mathcal{T}}_P^{nul}(f) \tag{36}$$

Then the following results hold:

i) $\mathcal{M}^{nul}_{f,\pi} = \{h \in \mathcal{O}_P \mid \tilde{\mathcal{T}}_P(f)|_h \geq \tilde{\mathcal{T}}_P^{nul}(f)\}$ *and hence it is logarithmic.*

ii) *Moreover, the residues of* $\pi^*\psi$ *at any point of* $\overline{V}_f^{[2]}$ *vanish.*

Proof. Note that proving (i) is equivalent to prove

(1) The 2-form ψ is logarithmic (with respect to V_f and π).

(2) Any pull-back $\pi_k^*\psi$ has poles along a subset of $\hat{V}_{f,k}$, and if the equality in (36) holds, then the form $\pi_k^*\psi$ has poles exactly along $\hat{V}_{f,k}$.

Let us calculate local equations for the pull-back of ψ after the first blow-up ε_1 (see (6) for a description of the local charts).

(37)
$$h\,\frac{dx\wedge dy}{f} \overset{\substack{x=\bar{u}^p,\ u=\bar{u}^e\\ y=\bar{u}^q v}}{\longleftarrow} \frac{p}{e}\hat{h}\ \bar{u}^{\nu_P(h)}\ \frac{\bar{u}^{p-1+q-e+1}d\bar{u}^e\wedge dv}{\bar{u}^{\nu_P(f)}\hat{f}} = \frac{p}{e}\hat{h}\ u^{\lambda}\frac{du\wedge dv}{\hat{f}}$$

where $\lambda = \dfrac{\nu_P(h)-\nu_P(f)+p+q-e}{e} \geq 0$ which implies

$$\nu_P(h) \geq \nu_P(f) - p - q + e.$$

Therefore, *i)* follows immediately by recursion on the infinitely near points to P.

Part *ii)* follows from the fact that such 2-forms have weight 1, whose residue in codimension 2 is zero.

□

Section § **III.2**

Construction of logarithmic trees: $\tilde{\mathcal{T}}_P^{\delta_1\delta_2}$

Our purpose in this section is to define new families of logarithmic trees. Throughout it, notations and definitions from (III.1.4), will be frequently used.

Notation (III.2.1). Let ψ be a logarithmic 2-form (with respect to $\mathcal{D} = V_f$ and the $\mathbf{Q}$-resolution π). By Definition (III.1.4) it makes sense to consider the residue of the pull-back $\pi^*(\psi)$ of ψ at an edge γ of $\tilde{\mathcal{T}}_P(f)$ which shall be denoted by

$$\left(\operatorname{Res}^{[2]}(\psi)\right)_\gamma .$$

Note also that any two local branches δ_1 and δ_2 of f are joined by a unique path in $\tilde{\mathcal{T}}_P(f)$ which shall be denoted by $\gamma(\delta_1, \delta_2)$.

The purpose in the sequel is to describe conditions under which a germ $h \in \mathcal{O}_P$ satisfies that the 2-form $h\ \frac{dx\wedge dy}{f}$:

- is logarithmic (with respect to V_f and π) and
- has zero residues outside the edges of the path $\gamma(\delta_1, \delta_2)$.

The following definition gives an idea of how trees come into play and what type of trees are useful for solving the problem.

Definition (III.2.2). Let δ_1 and δ_2 be local branches of $\mathcal{D} = \{f = 0\}$ at P. A weighted tree $\mathcal{T}$ will be said to be a *logarithmic tree for* δ_1 *and* δ_2 if it satisfies the following properties:

(1) The trees $\mathcal{T}$ and $\mathcal{T}_P(f)$ are isomorphic as trees, i.e. $\mathcal{T} \cong \mathcal{T}_P(f)$.
(2) The module $\mathcal{N} := \{h \in \mathcal{O}_P \mid \mathcal{T}_P(f)|_h \ \geq \ \mathcal{T}\}$ is logarithmic.
(3) If φ belongs to $M_{\mathcal{N}} := \{h \in \mathcal{O}_P \mid \mathcal{T}_P(f)|_h \ = \ \mathcal{T}\} \subset \mathcal{N}$ then

$$\left(\mathrm{Res}^{[2]}(\varphi\ \frac{dx \wedge dy}{f})\right)_\gamma \neq 0$$

if and only if γ is an edge of $\gamma(\delta_1, \delta_2)$.

III.2–1. Recursive method to construct $\tilde{\mathcal{T}}_P^{\delta_1\delta_2}$

Our aim in this section is to give a method to construct a logarithmic tree for δ_1 and δ_2 (Definition (III.2.2)). In order to do that, a recursive method is described, starting first with a non-weighted tree $\tilde{\mathcal{T}}$ ($\tilde{\mathcal{T}} \cong \tilde{\mathcal{T}}(f)_P$) and then assigning weights to its vertices.

During this process some of the weights assigned will be definitive but others will be subject of recursion. If definitive, they are denoted by $w(Q)$. If temporary, they are denoted by $w(\tilde{\mathcal{T}}_j^i, Q)$, where $\tilde{\mathcal{T}}_j^i$ is a subtree of $\tilde{\mathcal{T}}$. The finiteness of the recursion is based on the finiteness of the extended multiplicity trees.

Step 0: distinguished points.

Consider a non-weighted tree $\tilde{\mathcal{T}} \cong \tilde{\mathcal{T}}_P(f)$ and distinguish in it two maximal vertices δ_1 and δ_2. They shall be referred to as *distinguished points.*

Step 1: first decomposition.

Denote by $P_2 = \gcd(\delta_1, \delta_2)$ the greatest common descendant of the two distinguished vertices δ_1 and δ_2. Note that blowing up P_2 separates the branches δ_1 and δ_2.

Removing P_2 and its adjacent edges from $\tilde{\mathcal{T}}$ decomposes the tree into a finite number of subtrees $\tilde{\mathcal{T}}_0^2, \tilde{\mathcal{T}}_1^2, ..., \tilde{\mathcal{T}}_{n_2}^2$ as shown in Figure III.2.

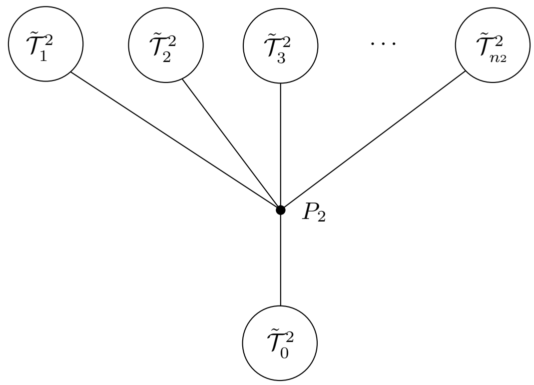


FIGURE III.2. First decomposition.

From these subtrees only $\tilde{\mathcal{T}}_0^2$ contains descendants of P_2. From the remaining subtrees, only one, say $\tilde{\mathcal{T}}_1^2$, contains δ_1. Another one, say $\tilde{\mathcal{T}}_2^2$, contains δ_2. Any remaining subtrees are denoted by $\tilde{\mathcal{T}}_3^2, ..., \tilde{\mathcal{T}}_{n_2}^2$. From now on, this decomposition will be called *the first decomposition*. For technical reasons it is convenient to denote $\tilde{\mathcal{T}}_P(f)$ by $\tilde{\mathcal{T}}^1$.

Step 2: weights for $|\tilde{\mathcal{T}}_0^2|$ and P_2.

At this step we are in conditions to define definitive weights at $|\tilde{\mathcal{T}}_0^2| \cup \{P_2\} \subset |\tilde{\mathcal{T}}|$ as follows:

$$w(Q) := \begin{cases} w(\tilde{\mathcal{T}}^1, Q) - p - q + e & \text{if } Q \in |\tilde{\mathcal{T}}_0^2| \\ w(\tilde{\mathcal{T}}^1, P_2) - p - q & \text{if } Q = P_2. \end{cases} \tag{38}$$

Remark **(III.2.3).** Let us denote by X_{k_2} the V-variety obtained after blowing up P_2. Remember that the exceptional divisor of such a blow-up is called E_{k_2} (Definition (III.1.4)).

Let $\psi = \varphi\ \frac{dx\wedge dy}{f}$ be a form satisfying,

(39)

$$\forall\ Q \in |\tilde{\mathcal{T}}_0^2|, \quad w(\mathcal{T}_P(f)|_\varphi, Q)\ \geq\ w(\tilde{\mathcal{T}}, Q) \quad \Big(= w(\tilde{\mathcal{T}}^1, Q) - p - q + e \text{ by } (38)\Big)\,.$$

By (III.1.18), the pull-back $\pi_{k_2-1}^*\psi$ will have poles along the strict transform $\hat{V}_{f,k_2-1}$. Assume also

$$w(\mathcal{T}_P(f)|_\varphi, P_2)\ \geq\ w(\tilde{\mathcal{T}}, P_2) \quad \Big(= w(\tilde{\mathcal{T}}^1, P_2) - p - q \text{ by } (38)\Big)\,. \tag{40}$$

Then, by (37), the pull-back $\pi_{k_2}^*\psi = \varepsilon_{k_2}^*(\pi_{k_2-1}^*\psi)$ has a local equation

$$\pi_{k_2}^*(\varphi)\ u^\lambda\ \frac{du_{k_2}\wedge dv_{k_2}}{\hat{f}_{k_2}} \tag{41}$$

where $\lambda \geq -1$ and E_{k_2} is defined by $u_{k_2} = 0$. Therefore, $\pi^*_{k_2}\psi$ has poles along $E_{k_2} \cup \hat{V}_{f,k_2}$.

Step 3: Analysis of secondary distinguished points.

In this step some weights will be assigned to the vertices of $\tilde{\mathcal{T}}^2_1, \tilde{\mathcal{T}}^2_2, ..., \tilde{\mathcal{T}}^2_{n_2}$. Recall that only $\tilde{\mathcal{T}}^2_1$ and $\tilde{\mathcal{T}}^2_2$, have a distinguished point. The remaining subtrees have no distinguished points.

Note that from Definition (III.1.4), e_{ij} denoted the unique double point on $\pi^*(f)$, if any, lying in the intersection of E_i and E_j. Let us collect all the points of the form $e_{k_2 *}$. These points shall be called *secondary distinguished points*. Note that there cannot be more than one secondary distinguished point on each subtree $\tilde{\mathcal{T}}^2_i$. The two possible cases are considered separately:

Step 3.1: no secondary distinguished points.

Suppose the subtree $\tilde{\mathcal{T}}^2_j$ does not contain any secondary distinguished point. This means that,

(1) The subtree $\tilde{\mathcal{T}}^2_j$ is simply a point of type $\circ$ or,
(2) E_{k_2} is already transversal to $\hat{V}_{f,k_2}$ at $P^2_j := \gcd(\tilde{\mathcal{T}}^2_j)$. This implies that $P^2_j \in X(d; a, b)$ is never the center of a weighted blow-up and hence, $\tilde{\mathcal{T}}^2_j$ is a point of type $\bullet$.

In this case, definitive weights are defined as:

(1) If the vertex is of type $\circ$, then

$$w(\circ) = d - 1 \tag{42}$$

(2) If the vertex is of type $\bullet$, then

$$w(\bullet) = \begin{cases} 0 & \text{if } \bullet \in |\tilde{\mathcal{T}}^2_j| \text{ is a distinguished point} \\ d & \text{otherwise.} \end{cases} \tag{43}$$

At this stage, the algorithm stops for $\tilde{\mathcal{T}}^2_j$.

Remark **(III.2.4).** Let us consider the form $\psi = \varphi \, \frac{dx \wedge dy}{f}$ from step 2, satisfying (39) and (40). As in step 3.1 above, let us assume E_{k_2} intersects transversally a branch of $\hat{V}_{f,k_2} \subset X_{k_2}$, at $P^2_j \in X(d; a, b)$. Hence, according to (41), the form $\pi^*_{k_2}\psi$ is logarithmic at P^2_j. In fact, if ψ satisfies (43), then $\pi^*_{k_2}\psi$ has a non-zero residue at P^2_j if and only if P^2_j is a distinguished point and ψ satisfies the equalities in (39) and (40).

Step 3.2: one secondary distinguished point.

Suppose the subtree $\tilde{\mathcal{T}}^2_j$ contains one (and hence only one) secondary distinguished point $e_{k_2 m_j}$. There is a unique path γ_j, which will be called *a distinguished path for* $\tilde{\mathcal{T}}_j^2$, joining $e_{k_2 m_j}$ and $P_j^2 = \gcd(\tilde{\mathcal{T}}_j^2)$. In this case, temporary and definitive weights are defined as:

(44)

$$w(\tilde{\mathcal{T}}_j^2, Q) = \begin{cases} w(\tilde{\mathcal{T}}^1, Q) + \nu_Q(E_{k_2}) & \text{if } Q \in |\gamma_j|, \quad j = 1,2 \\ w(\tilde{\mathcal{T}}^1, Q) & \text{if } Q \in |\tilde{\mathcal{T}}_j^2| \setminus |\gamma_j|, \quad j = 1,2 \end{cases}$$

$$w(Q) = \begin{cases} w(\tilde{\mathcal{T}}^1, Q) + \nu_Q(E_{k_2}) - p - q + e & \text{if } Q \in |\gamma_j|, \quad j = 3, ..., n_2 \\ w(\tilde{\mathcal{T}}^1, Q) - p - q + e & \text{if } Q \in |\tilde{\mathcal{T}}_j^2| \setminus |\gamma_j|, \quad j = 3, ..., n_2. \end{cases}$$

At this stage, the algorithm stops for the subtrees $\tilde{\mathcal{T}}_3^2, ..., \tilde{\mathcal{T}}_{n_2}^2$.

Remark **(III.2.5).** Let us consider the form $\psi = \varphi \, \frac{dx\wedge dy}{f}$ from step 2, satisfying (39) and (40). As in step 3.2 above, let us assume E_{k_2} does not intersect $\hat{V}_{f,k_2} \subset X_{k_2}$ transversally at $P_j^2 \in X(d; a, b)$. Note that $u_{k_2}\hat{f}_{k_2}$, where $u_{k_2} = 0$ defines E_{k_2}, is a reduced equation for the pull-back of $\hat{f}_{k_2-1}$ after blowing up the point P_2. Let us consider the multiplicity tree $\tilde{\mathcal{T}}_{P_j^2}(u_{k_2}\hat{f}_{k_2}) \cong \tilde{\mathcal{T}}_j^2$.

From (44), one has

$$\tilde{\mathcal{T}}_j^2 = \tilde{\mathcal{T}}_{P_j^2}(u_{k_2}\hat{f}_{k_2}) \quad \text{if } j = 1,2$$

and

$$\tilde{\mathcal{T}}_j^2 = \tilde{\mathcal{T}}_{P_j^2}^{nul}(u_{k_2}\hat{f}_{k_2}) \quad \left(= \tilde{\mathcal{T}}_{P_j^2}(u_{k_2}\hat{f}_{k_2}) - p - q + e\right) \quad \text{if } j = 3, ..., n_2.$$

Corollary (III.2.6). *As a simple consequence of Remarks (III.2.4) and (III.2.5) above, note that the subtrees* $\tilde{\mathcal{T}}_j^2$ *are isomorphic to extended multiplicity trees of* $\hat{V}_{f,k_2} \cup E_{k_2} \subset X_{k_2}$ *at the points* P_j^2.

Step 4: recursive method, *i*-th decomposition.

In order to use a recursive method, both secondary and non-secondary distinguished points will all be renamed as distinguished points.

Consider now the subtrees $\tilde{\mathcal{T}}_1^2$ and $\tilde{\mathcal{T}}_2^2$ separately. Each of these subtrees has two distinguished points: one of type $e_{k_2 m_i}$ and the other one of type δ_i, for $i = 1, 2$. From Remark (III.2.4), $\tilde{\mathcal{T}}_1^2$ and $\tilde{\mathcal{T}}_2^2$ become multiplicity trees with two distinguished vertices. Therefore one can decompose $\tilde{\mathcal{T}}_1^2$ and $\tilde{\mathcal{T}}_2^2$ into subtrees as in step 1.

Let us denote their components as in Figure III.14 (for $i = 3$) and start the process from step 1.

The decomposition of Figure III.2 is called *the first decomposition.* In general, a decomposition such as the one shown in Figure III.14 is called the $(i-1)$*-th decomposition.*

Clearly, this process will terminate, since going through step 1 strictly decreases the cardinality of $|\tilde{\mathcal{T}}_j^i|$.

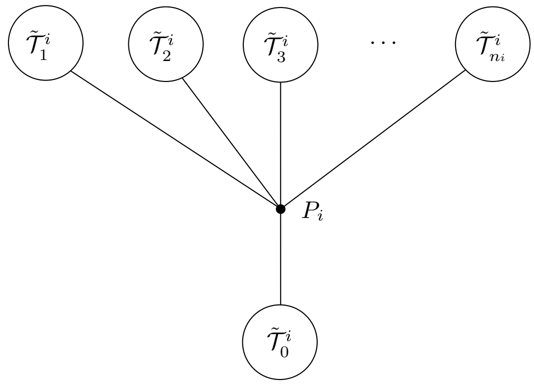


FIGURE III.3. $(i-1)$-th decomposition.

Definition (III.2.7). A subtree of $\tilde{\mathcal{T}}$ is said *of type 0* if it is $\tilde{\mathcal{T}}^i_0$ at a certain step. Analogously, is said to be *of type 1* (or *2*) if it is $\tilde{\mathcal{T}}^i_j$, for some $j \in \{1, 2\}$ (or $\tilde{\mathcal{T}}^i_j$, for some $j \geq 3$ respectively). The point P_i on an $(i-1)$-th decomposition is called a *central point of* $\tilde{\mathcal{T}}_P(f)$. Also, a subtree is called *simple* if it is of type 1 or 2 and is not decomposed further.

By construction, one has the following.

Lemma (III.2.8). *The tree* $\tilde{\mathcal{T}}^{\delta_i,\delta_j}_P(f)$ *is logarithmic for* δ_i *and* δ_j *at* P.

PROOF. It follows directly from the construction of $\tilde{\mathcal{T}}^{\delta_i,\delta_j}_P(f)$ and remarks (III.2.3), (III.2.4) and (III.2.5). □

SECTION § III.3
Some examples

In this section we will construct logarithmic trees for some examples in order to illustrate the process explained in §III.2–1, which is summarized in the following remark.

Remark **(III.3.1).** As a summary of the recursive process seen before we have that formulas (38), (42), (43) and (44) can be rewritten as

$$w(Q) := \begin{cases} w(\tilde{\mathcal{T}}^i_*, Q) - p - q + e & \text{if } Q \in |\tilde{\mathcal{T}}^{i+1}_0| \\ w(\tilde{\mathcal{T}}^i, P_{i+1}) - p - q & \text{if } Q = P_{i+1} \end{cases}$$

$$w(\circ) = d - 1$$

$$w(\bullet) = \begin{cases} 0 & \text{if } \bullet \in |\tilde{\mathcal{T}}^{i+1}_*| \text{ is a distinguished point} \\ d & \text{otherwise,} \end{cases}$$

$$w(\tilde{\mathcal{T}}_j^{i+1}, Q) = \begin{cases} w(\tilde{\mathcal{T}}_*^i, Q) + \nu_Q(E_{k_i}) & \text{if } Q \in |\gamma_j|, \quad j = 1,2 \\ w(\tilde{\mathcal{T}}_*^i, Q) & \text{if } Q \in |\tilde{\mathcal{T}}_j^{i+1}| \setminus |\gamma_j|, \quad j = 1,2 \end{cases}$$

$$w(Q) = \begin{cases} w(\tilde{\mathcal{T}}_*^i, Q) + \nu_Q(E_{k_i}) - p - q + e & \text{if } Q \in |\gamma_j|, \quad j = 3, ..., n_2 \\ w(\tilde{\mathcal{T}}_*^i, Q) - p - q + e & \text{if } Q \in |\tilde{\mathcal{T}}_j^{i+1}| \setminus |\gamma_j|, \quad j = 3, ..., n_2. \end{cases}$$

Example (III.3.2). Let $f = xy$ be a curve germ singularity on $X(d; a, b)$. It has only two local branches $\delta_1 = x$ and $\delta_2 = y$ at $P = [(0,0)]$ on $X(d; a, b)$.

After one (a, b)-weighted blow-up (recall (6)) one has (see Figure III.4)

$$X\widehat{(d; a, b)}_w = X\,(a; 1, \alpha) \cup X\,(b; \beta, 1) = X\,(a; 1, \alpha) \cup X\,\left(b; 1, \beta'\right). \tag{45}$$

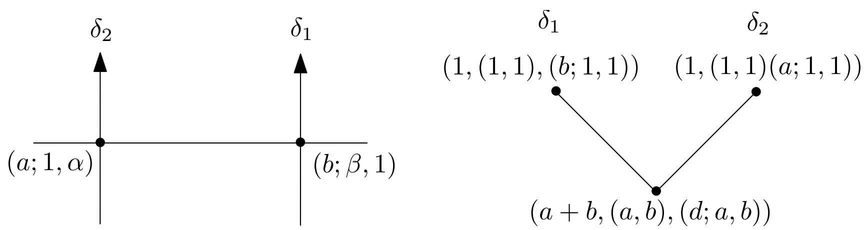


FIGURE III.4. **Q**-Resolution of $\mathcal{C}$ and $\tilde{\mathcal{T}}_0(xy)$.

One can observe the construction of $\tilde{\mathcal{T}}_0^{nul}(xy)$ and $\tilde{\mathcal{T}}_0^{\delta_1\delta_2}(xy)$ in Figure III.5.

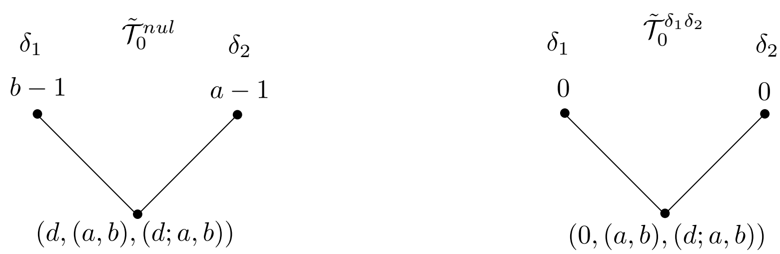


FIGURE III.5. $\tilde{\mathcal{T}}_0^{nul}(xy)$ and $\tilde{\mathcal{T}}_0^{\delta_1\delta_2}(xy)$.

Let us compute

$$\deg(\tilde{\mathcal{T}}_0^{\delta_1,\delta_2}(xy)) - \deg(\tilde{\mathcal{T}}_0^{nul}(xy)),$$

recalling the definition of degree seen in (34).

