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Bounds on the Number of Maximal Subgroups of Finite Groups: Applications

Adolfo Ballester-Bolinches [1], Ramón Esteban-Romero [1,*] and Paz Jiménez-Seral [2]

1 Departament de Matemàtiques, Universitat de València, Dr. Moliner 50, Burjassot, 46100 València, Spain; adolfo.ballester@uv.es
2 Departamento de Matemáticas, Universidad de Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza, Spain; paz@unizar.es
* Correspondence: ramon.esteban@uv.es

Abstract: The determination of bounds for the number of maximal subgroups of a given index in a finite group is relevant to estimate the number of random elements needed to generate a group with a given probability. In this paper, we obtain new bounds for the number of maximal subgroups of a given index in a finite group and we pin-point the universal constants that appear in some results in the literature related to the number of maximal subgroups of a finite group with a given index. This allows us to compare properly our bounds with some of the known bounds.

Keywords: finite group; maximal subgroup; probabilistic generation; primitive group

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1. Introduction

All groups considered in this paper will be finite.

Given a group G, one can ask how many elements one should choose uniformly and at random to generate G with a certain given probability. The fact that an ordered r-tuple $(g_1, \ldots, g_r)$ generates G is equivalent to the fact that $\{g_1, \ldots, g_r\}$ is not contained in any maximal subgroup M of G. The probability that $\{g_1, \ldots, g_r\}$ is contained in the maximal subgroup M of G is $1/|G : M|^r$. This makes it relevant, in this context, to analyse the number of maximal subgroups of a group G with a given index n. Let us call this number $\mathrm{m}_n(G)$.

Pak [1], motivated by potential applications for the product replacement algorithm, widely used to generate random elements in a finitely generated group, introduced the following invariant.

Definition 1. *Given a group G, we denote by $\mathcal{V}(G)$ the least positive integer k, such that the probability that G is generated by k random elements is at least* $1/\mathrm{e}$.

Pak conjectured that for a group G with minimum size of a generating system $\mathrm{d}(G)$, $\mathcal{V}(G) = \mathrm{O}(\mathrm{d}(G) \log\log|G|)$. Here and throughout this paper, the symbol log will be used to denote the logarithm to the base 2, and we follow the convention that $\log 0 = -\infty$, while we reserve ln to denote the natural logarithm, that is, the logarithm to the base e. Lubotzky [2], with the help of the number of chief factors in a given chief series, and Detomi and Lucchini [3], with the help of the number $\lambda(G)$ of non-Frattini chief factors in a given chief series of G and by considering their associate crowns, proved independently the validity of Pak's conjecture.

We have obtained upper bounds for $\mathrm{m}_n(G)$ and $\mathcal{V}(G)$ in [4] that improve some results of Lubotzky [2] and Detomi and Lucchini [3]. The bounds of [4] depend on the next invariants, associated to the different types of primitive quotient groups according to the theorem of Baer [5] (see also Theorem 1.1.7 in [6]) and to the crowns associated to abelian and non-abelian chief factors (see Chapter 1 in [6]). The first invariant is related to primitive quotient groups of type 1.

Definition 2. *Let G be a group and let $n > 1$ be a natural number. We denote by $\mathrm{cr}_n^{\mathfrak{A}}(G)$ the number of crowns associated to complemented abelian chief factors of order n of G, that is, the number of G-isomorphism classes of complemented abelian chief factors of G.*

Clearly, $\mathrm{cr}_n^{\mathfrak{A}}(G) = 0$ unless n is a power of a prime. The second invariant concerns non-abelian chief factors and is related to the primitive quotients of type 2.

Definition 3. *Let n be a natural number. The symbol $\mathrm{rs}_n(G)$ denotes the number of non-abelian chief factors A in a given chief series of G, such that the associated primitive group $G/\mathrm{C}_G(A)$ has a core-free maximal subgroup of index n.*

Our third and fourth invariants concern also non-abelian chief factors and contain information about the primitive quotients of type 3 of the group.