On the one hand,

$$\begin{aligned}\deg(\tilde{\mathcal{T}}_0^{nul}(xy)) &= \frac{d(d+a+b-d)}{2dab} + \frac{(a-1)(a-1+1+1-a)}{2a} \\ &+ \frac{(b-1)(b-1+1+1-b)}{2b} = \frac{a+b}{2ab} + \frac{a-1}{2a} + \frac{b-1}{2b} \\ &= 1,\end{aligned}$$

Recall Example (II.2.11) and Remark (III.1.16), as we already know

$$\deg(\tilde{\mathcal{T}}_0^{nul}(xy)) = \delta_0^w(xy).$$

On the other hand one has

$$\deg(\tilde{\mathcal{T}}_0^{\delta_1,\delta_2}(xy)) = 0.$$

Finally one gets

$$\deg(\tilde{\mathcal{T}}_0^{\delta_1,\delta_2}(xy)) = \deg(\tilde{\mathcal{T}}_0^{nul}(xy)) - 1.$$

Example (III.3.3). Let $f = xy(x^b + y^a)$. with a, b coprime, be a germ of curve singularity on $X(d;a,b)$. It has three local branches $\delta_1 = x$ and $\delta_2 = y$ and $\delta_3 = x^b + y^a$ at $P = [(0,0)]$ on $X(d;a,b)$.

After one (a,b)-weighted blow-up (recall (6)) one has (see Figure III.6)

(46) $$\widehat{X(d;a,b)}_w = X(a;1,\alpha) \cup X(b;\beta,1) = X(a;1,\alpha) \cup X\left(b;1,\beta'\right).$$

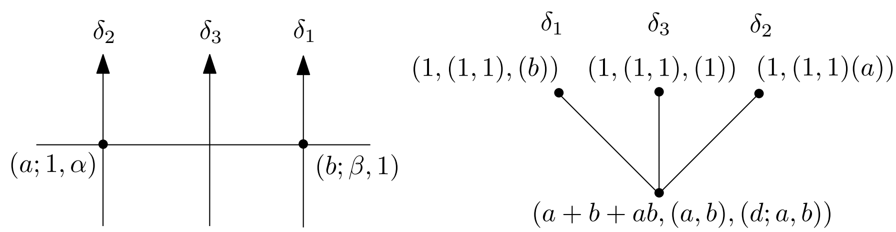


FIGURE III.6. **Q**-Resolution of $\mathcal{C}$ and $\tilde{\mathcal{T}}_0(xy)$.

One can observe the construction of $\tilde{\mathcal{T}}_0^{nul}(xy(x^b+y^a))$ and $\tilde{\mathcal{T}}_0^{\delta_i\delta_j}(xy(x^b+y^a))$ in Figure III.7.

Let us compute

$$\deg(\tilde{\mathcal{T}}_0^{\delta_1,\delta_2}(xy(x^b+y^a))) - \deg(\tilde{\mathcal{T}}_0^{nul}(xy(x^b+y^a))).$$

Recall the definition of degree seen in (34).

On the one hand,

$$\begin{aligned}\deg(\tilde{\mathcal{T}}_0^{nul}(xy(x^b+y^a))) &= \frac{(d+ab)(d+ab+a+b-d)}{2dab} + \frac{(a-1)}{2a} + \frac{(b-1)}{2b} \\ &= 1 + \frac{1}{2} + \frac{(ab+a+b)}{2d}.\end{aligned}$$

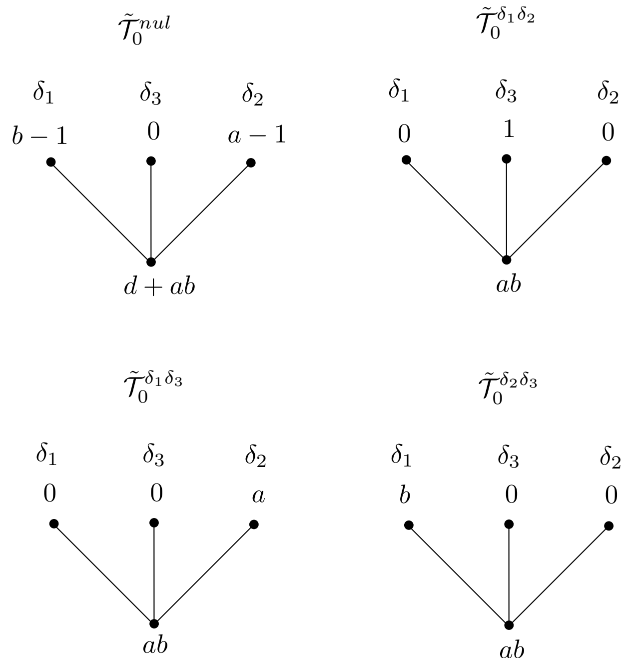


FIGURE III.7. $\tilde{\mathcal{T}}_0^{nul}(xy(x^b+y^a))$ and $\tilde{\mathcal{T}}_0^{\delta_i\delta_j}(xy(x^b+y^a))$.

On the other hand,

$$\begin{aligned}
\deg(\tilde{\mathcal{T}}_0^{\delta_1\delta_2}(xy(x^b+y^a))) &= \frac{(ab)(ab+a+b-d)}{2dab}+\frac{1(1+1+1-1)}{2}\\
&= \frac{(ab+a+b)}{2d}-\frac{1}{2}+1.
\end{aligned}$$

$$\begin{aligned}
\deg(\tilde{\mathcal{T}}_0^{\delta_1\delta_3}(xy(x^b+y^a))) &= \frac{(ab)(ab+a+b-d)}{2dab}+\frac{a(a+1+1-a)}{2a}\\
&= \frac{(ab+a+b)}{2d}-\frac{1}{2}+1.
\end{aligned}$$

$$\begin{aligned}
\deg(\tilde{\mathcal{T}}_0^{\delta_2\delta_3}(xy(x^b+y^a))) &= \frac{(ab)(ab+a+b-d)}{2dab}+\frac{b(b+1+1-b)}{2b}\\
&= \frac{(ab+a+b)}{2d}-\frac{1}{2}+1.
\end{aligned}$$

Finally one gets

$$\deg(\tilde{\mathcal{T}}_0^{\delta_1,\delta_2}(xy))=\deg(\tilde{\mathcal{T}}_0^{nul}(xy))-1.$$

Example (III.3.4). Assume $x^p+y^q=0$, with p, q coprime, defines a curve on a surface singularity of type $X(d;a,b)$. Note that $\nu=pq$. Also, since $x^p+y^q=0$ defines a set of zeros in $X(d;a,b)$ by hypothesis, this implies

$ap \equiv bq \mod d$ and hence $e := \gcd(d, ap - bq) = d$. Note that a simple (q,p)-blow-up will be a (strong) **Q**-resolution of the singular point. Let $f = (x^q + y^p)(x^q - y^p)$ be a germ of curve singularity at $[(0,0)]$ on $X(d; a, b)$. After one (p,q)-weighted blow-up one has (recall (6))

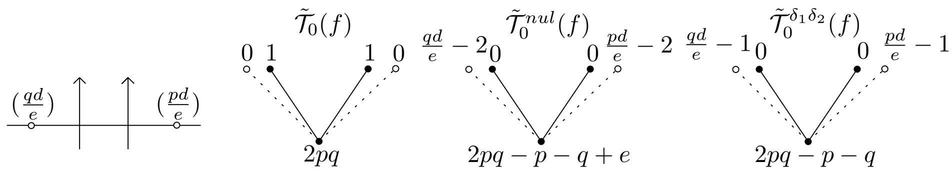


FIGURE III.8. Strong **Q**-resolution of f, $\tilde{\mathcal{T}}_0(f)$, $\tilde{\mathcal{T}}_0^{nul}(f)$ and$\tilde{\mathcal{T}}_0^{\delta_1\delta_2}(f)$.

Recall the notation used in Example (II.2.13). Let us compute

$$\begin{aligned}
\deg(\mathcal{T}_0^{\delta_1,\delta_2}(f)) - \deg(\mathcal{T}_0^{nul}(f)) &= \frac{(2pq-p-q)(2pq-e)}{2dpq} + \frac{\frac{qd}{e} - 1}{\frac{2qd}{e}} \\
&+ \frac{\frac{pd}{e} - 1}{\frac{2pd}{e}} - \frac{(2pq-p-q+e)(2pq)}{2dpq} \\
&= 1 - \frac{(p+q)}{2pq} + \frac{e(p+q)}{2dpq} - 2\frac{e}{d} \\
&\overset{e=d}{=} 1 - \frac{(p+q)}{2pq} + \frac{(p+q)}{2pq} - 2 = -1.
\end{aligned}$$

Finally one has

$$\deg(\tilde{\mathcal{T}}_0^{\delta_1,\delta_2}(f)) = \deg(\tilde{\mathcal{T}}_0^{nul}(f)) - 1.$$

Example (III.3.5). Let $f = xy(xy + (x^3 - y^2)^2)$ be a germ of curve singularity at $[(0,0)]$ on $X(7; 2, 3)$, recall §V.3–2. Figure III.9 is a possible graph for $\tilde{\mathcal{T}}_0(f)$. Let us denote by $\delta_1 = x$, $\delta_2 = y$ and δ_3, δ_4 the two local branches of $xy + (x^3 - y^2)^2$ at the origin (see Figure V.1).

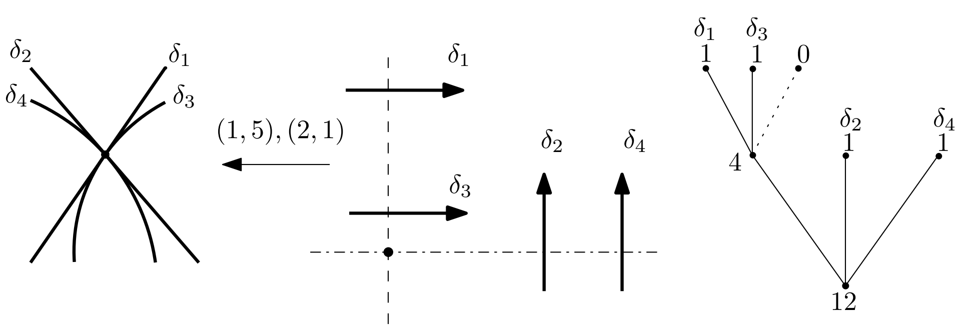


FIGURE III.9. **Q**-Resolution of $\mathcal{D}$ at $[0:0:1]$ and $\tilde{\mathcal{T}}_0(f)$.

One can observe the construction of $\tilde{\mathcal{T}}_0^{nul}(f)$ in Figure III.10, the one of $\tilde{\mathcal{T}}_0^{\delta_1\delta_2}(f)$ in two steps, $\tilde{\mathcal{T}}_0^{\delta_1\delta_4}(f)$, $\tilde{\mathcal{T}}_0^{\delta_2\delta_3}(f)$ and $\tilde{\mathcal{T}}_0^{\delta_3\delta_4}(f)$ in Figure III.11,the construction of $\tilde{\mathcal{T}}_0^{\delta_1\delta_3}(f)$ in Figure III.12 and finally the one of $\tilde{\mathcal{T}}_0^{\delta_2\delta_4}(f)$ in Figure III.13.

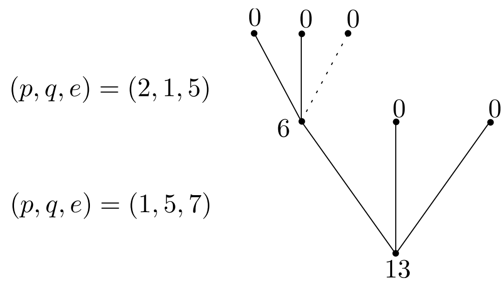


FIGURE III.10. $\tilde{\mathcal{T}}_0^{nul}(f)$.

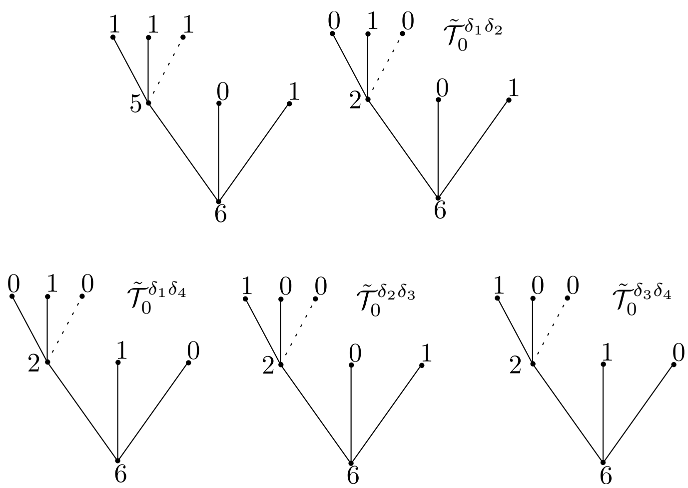


FIGURE III.11. Construction of $\tilde{\mathcal{T}}_0^{\delta_1\delta_2}(f)$ in two steps, $\tilde{\mathcal{T}}_0^{\delta_1\delta_4}(f)$, $\tilde{\mathcal{T}}_0^{\delta_2\delta_3}(f)$ and $\tilde{\mathcal{T}}_0^{\delta_3\delta_4}(f)$.

Let us compute

$$\deg(\tilde{\mathcal{T}}_0^{\delta_i,\delta_j}(f)) - \deg(\tilde{\mathcal{T}}_0^{nul}(f)).$$

One has

$$\deg(\tilde{\mathcal{T}}_0^{nul}(f)) = \frac{13(13+1+5-7)}{2\cdot 7\cdot 1\cdot 5} + \frac{6(6+2+1-5)}{2\cdot 5\cdot 2\cdot 1} = \frac{24}{7},$$

$$\begin{aligned}\deg(\tilde{\mathcal{T}}_0^{\delta_1\delta_2}(f)) &= \deg(\tilde{\mathcal{T}}_0^{\delta_1\delta_4}(f)) = \deg(\tilde{\mathcal{T}}_0^{\delta_2\delta_3}(f)) = \deg(\tilde{\mathcal{T}}_0^{\delta_3\delta_4}(f)) \\ &= \frac{6(6+1+5-7)}{2\cdot 7\cdot 1\cdot 5} + 1 + 1 = \frac{17}{7}.\end{aligned}$$

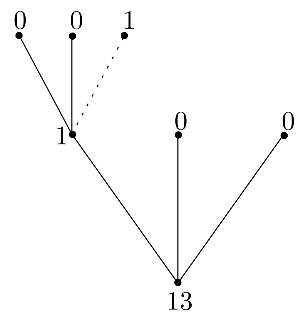


FIGURE III.12. Construction of $\tilde{\mathcal{T}}_0^{\delta_1\delta_3}(f)$.

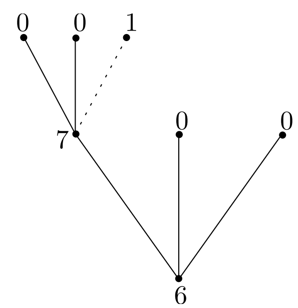


FIGURE III.13. Construction of $\tilde{\mathcal{T}}_0^{\delta_2\delta_4}(f)$.

$$\deg(\tilde{\mathcal{T}}_0^{\delta_1\delta_3}(f)) = \frac{13(13+1+5-7)}{2\cdot 7\cdot 1\cdot 5} + \frac{1(1+2+1-5)}{2\cdot 5\cdot 2\cdot 1} + \frac{1}{4} = \frac{17}{7},$$

$$\deg(\tilde{\mathcal{T}}_0^{\delta_2\delta_4}(f)) = \frac{6(6+1+5-7)}{2\cdot 7\cdot 1\cdot 5} + \frac{7(7+2+1-5)}{2\cdot 5\cdot 2\cdot 1} + \frac{1}{4} = \frac{17}{7},$$

Finally one gets

$$\deg(\tilde{\mathcal{T}}_0^{\delta_i,\delta_j}(f)) = \deg(\tilde{\mathcal{T}}_0^{nul}(f)) - 1, \ \forall\ 1 \leq i < j \leq 4.$$

SECTION § **III.4**

Relation between of $\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}(f)$ and $\tilde{\mathcal{T}}_P^{nul}(f)$

The following result (Lemma (III.4.3)) shows the relation between the degree of $\tilde{\mathcal{T}}_P^{nul}(f)$ and the degree of $\tilde{\mathcal{T}}_P^{\delta_i,\delta_j}(f)$ constructed above.

Remark **(III.4.1).** This Lemma generalizes [**CA02**, Lemma 2.35]. Notice that in our case the singular points of the exceptional divisors which appear will play an important role along all the proof and so the δ^w-invariant defined in Section II.3.

We have seen in subsection §III.2–1 a way to construct the weights of $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$ using a recursive process. In order to simplify the proof of the main result some notation will be used.

Notation (III.4.2). A vertex Q of a tree $\tilde{\mathcal{T}}$ will be said to have a defect $\mathbf{f}$, denoted by $\mathbf{f}(\tilde{\mathcal{T}}, Q)$ if

$$w(\tilde{\mathcal{T}}, Q) = w(\tilde{\mathcal{T}}_P(f), Q) + \mathbf{f}.$$

Also $\mathbf{f}(Q)$ will denote $\mathbf{f}(\tilde{\mathcal{T}}^{\delta_1,\delta_2}, Q)$.

Let $A \subset \overline{V}_f^{[2]}$ be a set of maximal points of $\tilde{\mathcal{T}}$. A function χ_A shall be defined as follows:

$$\chi_A = \#A \cap \{\delta_1, \delta_2\}.$$

Note that, if A and B are disjoint, then $\chi_A + \chi_B = \chi_{A\cup B}$. Also note that

$$\sum_{\bullet \text{ distinguished point}} \chi_{\{\bullet\}} = 2.$$

Lemma (III.4.3). *One has the following result,*

$$\deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}(f)) = \deg(\tilde{\mathcal{T}}_P^{nul}(f)) - 1. \tag{47}$$

Proof. For the sake of simplicity we will write $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$ for $\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}(f)$ and $\tilde{\mathcal{T}}^{nul}$ for $\tilde{\mathcal{T}}_P^{nul}(f)$. The result is equivalent to

$$\deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}) - \deg(\tilde{\mathcal{T}}^{nul}) = -1,$$

since the addition of non-infinitely near points does not affect the degree of either tree.

An inductive method will be used in order to prove equation (47). The idea consists in computing first the difference

$$\deg(\tilde{\mathcal{T}}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}})$$

for a tree that only decomposes into simple subtrees (base case of induction). Secondly, the same difference in any other subtree of type $\tilde{\mathcal{T}}^i$ (inductive step) will be studied. Finally, applying induction hypothesis the desired result will be concluded.

(A) **Consequences of the construction of $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$.**

Let us see a few remarks about simple subtrees of type $i \geq 1$. Some obvious consequences of the construction of $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$ are:

(A.1) Simple subtrees have either one or no distinguished vertices.

(A.2) Before a vertex belongs to a simple tree, it can never belong to any distinguished path.

(A.3) Before a vertex belongs to a simple tree, it always has to belong to trees of type 1.

(A.4) Any point of the tree $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$ can only belong to at most one distinguished path and be at most once a central point.

(IB) **Base case of induction: simple trees.**

Recall that a simple subtree is a tree in the induction process which has no further decomposition.

(IB.0) *If* $\tilde{\mathcal{T}} = \circ$.

It is easy to check by construction that if we are in a point of type $\circ$ then,

$$\deg(\tilde{\mathcal{T}}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}}) = \delta_\circ^w(E_k). \tag{48}$$

From now on we will only focus on points of type $\bullet$.

(IB.1) *Tree of type* 1.

(1) A subtree $\tilde{\mathcal{T}}$ of type 1 is simple if and only if it is a single vertex.

(2) In that case,

$$\deg(\tilde{\mathcal{T}}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}}) = \text{mult}_\bullet(\hat{V}_{f,k}, E_k) + \delta_\bullet^w(E_k) - 1 \tag{49}$$

Proof. For the first part (i), the "if" direction is obvious. There is only one direction left to prove, i.e. that any simple subtree of type 1 cannot have more than one vertex. Let $\tilde{\mathcal{T}}$ be a subtree of type 1. Let us denote by $m \in E_k \subset X_k$ the root of $\tilde{\mathcal{T}}$. If $\tilde{\mathcal{T}}$ has more than one vertex, then E_k is not transversal to $\hat{V}_{f,k}$ at m and hence, according to step 3, the vertex $e_{k,*}$ in $\tilde{\mathcal{T}}$ has to be distinguished. Therefore, $\tilde{\mathcal{T}}$ has two distinguished points and hence it is not simple.

For the second part (ii) is a direct consequence (43), if $\bullet = \delta_i,\ i = 1,2$ then

$$\deg(\tilde{\mathcal{T}}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}}) =$$

$$\frac{(w(\tilde{\mathcal{T}}^{\delta_1,\delta_2},\bullet))(w(\tilde{\mathcal{T}}^{\delta_1,\delta_2},\bullet)+p+q-e)}{2dpq} - \frac{(w(\tilde{\mathcal{T}}^{nul},\bullet))(w(\tilde{\mathcal{T}}^{nul},\bullet)+p+q-e)}{2dpq}$$

$$= \frac{0(0+1+1-d)}{2d} - \frac{(d-1)(d-1+1+1-d)}{2d}$$

$$= 0 - \frac{d-1}{2d} = \frac{1}{d} + \frac{d-1}{2d} - 1 = \text{mult}_\bullet(\hat{V}_{f,k}, E_k) + \delta_\bullet^w(E_k) - 1.$$

□

(IB.2) *Tree of type* 2 *with no distinguished points.*

A simple subtree $\tilde{\mathcal{T}}$ of type 2 with no distinguished points must be a single point.

By (43) such vertices must have defect 0. Moreover,

$$\deg(\tilde{\mathcal{T}}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}}) = \text{mult}_\bullet(\hat{V}_{f,k}, E_k) + \delta_\bullet^w(E_k). \tag{50}$$

Proof. Applying (43), one has

$$\deg(\tilde{\mathcal{T}}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}}) =$$

$$\frac{(w(\tilde{\mathcal{T}}^{\delta_1,\delta_2},\bullet))(w(\tilde{\mathcal{T}}^{\delta_1,\delta_2},\bullet)+p+q-e)}{2dpq} - \frac{(w(\tilde{\mathcal{T}}^{nul},\bullet))(w(\tilde{\mathcal{T}}^{nul},\bullet)+p+q-e)}{2dpq}$$

$$= \frac{d(d+1+1-d)}{2d} - \frac{(d-1)(d-1+1+1-d)}{2d}$$

$$= 1 - \frac{d-1}{2d} = \frac{1}{d} + \frac{d-1}{2d} = \mathrm{mult}_\bullet(\hat{V}_{f,k}, E_k) + \delta^w_\bullet(E_k).$$

□

(IB.3) *Tree of type* 2 *with one distinguished point.*
Let $\tilde{\mathcal{T}}$ be a simple subtree of type 2 with one (and hence unique) distinguished vertex, say Q. In this case, for vertices of type $\bullet$ one has:

(51) $$\mathbf{f}(\tilde{\mathcal{T}}, Q) = \begin{cases} \nu_Q(E_k) - p - q + e & \text{if } Q \in |\gamma| \\ -p - q + e & \text{if } Q \in |\tilde{\mathcal{T}}| \setminus |\gamma|, \end{cases}$$

where γ is the path joining Q and $m = \gcd(\tilde{\mathcal{T}})$. Otherwise, note that $\mathbf{f}(\circ) = d - 1$ Moreover,

(52)
$$\deg(\tilde{\mathcal{T}}) - \deg(\tilde{\mathcal{T}}_P^{nul}(f)|_{\tilde{\mathcal{T}}}) = \mathrm{mult}_m(\hat{V}_{f,k}, E_k) + \sum_{\bullet \in |\gamma|} \frac{\nu_\bullet(E_k)(\nu_\bullet(E_k) - p - q + e)}{2dpq}.$$

where $\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}}$ denotes the corresponding subtree of $\tilde{\mathcal{T}}^{nul}$ isomorphic to $\tilde{\mathcal{T}}$ and where $m \in E_k$.

Proof. Combining (A.2) with (44, one has (51). Moreover, using Proposition (I.3.7) and Theorem (II.2.1)

$$\deg(\tilde{\mathcal{T}}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}}) =$$

$$\sum_{\bullet \in |\gamma|} \frac{(\nu_\bullet(f) + \nu_\bullet(E_k))(\nu_\bullet(f) + \nu_\bullet(E_k) - p - q + e)}{2dpq} - \frac{\nu_\bullet(f)(\nu_\bullet(f) - p - q + e)}{2dpq}$$

$$= \sum_{\bullet \in |\gamma|} \frac{\nu_\bullet(f)\nu_\bullet(E_k)}{dpq} + \sum_{\bullet \in |\gamma|} \frac{\nu_\bullet(E_k)(\nu_\bullet(E_k) - p - q + e)}{2dpq}$$

$$= \mathrm{mult}_m(\hat{V}_{f,k}, E_k) + \sum_{\bullet \in |\gamma|} \frac{\nu_\bullet(E_k)(\nu_\bullet(E_k) - p - q + e)}{2dpq}.$$

□

Note that the difference of degrees between two isomorphic weighted trees can be calculated by decomposing each one into similar subtrees and then adding the differences of degrees for each pair of similar subtrees. Let $\tilde{\mathcal{T}}^i$ be a subtree of $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$ whose decomposition produces only simple subtrees $\tilde{\mathcal{T}}_1^{i+1}$, $\tilde{\mathcal{T}}_2^{i+1}$, $\tilde{\mathcal{T}}_3^{i+1}$, ... ,$\tilde{\mathcal{T}}_{n_i}^{i+1}$.

Items (IB.0), (IB.1), (IB.2) and (IB.3) allow for the calculation of the difference of degrees of $\tilde{\mathcal{T}}^i$ and $\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}^i}$. Since

$$\deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}|_{\tilde{\mathcal{T}}_0^{i+1}}) = \deg(\tilde{\mathcal{T}}_P^{nul}|_{\tilde{\mathcal{T}}_0^{i+1}}), \tag{53}$$

one has

$$\deg(\tilde{\mathcal{T}}^i) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}^i}) =$$

$$\mathbf{f}(P_{i+1}) + \sum_{\bullet \in E_k} \mathrm{mult}_\bullet(\hat{V}_{f,k}, E_k) - \chi_{A^i} + \sum_{m \in E_k} \delta_m^w(E_k) \tag{54}$$

$$= \mathbf{f}(P_{i+1}) + \left.\frac{\nu_{P_{i+1}}(f)e}{dpq}\right|_{P_{i+1}} - \chi_{A^i} + 1 - \left.\frac{(p+q)e}{2pqd}\right|_{P_{i+1}}. \tag{55}$$

where P_{i+1} is the central point of $\tilde{\mathcal{T}}^i$, E_k is the exceptional divisor resulting after blowing up P_{i+1}, and

$$A^i = \bigcup_{j=0}^{n_{i+1}} |\tilde{\mathcal{T}}_j^{i+1}|.$$

Notice that one has,

$$\mathbf{f}(P_{i+1}) = \frac{(\nu_{P_{i+1}} - p - q)(\nu_{P_{i+1}} - e)}{2dpq} - \frac{\nu_{P_{i+1}}(\nu_{P_{i+1}} - p - q + e)}{2dpq} = \tag{56}$$

$$= \left.\frac{(p+q)e}{2pqd}\right|_{P_{i+1}} - \left.\frac{\nu_{P_{i+1}}(f)e}{dpq}\right|_{P_{i+1}}.$$

In particular we obtain

$$\deg(\tilde{\mathcal{T}}^i) = \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}^i}) - 1.$$

(IS) **Inductive step.**

An obvious consequence of A.4 is that all central points P_{i+1} belong to exactly one distinguished path, except for P_2, which belongs to none. Hence

$$\mathbf{f}(\tilde{\mathcal{T}}^{\delta_1,\delta_2}, P_{i+1}) = \begin{cases} \nu_{P_{i+1}}(E_{k_i}) - p - q & \text{if } i \geq 2 \\ -p-q & \text{if } i = 1. \end{cases}$$

Using (44), the defects of $\tilde{\mathcal{T}}_0^{i+1}$ follow this pattern

$$(57)\quad \mathbf{f}(Q) = \begin{cases} \nu_Q(E_{k_i}) - p - q + e & \text{if } Q \in |\gamma(m_i, P_{i+1})| \setminus \{P_{i+1}\},\ i \geq 1 \\ -p-q & \text{if } Q = P_2 \\ \nu_Q(E_{k_i}) - p - q & \text{if } Q = P_{i+1},\ i \geq 2 \\ -p-q+e & \text{otherwise,} \end{cases}$$

where $m_i := \gcd(\tilde{\mathcal{T}}_0^{i+1}) = \gcd(\tilde{\mathcal{T}}^i)$. Denote by $\check{\gamma}_{i+1} := |\gamma(m_i, P_{i+1})| \setminus \{P_{i+1}\}$

$\tilde{\mathcal{T}}^i :=$

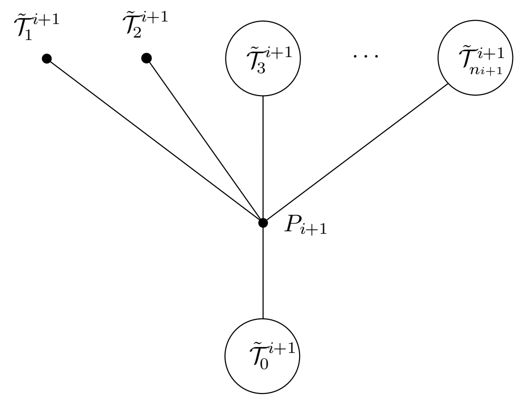


FIGURE III.14. Induction ($i \neq 1$).

Therefore

$$\deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}|_{\tilde{\mathcal{T}}^i}) - \deg(\tilde{\mathcal{T}}_P^{nul}|_{\tilde{\mathcal{T}}^i}) =$$

(58)

$$\deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}|_{\mathrm{Asc}(P_{i+1})}) - \deg(\tilde{\mathcal{T}}_P^{nul}|_{\mathrm{Asc}(P_{i+1})}) + \deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}|_{\tilde{\mathcal{T}}_0^{i+1}}) - \deg(\tilde{\mathcal{T}}_P^{nul}|_{\tilde{\mathcal{T}}_0^{i+1}}).$$

Since $w(\tilde{\mathcal{T}}^{\delta_1,\delta_2}, Q) = w(\tilde{\mathcal{T}}^{nul}, Q)$ for $Q \notin |\gamma(m_i, P_{i+1})|$ (see (57)) formula (58) equals

$$(59)\quad \deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}|_{\mathrm{Asc}(P_{i+1})}) - \deg(\tilde{\mathcal{T}}_P^{nul}|_{\mathrm{Asc}(P_{i+1})})$$

$$+ \sum_{\bullet \in \check{\gamma}_{i+1}} \frac{\nu_\bullet(f)\nu_\bullet(E_{k_i})}{dpq} + \sum_{\bullet \in \check{\gamma}_{i+1}} \frac{\nu_\bullet(E_{k_i})(\nu_\bullet(E_{k_i}) - p - q + e)}{2dpq}$$

Using induction (see (54) and (55)), formula (59) equals

$$= \mathbf{f}(P_{i+1}) + \left.\frac{\nu_{P_{i+1}}(f)e}{dpq}\right|_{P_{i+1}} - \chi_{A^i} + \sum_{\bullet \in E_{k_{i+1}}} \delta_\bullet(E_{k_{i+1}})$$

$$+ \sum_{\bullet \in \check{\gamma}_{i+1}} \frac{\nu_\bullet(f)\nu_\bullet(E_{k_i})}{dpq} + \sum_{\bullet \in \check{\gamma}_{i+1}} \frac{\nu_\bullet(E_{k_i})(\nu_\bullet(E_{k_i}) - p - q + e)}{2dpq}$$

$$= \mathbf{f}(P_{i+1}) + \left.\frac{\nu_{m_i}(f)e}{dpq}\right|_{m_i} + 1 - \left.\frac{(p+q)e}{2pqd}\right|_{m_i} - \chi_{A^i}.$$

where m_i is the root of $\tilde{\mathcal{T}}^i$, E_{k_i} is the exceptional divisor containing m_i, and $\gamma(m_i, P_{i+1})$ is the path joining m_i and P_{i+1}.

(IH) **Inductive Hypothesis.**

Descending in the chain of subtrees $\tilde{\mathcal{T}}^i$, one will eventually reach $\tilde{\mathcal{T}}^1 = \tilde{\mathcal{T}}_P(f)$. From the above calculations, (57), and the fact that

$$\chi_{A^1} = \sum_{d \text{ distinguished point}} \chi_{\{d\}} = 2,$$

one can check that

$$\deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}) - \deg(\tilde{\mathcal{T}}^{nul}) = \deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}|_{\tilde{\mathcal{T}}^1}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}^1})$$
$$= \deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}|_{\mathrm{Asc}(P_2)}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\mathrm{Asc}(P_2)}) + \deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}|_{\tilde{\mathcal{T}}_0^2}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}_0^2})$$
$$\overset{(53)}{=} \deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}|_{\mathrm{Asc}(P_2)}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\mathrm{Asc}(P_2)})$$
$$\overset{(55)}{=} \mathbf{f}(P_2) + \left.\frac{\nu_{P_2}(f)e}{dpq}\right|_{P_2} + 1 - \left.\frac{(p+q)e}{2pqd}\right|_{P_2} - \chi_{A^1}$$
$$\overset{(56)}{=} \left.\frac{(p+q)e}{2pqd}\right|_{P_2} - \left.\frac{\nu_{P_2}(f)e}{dpq}\right|_{P_2} + \left.\frac{\nu_{P_2}(f)e}{dpq}\right|_{P_2} + 1 - \left.\frac{(p+q)e}{2pqd}\right|_{P_2} - \chi_{A^1} = -1$$

Finally one has the desired result,

$$\deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}) = \deg(\tilde{\mathcal{T}}^{nul}) - 1.$$

□

Remark **(III.4.4).** Notice that in the case of a germ$(f,P) \subset (\mathbb{C}^2, P)$ one has $\deg(\mathcal{T}_P^{\delta_1,\delta_2}(f)) - \deg(\mathcal{T}_P^{nul}(f)) = \deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}(f)) - \deg(\tilde{\mathcal{T}}_P^{nul}(f))$ as a consequence of the construction of the trees, Lemma (III.4.3) and [**CA02**, Lemma 2.35].

Section § III.5
Number of local conditions

Let us construct the following submodule.

Definition (III.5.1). Let $\mathcal{D} = \{f = 0\}$ be a germ in $P \in X(d; a, b)$, where $f \in \mathcal{O}_P(k)$. Consider π a **Q**-resolution of $(\mathcal{D}, P)$.

Let us define $\mathcal{M}_{\mathcal{D},\pi}^{\delta_i\delta_j}$ the submodule of $\mathcal{M}_{\mathcal{D},\pi}^{log}$ consisting of all $h \in \mathcal{O}_P$ such that the 2-form

$$\omega = h\,\frac{dx \wedge dy}{f}$$

has zero residues outside the path $\gamma(\delta_1, \delta_2)$.

Remark **(III.5.2).** Note that as a consequence of Definitions (II.3.8), (III.5.1), and the construction of the trees $\tilde{\mathcal{T}}_P^{nul}$ and $\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}$, one has:

$$\mathcal{M}_{\mathcal{D},\pi}^{nul} = \{h \in \mathcal{O}_P \mid \tilde{\mathcal{T}}_P(\mathcal{D},\pi)|_h \geq \tilde{\mathcal{T}}_P^{nul}(\mathcal{D},\pi)\}.$$

$$\mathcal{M}_{\mathcal{D},\pi}^{\delta_i\delta_j} = \{h \in \mathcal{O}_P \mid \tilde{\mathcal{T}}_P(\mathcal{D},\pi)|_h \geq \tilde{\mathcal{T}}_P^{\delta_i\delta_j}(\mathcal{D},\pi)\}.$$

Remark **(III.5.3).** Note that the converse in Remark (III.1.11) is not true since $\mathcal{M}_{\mathcal{D},\pi}^{nul} \subseteq \mathcal{M}_{\mathcal{D},\pi}^{\delta_i\delta_j}$ but $\tilde{\mathcal{T}}_P^{nul}(\mathcal{D},\pi)\} \not\geq \tilde{\mathcal{T}}_P^{\delta_i\delta_j}(\mathcal{D},\pi)$ in general as Example (III.3.3) shows.

Definition (III.5.4). Let $\mathcal{D} = \{f = 0\}$ be a germ in $P \in X(d;a,b)$. Let us define the following dimension,

$$K_P^{\delta_i\delta_j}(\mathcal{D}) = K_P^{\delta_i\delta_j}(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\mathcal{M}_{\mathcal{D},\pi}^{\delta_i\delta_j}}.$$

Remark **(III.5.5).** Note that $K_P^{\delta_i\delta_j}(f)$ turns out to be a finite number (see Remark (III.5.2)) independent on the chosen **Q**-resolution. Intitutively, the number $K_P^{\delta_i\delta_j}(f)$ can be interpreted as the minimal number of conditions required to a generic germ $h \in \mathcal{O}_P(s)$ so that $h \in \mathcal{M}_{\mathcal{D},\pi}^{\delta_i\delta_j}(s)$.

Proposition (III.5.6). *Let $\{f = 0\}$ be an analytic germ of curve singularity at $[0]$ on $X(d;a,b)$. Consider δ_1, δ_2 any two different local branches of f at $[0]$, then*

$$K_0^{\delta_1\delta_2}(f) = K_0(f) - 1.$$

Proof. Recall the notation used in §II.3–2. By Remark (II.3.1) and the discussion after it, we can assume that

$$f(x,y) = y^r + \sum_{i>0\leq j<r} a_{ij}x^iy^j \in \mathbb{C}\{x\}[y]$$

in $X(p;-1,q)$ ($p = d, q \equiv -ba^{-1} \mod d$). Consider $g \in \mathbb{C}\{x,y\}$ the reduced germ obtained after applying Lemma (II.3.2) to f. Denote by $\pi_{(p,q)}$ the blowing-up at the origin of $(\mathbb{C}^2, 0)$. Note that $\nu_g = qr$, and thus $\delta_r^{(p,q)} = \delta_{\pi_{(p,q)}}^w(g)$ (see (26) and (16)). Consider a form $\alpha := \phi\frac{dx\wedge dy}{g}$, with $\phi \in \mathbb{C}\{x,y\}$ and let us calculate the local equations for the pull-back of α after $\pi_{(p,q)}$:

$$\phi\frac{dx \wedge dy}{g} \overset{\pi_{(p,q)}}{\longleftarrow} x^{\nu_\phi+p+q-1-\nu_g}h\frac{dx\wedge dy}{f}. \tag{60}$$

From (32) one has

$$K_0(f) = \delta_0^w(f) + \delta_r^{(p,q)} - A_r^{(p,q)}. \tag{61}$$

Following a similar argument as in the proof of Theorem (II.3.13) (see (30)), we will study the relation between $K_0^{\delta_1\delta_2}(f)$ and $K_0^{\tilde{\delta}_1\tilde{\delta}_2}(g)$, where $\tilde{\delta}_i$ are the corresponding branches of g. If π is a resolution of f, then the composition of π and $\pi_{(p,q)}$ gives a resolution of g, in particular by Corollary (II.2.6) one has

$$\delta_0(g) = \delta_r^{(p,q)} + \delta_0^w(f). \tag{62}$$

Also, by the construction of $\mathcal{T}^{\tilde{\delta}_1\tilde{\delta}_2}(g)$ one immediately obtains:

$$K_0^{\tilde{\delta}_1\tilde{\delta}_2}(g) = A_r^{(p,q)} + K_0^{\delta_1\delta_2}(f). \tag{63}$$

On the other hand, by Lemma (III.4.3), Remark (III.1.17), and (62)

$$K_0^{\tilde{\delta}_1\tilde{\delta}_2}(g) = \delta_0(g) - 1 = \delta_r^{(p,q)} + \delta_0^w(f) - 1, \tag{64}$$

therefore from (63) and (64) we have,

$$K_0^{\delta_1\delta_2}(f) = \delta_0^w(f) + \delta_r^{(p,q)} - A_r^{(p,q)} - 1. \tag{65}$$

Finally using (61) and (65) one concludes the desired result.

□

IV

Global Invariants: Adjunction-like Formula on $\mathbb{P}^2_w$

Consider an irreducible curve $\mathcal{C} \subset \mathbb{P}^2$ of degree d, it is a classical result (cf. [**Nor55, Mil68, Ser59, BK86, CA00**]) that the genus of its normalization considered as a compact oriented Riemann surface (also denoted by $g(\mathcal{C})$) depends only on d and the local type of its singularities as follows:

$$g(\mathcal{C}) = g_d - \sum_{P \in \operatorname{Sing}(\mathcal{C})} \delta_P,$$

where $g_d = \frac{(d-1)(d-2)}{2}$.

In §IV.1 we apply our previous results seen in Chapter II to generalize such a formula for the genus of a weighted-projective curve from its degree and the local type of its singularities (see [**CAMO13**]). This question is solved in Theorem (IV.1.12) (see also [**Dol82, OW71**] for the quasi smooth case).

As an application of such a formula (see Chapter IV, Corollary (IV.4.4)) this provides with a more concrete and computable version of an Adjunction-like Formula for divisors on quotient surface singularities, in particular on weighted projective planes. This means a formula relating the genus of a generic curve of quasi-homogeneous degree d, and the dimension of the space of polynomials of degree $d + \deg K$ (note that $\deg K = -|w| = -(w_0 + w_1 + w_2)$), with K the canonical divisor in $\mathbb{P}^2_w$ (this dimension will be denoted by $D_{d-|w|,w}$).

Section § **IV.1**

A genus formula for weighted projective curves

Denote by $\mathbb{P}^2_w$ the weighted projective space of weight $w = (w_0, w_1, w_2)$ written in a normalized form, that is, $\gcd(w_i, w_j) = 1$, for any pair of weights $i \neq j$ (cf. [**Dol82**]). Denote by $P_0 := [1:0:0]_w$, $P_1 := [0:1:0]_w$ and $P_2 := [0:0:1]_w$ the three vertices of $\mathbb{P}^2_w$. Denote also by $\bar{w} := w_0 w_1 w_2$, $w_{ij} := w_i w_j$ and $|w| := \sum_i w_i$.

Note that the condition $\gcd(w_i, w_j) = 1$ can be assumed without loss of generality, since a polynomial $F(X_0, X_1, X_2)$ defining a zero set on $\mathbb{P}^2_w$ with $w = \big(w_0, w_1, w_2\big)$ such that $X_i \nmid F$ is transformed into another polynomial $F(X_0^{\frac{1}{d_0}}, X_1^{\frac{1}{d_1}}, X_2^{\frac{1}{d_2}})$ on $\mathbb{P}^2(p_0, p_1, p_2)$ by the normalizing isomorphism defined in Proposition (I.4.3).

Definition (IV.1.1). Consider $\mathcal{C} \subset \mathbb{P}^2_w$ a curve of degree d given by a reduced equation $F = 0$. We define

$$\operatorname{Sing}(\mathcal{C}) = \mathcal{C} \cap \left(\{\partial_x F = \partial_y F = \partial_z F = 0\} \cup \{P_0, P_1, P_2\}\right).$$

Thus, we say $\mathcal{C}$ is *smooth* if $\operatorname{Sing}(\mathcal{C}) = \emptyset$. Also, we say $\mathcal{C}$ is *transversal w.r.t. the axes* if $\{P_0, P_1, P_2\} \cap \mathcal{C} = \emptyset$ and for any $P \in \mathcal{C} \cap \{X_i = 0\}$ the local equations of $\mathcal{C}$ and $\{X_i = 0\}$ are given by $uv = 0$.

Formulas for the genus of quasi-smooth curves can be found in [**Dol82**]. As an introductory example, let us compute the genus of a smooth curve $\mathcal{C} \subset \mathbb{P}^2_w$ of degree d transversal w.r.t. the axes. Note that only for some d such curves exist (see Remark (IV.1.5) and Lemma (IV.1.6)).

Consider the covering

$$\begin{array}{cccc} \phi: & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2_w \\ & [X_0 : X_1 : X_2] & \mapsto & [X_0^{w_0} : X_1^{w_1} : X_2^{w_2}]_w \end{array}$$

Note that $\phi^*(\mathcal{C})$ is a smooth projective curve of degree d. Therefore $\chi(\phi^*(\mathcal{C})) = 2 - (d-1)(d-2)$ and $\chi(\mathcal{C}) = 2 - 2g(\mathcal{C})$. Using the Riemann-Hurwitz formula one obtains

$$2-(d-1)(d-2) = \bar{w}\left(\chi(\mathcal{C}) - d\sum_i \frac{1}{w_{jk}}\right) + 3d = 2\bar{w} - 2\bar{w}g(\mathcal{C}) - d\sum_i w_i + 3d,$$

where $\chi(\mathcal{C}) - d\sum_i \frac{1}{w_{jk}}$ is the Euler characteristic of $\mathcal{C} \setminus (\mathcal{C} \cap \{X_0X_1X_2 = 0\})$, $\bar{w}$ is the degree of the covering, and $3d$ the cardinality of $\phi^{-1}(\mathcal{C} \cap \{X_0X_1X_2 = 0\})$. Therefore,

$$2\bar{w}g(\mathcal{C}) = d^2 - d|w| + 2\bar{w}.$$

This justifies the following.

Definition (IV.1.2) ([**CAMO13**]). For a given $d \in \mathbb{N}$ and a normalized weight list $w \in \mathbb{N}^3$ the *virtual genus* associated with d and w is defined as

$$g_{d,w} := \frac{d(d - |w|)}{2\bar{w}} + 1.$$

Remark **(IV.1.3).** Note that this number, $g_{d,w}$, is a generalization of the combinatorial number $g_d = \binom{d-1}{2}$ for $w = (1, 1, 1)$.