Definition 4. *Let n be a natural number. The symbol $\mathrm{ro}_n(G)$ denotes the number of non-abelian chief factors A in a given chief series of G, such that A has order n.*

Definition 5. *Let n be a natural number. The symbol $\mathrm{rm}_n(G)$ denotes the maximum of the lengths of the G-crowns associated to non-abelian chief factors of order n of G.*

Denote by $\mathbb{T}$ the set of all prime powers greater than 1 and by $\mathbb{S}$ the set of all powers greater than 1 of the orders of non-abelian simple groups. The main results of [4] are the following ones.

Theorem 1 (Theorem B in [4])**.** *The number $\mathrm{m}_n(G)$ of maximal subgroups of index n of a d-generated group G satisfies the following bounds:*

$$\begin{aligned}
&\mathrm{m}_n(G) \le (n^d - 1)\mathrm{cr}_n^{\mathfrak{A}}(G) + n^2\mathrm{rs}_n(G) && \text{if } n \in \mathbb{T},\\
&\mathrm{m}_n(G) \le n^2\mathrm{rs}_n(G) + n^2\binom{\mathrm{rm}_n(G)\,\mathrm{ro}_n(G)}{2} \le n^2\mathrm{rs}_n(G) + n^{d+2}\binom{\mathrm{ro}_n(G)}{2} && \text{if } n \in \mathbb{S},\\
&\mathrm{m}_n(G) \le n^2\mathrm{rs}_n(G) && \text{if } n \notin \mathbb{S} \cup \mathbb{T}.
\end{aligned}$$

Theorem 2. *Let G be a d-generated non-trivial group. Then, for*

$$\begin{aligned}
\eta(G) := \max\Big\{ & d + 2.02 + \max_{n\in\mathbb{T}}\{\log_n 2 + \log_n \mathrm{cr}_n^{\mathfrak{A}}(G)\},\\
& 4.02 + \max_n\{\log_n 2 + \log_n \mathrm{rs}_n(G)\},\\
& 4.02 + \max_{n\in\mathbb{S}}\{\log_n \mathrm{rm}_n(G) + \log_n \mathrm{ro}_n(G)\}\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\kappa(G) := \max\Big\{ & d + 2.02 + \max_{n\in\mathbb{T}}\{\log_n 2 + \log_n \mathrm{cr}_n^{\mathfrak{A}}(G)\},\\
& 4.02 + \max_n\{\log_n 2 + \log_n \mathrm{rs}_n(G)\},\\
& 4.02 + d + \max_{n\in\mathbb{S}}\{\log_n \mathrm{ro}_n(G)\}\Big\},
\end{aligned}$$

we find that

$$\mathcal{V}(G) \le \eta(G) \le \kappa(G).$$

Other bounds for $\mathcal{V}(G)$ can be found in [7]. They depend on an invariant, defined there only for non-abelian characteristically simple groups, but that can be also defined for elementary abelian groups.

Definition 6. *Let G be a group. For a characteristically simple group A, that is, a direct product of copies of a simple group S, $\mathrm{r}_A(G)$ denotes the largest number r such that G has a normal section that is the direct product of r non-Frattini chief factors of G that are isomorphic (not necessarily G-isomorphic) to A.*

The main theorem of [8] gives a simpler reinterpretation of the invariant $\mathrm{r}_A(G)$ as the number of non-Frattini chief factors isomorphic to A in a given chief series of G.

The following result was proved in [7]. We present here a corrected version available in [9] due to a misprint in the originally published version.

Theorem 3 (Theorem 9.5 in [9])**.** *Let G be a d-generated group. Then,*

$$\max\left\{d, \max_{n\geq 5} \frac{\log \mathrm{rk}_n(G)}{c_7 \log n} - 5\right\} \leq \mathcal{V}(G) \leq cd + \max_{n\geq 5} \frac{\log \max\{1, \mathrm{rk}_n(G)\}}{\log n} + 3,$$

where c and c_7 are two absolute constants.

Here, the symbol $\mathrm{rk}_n(G)$ denotes the maximum of the numbers $\mathrm{r}_A(G)$, where A runs over the non-abelian characteristically simple groups A with $\mathrm{l}(A) \leq n$, where $\mathrm{l}(X)$ denotes the least degree of a faithful transitive permutation representation of a group X, that is, the smallest index of a core-free subgroup of X.

The lower bound in Theorem 3 depends on the following result.