Let us see a lemma which will be useful in future results to understand the genus in the non-irreducible case.

Lemma (IV.1.4). *Let $d_i, n \in \mathbb{N}$ with $i = 1, \ldots, n$ and a normalized weight list $w \in \mathbb{N}^3$. Consider $d = d_1 + \cdots + d_n$ then,*

$$g_{d,w} = \sum_{i=1}^{n} g_{d_i,w} + \sum_{i \neq j} \frac{d_i d_j}{\bar{w}} - (n - 1).$$

Proof. If $d = d_1 + d_2$, after straightforward computation one has

$$g_{d,w} = g_{d_1,w} + g_{d_2,w} + \frac{d_1 d_2}{\bar{w}} - 1.$$

It is enough to proceed by induction to conclude the desired result. □

Remark **(IV.1.5)** ([**CAMO13**]). Note that $g_{d,w}$ is always defined regardless of whether or not there actually exist smooth curves of degree d in $\mathbb{P}^2_w$. For instance, it is easy to see that there are no smooth curves of degree 5 in $\mathbb{P}^2_{(2,3,5)}$ (see Remark (IV.1.9)).

In general, by the discussion above, if $g_{d,w}$ is not a positive integer, then no smooth curves in $\mathbb{P}^2_w$ of degree d transversal w.r.t. the axes can exist. However, this is not a sufficient condition, since $g_{40,(2,3,5)} = 20$, but all curves of degree 40 need to pass through at least one vertex.

The characterization is given by the following.

Lemma (IV.1.6) ([**CAMO13**]). *Given d and w as above, then the space of smooth curves of degree d transversal w.r.t. the axes in $\mathbb{P}^2_w$ is non-empty if and only if $\bar{w} \,|\, d$. Moreover, any smooth curve can be deformed into a smooth curve of the same degree and transversal w.r.t. the axes.*

Proof. Let F be a weighted homogeneous polynomial of degree d whose set of zeros defines $\mathcal{C}$. The condition $P_i \notin \mathcal{C}$ implies that F contains a monomial of type $\lambda_i X_i^{d_i}$, $\lambda_i \neq 0$, $i = 0, 1, 2$. Therefore $w_i d_i = d$, which implies the result since $\gcd(w_i, w_j) = 1$ by hypothesis on the weights w.

For the converse, assume $\bar{w}|d$, then $X_0^{\frac{d}{w_0}} + X_1^{\frac{d}{w_1}} + X_2^{\frac{d}{w_2}}$ is a smooth curve of degree d transversal w.r.t. the axes in $\mathbb{P}^2_w$.

The *moreover* part is a consequence of the fact that if $\mathcal{C}$ is smooth then $\bar{w}|d$ and hence $\mathcal{C} + \lambda_0 X_0^{\frac{d}{w_0}} + \lambda_1 X_1^{\frac{d}{w_1}} + \lambda_2 X_2^{\frac{d}{w_2}}$ is transversal w.r.t. the axes for an appropriate generic choice of λ_i. □

Remark **(IV.1.7).** A generic curve $\mathcal{C}$ of degree d in $\mathbb{P}^2_w$ is not smooth in general (recall the definition of $\operatorname{Sing}(\mathcal{C})$ seen in Definition (IV.1.1)).

PROOF. It is enough to apply Lemma (IV.1.6). □

Corollary (IV.1.8) ([**CAMO13**]). *If $\mathcal{C}$ is a smooth weighted curve in $\mathbb{P}^2_w$ of degree d, then $g(\mathcal{C}) = g_{d,w}$.*

PROOF. By Lemma (IV.1.6) one can assume that $\mathcal{C}$ is transversal w.r.t. the axes. The result follows immediately from the discussion above. □

Remark **(IV.1.9)** ([**CAMO13**]). Note that Corollary (IV.1.8) does not apply to quasi-smooth weighted curves, that is, curves whose equation is a weighted homogeneous polynomial whose only singularity in $\mathbb{C}^3$ is $\{0\}$. For instance, the curve $\mathcal{C} := \{X_0X_1 = X_2\} \subset \mathbb{P}^2_{(2,3,5)}$ of degree 5 can be parametrized by the map $\mathbb{P}^1 \to \mathbb{P}^2_{(2,3,5)}$, given by $[t : s] \mapsto [t^2 : s^3 : t^2s^3]_w$. Hence it is rational and $g(\mathcal{C}) = g(\mathbb{P}^1) = 0$. However, $g_{5,(2,3,5)} = \frac{7}{12}$. As a consequence of Corollary (IV.1.8), there are no smooth curves of degree 5 in $\mathbb{P}^2_{(2,3,5)}$ (see Remark (IV.1.5)).

Remark **(IV.1.10).** Let $V(F)$ be a curve of degree d in $\mathbb{P}^2_w$. Then

$$d \not\equiv 0 \mod w_i \Rightarrow F(P_i) = 0.$$

Moreover, if $V(F)$ is generic then

$$d \not\equiv 0 \mod w_i \Leftrightarrow F(P_i) = 0.$$

PROOF. For the first part notice that if $F(P_i) \neq 0$ then $F = X_i^a + \ldots$, $a \in \mathbb{N}$ with $aw_i = d$ and so $d \equiv 0 \mod w_i$. For the second one it is enough to prove that

$$F(P_i) = 0 \Rightarrow d \not\equiv 0 \mod w_i.$$

Since $V(F)$ is generic, one can write

$$F = \sum_{w_0 i + w_1 j + w_2 k = d} \alpha_{ijk} X_0^i X_1^j X_2^k,$$

with $\alpha_{ijk} \in \mathbb{C}$. If $F(P_i) = 0$, then the monomial X_i^a, $a \in \mathbb{N}$ with $aw_i = d$ cannot belong to F, and thus $d \not\equiv 0 \mod w_i$. □

Example (IV.1.11) ([**CAMO13**]). Consider Fermat curves of type

$$\mathcal{C} := \{X_0^{aw_1w_2} + X_1^{aw_0w_2} + X_2^{aw_0w_1} = 0\} \subset \mathbb{P}^2_w,$$

$w = (w_0, w_1, w_2)$. Note that $d = a\bar{w}$ and hence applying Corollary (IV.1.8) one obtains

$$g(\mathcal{C}) = g_{d,w} = \frac{a}{2}(a\bar{w} - |w|) + 1,$$

(see [**Ker07**] for the special case $w_0 = 1$).

Now we are in conditions to prove the main theorem of this section.

Theorem (IV.1.12) ([**CAMO13**])**.** *Let $\mathcal{C} \subset \mathbb{P}^2_w$ be an irreducible curve of degree $d > 0$, then*

$$g(\mathcal{C}) = g_{d,w} - \sum_{P \in \operatorname{Sing}(\mathcal{C})} \delta^w_P. \tag{66}$$

Proof. Let $F \in \mathcal{C}[X_0, X_1, X_2]$ be a defining equation for $\mathcal{C}$. Note that $F^{\bar{w}}$ defines a function. Also, from the proof of Lemma (IV.1.6) one can obtain an algebraic family of smooth curves $\mathcal{C}_t = \{F_t = 0\}$, $t \in (0, 1]$ of degree $d\bar{w}$ whose defining polynomials F_t degenerate to $F^{\bar{w}} = F_0$ such that $\mathcal{C}_t$ and $\mathcal{C}$ intersect transversally and outside the axes. Therefore, by Corollary (IV.1.8),

$$g(\mathcal{C}_t) = g_{d\bar{w},w} = \frac{d\bar{w}}{2\bar{w}}(d\bar{w} - |w|) + 1 = \frac{d}{2}(d\bar{w} - |w|) + 1.$$

On the other hand, define $I_t := \mathcal{C} \cap \mathcal{C}_t$. Using Bézout's Theorem (I.4.7),

$$(\mathcal{C} \cdot \mathcal{C}_t) = \sum_{P_t \in I_t} (\mathcal{C} \cdot \mathcal{C}_t)_{P_t} = \frac{1}{\bar{w}} d^2 \bar{w} = d^2.$$

Since $\mathcal{C}_t$ and $\mathcal{C}$ intersect transversally outside the axes at smooth points one has $(\mathcal{C} \cdot \mathcal{C}_t)_{P_t} = 1$ (see Example (I.3.6)) and hence $\# I_t = d^2$.

For each $P \in \operatorname{Sing}(\mathcal{C})$, consider B^S_P a regular neighborhood of $\mathcal{C}$ at P and for each $P_t \in I_t$ consider $B^I_{P_t}$ a regular neighborhood of $\mathcal{C} \cup \mathcal{C}_t$ at P_t.

Note that, outside $B := \bigcup_{P \in \operatorname{Sing}(\mathcal{C})} B^S_P \cup \bigcup_{P_t \in I_t} B^I_{P_t}$, $\mathcal{C}_t$ provides a $\bar{w} : 1$ covering of $\mathcal{C}$, that is,

$$\chi(\mathcal{C}_t \setminus B \cap \mathcal{C}_t) = \bar{w} \cdot \chi(\mathcal{C} \setminus B \cap \mathcal{C}).$$

Also, in each B^S_P, the curve $\mathcal{C}_t$ is the disjoint union of $\bar{w}$ Milnor fibers of $(\mathcal{C}, P)$. Therefore

(67)

$$\begin{aligned}\chi(\mathcal{C}_t) &= \chi(\mathcal{C}_t \setminus B \cap \mathcal{C}_t) + \textstyle\sum_{P \in \operatorname{Sing}(\mathcal{C})} \chi(B^S_P \cap \mathcal{C}_t) + \sum_{P_t \in I_t} \chi(B^I_{P_t} \cap \mathcal{C}) \\ &= \bar{w} \cdot \chi(\mathcal{C} \setminus B \cap \mathcal{C}) + \bar{w} \cdot \left[\textstyle\sum_{P \in \operatorname{Sing}(\mathcal{C})} 2\delta^w_P + \sum_{P \in \operatorname{Sing}(\mathcal{C})} r_P\right] + d^2.\end{aligned}$$

On the other hand

$$\chi(\mathcal{C}_t) = 2 - 2\left(\frac{d}{2}(d\bar{w} - |w|) + 1\right) = d|w| - d^2\bar{w} \tag{68}$$

and

$$\chi(\mathcal{C}) = \begin{cases} 2-2g(\mathcal{C}) - \sum_{P\in \operatorname{Sing}(\mathcal{C})}(r_P-1) \\ \chi(\mathcal{C}\setminus B\cap \mathcal{C}) + d^2 + \#\operatorname{Sing}(\mathcal{C}). \end{cases}$$

Therefore

$$\chi(\mathcal{C}\setminus B\cap\mathcal{C}) = 2-2g(\mathcal{C}) - d^2 - \sum_{P\in\operatorname{Sing}(\mathcal{C})} r_P. \tag{69}$$

Substituting (68) and (69) in (67) one obtains

$$d|w| - d^2\bar{w} = \bar{w}\cdot\left[2-2g(\mathcal{C}) - d^2 - \sum_{P\in\operatorname{Sing}(\mathcal{C})} r_P + \sum_{P\in\operatorname{Sing}(\mathcal{C})} 2\delta^w_P + \sum_{P\in\operatorname{Sing}(\mathcal{C})} r_P\right] + d^2,$$

which after simplification becomes

$$2\bar{w}g(\mathcal{C}) = d^2 - d|w| + 2\bar{w} - 2\bar{w}\cdot\left(\sum_{P\in\operatorname{Sing}(\mathcal{C})}\delta^w_P\right)$$

and results into the desired formula. □

Remark **(IV.1.13).** From Remark (III.1.16) one can rewrite formula (66) as follows

$$g(\mathcal{C}) = g_{d,w} - \sum_{P\in\operatorname{Sing}\mathcal{C}} \delta^w_P = g_{d,w} - \sum_{P\in\operatorname{Sing}\mathcal{C}} \deg(\mathcal{T}^{nul}_P(\mathcal{C})), \tag{70}$$

where $\mathcal{C}$ is an irreducible curve on $\mathbb{P}^2_w$ of quasi-homogeneous degree d.

Remark **(IV.1.14).** Let $\mathcal{C} = \cup_i \mathcal{C}_i \subset \mathbb{P}^2_w$, with $\mathcal{C}_i$ irreducible curves, then

$$g(\mathcal{C}) = \sum_i g(\mathcal{C}_i).$$

Remark **(IV.1.15).** Notice that

$$g(\mathcal{C}) = h^0(\hat{\mathcal{C}};\Omega^1).$$

Example (IV.1.16). Let us consider the curve $\mathcal{C} = \{X_0X_1 - X_2\} \subset \mathbb{P}^2_w$, with $w=(a,b,a+b)$. Note that $\mathbb{P}^1 \to \mathbb{P}^2_w$ given by $[t:s]\mapsto [t^a : s^b : t^a s^b]$ is an isomorphism and hence $g(\mathcal{C}) = g(\mathbb{P}^1) = 0$ (see discussion in Remark (IV.1.9)). In order to use Theorem (IV.1.12) one needs to compute the virtual genus of $\mathcal{C}$

$$g_{d,w} = \frac{2ab-a-b}{2ab}.$$

On the other hand $\operatorname{Sing}(\mathcal{C}) = \{P_0, P_1\}$. Note that both singularities P_0 and P_1 are of type $x^p - y^q$ with $(p,q)=(1,1)$ in their respective quotient-singularity charts ($P_0 \in X(a;b,a+b)$ and $P_1\in X(b;a,a+b)$) and thus, formula (18) implies:

$$\delta^w_{P_0} = \frac{1}{2}\frac{(1-1-1+a)}{a} = \frac{a-1}{2a}$$

and

$$\delta^w_{P_1} = \frac{1}{2}\frac{(1-1-1+b)}{b} = \frac{b-1}{2b}.$$

Therefore, according to Theorem (IV.1.12)

$$g(\mathcal{C}) = g_{d,w} - \delta^w_{P_0} - \delta^w_{P_1} = \frac{2ab-a-b}{2ab} - \frac{a-1}{2a} - \frac{b-1}{2b} = 0.$$

Example (IV.1.17). Let us consider now the curve $\mathcal{C} = \{X_0X_1 - X_2^2\} \subset \mathbb{P}^2_w$, with $w = (2k-1, 2k+1, 2k)$. In order to use Theorem (IV.1.12) one needs to compute the virtual genus of $\mathcal{C}$

$$g_{4k,w} = \frac{4k(4k-6k)}{2(4k^2-1)2k} + 1 = 1 - \frac{2k}{(4k^2-1)}.$$

On the other hand $\operatorname{Sing}(\mathcal{C}) = \{P_0, P_1\}$, $P_0 \in X(2k-1; 2k+1, 2k)$ and $P_1 \in X(2k+1; 2k-1, 2k)$.

Using Example (II.2.7) one has,

$$\delta^w_{P_0} = \frac{2-1-2+(2k-1)}{2(2k-1)} = \frac{k-1}{2k-1}$$

and

$$\delta^w_{P_1} = \frac{2-1-2+(2k+1)}{2(2k+1)} = \frac{k}{2k+1}.$$

Therefore, according to Theorem (IV.1.12)

$$g(\mathcal{C}) = 1 - \frac{2k}{(4k^2-1)} - \frac{k-1}{2k-1} - \frac{k}{2k+1} = 0.$$

Example (IV.1.18) ([**CAMO13**])**.** In the following example (recall Example (I.2.8)), the curve $\mathcal{C}$ is tangent to one of the axes. From the point of view of this work, such points are not special and do not contribute to the genus of $\mathcal{C}$ since they are smooth in $\mathcal{C}$. Note that this is one of the main differences with the approach shown in [**ABFdBLMH10**].

Let us consider the curve $\mathcal{C} = \{X_0X_1X_2 + (X_0^3 - X_1^2)^2\} \subset \mathbb{P}^2_w$ of quasi-homogeneous degree $d = 12$, with $w = (2, 3, 7)$. Note that

$$g_{12,w} = \frac{12(12-12)}{2\bar{w}} + 1 = 1.$$

On the other hand, $\operatorname{Sing}(\mathcal{C}) = \{P_2\}$, which is a quotient singularity of local type $xy + (x^2 - y^3)^2$ in $X(7; 2, 3)$. In order to obtain $\delta^w_{P_2}$ one can perform, for instance, a blow-up of type $(1, 5)$. The multiplicity of the exceptional divisor is 6 and hence

$$\frac{\nu(\nu-p-q+e)}{2dpq} = \frac{6(6-1-5+7)}{2\cdot 7\cdot 1\cdot 5} = \frac{3}{5}. \tag{71}$$

After this first blow-up, the two branches separate and the strict preimage becomes a smooth branch (at a smooth point of the surface) and a singularity of type $x^p - y^q$, where $(p,q) = (2,1)$, in $X(5;2,1)$. Using formula (18) one obtains:

$$\frac{\nu(\nu - p - q + e)}{2dpq} = \frac{pq - p - q + d}{2d} = \frac{2 - 2 - 1 + 5}{2 \cdot 5} = \frac{2}{5}. \tag{72}$$

Combining (71) and (72) one obtains

$$\delta^w_{P_2} = \frac{3}{5} + \frac{2}{5} = 1.$$

Therefore $g(\mathcal{C}) = 1 - 1 = 0$ according to Theorem (IV.1.12).

Section § IV.2
Computing the continuous discretely

Consider an irreducible curve $\mathcal{C} \subset \mathbb{P}^2_w$ of quasi-homogeneous degree d. During the rest of this chapter we will focus our efforts on obtaining an Adjunction-like Formula relating the genus of a generic curve of quasi-homogeneous degree d, and the dimension of the space of polynomials of degree $d + \deg K$, with K the canonical divisor in $\mathbb{P}^2_w$ (this dimension will be denoted by $D_{d-|w|,w}$).

Notation (IV.2.1). Consider $w_0, w_1, w_2, k \in \mathbb{N}$. We will use the following notation,

$$D_{k,w} := \# \left\{ (x,y,z) \in \mathbb{N}^3 \ | w_0 x + w_1 y + w_2 z = k \right\}.$$

Remark **(IV.2.2).** Notice that, with the previous notation, one has

$$D_{k-|w|,w} = h^0(\mathbb{P}^2_w; \mathcal{O}(k - |w|)).$$

IV.2–1. A preliminary example

Let us start this section with one basic illustrative example. Let us compute the number of solutions $(a,b,c) \in \mathbb{N}^3$ of the equation

$$aw_0 + bw_1 + cw_2 = k\bar{w}$$

with $w_0, w_1, w_2, k \in \mathbb{N}$ fixed, or equivalently, the number of monomials in $\mathcal{O}_{\mathbb{P}^2_w}$ of quasi-homogeneous degree $k\bar{w}$ (recall the notation used in §IV.1). This number will be denoted by $D_{k\bar{w},w}$. Notice that this is equivalent to computing the number of natural solutions (a,b,c) to $aw_0 + bw_1 = (kw_{01} - c)w_2$ (recall the notation used in the beginning of Section IV.1). To do this, consider the following sets:

$$\tilde{A} := \left\{ (a,b) \in \mathbb{N}^2, a > 1, b > 1 \ | aw_0 + bw_1 = \alpha w_2,\ \alpha = 0, \ldots, kw_{01} \right\},$$

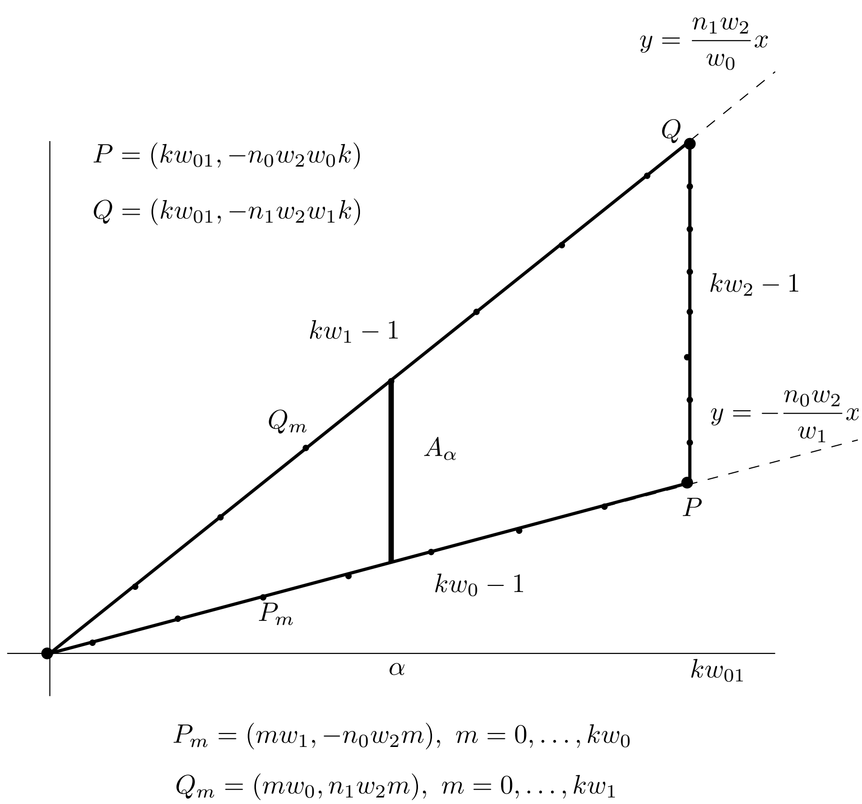


Figure IV.1.

$\tilde{B} := \left\{(a,0) \in \mathbb{N}^2, a > 1 \text{ or } (0,b) \in \mathbb{N}^2, b > 1 | aw_0 = \alpha w_2,\ \alpha = 0, \dots, kw_{01}\right\}.$ Denote by $A = \#\tilde{A}$ and $B = \#\tilde{B}$. With the previous notation one has $D_{k\bar{w},w} = A + B + 1$. To compute the cardinal of A take two integers $n_0,\ n_1$ such that $n_0 w_0 + n_1 w_1 = 1$ with $n_1 > 0$ and $n_0 \leq 0$ (it can always be done because the weights are pairwise coprime). Consider

$$A_\alpha := \# \left\{ \lambda \in \mathbb{N} \mid -\frac{n_0 \alpha w_2}{w_1} < \lambda < \frac{n_1 \alpha w_2}{w_0} \right\}$$

Note that by virtue of Pick's theorem the area of the triangle is equal to the number of natural points in its interior I plus one half the number of points in the boundary plus one. The area of the triangle equals $\frac{k^2\bar{\omega}}{2}$, thus

$$\frac{k^2\bar{\omega}}{2} = I + \frac{k|w|}{2} + 1,$$

therefore

$$A = I + (kw_2 - 1) = \left(\frac{k^2\bar{w}}{2} - \frac{k|w|}{2} + 1 \right) + (kw_2 - 1) = \frac{1}{2}(k^2\bar{w} - k|w|) + kw_2.$$

It is easy to check that $B = kw_0 + kw_1$, then we have

$$D_{k\bar{w},w} = \frac{1}{2}k\left(k\bar{w} + |w|\right) + 1$$

It is known that the genus of a smooth curve on $\mathbb{P}^2_w$ of degree d transversal w.r.t. the axes is (Definition (IV.1.2))

$$g_{d,w} = \frac{d(d-|w|)}{2\bar{w}} + 1.$$

We want to find d such that $D_{k\bar{w},w} = g_{d,w}$. To do that it is enough to solve the equation

$$\frac{1}{2}k\left(k\bar{w} + |w|\right) + 1 = \frac{d(d-|w|)}{2\bar{w}} + 1.$$

One finally gets that

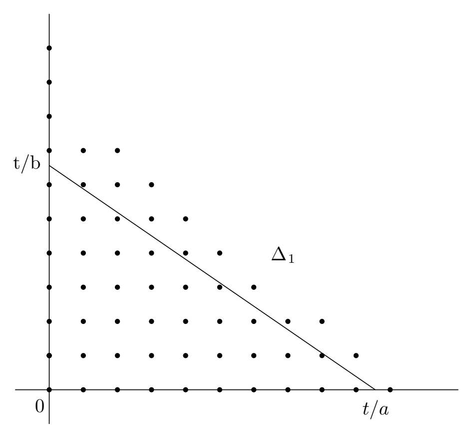
$$D_{k\bar{w},w} = g_{|w|+k\bar{w},w}.$$

An important by-product of this section is that only to compute $D_{k\bar{w},w}$, the computations have been, in some way, a bit "tricky". The natural question now is, how can $D_{d,w}$ be computed for an arbitrary degree d? In the following section this question will be solved.