Lemma 1 (Corollary 9.3 in [9])**.** *Let G be a group. Then, $\mathrm{m}_x(G) \geq \mathrm{rk}_n(G)/n^{c_7}$ for some $x \leq n^{c_7}$.*

Our aim in this paper is twofold. In the first place, we establish significant improvements of the lower bound of Theorem 3. We prove:

Theorem 4. *Let G be a d-generated non-trivial group. Then,*

$$\max\{d, \max_{A} \frac{\log \mathrm{r}_A(G)}{2 \log \mathrm{l}(A)} - 2.63\} \leq \mathcal{V}(G).$$

We obtain our lower bound by means of an improved version of Lemma 1. This is done in two ways. On the one hand, we show that the constant c_7 can be taken to be 2 and, on the other hand, we obtain a larger lower bound for the number of maximal subgroups of index x. Moreover, if we impose some restrictions to the composition factors of the group, the bounds are further improved (see Theorem 13).

Theorem 5. *Let G be a group. Then, $\mathrm{m}_x(G) \geq x\lceil (2/3)\mathrm{rk}_n(G)\rceil$ for some $x \leq n^2$.*

Here, the symbol $\lceil x \rceil$ denotes the excess integer part of x, that is, the smallest integer number n, such that $x \leq n$. The symbol $\lfloor x \rfloor$ will denote the defect integer part of x, that is, the largest integer number n, such that $n \leq x$.

The proof of Theorem 5 will be presented in Section 2. It depends on some results proved in [10] about maximal subgroups of small index in almost finite groups.

In the second place, we compare the bounds of Theorem 1 with the ones of Theorem 3. This is only possible if we precise the values of the constants appearing in Theorem 3. As far as we know, the values of the constants c and c_7 in Theorem 3 have not been estimated in the literature. We obtain, in Section 3, the value of the constant c of Theorem 3 and we prove the following result.

Theorem 6. *Let G be a d-generated group. Then,*

$$\eta(G) \leq cd + \max_{n \geq 5} \frac{\log \max\{1, \mathrm{rk}_n(G)\}}{\log n} + 3,$$

where $c = 375.06$.

A slight variation of the proof of Theorem 6 gives smaller values for some of the constants (see Theorem 20). We do it in Section 3.

The following example, that appears in Example 3.4 in [4], depends on a couple of constructions of subdirect products also introduced in that paper. In this example, we see that our bound for $\mathcal{V}(G)$ improves dramatically the one of [7].

Example 1. *There are three isomorphism classes of 2-generated primitive groups of type 1 with socle of order 8, namely $G_1 = [\mathrm{C}_2^3]\mathrm{C}_7$, $G_2 = [\mathrm{C}_2^3][\mathrm{C}_7]\mathrm{C}_3$ and $G_3 = [\mathrm{C}_2^3]\mathrm{GL}_3(2)$. We can construct 2-generated groups $\hat{G}_1$, $\hat{G}_2$, and $\hat{G}_3$ with all possible crowns whose associated primitive quotients are isomorphic to G_1, G_2, and G_3, respectively, by using Construction 3.3 in [4]. By using Construction 3.2 in [4], we can construct a subdirect product S of $\hat{G}_1$, $\hat{G}_2$, and $\hat{G}_3$ in such a way the generating pairs of all these three groups are identified.*

Note that

$$\begin{aligned} &\mathrm{cr}_8^{\mathfrak{A}}(\hat{G}_1) = \mathrm{cr}_8^{\mathfrak{A}}(\hat{G}_2) = 16, && \mathrm{cr}_8^{\mathfrak{A}}(\hat{G}_3) = 114, && \mathrm{cr}_7^{\mathfrak{A}}(\hat{G}_1) = 1, \\ &\mathrm{cr}_7^{\mathfrak{A}}(\hat{G}_2) = 8, && \mathrm{cr}_3^{\mathfrak{A}}(\hat{G}_2) = 1, && \mathrm{r}_{\mathrm{GL}_3(2)}(\hat{G}_3) = 57; \end{aligned}$$

the other values of the ranks and the numbers of abelian crowns are zero. Therefore, $\mathrm{cr}_3^{\mathfrak{A}}(S) = 1$, $\mathrm{cr}_7^{\mathfrak{A}}(S) = 9$, $\mathrm{cr}_8^{\mathfrak{A}}(S) = 146$, $\mathrm{r}_{\mathrm{GL}_3(2)}(S) = 57$; the other values are zero (in fact, S coincides with the direct product $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3$). Since the indices of the maximal subgroups of $\mathrm{GL}_3(2)$ are 7 and 8 (see for instance [11]), we conclude that $\mathrm{rs}_7(S) = \mathrm{rs}_8(S) = 57$ and $\mathrm{ro}_{168}(S) = \mathrm{rm}_{168}(S) = 57$. The crowns of chief factors of order 8 in $\hat{G}_1$ are minimal normal subgroups, in $\hat{G}_2$ they are products of three minimal normal subgroups, while in $\hat{G}_3$ they are products of two minimal normal subgroups. The crowns of chief factors of order 7 in $\hat{G}_1$ and $\hat{G}_2$ coincide with the corresponding chief factors. There is a unique crown composed of two central chief factors of order 3. Hence, S has $16 + 16 \times 3 + 114 \times 2 = 292$ chief factors of order 8, 8 chief factors of order 7, 2 chief factors of order 3, and 57 chief factors isomorphic to $\mathrm{GL}_3(2)$.

Bearing in mind that $c = 375.06$, the bound of [7] is $\mathcal{V}(S) \leq 3 + 2c + \log_7 57$ and $3 + 2c + \log_7 57 \geq 5.07 + 2c \approx 755.198$. However, by Theorem 2 we obtain the bound

$$\begin{aligned} \mathcal{V}(S) \leq \max\{ &d + 2.02 + \log_3 2 + \log_3 \mathrm{cr}_3^{\mathfrak{A}}(S), \\ &d + 2.02 + \log_7 2 + \log_7 \mathrm{cr}_7^{\mathfrak{A}}(S), \\ &d + 2.02 + \log_8 2 + \log_8 \mathrm{cr}_8^{\mathfrak{A}}(S), \\ &4.02 + \log_7 2 + \log_7 \mathrm{rs}_7(S), \\ &4.02 + \log_8 2 + \log_8 \mathrm{rs}_8(S), \\ &4.02 + \log_{168} \mathrm{rm}_{168}(G) + \log_{168} \mathrm{ro}_{168}(G)\}, \end{aligned}$$

that is,

$$\begin{aligned} \mathcal{V}(S) \leq \max\{ &4.02 + \log_3 2 + \log_3 1, \\ &4.02 + \log_7 2 + \log_7 9, \\ &4.02 + \log_8 2 + \log_8 146, \\ &4.02 + \log_7 2 + \log_7 57, \\ &4.02 + \log_8 2 + \log_8 57, \\ &4.02 + 2\log_{168} 57\} \leq 6.75. \end{aligned}$$

As an example of application of Theorem 2, let us analyse the bounds a group with no abelian chief factor. We only have to compare $\mathrm{rs}_n(G)$ and $\mathrm{ro}_n(G)$ with $\mathrm{rk}_n(G)$. Let us start with $\mathrm{rs}_n(G)$. Let us denote by $s(n)$ the number of possible isomorphism types of socles of primitive groups of type 2 that possess a core-free maximal subgroup of index n. Given a chief series of G, let $\mathcal{C}_n$ denote the set of the non-abelian chief factors A of G in this series, such that the primitive group $G/\mathrm{C}_G(A)$ has a core-free maximal subgroup of index n. We have that $\mathrm{rs}_n(G) = |\mathcal{C}_n|$. If $A \in \mathcal{C}_n$, since $G/\mathrm{C}_G(A)$ can be embedded in $\mathrm{Sym}(n)$, we find that $\mathrm{l}(A) \leq n$. Let $A_1, \ldots, A_t$ be the different isomorphism classes of chief factors in $\mathcal{C}_n$. Then, $t \leq s(n)$. Suppose that A is isomorphic to A_i with $1 \leq i \leq t$. Then, the number of chief factors F isomorphic to A_i such that $G/\mathrm{C}_G(F)$ has a core-free maximal subgroup of index n is bounded by $\mathrm{r}_{A_i}(G)$, which coincides with the number of chief factors of G isomorphic to A_i by the main result of [8]. Hence,

$$\begin{aligned}\mathrm{rs}_n(G) = |\mathcal{C}_n| &\leq \mathrm{r}_{A_1}(G) + \cdots + \mathrm{r}_{A_t}(G)\\ &\leq t \max\{\mathrm{r}_{A_i}(G) \mid 1 \leq i \leq t\}\\ &\leq s(n)\mathrm{rk}_n(G).\end{aligned}$$

This proves the next result.