IV.2–2. Counting points

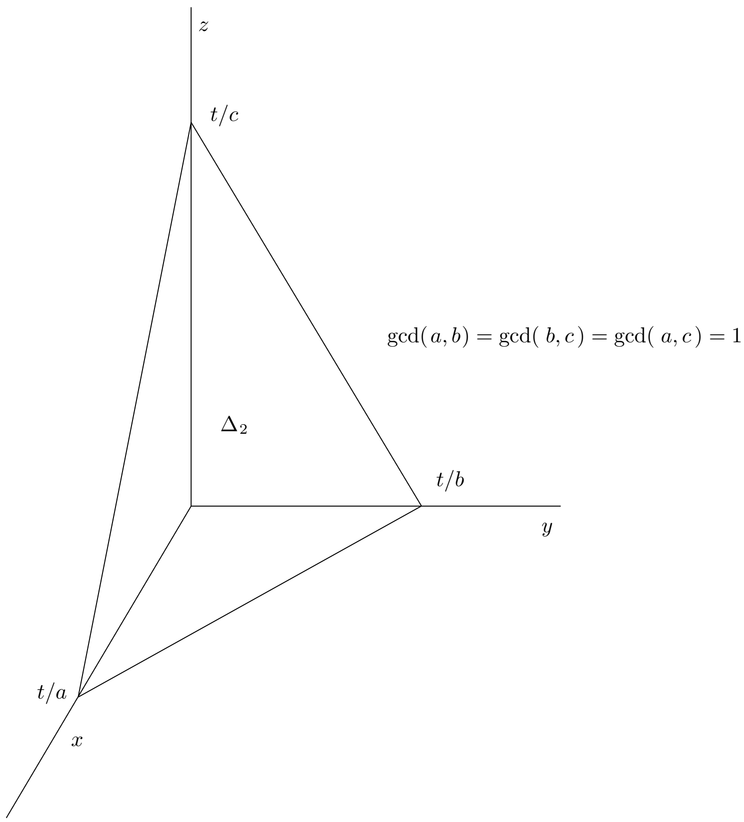
The main reference used in this section is [**BR07**]. Let a, b, c, t be positive integers with $\gcd(a,b) = \gcd(a,c) = \gcd(b,c) = 1$. The aim of this section is to give a way to compute the cardinal of the following two sets,

$$\Delta_1 := \{(x,y) \in \mathbb{N}^2 |\ ax + by \leq t\},$$



Note that $\#\Delta_1$ cannot be computed by means of Pick's Theorem unless t is divisible by a and b.

$$\Delta_2 := \{(x,y,z) \in \mathbb{N}^3 |\ ax + by + cz = t\},$$



Denote by $L_{\Delta_i}(t)$ the cardinal of Δ_i. Let us consider the following notation,

Notation (IV.2.3). If we denote by $\xi_a := e^{\frac{2i\pi}{a}}$, consider

(73)
$$\begin{aligned} p_{\{a,b,c\}}(t) &:= \mathrm{poly}_{\{a,b,c\}}(t) + \tfrac{1}{a}\sum_{k=1}^{a-1}\tfrac{1}{(1-\xi_a^{kb})(1-\xi_a^{kc})\xi_a^{kt}} \\ &+ \tfrac{1}{b}\sum_{k=1}^{b-1}\tfrac{1}{(1-\xi_b^{ka})(1-\xi_b^{kc})\xi_b^{kt}} + \tfrac{1}{c}\sum_{k=1}^{c-1}\tfrac{1}{(1-\xi_c^{ka})(1-\xi_c^{kb})\xi_c^{kt}}. \end{aligned}$$

with

(74)
$$\mathrm{poly}_{\{a,b,c\}}(t) := \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{3(ab+ac+bc)+a^2+b^2+c^2}{12abc}.$$

Remark **(IV.2.4).** Notice that in particular, one has

(75)
$$p_{\{a,b,1\}}(t) = \mathrm{poly}_{\{a,b,1\}}(t) + \frac{1}{a}\sum_{k=1}^{a-1}\frac{1}{(1-\xi_a^{kb})(1-\xi_a^{k})\xi_a^{kt}} + \frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{(1-\xi_b^{ka})(1-\xi_b^{k})\xi_b^{kt}}.$$

with

(76)
$$\mathrm{poly}_{\{a,b,1\}}(t) = \frac{t^2}{2ab} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{a} + \frac{1}{b}\right) + \frac{3(ab+a+b)+a^2+b^2+1}{12ab}.$$

Theorem (IV.2.5) ([**BR07**])**.** *One has the following result,*

$$L_{\Delta_1}(t) = p_{\{a,b,1\}}(t) \text{ and}$$

$$L_{\Delta_2}(t) = p_{\{a,b,c\}}(t).$$

Section § IV.3
Dedekind Sums

In this section we are going to define the Dedekind sums giving some properties which will be particularly useful for future results. See [**RG72**] and [**BR07**] for a more detailed exposition.

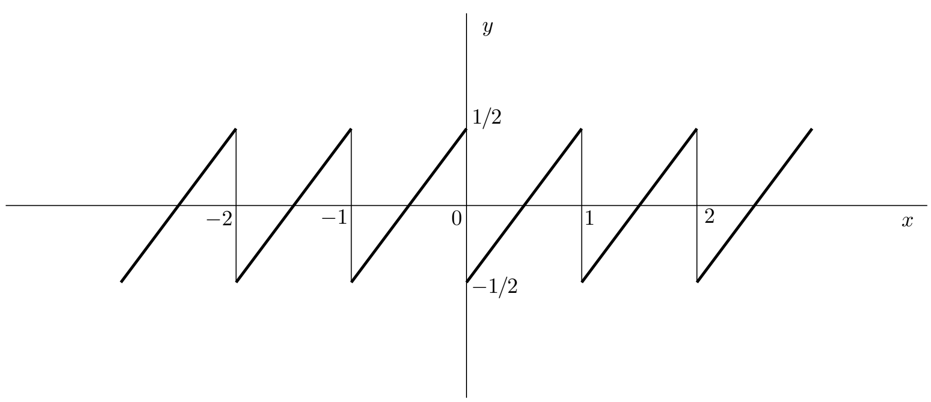
Definition (IV.3.1). Let a, b be integers, $\gcd(a,b) = 1$, $b \geq 1$. The Dedekind sum $s(a,b)$ is defined as follows

(77)
$$s(a,b) := \sum_{j=1}^{b-1} \left(\left(\frac{ja}{b}\right)\right)\left(\left(\frac{j}{b}\right)\right),$$

where the symbol $((x))$ denotes

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer,} \end{cases}$$

with $[x]$ the greatest integer not exceeding x. This is the well-known saw-tooth function of period 1,



which at the points of discontinuity takes the mean value between the limits from the right and from the left.

Let us see some properties of this kind of sums (see [**BR07**, Corolary 8.5] or [**RG72**, Theorem 2.1] for further details).

Theorem (IV.3.2) (Reciprocity Theorem, [**BR07, RG72**]). *Let a and b be two coprime integers. Then*

$$s(a,b)+s(b,a)=-\frac{1}{4}+\frac{1+a^2+b^2}{12ab}.$$

Let us express the sum (77) in terms of ath-roots of the unity (see for instance [**BR07**, Example 8.1] or [**RG72**, Chapter 2, (18b)] for further details).

Proposition (IV.3.3) ([**BR07, RG72**]). *Let a, b be integers,* $\gcd(a,b)=1$, $b\geq 1$, *denote by ξ_b a primitive bth-root of the unity. The Dedekind sum $s(a,b)$ can be written as follows:*

$$s(a,b)=\frac{b-1}{4b}-\frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{(1-\xi_b^{ka})(1-\xi_b^{k})}.$$

IV.3–1. Arithmetic properties

The reciprocity law of the Dedekind sums always contains two (and in some generalizations three and even more) Dedekind sums. We focus our attention now on a single Dedekind sum and its properties (see [**RG72**, Chapter 3] for a more detailed exposition).

Since $((-x))=-((x))$ it is clear that

$$s(-a,b)=-s(a,b)$$

and also

$$s(a,-b)=s(a,b).$$

If we define a' by $a'a\equiv 1 \mod b$ then

$$s(a',b)=s(a,b).$$

Proposition (IV.3.4) ([**BR07, RG72**]). *Let a, b, c be integers,* $\gcd(a,b)=\gcd(a,c)=\gcd(b,c)=1$. *Define a' by $a'a\equiv 1 \mod b$, b' by $b'b\equiv 1 \mod c$ and c' by $c'c\equiv 1 \mod a$. Then*

$$s(bc',a)+s(ca',b)+s(ab',c)=-\frac{1}{4}+\frac{a^2+b^2+c^2}{12abc}.$$

Proof. It is a direct consequence using Remark (IV.3.6) and Rademacher's reciprocity law (Corollary (IV.3.7)) in the next section. □

IV.3–2. Fourier-Dedekind Sums

Definition (IV.3.5) ([**BR07**])**.** Let $a_1, \ldots, a_m, n \in \mathbb{Z}$, $b \in \mathbb{N}$, then the Fourier-Dedekind sum is defined as follows:

$$s_n(a_1, \ldots, a_m; b) := \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{(1-\xi_b^{ka_1})(1-\xi_b^{ka_2}) \cdots (1-\xi_b^{ka_m})}.$$

Let us see some interesting properties of these sums.

Remark **(IV.3.6)** ([**BR07**])**.** Let $a, b, c \in \mathbb{Z}$ then

(1) For all $n \in \mathbb{Z}$, $s_n(a, b; 1) = 0$.
(2) For all $n \in \mathbb{Z}$, $s_n(a, b; c) = s_n(b, a; c)$.
(3) One has $s_0(a, 1; b) = -s(a, b) + \frac{b-1}{4b}$.
(4) If we denote by a' the inverse of a modulo c, then $s_0(a, b; c) = -s(a'b, c) + \frac{c-1}{4c}$.

Proof. The different proofs for these results can be found in [**BR07**, Chapter 8]. □

With this notation we can express (73) and (75) as follows

$$p_{\{a,b,1\}}(t) = \text{poly}_{\{a,1,b\}}(t) + s_{-t}(a, 1; b) + s_{-t}(1, b; a). \tag{78}$$

$$p_{\{a,b,c\}}(t) = \text{poly}_{\{a,b,c\}}(t) + s_{-t}(a, b; c) + s_{-t}(b, c; a) + s_{-t}(a, c; b). \tag{79}$$

As a consequence of Zagier reciprocity in dimension 3 (see [**BR07**, Theorem 8.4]) one has the following result.

Corollary (IV.3.7) (Rademacher's reciprocity law, [**BR07**])**.** *Making* $t = 0$ *in the above expression we get*

$$1 - \text{poly}_{\{a,b,c\}}(0) = s_0(a, b; c) + s_0(c, b; a) + s_0(a, c; b) = -\frac{1}{4} + \frac{a^2 + b^2 + c^2}{12abc}.$$

Section § **IV.4**

An Adjunction-like Formula on $\mathbb{P}^2_w$

Let us see how to compute an Adjunction-like Formula for curves on $\mathbb{P}^2_w$ relating $g(\mathcal{C})$ and $h^0(\mathbb{P}^2_w; \Omega^2(d))$.

Recall the notation used in the beginning of Section IV.1. Let w_0, w_1, w_2 be pairwise coprime integers, $d \in \mathbb{N}$ and denote by $\bar{w} = w_0 w_1 w_2$, $|w| = w_0 + w_1 + w_2$ where $w = (w_0, w_1, w_2)$.

Note also that the degree of the canonical divisor $K_{\mathbb{P}^2_w}$ in $\mathbb{P}^2_w$ is $-|w|$ (Definition (I.4.5)). Suppose there exists a non-singular curve $\mathcal{C}$ of degree d

(recall that this is not always possible, see Remark (IV.1.10)), the classical Adjunction Formula says,

$$K_{\mathcal{C}} = (K_{\mathbb{P}^2_w} + \mathcal{C})|_{\mathcal{C}}. \tag{80}$$

Note that, equating degrees in both sides of (80), by the Weighted Bézout's Theorem (Proposition (I.4.7)) one has

$$2g(\mathcal{C}) - 2 = \deg(K_{\mathbb{P}^2_w} + \mathcal{C})|_{\mathcal{C}} = \frac{\deg(\mathcal{C})\deg(K_{\mathbb{P}^2_w} + \mathcal{C})}{\bar{w}} = \frac{d(d-|w|)}{\bar{w}}.$$

This can be another approach to the combinatorial number $g_{d,w}$ seen in Definition (IV.1.2).

Remark **(IV.4.1).** Note that one can always consider a smooth curve of degree $k\bar{w}$ (see Remark (IV.1.10)). In that case, by §IV.2–1 and Remark (IV.2.2) one has

$$h^0(\mathbb{P}^2_w; \mathcal{O}(k\bar{w} - |w|)) = D_{k\bar{w}-|w|,w} = g_{k\bar{w},w}.$$

Now we are going revisit this last equatity in the case $\mathcal{C}$ is a singular curve.

Definition (IV.4.2). Let $\mathcal{C} \subset \mathbb{P}^2_w$ be a reduced curve of degree d, the *number of global conditions of* $\mathcal{C}$ is defined as follows (recall Definition (II.3.9))

$$K(\mathcal{C}) := \sum_{P \in \mathrm{Sing}(\mathcal{C})} K_P(f).$$

One has the following result.

Theorem (IV.4.3). *Let us consider positive integers* $p_i := w_i$ *and* $q_i := -w_j^{-1} w_k \mod w_i \in \mathbb{N}$ *with* $j < k$ *(recall that* $X(w_i; w_j, w_k) = X(p_i; -1, q_i)$*),* $r_i = w_k^{-1} d \mod w_i \in \mathbb{N}$. *Recall that*

$$A_{r_i}^{(p_i,q_i)} = \#\left\{(x,y) \in \mathbb{N}^2 \;|p_i x + q_i y \leq q_i r_i, x, y \geq 1\right\},$$

$$\delta_{r_i}^{(p_i,q_i)} = \frac{r_i(p_i r_i - p_i - q_i + 1)}{2p_i}.$$

Then

$$D_{d-|w|,w} = g_{d,w} + \sum_{i=0}^{2}\left(\delta_{r_i}^{(p_i,q_i)} - A_{r_i}^{(p_i,q_i)}\right).$$

Proof. From the definitions note that

$$D_{d-|w|,w} = p_{\{w_0,w_1,w_2\}}(d-|w|) =: p_{\{w\}}(d-|w|).$$

$$A_{r_i}^{(p_i,q_i)} = p_{\{p_i,q_i,1\}}(q_i r_i - p_i - q_i).$$

On the one hand from (79) and a direct computation one obtains

$$\begin{aligned} D_{d-|w|,w} - g_{d,w} &= -1 + \operatorname{poly}_{\{w\}}(0) + s_{|w|-d}(w_1, w_2; w_0) \\ &+ s_{|w|-d}(w_0, w_2; w_1) + s_{|w|-d}(w_0, w_1; w_2). \end{aligned}$$

Recall that (Definition (IV.3.5))

$$s_{|w|-d}(w_j, w_k; w_i) = \frac{1}{w_i} \sum_{l=1}^{w_i-1} \frac{1}{(1-\xi_{w_i}^{lw_j})(1-\xi_{w_i}^{lw_k})\xi_{w_i}^{l(d-|w|)}}.$$

Notice that (Corollary (IV.3.7))

$$-1 + \operatorname{poly}_{\{w\}}(0) = -\left(s_0(w_1, w_2; w_0) + s_0(w_0, w_2; w_1) + s_0(w_0, w_1; w_2)\right),$$

so we get

$$D_{d-|w|,w} - g_{d,w} = \sum_{i=0}^{2} \left(s_{|w|-d}(w_j, w_k; w_i) - s_0(w_j, w_k; w_i)\right). \tag{81}$$

On the other hand, from (78), and tedious computations one obtains

$$\begin{aligned} \delta_{r_i}^{(p_i,q_i)} - A_{r_i}^{(p_i,q_i)} &= \frac{p_i+q_i}{2p_iq_i} - \operatorname{poly}_{\{p_i,q_i,1\}}(0) \\ &- \left(s_{p_i+q_i-q_ir_i}(p_i, 1; q_i) + s_{p_i+q_i-q_ir_i}(q_i, 1; p_i)\right), \end{aligned} \tag{82}$$

with

$$s_{p_i+q_i-q_ir_i}(q_i, 1; p_i) = \frac{1}{p_i} \sum_{k=1}^{p_i-1} \frac{1}{(1-\xi_{p_i}^{kq_i})(1-\xi_{p_i}^{k})\xi_{p_i}^{k(q_ir_i-p_i-q_i)}},$$

and

$$\begin{aligned} s_{p_i+q_i-q_ir_i}(p_i, 1; q_i) &= \frac{1}{q_i} \sum_{k=1}^{q_i-1} \frac{1}{(1-\xi_{q_i}^{kp_i})(1-\xi_{q_i}^{k})\xi_{q_i}^{k(q_ir_i-p_i-q_i)}} \\ &= -\frac{1}{q_i} \sum_{k=1}^{q_i-1} \frac{1}{(1-\xi_{q_i}^{-kp_i})(1-\xi_{q_i}^{k})}, \end{aligned}$$

which implies using Proposition (IV.3.3)

$$s_{p_i+q_i-q_ir_i}(p_i, 1; q_i) = s_{p_i}(p_i, 1; q_i) = s(-p_i, q_i) - \frac{q_i-1}{4q_i}. \tag{83}$$

Since by hypothesis $p_i = w_i$, $q_i = -w_j^{-1}w_k \mod w_i$ and $r_i = w_k^{-1}d \mod w_i$, therefore

$$s_{p_i+q_i-q_ir_i}(q_i,1;p_i) = \tfrac{1}{w_i}\sum_{\ell=1}^{w_i-1} \tfrac{1}{(1-\xi_{w_i}^{\ell q_i})(1-\xi_{w_i}^{\ell})\xi_{w_i}^{\ell(q_ir_i-w_i-q_i)}}$$

$$= \tfrac{1}{w_i}\sum_{\ell=1}^{w_i-1} \tfrac{1}{(1-\xi_{w_i}^{\ell(-w_j^{-1}w_k)})(1-\xi_{w_i}^{\ell})\xi_{w_i}^{\ell(-w_j^{-1}d+w_j^{-1}w_k)}}$$

(84)
$$\overset{\ell=-w_j\bar\ell}{=} \tfrac{1}{w_i}\sum_{\bar\ell=1}^{w_i-1} \tfrac{1}{(1-\xi_{w_i}^{\bar\ell w_k})(1-\xi_{w_i}^{-\bar\ell w_j})\xi_{w_i}^{\bar\ell(d-w_k)}}$$

$$= -\tfrac{1}{w_i}\sum_{\bar\ell=1}^{w_i-1} \tfrac{1}{(1-\xi_{w_i}^{\bar\ell w_k})(1-\xi_{w_i}^{\bar\ell w_j})\xi_{w_i}^{\bar\ell(d-|w|)}} = -s_{|w|-d}(w_j,w_k;w_i).$$

Thus

(85) $$s_{|w|-d}(w_j,w_k;w_i) = -s_{p_i+q_i-q_ir_i}(q_i,1;p_i),$$

for $i = 0,\ldots,2$.

Using (82) it only remains to show that the right-hand side of (81) satisfies

(86) $$\begin{aligned} s_{|w|-d}(w_j,w_k;w_i) - s_0(w_j,w_k;w_i) &= \frac{p_i+q_i}{2p_iq_i} - \mathrm{poly}_{\{p_i,1,q_i\}}(0) \\ &- \left(s_{p_i+q_i-q_ir_i}(p_i,1;q_i) + s_{p_i+q_i-q_ir_i}(q_i,1;p_i)\right). \end{aligned}$$

Applying (85) this reduces to showing

(87) $$-s_0(w_j,w_k;w_i) = \frac{p_i+q_i}{2p_iq_i} - \mathrm{poly}_{\{p_i,1,q_i\}}(0) - s_{p_i+q_i-q_ir_i}(p_i,1;q_i).$$

For the left-hand side we use Remark (IV.3.6)(4) and obtain

$$s_0(w_j,w_k;w_i) = -s(-q_i,p_i) + \frac{p_i-1}{4p_i} = s(q_i,p_i) + \frac{p_i-1}{4p_i}.$$

For the right-hand side, using Corollary (IV.3.7) and (83) we have,

$$\mathrm{poly}_{\{p_i,1,q_i\}}(0)+s_{p_i+q_i-q_ir_i}(p_i,1;q_i) = (1 - s_0(q_i,1;p_i) - s_0(p_i,1;q_i))+s_{p_i}(p_i,1;q_i),$$

which, by Remark (IV.3.6)(3) and (83) becomes

$$\left(1 + +s(q_i,p_i) - \frac{p_i-1}{4p_i} + s(p_i,q_i) - \frac{q_i-1}{4q_i}\right) + s(-p_i,q_i) - \frac{q_i-1}{4q_i}.$$

Combining these equalities into (87) one obtains the result. □

Corollary (IV.4.4) (Adjunction-like Formula)**.** *Let $\mathcal{C} \subset \mathbb{P}^2_w$ be a reduced curve of degree d, then*

$$h^0(\mathbb{P}^2_w;\mathcal{O}(d-|w|)) = D_{d-|w|,w} = g_{d,w} - \sum_{P\in\mathrm{Sing}(\mathcal{C})} \delta^w_P + K(\mathcal{C}).$$

PROOF. Consider $\mathcal{C} \subset \mathbb{P}^2_w$ an irreducible curve of degree d, let us see that

$$D_{d-|w|,w} = g(\mathcal{C}) + K(\mathcal{C}).$$

To observe that it is enough to apply Theorem (IV.4.3) and recall the characterization of $K_P(\mathcal{C})$ in the proof of Theorem (II.3.13) (see (32)).

$$\begin{aligned} g(\mathcal{C}) &= g_{d,w} - \sum_{P \in \mathrm{Sing}(\mathcal{C})} \delta^w_P \\ &= \underbrace{g_{d,w} + \sum_{i=0}^{2} \left(\delta^{(p_i,q_i)}_{r_i} - A^{(p_i,q_i)}_{r_i} \right)}_{D_{d-|w|,w}} - \underbrace{\left(\sum_{P \in \mathrm{Sing}(\mathcal{C})} \delta^w_P + \sum_{i=0}^{2} \left(\delta^{(p_i,q_i)}_{r_i} - A^{(p_i,q_i)}_{r_i} \right) \right)}_{K(\mathcal{C})}. \end{aligned}$$

The second equality in the previous identity always holds and therefore if one considers $\mathcal{C} \subset \mathbb{P}^2_w$ a reduced curve of degree d, then

$$h^0(\mathbb{P}^2_w; \mathcal{O}(d - |w|)) = D_{d-|w|,w} = g_{d,w} - \sum_{P \in \mathrm{Sing}(\mathcal{C})} \delta^w_P + K(\mathcal{C}).$$

□

Let us see an example of the previous result.

Example (IV.4.5). Recall Example (I.2.8). Consider $\mathcal{C} = V(F)$ in $\mathbb{P}^2_w$ with $w = (2,3,7)$ and $F = (xyz + (x^3 - y^2)^2)$. One has $\deg_w(\mathcal{C}) = 12$. We know (Example (IV.1.18)) that $\delta^w_{[0.0:1]} = 1$, $g_{12,w} = 1$ and $g(\mathcal{C}) = 0$. The form $\tau = \frac{\Omega^2}{F}$ is logarithmic but it is not holomorphic outside the strict transform $\hat{F}$ (see (88)), so in particular $1 \notin \mathcal{M}^{nul}_{F,\pi}$. We need to study the number of conditions required for a generic germ $h \in \mathcal{O}_P$, $P \in X(7;2,3)$, so that $h \in \mathcal{M}^{nul}_{F,\pi}$. Notice that in the present case $\mathcal{O}_P = \mathbb{C}\{x,y\}^{\boldsymbol{G}_7} = \mathbb{C}\{x^7, y^7, x^2y\}$. Let us see that, for example, $y^7\tau$ is holomorphic outside $\hat{F}$ and therefore $y^7 \in \mathcal{M}^{nul}_{F,\pi}$,

$$y^7 \; 7\frac{dx \wedge dy}{xy + (x^3 - y^2)^2} \overset{\substack{x = u_1 \bar{v}_1 \\ y = \bar{v}_1^5,\ v_1 = \bar{v}_1^7}}{\longleftarrow} v_1^5 \; 5\frac{du_1 \wedge dv_1}{v_1(u_1 + (u_1^3 - v_1)^2)}$$

$$\overset{\substack{u_1 = \bar{u}_2^2,\ u_2 = \bar{u}_2^5 \\ v_1 = \bar{u}_2 v_2}}{\longleftarrow} u_2^5 v_2 \; 2\frac{du_2 \wedge dv_2}{u_2 v_2 (1 + v_2^2)}, \tag{88}$$

Proceeding in analogous way, $x^7, x^2y \in \mathcal{M}^{nul}_{F,\pi}$ and hence by Definition (II.3.9) one has

$$K(\mathcal{C}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_P}{\mathcal{M}^{nul}_{\mathcal{C},\pi}} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}^{\boldsymbol{G}_7}}{\mathcal{O}_P(x^7, y^7, x^2y)} = 1.$$

Finally, it is straightforward to check that $D_{12-|w|,w} = D_{0,w} = 1$ and therefore one has the result previously seen in Corollary (IV.4.4),

$$1 = D_{0,w} = g_{12,w} - \delta^w_{[0.0.1]} + K(\mathcal{C}) = 1 - 1 + 1.$$

V

Structure of $H^\bullet(\mathbb{P}^2_w \setminus \mathcal{R}; \mathbb{C})$

In [**CAM12**], Cogolludo-Agustín and Matei determine an explicit presentation by generators and relations of the cohomology algebra $H^\bullet(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ of the complement of an algebraic curve $\mathcal{C}$ in the complex projective plane $\mathbb{P}^2$, via the study of log-resolution logarithmic forms on $\mathbb{P}^2$.

Our aim in this Chapter is to extend this result for rational arrangements in $\mathbb{P}^2_w$ (see Definition (V.1.1) below). As a first approach , in §V.2, we present a basis for $H^1(\mathbb{P}^2_w \setminus \mathcal{C}; \mathbb{C})$. In §V.3 some examples of the computation of the ring structure of $H^2(\mathbb{P}^2_w \setminus \mathcal{C}; \mathbb{C})$ are provided. Finally, in §V.4, a holomorphic presentation for $H^2(\mathbb{P}^2_w \setminus \mathcal{R}; \mathbb{C})$, for a rational arrangement $\mathcal{R}$, is given.

Section § V.1
The spaces $H^k(\mathbb{P}^2_w \setminus \mathcal{D}; \mathbb{C})$ and the residue maps

Let us start with some basic definitions and useful notation.

Definition (V.1.1).
A reduced $\mathbb{Q}$-divisor in $\mathbb{P}^2_w$ will be called an *arrangement*. If all irreducible components $\mathcal{C}_0, ..., \mathcal{C}_n$ in an arrangement $\mathcal{D}$ are rational curves ($g(\mathcal{C}_i) = 0$), we shall say $\mathcal{D}$ is a *rational arrangement*.

Notation (V.1.2). From now on, $\mathcal{D}$ will denote an arrangement in $\mathbb{P}^2_w$, and $\mathcal{R}$ a rational arrangement. The complement of $\mathcal{D}$ (resp. $\mathcal{R}$) will be sometimes denoted by $X_\mathcal{D}$ (resp. $X_\mathcal{R}$) for simplicity.

Let us fix $\pi : \overline{X}_\mathcal{D} \longrightarrow \mathbb{P}^2_w$ a **Q**-resolution of the singularities of an arrangement $\mathcal{D}$ so that the reduced $\mathbb{Q}$-divisor $\overline{\mathcal{D}} = (\pi^*(\mathcal{D}))_{red}$ is a union of **Q**-smooth divisors on $\overline{X}_\mathcal{D}$ with **Q**-normal crossings as described in Chapter I.

Let us see two technical results which will be used in the subsequent sections. Proofs are omitted because they are similar to the correspondent results in $\mathbb{P}^2$, see [**CA02**, Propositions 2.2 and 2.5] for further details.

Proposition (V.1.3). *Let $\mathcal{D}$ be an arrangement in $\mathbb{P}^2_w$, then*

$$H^2(X;\mathbb{C}) \cong H_1(\mathcal{D};\mathbb{C}), \text{ and}$$
$$H^1(X_{\mathcal{D}};\mathbb{C}) \cong H_2(\mathcal{D};\mathbb{C})/\mathbb{C} = \mathbb{C}^{r+1}/\mathbb{C}.$$

(V.1.4). Let Y be a topological space. In what follows we will denote by $h_i(Y)$ (resp. $h^i(Y)$) the dimension of the vector space $H_i(Y;\mathbb{C})$ (resp. $H^i(Y;\mathbb{C})$). Note that, by the Universal Coefficient Theorem, $h_i(Y) = h^i(Y)$.

In the following proposition the residue maps are applied to the case of rational arrangements.

Proposition (V.1.5). *Let $\mathcal{D}$ be an arrangement in $\mathbb{P}^2_w$. Then there is an injection*

$$H^1(X_{\mathcal{D}};\mathbb{C}) \overset{\mathrm{Res}^{[1]}}{\hookrightarrow} H^0(\overline{\mathcal{D}}^{[1]};\mathbb{C})$$

and a map

$$H^2(X_{\mathcal{D}};\mathbb{C}) \overset{\mathrm{Res}^{[2]}}{\to} H^0(\overline{\mathcal{D}}^{[2]};\mathbb{C})$$

via the Poincaré residue operator (recall Definition (I.5.7)). Moreover, $\mathrm{Res}^{[2]}$ *is injective if and only if $\mathcal{D}$ is a rational arrangement.*

Section § V.2
Logarithmic 1-forms: a basis for $H^1(\mathbb{P}^2_w \setminus \mathcal{D};\mathbb{C})$

The aim of this section is to compute a basis for $H^1(\mathbb{P}^2_w \setminus \mathcal{D};\mathbb{C})$ generalizing [**CA02**, Theorem 2.11].

Notation (V.2.1). In what follows we shall consider a system of coordinates $[X:Y:Z]$ in $\mathbb{P}^2_w$. If one writes $\mathcal{D} := \{D = 0\}$, D can be expressed as a product $C_0 \cdot C_1 \cdot \ldots \cdot C_n$ where $\mathcal{C}_i := \{C_i = 0\}$, C_i are irreducible components of D. Denote also $\mathcal{C}_{ij} := \{C_iC_j = 0\}$.

Definition (V.2.2). One can consider the following differential forms

$$\sigma_{ij} := d\left(\log \frac{C_i^{d_j}}{C_j^{d_i}}\right) = d_j\ d(\log C_i) - d_i\ d(\log C_j). \tag{89}$$

where $i,j = 0,...,n$, $d_i := \deg_w(\mathcal{C}_i)$.

Note that, since any two determinations of $\log \frac{C_i^{d_j}}{C_j^{d_i}}$ differ by a constant, their differential is well defined.

Lemma (V.2.3). *The holomorphic 1-forms σ_{ij} are well defined on $\mathbb{P}^2_w \setminus \mathcal{C}_{ij}$.*

Proof. This is a consequence of the following two facts:

(1) Each σ_{ij} is invariant under the $\mathbb{C}^*$-action
$$\lambda \cdot (X, Y, Z) = (\lambda^{w_0} X, \lambda^{w_1} Y, \lambda^{w_2} Z).$$

(2) They vanish on the space tangent to the fibers of the natural projection
$$\begin{array}{ccc} \mathbb{C}^3 \setminus \{C_j = 0\} & \xrightarrow{p} & \mathbb{P}^2_w \setminus \mathcal{C}_j \\ (X, Y, Z) & \mapsto & [X : Y : Z]_w \end{array}$$
at any point.

Part (1) is straightforward. Let us prove (2). The vector
$$E(X, Y, Z) = w_0 X \frac{\partial}{\partial X} + w_1 Y \frac{\partial}{\partial Y} + w_2 Z \frac{\partial}{\partial Z}$$
generates the space tangent to the fibers of j at (X, Y, Z). Hence one has to check
$$\sigma_{ij}(E(X, Y, Z)) = 0.$$
Note that
$$\begin{aligned} \sigma_{ij}(E(X, Y, Z)) \quad = \quad & d_j \left(w_0 X \frac{C_{i,X}}{C_i} + w_1 Y \frac{C_{i,Y}}{C_i} + w_2 Z \frac{C_{i,Z}}{C_i} \right) \\ & - d_i \left(w_0 X \frac{C_{j,X}}{C_j} + w_1 Y \frac{C_{j,Y}}{C_j} + w_2 Z \frac{C_{j,Z}}{C_j} \right), \end{aligned}$$
where $C_{i,X}$, $C_{i,Y}$ and $C_{i,Z}$ are the derivatives of C_i with respect to X, Y and Z respectively. Finally, by the Euler identity
$$\sum_i w_i X_i \frac{\partial F}{\partial X_i} = \deg_w F,$$
$$w_0 X C_{i,X} + w_1 Y C_{i,Y} + w_2 Z C_{i,Z} = d_i C_i,$$
and therefore
$$\sigma_{ij}(E(X, Y, Z)) = d_j d_i \frac{C_i}{C_i} - d_i d_j \frac{C_j}{C_j} = d_j d_i - d_i d_j = 0.$$
□

Definition (V.2.4). From the discussion above, σ_{ij} defines a global differential 1-form on $X_{\mathcal{D}}$ for any $i, j = 0, \ldots, n$.

Remark **(V.2.5).** With the previous notation, the following equalities hold:

(1) $\sigma_{ij} = -\sigma_{ji}$.

(2) $d_k\sigma_{ij} + d_i\sigma_{jk} + d_j\sigma_{ki} = 0$

Proof. Straightforward computation. Direct consequences of the 1-form σ_{ij} definition (89). □

Proposition (V.2.6). *The pull-back $\pi^*\sigma_{ij}$ defines a logarithmic 1-form on $\overline{X}_{\mathcal{D}}$.*

Proof. Since the 1-forms σ_{ij} are C^∞ on $X_{\mathcal{D}}$ the statement is trivial on $X_{\mathcal{D}}$. Hence, it is enough to check the statement locally at the points of $\overline{\mathcal{D}} = \overline{X}_{\mathcal{D}} \setminus X_{\mathcal{D}}$.

Let $P \in \overline{\mathcal{D}}$ be a point of type $X(d;a,b)$ on the inverse image of $\mathcal{D}$. Hence, by Remark (I.2.3) the pull-back of σ_{ij} by π can be written locally at P as a multiple of

$$\frac{d(x^n y^m)}{x^n y^m} = n\frac{dx}{x} + m\frac{dy}{y}$$

which is logarithmic at P. □

Finally, the injectivity of the residue map $\mathrm{Res}^{[1]}$ (Proposition (V.1.5)) will prove that, fixing $k \in \{0,\ldots,n\}$, the forms σ_{ik}, $i \neq k$ define a basis for $H^1(X_{\mathcal{D}};\mathbb{C})$. From Corollary (I.5.12), if no ambiguity seems likely to arise, we will use σ_{ik} instead of $\pi^*\sigma_{ik}$.

Theorem (V.2.7). *The cohomology classes of*

$$\mathcal{B}_1(\mathcal{D}) := \{\sigma_{ik}\}_{i\neq k}$$

for a fixed k, constitute a basis for $H^1(X_{\mathcal{D}};\mathbb{C})$.

Proof. By the de Rham theorem, the classes $[\sigma_{ik}]$ with respect to d-cohomology are elements of $H^1(X_{\mathcal{D}};\mathbb{C})$. Moreover, their residues can be obtained as follows

$$\left(\mathrm{Res}^{[1]}_\pi \sigma_{ik}\right)_{\hat{\mathcal{C}}_j} = \begin{cases} d_k & \text{if } i = j \neq k \\ 0 & \text{if } i \neq j \neq k \\ -d_i & \text{if } j = k \end{cases}$$

where $\hat{\mathcal{C}}_j \in \overline{\mathcal{D}}^{[1]}$ is the strict transform of $\mathcal{C}_j$. Since the images are linearly independent in $H^0(\overline{\mathcal{D}}^{[1]};\mathbb{C})$, the forms σ_{ik} are also linearly independent in $H^1(X_{\mathcal{D}};\mathbb{C})$. Finally, by Proposition (V.1.3), $\mathcal{B}_1(\mathcal{D})$ has maximal cardinality. □

Section § **V.3**

Two examples: ring structure of $H^\bullet(\mathbb{P}^2_w \setminus \mathcal{C};\mathbb{C})$

For line arrangements in $\mathbb{P}^2$, the 2-forms $\frac{d\ell_i}{\ell_i}\wedge\frac{d\ell_j}{\ell_j}$ generate $H^2(X_{\mathcal{L}};\mathcal{C})$, but in general it is no longer true for rational arrangements in $\mathbb{P}^2$. For instance, let $\mathcal{C}=\mathcal{C}_0\cup\mathcal{C}_1\cup\mathcal{C}_2$ be a plane quartic, where $\mathcal{C}_1$ is a conic, $\mathcal{C}_2$ is a line tangent to $\mathcal{C}_1$, and $\mathcal{C}_0$ is a transversal line. One has $h^1(X_{\mathcal{C}})=h^2(X_{\mathcal{C}})=2$ and hence, by dimension reasons, $H^2(X_{\mathcal{C}};\mathbb{C})$ cannot be generated by $\wedge^2 H^1(X_{\mathcal{C}};\mathbb{C})$.

As a first approach to the general problem of curves in $\mathbb{P}^2_w$ we will describe the ring structure of $H^\bullet(X_{\mathcal{C}};\mathbb{C})$ for some particular examples.

V.3–1. Ring structure of $H^\bullet(\mathbb{P}^2_w \setminus \{xyz=0\};\mathbb{C})$

As a first example, let us compute the ring structure of $H^\bullet(\mathbb{P}^2_w \setminus \mathcal{D};\mathbb{C})$ being $\mathcal{D}=V(xyz)$.

First of all the Euler characteristic $\chi(\mathbb{P}^2_w\setminus\mathcal{D})$ is computed,

$$\begin{aligned} 3=\chi(\mathbb{P}^2_w) &= \chi(\mathbb{P}^2_w\setminus\mathcal{D})+(\chi(V(z))-2)\\ &+ (\chi(V(y))-2)+(\chi(V(x))-2)+3, \end{aligned}$$

then $\chi(\mathbb{P}^2_w\setminus\mathcal{D})=0$.

On the other hand $h^1(\mathbb{P}^2_w\setminus\mathcal{D})=h_2(\mathcal{D})-1=2$ and one has

$$\chi(\mathbb{P}^2_w\setminus\mathcal{D})=\dim H^0(\mathbb{P}^2_w,\mathbb{P}^2_w\setminus\mathcal{D})-h^1(\mathbb{P}^2_w\setminus\mathcal{D})+h^2(\mathbb{P}^2_w\setminus\mathcal{D}).$$

We conclude that $h^1(\mathbb{P}^2_w\setminus\mathcal{D})=2$ and $h^2(\mathbb{P}^2_w\setminus\mathcal{D})=1$ so we need to find two logarithmic 1-forms and one logarithmic 2-form invariant under the Euler operator to generate our space. Consider the following global logarithmic 1-forms (see Lemma (V.2.3)):

$$\begin{aligned} \sigma_{01} &= w_1\frac{dx}{x}-w_0\frac{dy}{y},\\ \sigma_{02} &= w_2\frac{dx}{x}-w_0\frac{dz}{z},\\ \sigma_{12} &= w_2\frac{dy}{y}-w_1\frac{dz}{z}. \end{aligned}$$

The following relation is easily checked (see Remark (V.2.5)).

$$R_0 := w_2\sigma_{01}+w_0\sigma_{12}+w_1\sigma_{20}=0.$$

By Theorem (V.2.7), σ_{01} and σ_{02} are generators of $H^1(\mathbb{P}^2_w\setminus\mathcal{D})$. Consider now the 2-form

$$\tau := \frac{\Omega^2}{xyz} = w_2\ \frac{dx\wedge dy}{xy}+w_0\ \frac{dy\wedge dz}{yz}+w_1\frac{dz\wedge dx}{zx},$$

which is clearly logarithmic (recall the definition of the weighted volume form in (9) $\Omega^2 := w_2\ zdx\wedge dy + w_0\ xdy\wedge dz + w_1\ ydz\wedge dx$).

By direct computation, the following relations hold:

$$\begin{aligned} R_1 &:= \sigma_{01}\wedge\sigma_{12} = w_1\tau, \\ R_2 &:= \sigma_{01}\wedge\sigma_{02} = w_0\tau, \\ R_3 &:= \sigma_{02}\wedge\sigma_{12} = w_2\tau. \end{aligned} \tag{90}$$

Summarizing all the results we get the ring structure of $H^\bullet(\mathbb{P}^2_w \setminus \{xyz = 0\};\mathbb{C})$. The generators are:

$$\begin{aligned} H^\bullet(\mathbb{P}^2_w \setminus \{xyz=0\};\mathbb{C}) &= H^0(\mathbb{P}^2_w\setminus\mathcal{D};\mathbb{C})\oplus H^1(\mathbb{P}^2_w\setminus\mathcal{D};\mathbb{C})\oplus H^2(\mathbb{P}^2_w\setminus\mathcal{D};\mathbb{C}) \\ &= \langle 1\rangle\oplus\langle\sigma_{01},\sigma_{02}\rangle\oplus\langle\tau\rangle, \end{aligned}$$

with the relation

$$\sigma_{01}\wedge\sigma_{02} = w_0\tau.$$

Note that $H^\bullet(\mathbb{P}^2_w \setminus \{xyz = 0\};\mathbb{C})$ is independent of w.

V.3–2. Ring structure of $H^\bullet(\mathbb{P}^2_w \setminus \{xyz(xyz + (x^3-y^2)^2) = 0\};\mathbb{C})$

As a second example (recall Example (I.2.8) and Figure I.4), let us see how to compute the ring structure of $H^\bullet(\mathbb{P}^2_w\setminus\mathcal{D};\mathbb{C})$ being $\mathcal{D} = V(xyz(xyz+(x^3-y^2)^2))$ and $w = (2,3,7)$. Denote by $\mathcal{C}_0 = \{x = 0\}$, $\mathcal{C}_1 = \{y = 0\}$, $\mathcal{C}_2 = \{z = 0\}$ and $\mathcal{C}_3 = V(F) = \{xyz + (x^3-y^2)^2 = 0\}$. Under the previous notation $\mathcal{D} = \mathcal{C}_0\cup\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3$.