Proposition 1. *Let G be a group and $n \geq 5$. Then,*

$$\mathrm{rs}_n(G) \leq s(n)\mathrm{rk}_n(G).$$

Therefore, to compare $\mathrm{rs}_n(G)$ with $\mathrm{rk}_n(G)$, our first interest is to obtain a bound for $s(n)$. This will be done in Section 3, where we obtain the following result.

Theorem 7. *$s(n) \leq n^{1.218}$ for all n.*

To compare $\mathrm{rk}_n(G)$ with $\mathrm{ro}_n(G)$, it is enough to take into account the following result formulated as a question by Cameron [12], who attributed its proof to Teague in Note (ii) at the end of his paper and generalises the well-known fact, derived from the classification of finite simple groups, that for each natural number there are at most two simple groups of that order.

Theorem 8 (see Theorem 6.1 in [13])**.** *Let S and T be non-isomorphic finite simple groups. If $|S^a| = |T^b|$ for some natural numbers a and b, then $a = b$ and S and T are either $\mathrm{PSL}_3(4)$ or $\mathrm{PSL}_4(2)$, or are $\mathrm{O}_{2n+1}(q)$ and $\mathrm{PSp}_{2n}(q)$ for some $n \geq 3$ and some odd q.*

From this, we conclude that $\mathrm{ro}_n(G) \leq 2\,\mathrm{rk}_n(G)$, and $\mathrm{ro}_n(G) \leq \mathrm{rk}_n(G)$ unless n is one of the order of the exceptional groups of Theorem 8. Since $|\mathrm{PSL}_3(4)| = 20\,160$, we find that

$$\begin{aligned}\mathcal{V}(G) \leq \kappa(G) &= \max\{4.02 + \max_n\{\log_n 2 + \log_n \mathrm{rs}_n(G)\}, 4.02 + d + \max_{n\in\mathbb{S}}\{\log_n \mathrm{ro}_n(G)\}\}\\ &\leq 4.02 + \max\{\log_5 2 + 1.218 + \max_n\{\log_n \mathrm{rk}_n(G)\},\\ &\qquad d + \log_{20\,160} 2 + \max_{n\in\mathbb{S}}\{\log_n \mathrm{rk}_n(G)\}\}\\ &\leq 4.09 + d + \max_{n\in\mathbb{S}}\{\log_n \mathrm{rk}_n(G)\}.\end{aligned}$$

This proves:

Theorem 9. *Let G be a d-generated group with no abelian chief factors. Then, $\mathcal{V}(G) \leq 4.09 + d + \max_{n\in\mathbb{S}}\{\log_n \mathrm{rk}_n(G)\}$.*

Example 2. *If G is a group such as in Theorem 9, with no abelian chief factors, the upper bound of Theorem 3 gives that $\mathcal{V}(G) \leq 3 + cd + \max_{n\geq 5} \log_n \max\{1, \mathrm{rk}_n(G)\}$. Since $c = 375.06$, our upper bound improves dramatically the one of [7].*

For soluble groups, the contribution of the abelian chief factors to $\mathcal{V}(G)$ in Theorem 3 is contained in the term cd. We have the following improvement of Theorem 19 for soluble groups, that will be proved in Section 3.

Theorem 10. *Let G be a soluble d-generated group. The number of inequivalent irreducible G-modules of size n is at most $n^{\hat{c}_6 d+\hat{k}_6}$, where $\hat{c}_6 = 133.772$ and $\hat{k}_6 = 66$. In particular, $\mathcal{V}(G) \leq 2.02 + (\hat{c}_6 + 1)d + \hat{k}_6$.*

Now, let us compare $\mathrm{cr}_n^{\mathfrak{A}}(G)$ with $\mathrm{rk}_n(G)$. First of all, note that $\mathrm{cr}_n^{\mathfrak{A}}(G)$ is bounded by the number of irreducible G-modules of size n. As a result of combining Propositions 6.1 and 7.1 and Lemma 7.2 of [7], we obtain the following stronger form of Corollary 7.3 in [7] that appears as an intermediate step in the proof of Theorem 3.