We have already seen that $g(\mathcal{C}_3) = 0$ (see Example (IV.1.18)). Also note that $\#\operatorname{Sing}(\mathcal{D})\cap\mathcal{C}_0 = \#\operatorname{Sing}(\mathcal{D})\cap\mathcal{C}_1 = \#\operatorname{Sing}(\mathcal{D})\cap\mathcal{C}_3 = 2$, $\#\operatorname{Sing}(\mathcal{D})\cap\mathcal{C}_2 = 3$. and $\chi(\mathcal{C}_3\setminus\{[0:0:1],[1:1:0]\}) = -1$ since $\mathcal{C}_3$ has two branches at $[0:0:1]$. Finally we know that

$$\begin{aligned} 3 = \chi(\mathbb{P}^2_w) &= \chi(\mathbb{P}^2_w\setminus\mathcal{D}) + (\chi(\mathcal{C}_0)-2) + (\chi(\mathcal{C}_1)-2) \\ &+ (\chi(\mathcal{C}_2)-3) + (\chi(\mathcal{C}_3)-2) + 4, \end{aligned}$$

then $\chi(\mathbb{P}^2_w\setminus\mathcal{D}) = 1$. One also has that $\dim H^0(\mathbb{P}^2_w,\mathbb{P}^2_w\setminus\mathcal{D}) = 1$ and $h^1(\mathbb{P}^2_w\setminus\mathcal{D}) = h^2(\mathcal{D}) - 1 = 3$, therefore

$$\chi(\mathbb{P}^2_w\setminus\mathcal{D}) = \dim H^0(\mathbb{P}^2_w,\mathbb{P}^2_w\setminus\mathcal{D}) - h^1(\mathbb{P}^2_w\setminus\mathcal{D}) + h^2(\mathbb{P}^2_w\setminus\mathcal{D}) \Rightarrow h^2(\mathbb{P}^2_w\setminus\mathcal{D}) = 3.$$

We need to find three global logarithmic 1-forms and three independent 2-forms to generate our space. Consider the following three global 1-forms,

$$\begin{aligned} \sigma_{01} &= 3\frac{dx}{x} - 2\frac{dy}{y}, \\ \sigma_{02} &= 7\frac{dx}{x} - 2\frac{dz}{z}, \end{aligned}$$

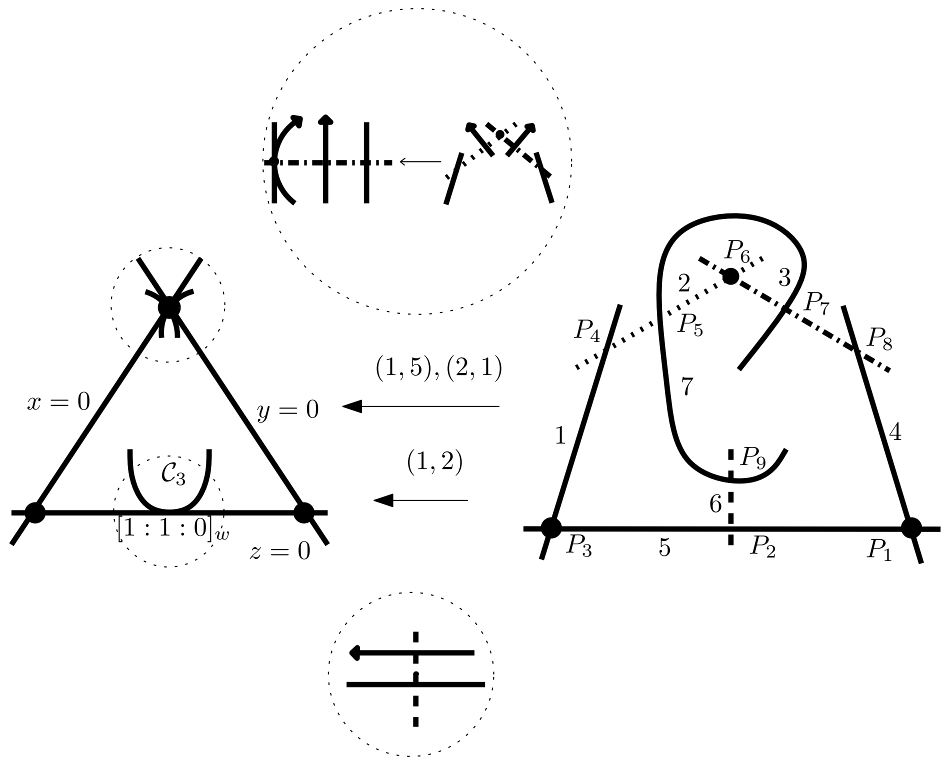


FIGURE V.1. **Q**-Resolution of $\mathcal{D}$.

$$\sigma_{03} = 12\frac{dx}{x} - 2\frac{dF}{F}.$$

By Theorem (V.2.7), they constitute a basis for $H^1(\mathbb{P}^2_w \setminus \mathcal{D};\mathbb{C})$. Consider the following three well-defined global 2-forms,

$$\tau_0 := \frac{\Omega^2}{xyz},$$

$$\tau_1 := \frac{(x^3-y^2)x^2}{yzF}\Omega^2,$$

$$\tau_2 := \frac{(x^3-y^2)y}{xzF}\Omega^2.$$

The following relations hold,

$$\sigma_{01} \wedge \sigma_{02} \overset{(90)}{=} 2\tau_0, \tag{91}$$

$$\sigma_{01} \wedge \sigma_{03} = 2\tau = 2(\tau_0 - \tau_1 + \tau_2), \tag{92}$$

$$\sigma_{02} \wedge \sigma_{03} = 8\tau_2 - 2\tau = 2(-\tau_0 + \tau_1 + 3\tau_2), \tag{93}$$

so in particular, by construction, the forms τ_0, τ_1 and τ_2 are logarithmic.

Consider also the well defined 2-form

$$\tau := \frac{\Omega^2}{F},$$

noticing that

$$\tau = \tau_0 - \tau_1 + \tau_2.$$

The forms τ_i are independent. To check that it is enough to observe their residues (recall Definition (I.5.14)) at different points in Table 1. Let us compute three particular illustrative examples of the computation of the residues at different points. First of all we have to establish an order for the divisors in the **Q**-resolution (see Figure V.1).

Let us compute $\operatorname{Res}^{[2]}_{P_6}(\tau_0)$. After two weighted blow-ups (recall Example (I.2.8)) one has

$$7\frac{dx\wedge dy}{xy} \quad \overset{\substack{x=u_1\bar v_1\\ y=\bar v_1{}^5,\ v_1=\bar v_1^7}}{\longleftarrow} \quad 5\frac{du_1\wedge dv_1}{u_1v_1} \quad \overset{\substack{u_1=\bar u_2^2,\ u_2=\bar u_2^5\\ v_1=\bar u_2 v_2}}{\longleftarrow} \quad 2\frac{du_2\wedge dv_2}{u_2v_2}.$$

Therefore, locally at $P_6 \in X(2;1,1)$ the form can be written as

$$\tau_0 = 2\frac{du_2\wedge dv_2}{u_2v_2}.$$

Then,

$$\operatorname{Res}^{[2]}_{P_6}(\tau_0) = \operatorname{Res}^{[2]}_{0} 2\frac{du_2\wedge dv_2}{u_2v_2} = \frac{2}{2}(-1)^{\sigma(1,4,5,6,7,3,2)} = (-1)^9 = -1.$$

Let us compute now $\operatorname{Res}^{[2]}_{P_7}(\tau_1)$. After a weighted blow-up of type $(1,5)$ one has

$$7\frac{(x^3-y^2)x^2\ dx\wedge dy}{y(xy+(x^3-y^2)^2)} \quad \overset{\substack{x=\bar u,\ u=\bar u^7\\ y=\bar u^5 v}}{\longleftarrow} \quad \frac{(1-uv^2)\ du\wedge dv}{uv(v+(1-uv^2)^2)}.$$

Hence, locally at P_7 (smooth point of $\bar X_{\mathcal{D}}$) the form τ_1 can be written

$$\tau_1 = \frac{(1-uv^2)\ du\wedge dv}{uv(v+(1-uv^2)^2)}$$

$$\begin{aligned}
\operatorname{Res}^{[2]}_{P_7}(\tau_1) &= \operatorname{Res}^{[2]}_{(0,-1)} \frac{(1-uv^2)\ du\wedge dv}{uv(v+(1-uv^2)^2)}\\
&= \operatorname{Res}^{[2]}_{(0,0)} \frac{(1-u(v-1)^2)\ du\wedge d(v-2uv^2+u^2v^4)}{u(v-1)(v-2uv^2+u^2v^4)}\\
&= (-1)(-1)^{\sigma(1,2,4,5,6,3,7)} = (-1)(-1)^3 = 1.
\end{aligned}$$

As a final example, let us compute $\operatorname{Res}^{[2]}_{P_2}(\tau_2)$. In the first chart $(X(2;1,1))$ the form τ_2 can be written in the following way

$$\tau_2 = 2\frac{(1-y^2)y dy\wedge dz}{z(yz+(1-y^2)^2)}.$$

Notice that the ideal associated at $P_2 = [1:1:0]$ is (y^2-1, z^2, yz) (in the ring of polynomials invariant in $[y,z]$) and so, calling $u = y^2-1$, $v = z^2$ and $w = yz$, the local ring at $P_2 = [1:1:0]$ is

$$\mathcal{O}_{P_2} = \frac{\mathbb{C}\{u,v,w\}}{((u+1)v - w^2)} = \mathbb{C}\{u,w\}.$$

After this change of coordinates one has that locally at P_2

$$\tau_2 = 2\frac{(1-y^2)ydy\wedge dz}{z(yz+(1-y^2)^2)} = \frac{-u\ du\wedge dw}{w(w+u^2)}.$$

$$-\frac{u\ du\wedge dw}{w(w+u^2)} \overset{\substack{u=u_1\\ w=u_1^2w_1}}{\longleftarrow} -\frac{du_1\wedge dw_1}{u_1w_1(w_1+1)}.$$

After another suitable change of coordinates one has

$$\operatorname{Res}^{[2]}_{P_2}(\tau_2) = \operatorname{Res}^{[2]}_0 - \frac{du_1\wedge dw_1}{u_1w_1(w_1+1)} = (-1)(-1)^{\sigma(1,2,3,4,7,6,5)} = (-1)(-1)^3 = 1.$$

The computations are similar in the rest of cases. The results obtained are shown in Table 1.

$\operatorname{Res}^{[2]}_{P_i}$	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
τ_0	1	0	1	1	0	−1	0	1	0
τ_1	1	1	0	0	0	0	1	1	1
τ_2	0	1	−1	−1	1	0	0	0	1
τ	0	0	0	0	1	−1	−1	0	0

TABLE 1. $\operatorname{Res}^{[2]}_{P_i}$

Summarizing all the results we get the ring structure of $H^\bullet(\mathbb{P}^2_w\setminus\{xyz(xyz+(x^3-y^2)^2)=0\};\mathbb{C})$. The generators are:

$$\begin{aligned} H^\bullet(\mathbb{P}^2_w\setminus\mathcal{D};\mathbb{C}) &= H^0(\mathbb{P}^2_w\setminus\mathcal{D};\mathbb{C})\oplus H^1(\mathbb{P}^2_w\setminus\mathcal{D};\mathbb{C})\oplus H^2(\mathbb{P}^2_w\setminus\mathcal{D};\mathbb{C}) \\ &= \langle 1\rangle \oplus \langle \sigma_{01},\sigma_{02},\sigma_{03}\rangle \oplus \langle \tau_0,\tau_1,\tau_2\rangle, \end{aligned}$$

with the relations (91), (92) and (93) already seen.

Remark **(V.3.1).** Notice that given a logarithmic 2-form one can compute its coordinates respect to a basis by means of the Res^2 at the different points.

Section § V.4
A holomorphic presentation for $H^2(\mathbb{P}^2_w \setminus \mathcal{R};\mathbb{C})$

Let $\mathcal{R}$ be a rational arrangement. Let us fix a resolution $\pi : \overline{X}_\mathcal{R} \to \mathbb{P}^2_w$ as in (V.1.1). The purpose of this section is to give a presentation of the space

$$H^2(W_2, d) \cong H^2(X_\mathcal{R};\mathbb{C})$$

of global C^∞ 2-forms of weight 2, module closed forms, on $\overline{X}_\mathcal{R}$, using only holomorphic forms as representatives.

According to Lemmas (III.2.8) and (III.1.18) i), logarithmic trees provide a simple way to construct logarithmic sheaves.

For the sake of simplicity we will introduce the following notation which will be used in the future.

Notation (V.4.1). Let $\mathcal{C}_i,\ \mathcal{C}_j,\ \mathcal{C}_k$ be three curves in $\mathbb{P}^2_w$ (not necessarily different). We will denote by $\mathcal{C}_{ijk}$ the union $\mathcal{C}_i \cup \mathcal{C}_j \cup \mathcal{C}_k$ and consider C_{ijk} a reduced equation for $\mathcal{C}_{ijk}$. We also use $d_{ijk} := \deg_w \mathcal{C}_{ijk}$.

For instance, if $i = j = k$, $\mathcal{C}_{ijk} = \mathcal{C}_i$, $C_{ijk} = C_i$ and $d_{ijk} = \deg_w(\mathcal{C}_i)$.

Let us see a lemma which will be of particular interest.

Lemma (V.4.2). *Let $\mathcal{C}_i = \{C_i = 0\}$, $i = 1\dots,n \in \mathbb{N}$ be reduced $\mathbb{Q}$-divisors on $\mathbb{P}^2_w$ without common components, $g(\mathcal{C}_i) = 0$ and $d_i = \deg_w \mathcal{C}_i$. Consider $\mathcal{R} = \{C_1 \cdots C_n = 0\}$ a rational arrangement with $d = \deg_w \mathcal{R} = d_1+\dots+d_n$. Then*

$$g_{d,w} - \sum_{P\in \operatorname{Sing}\mathcal{C}} \delta^w_P = 1-n.$$

Proof. Denote $\delta^w(\mathcal{C}_i) = \sum_{P\in\operatorname{Sing}\mathcal{C}_i} \delta^w_P(\mathcal{C}_i)$, one has

$$0 = g(\mathcal{R}) = \sum_{i=1}^n g(\mathcal{C}_i) = \sum_{i=1}^n (g_{d_i,w} - \delta^w(\mathcal{C}_i)). \tag{94}$$

On the one hand, after applying Lemma (II.2.10) one has

$$\delta^w_P(\mathcal{R}) = \sum_{i=1}^n \delta^w_P(C_i) + \sum_{i\neq j} (C_i\cdot C_j)_P. \tag{95}$$

On the other hand, by Lemma (IV.1.4)

$$g_{d,w} = \sum_{i=1}^n g_{d_i,w} + \sum_{i\neq j} \frac{d_i d_j}{\bar{w}} - (n-1). \tag{96}$$

From equations (94), (95), (96) and Weighted Bézout's Theorem on $\mathbb{P}^2_w$ (Theorem (I.4.7)) one has

$$g_{d,w} - \sum_{P\in \operatorname{Sing}\mathcal{C}} \delta^w_P = 1-n.$$

□

Definition (V.4.3). Let $\mathcal{R} = \bigcup_i \mathcal{R}_i$ be a rational arrangement and π a **Q**-resolution of singularities for $\mathcal{R}$. For every triple $(\mathcal{R}_i, \mathcal{R}_j, \mathcal{R}_k)$, not necessarily $i \neq j \neq k$, let us take three points $P_1 \in \operatorname{Sing}(\mathcal{R}_i \cap \mathcal{R}_j)$, $P_2 \in \operatorname{Sing}(\mathcal{R}_j \cap \mathcal{R}_k)$ and $P_3 \in \operatorname{Sing}(\mathcal{R}_i \cap \mathcal{R}_k)$. For every P_l choose two local branches, $\delta^{i_l}_l$ of $\mathcal{R}_i$ and $\delta^{j_l}_l$ of $\mathcal{R}_j$. Consider

$$\Delta := \left[(P_1, \delta^{i_1}_1, \delta^{j_1}_1),\ (P_2, \delta^{j_2}_2, \delta^{k_2}_2),\ (P_3, \delta^{k_3}_3, \delta^{i_3}_3)\right].$$

Let us construct a sheaf $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ associated with Δ. Let $Q \in \mathcal{R}_{ijk}$, one has the following module

$$(\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi})_Q := \left\{ \begin{array}{ll} \mathcal{O}_Q & \text{if } Q \notin \operatorname{Sing}(\mathcal{R}_{ijk}) \\ (\mathcal{M}^{nul}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } P_l \neq Q \in \operatorname{Sing}(\mathcal{R}_{ijk}) \\ (\mathcal{M}^{\delta^i_l,\delta^j_l}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } Q = P_l \text{ with } \delta^i_l \neq \delta^j_l \\ (\mathcal{M}^{nul}_{\mathcal{R}_{ijk},\pi})_Q & \text{if } Q = P_l \text{ with } \delta^i_l = \delta^j_l \end{array} \right\}.$$

This module lead us to the corresponding sheaf $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ which will be called *sheaf of Δ-logarithmic forms along $\mathcal{R}_{ijk}$ w.r.t. π.*

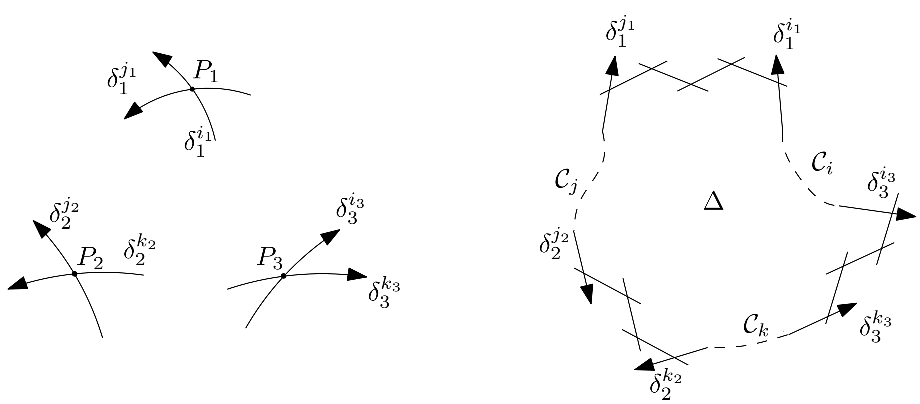


FIGURE V.2. Δ in $H^1(\bar{\mathcal{R}}_{ijk};\mathbb{C})$.

Remark **(V.4.4).** The previous sheaf $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk},\pi}$ does not depend on the choice of the resolution π. For a given $\mathcal{R}$ we will simply write $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}$ if no ambiguity seems no likely to arise.

Proposition (V.4.5). *Let $\mathcal{R}$ be a rational arrangement in $\mathbb{P}^2_w$ as in Definition (V.4.3), then*

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)) > 0.$$

Proof. Consider a well-defined global 2-form $\omega = H\dfrac{\Omega^2}{R_{ijk}}$, being H a polynomial of quasi-homogeneous degree $d_{ijk}-|w|$ such that

$$H \in H^0(\mathbb{P}^2_w, \mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)) := \{P \in \mathcal{P}_{d_{ijk}-|w|,w} | P_Q \in (\mathcal{M}^\Delta_{\mathcal{R}_{ijk}})_Q\}.$$

One has the following short exact sequence

$$0 \longrightarrow \mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|) \longrightarrow \mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|) \longrightarrow \frac{\mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)}{\mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)} \longrightarrow 0.$$

Since $H^1(\mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)) = 0$ and (see [**Ful93**, §3.5] or [**RT11**, Proposition 4.6] for a direct approach from toric point of view), one has

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)) - \dim H^0(\mathbb{P}^2_w, \mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)) + \dim H^0\left(\mathbb{P}^2_w, \frac{\mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)}{\mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)}\right)$$

$$= \dim H^1(\mathbb{P}^2_w, \mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)),$$

thus,

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)) - \dim H^0(\mathbb{P}^2_w, \mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)) + \dim H^0\left(\mathbb{P}^2_w, \frac{\mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)}{\mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)}\right) \geq 0.$$

Note that the sheaf $\dfrac{\mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)}{\mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)}$ is supported on a finite number of points. Hence one has,

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)) - \dim H^0(\mathbb{P}^2_w, \mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)) + \sum_{Q \in \operatorname{Sing} \mathcal{R}_{ijk}} \dim\left(\frac{\mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)}{\mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)}\right)_Q \geq 0.$$

Notice that

$$\sum_{Q \in \operatorname{Sing} \mathcal{R}_{ijk}} \dim\left(\frac{\mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)}{\mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)}\right)_Q = \sum_{Q \in \operatorname{Sing} \mathcal{R}_{ijk}} K_Q(\mathcal{R}_{ijk}) = K(\mathcal{R}_{ijk}).$$

Hence

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)) - \dim H^0(\mathbb{P}^2_w, \mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)) + K(\mathcal{R}_{ijk}) \geq 0.$$

Because of $\dim H^0(\mathbb{P}^2_w, \mathcal{O}_{\mathbb{P}^2_w}(d_{ijk}-|w|)) = D_{d_{ijk}-|w|,w}$, the inequality one has is the following

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^\Delta_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)) - D_{d_{ijk}-|w|,w} + K(\mathcal{R}_{ijk}) \geq 0.$$

By Corollary (IV.4.4) and Lemma (III.4.3)

$$K(\mathcal{R}_{ijk}) = D_{d_{ijk}-|w|,w} - g_{d_{ijk},w} + \sum_{P \in \operatorname{Sing}(\mathcal{R}_{ijk})} \delta^w_P - \#\{i,j,k\},$$

where $\#\{i,j,k\}$ denotes the cardinal of the set. Thus

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)) - g_{d_{ijk},w} + \sum_{P\in \mathrm{Sing}(\mathcal{R}_{ijk})} \delta^w_P - \#\{i,j,k\} \geq 0.$$

Finally applying Lemma (V.4.2)

$$\dim H^0(\mathbb{P}^2_w, \mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)) \geq 1 - \#\{i,j,k\} + \#\{i,j,k\} = 1 > 0.$$

□

Remark **(V.4.6).** Notice that Definition(V.4.3) and Proposition (V.4.5) can be automatically extended in the case $\mathcal{C}_{ijk}$ has only one or two components which allows the existance of a one-cicle defined by Δ in $H^1(\bar{\mathcal{R}};\mathbb{C})$.

Therefore, it can be concluded that the logarithmic sheaf $\mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk} - |w|)$ has non-trivial global sections. After this result we are in conditions to prove the following theorem.

Theorem (V.4.7). *Let* $\mathcal{R} = \bigcup_i \mathcal{R}_i$ *be a rational arrangement in* $\mathbb{P}^2_w$ *and* π *a* **Q**-*resolution of singularities for* $\mathcal{R}$. *Let* H *be a polynomial of quasi-homogeneous degree* $d_{ijk} - |w|$, *such that*

$$H \in H^0(\mathbb{P}^2_w, \mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk}-|w|)).$$

The well-defined global 2*-forms with* $\omega = H\dfrac{\Omega^2}{R_{ijk}}$ *form a holomorphic presentation of* $H^2(\mathbb{P}^2_w \setminus \mathcal{R},\mathbb{C})$.

Proof. Proposition (V.4.5) assures the existance of such 2- forms $\omega = H\dfrac{\Omega^2}{R_{ijk}}$, satisfaying $H \in H^0(\mathbb{P}^2_w, \mathcal{M}^{\Delta}_{\mathcal{R}_{ijk}}(d_{ijk}-|w|))$. Consider the map

$$\begin{array}{ccc} H^2(X_{\mathcal{R}};\mathbb{C}) & \overset{{}^2R^{[1]}}{\longleftrightarrow} & H^1(\bar{\mathcal{R}}^{[1]};\mathbb{C}) \\ \pi^*\omega & \longleftrightarrow & \Delta \end{array} \tag{97}$$

with ${}^2R^{[1]}$ the generalization of the Poincaré Residue Operator (see §1.3 in [**CA02**] and §I.5) which is injective (Proposition (V.1.5)).

One has that under the residue map ${}^2R^{[1]}$, $\mathrm{Im}(H^2(X_{\mathcal{R}};\mathbb{C})) \cong H_1(\mathcal{R},\mathbb{C})$, by Proposition (V.1.3) one has

$$h^2(X_{\mathcal{R}};\mathbb{C})) = h_1(\mathcal{R},\mathbb{C}).$$

From the construction made in Proposition (V.4.5) (see FigureV.2), this map (97) is clearly surjective. This gives us a set of 2-forms which generates $H^2(X_{\mathcal{R}};\mathbb{C})$. □

Corollary (V.4.8). *Following the proof in Theorem (V.4.7) one has a method to find the relations among the generators in* $H^2(X_{\mathcal{R}};\mathbb{C})$ *by means of the relations in* $H^1(\bar{\mathcal{R}}^{[1]};\mathbb{C})$.