Corollary 1. *Let G be a d-generated group. There exists a constant c_6, such that the number of irreducible G-modules of size n is at most*

$$n^{c_6 d} \max\{1, \mathrm{rk}_n(G)\}.$$

The value of the constant c_6 is not specified in [7]. Again, we precise this bound in Section 3.

At a first glance, we can use Corollary 1 to obtain that Theorem 3 follows from Theorem 2 and the results of Sections 5 and 6 in [7]. By Corollary 1,

$$\log_n \mathrm{cr}_n^{\mathfrak{A}}(G) \leq c_6 d + \log_n \max\{1, \mathrm{rk}_n(G)\}.$$

On the other hand, we know by Proposition 1 that $\mathrm{rs}_n(G) \leq s(n)\mathrm{rk}_n(G)$. By Theorem 7, $\log_n s(n) \leq 1.218$. Finally, by Theorem 8, $\mathrm{ro}_n(G) \leq 2\mathrm{rk}_n(G)$. We conclude that

$$\begin{aligned}
\mathcal{V}(G) \leq \max\Big\{ & d + 2.02 + \max_{n\in\mathbb{T}}\{\log_n 2 + c_6 d + \log_n \max\{1, \mathrm{rk}_n(G)\}\}, \\
& 4.02 + \max_{n}\{\log_n 2 + 1.218 + \log_n \max\{1, \mathrm{rk}_n(G)\}\} \\
& 4.02 + d + \max_{n\in\mathbb{S}}\{\log_n 2 + \log_n \max\{1, \mathrm{rk}_n(G)\}\}\Big\}. \\
\leq\ & d + 2.02 + \log_2 2 + c_6 d + \max \log_n \max\{1, \mathrm{rk}_n(G)\} \\
=\ & (c_6 + 1)d + 3.02 + \max \log_n \max\{1, \mathrm{rk}_n(G)\}.
\end{aligned}$$

Hence, Theorem 3 can be obtained as a consequence of Theorem 2.

Unless otherwise stated, we will follow the notation of the books [6,14]. Detailed information about primitive groups and chief factors, crowns, and precrowns of a group can be found in Chapter 1 in [6].

2. Lower Bounds for the Number of Maximal Subgroups of a Given Index in a Group

In this section we will obtain the inequality of Theorem 4. For this, we will need the results of [10] on the existence of conjugacy classes of subgroups of small indices in almost simple groups and some arithmetical properties about the smallest index of a core-free maximal subgroup of an almost simple group. These results are needed to obtain lower bounds for the number of maximal subgroups of a given index in a group G and for $\mathcal{V}(G)$. The results depend on some classes of simple groups that we define now.

Notation 1. *Let $\mathfrak{X}$ be the class of simple groups composed of the following groups:*

1. *The linear groups* $\mathrm{PSL}_3(q)$*, where* $q = p^f > 3$ *is a power of a prime* p *with* f *odd;*
2. *The linear groups* $\mathrm{PSL}_n(q)$*, with* q *a prime power and* $n = 5$ *or* $n \geq 7$*;*
3. *The symplectic groups* $\mathrm{PSp}_4(2^f)$*,* $f \geq 2$*.*

Notation 2. *Let $\mathfrak{Y}$ be the class of simple groups composed of the following groups:*

1. *The Mathieu group* M_{12}*;*
2. *The O′Nan group* $\mathrm{O'N}$*;*
3. *The Tits group* ${}^2F_4(2)'$*;*
4. *The linear groups* $\mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2)$*,* $\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$*,* $\mathrm{PSL}_2(11)$*,* $\mathrm{PSL}_3(3)$*;*
5. *The linear groups* $\mathrm{PSL}_3(q_0^2)$*, with* q_0 *a prime power;*
6. *The linear groups* $\mathrm{PSL}_4(q)$*,* q *a prime power;*
7. *The linear groups* $\mathrm{PSL}_6(q)$*,* q *a prime power;*
8. *The unitary group* $\mathrm{PSU}_3(5)$*;*
9. *The orthogonal groups* $\mathrm{O}_8^+(q)$*,* q *a prime power;*
10. *The orthogonal groups* $\mathrm{O}_n^+(3)$*,* $n \geq 10$*;*
11. *The exceptional groups of Lie type* $G_2(3^f)$*,* $f \geq 1$*;*
12. *The exceptional groups of Lie type* $F_4(2^f)$*,* $f \geq 1$*;*
13. *The exceptional groups of Lie type* $E_6(q)$*,* q *a prime power.*

If R is a primitive group, we denote by $\mathrm{l}^*(R)$ the smallest index of a core-free maximal subgroup of R.