Example (V.4.9). Let us see how to compute the ring structure of $H^\bullet(\mathbb{P}^2_w \setminus \mathcal{D};\mathbb{C})$ being $\mathcal{D} = V(xyz(xyz+(x^3-y^2)^2))$ and $w=(2,3,7)$ using the results seen in this last section. Recall that this is a rational arrangement (Example (IV.1.18)). We have studied a **Q**-resolution of $\mathcal{D}$ in Example (I.2.8) (see Figure V.3), computed part of its different logarithmic trees in Examples (III.3.2) and (III.3.5), and analyzed in detail its cohomology ring in §V.3–2.

Denote by δ_i^j the different local branches at the points P_j as shown in Figure V.3.

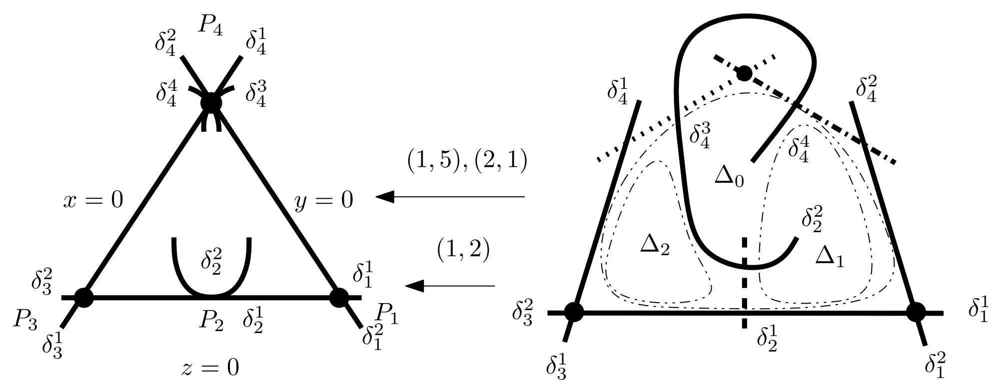


FIGURE V.3. Representation of Δ_i.

Let us consider for example the following three Δ_i's:

$$\Delta_0 := \left[(P_1,\delta_1^1,\delta_1^2),\ (P_3,\delta_3^1,\delta_3^2),\ (P_4,\delta_4^1,\delta_4^2)\right],$$

$$\Delta_1 := \left[(P_1,\delta_1^1,\delta_1^2),\ (P_2,\delta_2^1,\delta_2^2),\ (P_4,\delta_4^2,\delta_4^4)\right],$$

and

$$\Delta_2 := \left[(P_2,\delta_2^1,\delta_2^2),\ (P_3,\delta_3^1,\delta_3^2),\ (P_4,\delta_4^1,\delta_4^3)\right].$$

Recall the three 2-forms defined in §V.3–2 ,

$$\tau_0 := \frac{\Omega^2}{xyz} = H_0\ \frac{\Omega^2}{xyz},$$

$$\tau_1 := \frac{(x^3-y^2)x^2}{yzF}\Omega^2 = H_1\ \frac{\Omega^2}{yzF},$$

and

$$\tau_2 := \frac{(x^3-y^2)y}{xzF}\Omega^2 = H_2\ \frac{\Omega^2}{xzF}.$$

We already know that τ_0,τ_1 and τ_2 constitute a basis for $H^\bullet(\mathbb{P}^2_w \setminus \mathcal{D};\mathbb{C})$ (recall §V.3–2).

Let us see $H_0 \in \mathcal{M}^{\Delta_0}_{xyz}(0)$, $H_1 \in \mathcal{M}^{\Delta_1}_{yzf}(10)$ and $H_2 \in \mathcal{M}^{\Delta_2}_{xzf}(9)$ (recall Definition (V.4.3)).

Let us consider the germs h_{ij}, $i = 0, 1, 2$ at the different points P_j, $j = 1, \ldots, 4$, shown in Table 2. Denote by $u_1 = (yz + (1-y^2)^2) \in \mathcal{O}_{P_1}$ and $u_2 = (xz + (x^3-1)^2) \in \mathcal{O}_{P_3}$ unities in their correspondent local rings. One has the following Table 2:

h_{ij}	P_1	P_2	P_3	P_4
H_0	1	$(w+u^2)(u+1)^{-1}$	1	$xy + (x^3-y^2)^2$
H_1	$u_1^{-1}\ (1-y^2)$	$-u(u+1)^{-1}$	$u_2^{-1}\ x^3(x^3-1)$	$x^3(x^3-y^2)$
H_2	$u_1^{-1}\ (1-y^2)y^2$	$-u$	$u_2^{-1}\ (x^3-1)$	$y^2(x^3-y^2)$

TABLE 2.

Remark (III.5.2) gives us a characterization of the modules in the sheaf of Definition (V.4.3) by means of logarithmic trees. To check the hypothesis one has to construct the different appropriate logarithmic trees seen in Chapter III (recall Examples (III.3.2), (III.3.5) and Figure V.4) and compare it with the restrictions $\mathcal{T}_{P_j}|_{h_{ij}}$ to the different germs in Table 2 (recall Definition (III.1.14)).

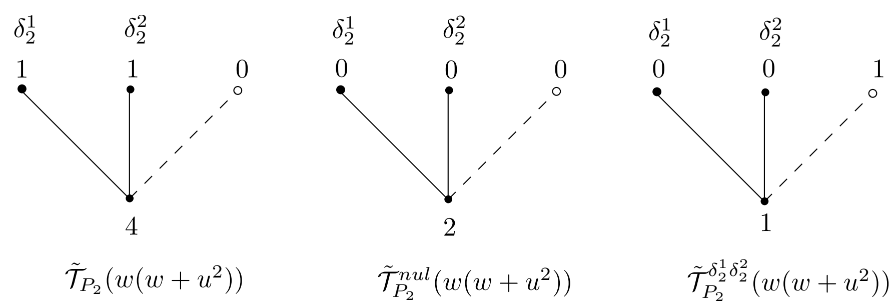


FIGURE V.4. Different logarithmic trees for $w(w+u^2)$ in $\mathbb{C}^2$.

The different restrictions of the logarithmic trees are constructed in Figures V.5 to V.8.

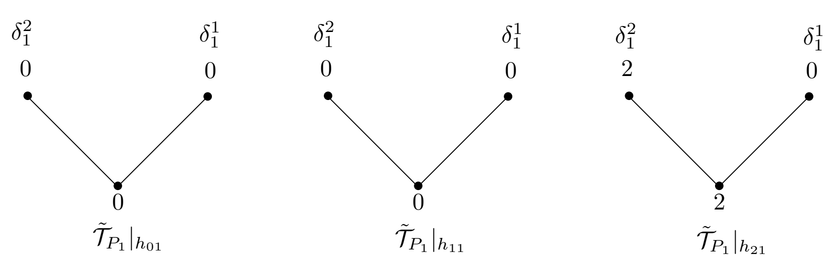


FIGURE V.5. Restriction of $\tilde{\mathcal{T}}_{P_1}$ to h_{i1}.

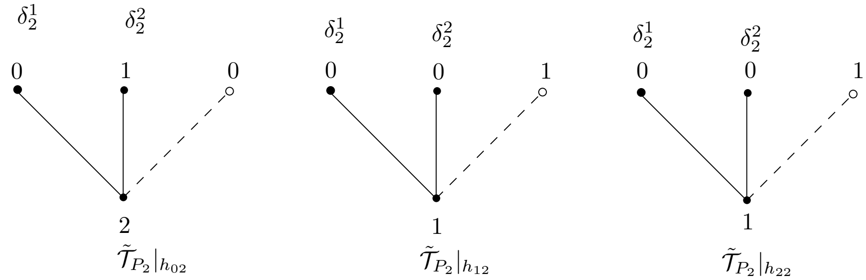


FIGURE V.6. Restriction of $\tilde{\mathcal{T}}_{P_2}$ to h_{i2}.

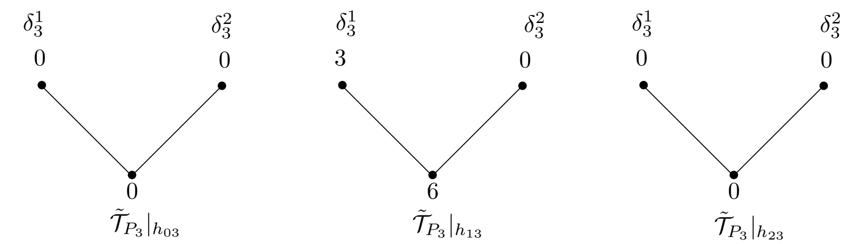


FIGURE V.7. Restriction of $\tilde{\mathcal{T}}_{P_3}$ to h_{i3}.

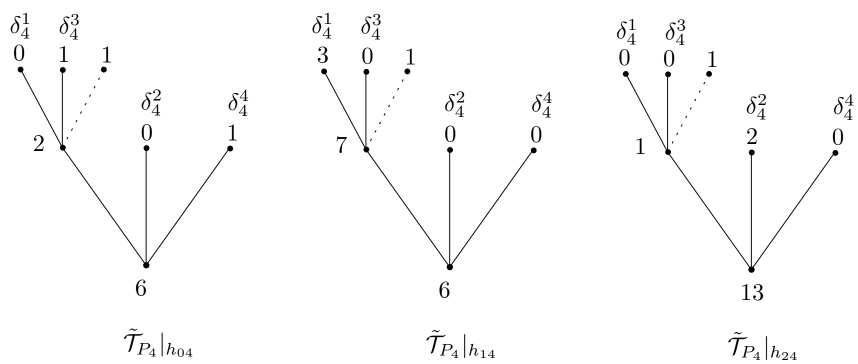


FIGURE V.8. Restriction of $\tilde{\mathcal{T}}_{P_4}$ to h_{i4}.

Looking at the construction of the trees in Figures V.5 to V.8 one has the following facts:

- $\tilde{\mathcal{T}}_{P_1}|_{h_{01}} \geq \tilde{\mathcal{T}}_{P_1}^{\delta_1^1\delta_1^2}$, $\tilde{\mathcal{T}}_{P_2}|_{h_{02}} \geq \tilde{\mathcal{T}}_{P_2}^{nul}$, $\tilde{\mathcal{T}}_{P_3}|_{h_{03}} \geq \tilde{\mathcal{T}}_{P_3}^{\delta_3^1\delta_3^2}$ and $\tilde{\mathcal{T}}_{P_4}|_{h_{04}} \geq \tilde{\mathcal{T}}_{P_4}^{\delta_4^1\delta_4^2}$.
- $\tilde{\mathcal{T}}_{P_1}|_{h_{11}} \geq \tilde{\mathcal{T}}_{P_1}^{\delta_1^1\delta_1^2}$, $\tilde{\mathcal{T}}_{P_2}|_{h_{12}} \geq \tilde{\mathcal{T}}_{P_2}^{\delta_1^1\delta_1^2}$, $\tilde{\mathcal{T}}_{P_3}|_{h_{13}} \geq \tilde{\mathcal{T}}_{P_3}^{nul}$ and $\tilde{\mathcal{T}}_{P_4}|_{h_{14}} \geq \tilde{\mathcal{T}}_{P_4}^{\delta_4^2\delta_4^4}$.
- $\tilde{\mathcal{T}}_{P_1}|_{h_{21}} \geq \tilde{\mathcal{T}}_{P_1}^{\delta_1^1\delta_1^2}$, $\tilde{\mathcal{T}}_{P_2}|_{h_{22}} \geq \tilde{\mathcal{T}}_{P_2}^{\delta_1^1\delta_1^2}$, $\tilde{\mathcal{T}}_{P_3}|_{h_{23}} \geq \tilde{\mathcal{T}}_{P_3}^{\delta_3^1\delta_3^2}$ and $\tilde{\mathcal{T}}_{P_4}|_{h_{24}} \geq \tilde{\mathcal{T}}_{P_4}^{\delta_4^1\delta_4^3}$.

Finally one concludes that $H_0 \in \mathcal{M}^{\Delta_0}_{xyz}(0)$, $H_1 \in \mathcal{M}^{\Delta_1}_{yzf}(10)$ and $H_2 \in \mathcal{M}^{\Delta_2}_{xzf}(9)$.

CONCLUSION AND FUTURE WORK

Along these lines we will give a non-exhaustive list of applications of the results obtained in this thesis as well as the problems we would like to study for possible future work.

The results here obtained are essential in the study of resonance varieties and formality. The cohomology ring provides a useful way to compute resonance varieties (see for instance [**CA02**]). This present work can be also used in order to study the formality of the complement of curves in weighted projective planes following a similar philosophy as in the works of [**Bri73, OS80**] and [**CAM12**] where the authors prove that the complements of hyperplane arrangements and curves in $\mathbb{P}^2$ are formal spaces.

As we have already mentioned, in [**CAM12**], Cogolludo-Agustín and Matei determine an explicit presentation by generators and relations of the cohomology algebra $H^\bullet(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ of the complement to an algebraic curve $\mathcal{C}$ in $\mathbb{P}^2$. The existence of genus makes the holomorphic classes not to be enough to generate the cohomology ring, and so, anti-holomorphic forms are required. The main result in this thesis has been presented for rational curves in $\mathbb{P}^2_w$ (Chapter V) generalizing the result in [**CA02**]. Our intention in future works will be to get an analogous result to [**CAM12**] for algebraic curves $\mathcal{C}$ (not necessarily rational) in the weighted projective plane $\mathbb{P}^2_w$.

Another invariants of the pair $(\mathbb{P}^2_w, \mathcal{C})$ we would like to study in a closed future would be the *sequence of characteristic varieties* of $\mathcal{C}$ and the *fundamental group of the complement* of an algebraic curve $\mathcal{C} \in \mathbb{P}^2_w$.

The *sequence of characteristic varieties* of $\mathcal{C} \in \mathbb{P}^2_w$, $\{\mathrm{Char}_k(\mathcal{C})\}_{k\in\mathbb{N}}$ was first studied by Libgober in [**Lib01**] for curves in $\mathbb{P}^2$. It is used to obtain information about all abelian covers of the complex projective plane $\mathbb{P}^2$ ramified along the curve $\mathcal{C}$. This study can be generalized in the case of $\mathcal{C} \in \mathbb{P}^2_w$.

The *fundamental group of the complement* of an algebraic curve $\mathcal{C} \in \mathbb{P}^2_w$ is a very important topological invariant of the pair $(\mathbb{P}^2_w, \mathcal{C})$. In the case of $\mathbb{P}^2$, techniques for computing a finite presentation of these groups were first carried out by Zariski ([**Zar95**]) and van Kampen ([**Kam33**]). We have used a variation of this method to compute the fundamental group for certain weighted projective curves in $\mathbb{P}^2_w$ in some particular examples. We plan to find more general and effective methods for this computation. Along this time, the study of Zariski-Van Kampen Theorem and braid monodromy has lead us to two works ([**ACO12a**] and [**ACO12b**]) related, in some way, with our future purpose. In [**ACO12a**], we present a new method for computing fundamental groups of curve complements using a variation of the Zariski-Van Kampen method on general ruled surfaces. In [**ACO12b**] we describe a method to reconstruct the braid monodromy of the preimage of a curve by a Kummer cover.

CONCLUSIÓN Y TRABAJO FUTURO (Spanish)

A lo largo de estas líneas vamos a dar una lista de algunas de las aplicaciones que tienen los resultados obtenidos en esta tesis así como de los problemas que preveemos atacar de cara a posibles trabajos futuros.

Los resultados aquí obtenidos son esenciales para entender las variedades de resonancia y la formalidad de los espacios estudiados. El anillo de cohomología proporciona una forma útil de calcular las variedades de resonancia (ver por ejemplo [**CA02**]). Este trabajo se puede utilizar también para estudiar la formalidad del complementario de curvas en planos proyectivos ponderados siguiendo una filosofía similar a la de los trabajos de [**Bri73, OS80**] y [**CAM12**] donde los autores prueban que los complementarios de configuraciones de hiperplanos y curvas en $\mathbb{P}^2$ son espacios formales.

Como ya hemos mencionado, en [**CAM12**], Cogolludo-Agustín y Matei determinan una presentación explícita dando los generadores y relaciones del álgebra de cohomología $H^\bullet(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ del complementario de una curva algebraica $\mathcal{C}$ en $\mathbb{P}^2$. La existencia de género hace que no sea suficiente trabajar con formas holomorfas para generar el anillo de cohomología, de ahí que sea necesario introducir formas anti-holomorfas. El resultado principal de esta tesis se presenta para el caso de curvas racionales en $\mathbb{P}^2_w$ (Capítulo V), generalizando el resultado obtenido en [**CA02**]. Nuestra intención en trabajos futuros será obtener un análogo a [**CAM12**] para curvas algebraicas $\mathcal{C}$ (no necesariamente racionales) en el plano proyectivo ponderado $\mathbb{P}^2_w$.

Otros invariantes del par $(\mathbb{P}^2_w, \mathcal{C})$ que nos gustaría estudiar en un futuro próximo serían la *sucesión de variedades características* de $\mathcal{C}$ y el *grupo fundamental del complementario* de una curva algebraica $\mathcal{C} \in \mathbb{P}^2_w$.

La *sucesión de variedades características* de $\mathcal{C} \in \mathbb{P}^2_w$, $\{\mathrm{Char}_k(\mathcal{C})\}_{k\in\mathbb{N}}$ fue estudiada por Libgober en [**Lib01**] para el caso de curvas en $\mathbb{P}^2$. Se usa para obtener información acerca de todas las posibles cubiertas abelianas del plano proyectivo complejo $\mathbb{P}^2$ que ramifican a lo largo de la curva $\mathcal{C}$. Este estudio se puede generalizar para el caso de $\mathcal{C} \in \mathbb{P}^2_w$.

El *grupo fundamental del complementario* de una curva algebraica $\mathcal{C} \in \mathbb{P}^2_w$ es uno de los invariantes topológicos más importantes del par $(\mathbb{P}^2_w, \mathcal{C})$. En el caso de $\mathbb{P}^2$, técnicas para calcular una presentación finita de estos grupos fueron estudiadas por primera vez por Zariski ([**Zar95**]) y van Kampen ([**Kam33**]). Hemos utilizado variantes de este método para calcular el grupo fundamental de ciertas curvas proyectivas ponderadas en $\mathbb{P}^2_w$ para varios ejemplos. En un futuro próximo queremos encontrar métodos más generales y efectivos para este cálculo. Durante este tiempo, el estudio del Teorema de Zariski-Van Kampen y de la monodromía de trenzas nos ha llevado a la consecución de dos trabajos ([**ACO12a**] y [**ACO12b**]) que están relacionados, en cierto modo, con nuestro propósito futuro. En [**ACO12a**], presentamos un método para calcular grupos fundamentales del complementario de curvas sobre superficies regladas usando una variación del método de Zariski-Van Kampen. En [**ACO12b**] describimos un método para reconstruir la monodromía de trenzas de la preimagen de una curva por una cubierta de Kummer.

CONCLUSION ET TRAVAUX FUTURS (French)

On donne une liste non exhaustive des applications des résultats obtenus dans cette thèse et une liste des problèmes que l'on prévoit d'aborder dans les travaux à venir.

Les résultats obtenus ici sont indispensables pour comprendre les variétés de résonance et la formalité des espaces étudiés. L'anneau de cohomologie fournit un moyen utile de calculer les variétés de résonance (voir par exemple [**CA02**]). Ce travail peut également être utilisée pour étudier la formalité des complémentaires de courbes dans les plans projectifs pondérés suivant l'esprit du travail de [**Bri73, OS80**] et [**CAM12**] où les auteurs montrent que les complémentaires des configurations d'hyperplans et des courbes dans $\mathbb{P}^2$ sont des espaces formels.

On a signalé un résultat de Cogolludo-Agustín et Matei ([**CAM12**]) où ils obtiennent une présentation explicite par générateurs et relations de l'algèbre de cohomologie $H^\bullet(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{C})$ du complémentaire d'une courbe algébrique $\mathcal{C}$ dans $\mathbb{P}^2$. S'il y a des composantes de genre positif, les classes holomorphes ne suffissent pas pour engendrer l'anneau de cohomologie, et dans ce cas, des formes anti-holomorphes sont nécessaires. Le résultat principal de cette thèse est présenté pour le cas des courbes rationnelles dans $\mathbb{P}^2_w$ (Chapitre V), généralisant le résultat obtenu dans [**CA02**]. Notre intention dans le futur est d'obtenir un analogue de [**CAM12**] pour les courbes algébriques $\mathcal{C}$ (pas nécessairement rationnelles) dans le plan projectif pondéré $\mathbb{P}^2_w$.

Un autre invariant de la paire $(\mathbb{P}^2_w, \mathcal{C})$ à étudier prochainement est la *suite des variétés caractéristiques* $\mathcal{C}$ et le *groupe fondamental du complémentaire* d'une courbe algébrique $\mathcal{C} \in \mathbb{P}^2_w$.

La *suite* $\{\mathrm{Char}_k(\mathcal{C})\}_{k\in\mathbb{N}}$ *des variétés caractéristiques* a été étudié par Libgober dans [**Lib01**] pour les courbes de $\mathbb{P}^2$. Elle est utilisée pour obtenir des renseignements sur tous les revêtements abéliens du plan projectif complexe $\mathbb{P}^2$ ramifiés le long de la courbe $\mathcal{C}$. Cet étude peut être généralisée au cas de $\mathcal{C} \in \mathbb{P}^2_w$.

Le *groupe fondamental du complémentaire* d'une courbe algébrique $\mathcal{C} \in \mathbb{P}^2_w$ est l'un des plus importants invariants topologiques de la paire $(\mathbb{P}^2_w, \mathcal{C})$. Dans le cas de $\mathbb{P}^2$, la méthode de calcul d'une présentation finie de ces groupes provient des travaux de Zariski ([**Zar95**]) et van Kampen ([**Kam33**]). On a utilisé des variations de cette méthode pour calculer le groupe fondamental des certaines courbes projectives pondérées dans $\mathbb{P}^2_w$ dans quelques exemples. On envisage de trouver des méthodes plus générales et effectives pour ce calcul. Au cours de ce travail, l'étude du Théorème de Zariski-Van Kampen et de la monodromie de tresses nous a conduit à deux travaux ([**ACO12a**] et [**ACO12b**]) reliés, d'une certaine façon, à cet objectif. Dans [**ACO12a**], on présente une méthode pour calculer les groupes fondamentaux du complémentaire des courbes sur des surfaces réglées en utilisant une variante de la méthode de Zariski-Van Kampen. Dans [**ACO12b**] on décrit une méthode pour reconstruire la monodromie de tresses de la préimage d'une courbe par un revêtement de Kummer.

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