We say that a maximal subgroup of a simple group S is *ordinary* if its conjugacy class in S coincides with its conjugacy class in $\mathrm{Aut}(S)$.

Theorem 11 (Theorem A in [10])**.** *Let S be a simple group and let R be an almost simple group, such that* $\mathrm{Soc}(R) \cong S$*. We can assume that* $S \leq R \leq A = \mathrm{Aut}(S)$*.*

1. *If S belongs to $\mathfrak{X}$, then S has at least two conjugacy classes of maximal subgroups of the smallest index $\mathrm{l}(S)$ and there exists a number $v_S \leq \mathrm{l}(S)^2$, depending only on S, such that R has at least two conjugacy classes of core-free maximal subgroups with index $\mathrm{l}(S)$ or one conjugacy class of core-free maximal subgroups with index v_S;*
2. *If S belongs to $\mathfrak{Y}$, then S has at least two conjugacy classes of maximal subgroups of the smallest index $\mathrm{l}(S)$ and there exists a number $v_S \leq \mathrm{l}(S)^2$, depending only on S, such that R has a conjugacy class of core-free maximal subgroups with index v_S;*
3. *If S does not belong to $\mathfrak{X} \cup \mathfrak{Y}$, then S has a conjugacy class of ordinary maximal subgroups. In particular, the smallest index of a core-free maximal subgroup of R is also $\mathrm{l}(S)$;*
4. *In all cases, $\mathrm{l}(S)^2 < |S|$ and $|\mathrm{Out}\, S| \leq 3 \log \mathrm{l}(S)$;*
5. *If, in addition, $S \not\cong \mathrm{Alt}(6)$, S is not of the form $\mathrm{PSL}_m(q)$ with $q = p^f$, $m \geq 3$, and $p \in \{2,3,5,7\}$, or $m = 2$ and $q = 3^f$, S is not of the form $\mathrm{PSU}_m(q)$ with $q = p^f$ and $p \in \{2,3,5,7\}$, and $S \not\cong \mathrm{O}_8^+(q)$ with $q = p^f$ and $p \in \{3,5,7,11,13\}$, then $|\mathrm{Out}\, S| \leq \log \mathrm{l}(S)$.*

Lemma 2. *Suppose that G is a primitive group of type 2 with socle isomorphic to S^k, where S is a non-abelian simple group.*

1. *If $S \in \mathfrak{X}$, then G has at least $2\mathrm{l}(S)^k$ maximal subgroups of index $\mathrm{l}(S)^k$ or at least v_S^k maximal subgroups of index v_S^k, where v_S is defined in Theorem 11;*
2. *If $S \in \mathfrak{Y}$, then G has at least $v_S^k \geq \mathrm{l}(S)^k$ maximal subgroups of index v_S^k, where v_S is defined in Theorem 11;*
3. *If $S \notin \mathfrak{X} \cup \mathfrak{Y}$, then G has at least $\mathrm{l}(S)^k$ maximal subgroups of index $\mathrm{l}(S)^k$.*

Proof. Suppose that G is not an almost simple group and that $\mathrm{Soc}(G) = S_1 \times \cdots \times S_r$, where $\{S_1, \ldots, S_r\}$ is the set of all conjugate subgroups of a simple normal subgroup S_1 of $\mathrm{Soc}(G)$, write $N = \mathrm{N}_G(S_1)$ and $K = S_2 \times \cdots \times S_r$. By a result of Gross and Kovács ([15], see also

Theorem 1.1.35 in [6]), there exists a bijection between, on the one hand, the conjugacy classes in G of supplements U of $\mathrm{Soc}(G)$ in G, such that $U \cap \mathrm{Soc}(G) = (U \cap S_1) \times \cdots \times (U \times S_r)$ and, on the other hand, the conjugacy classes in N/K of supplements L/K of $\mathrm{Soc}(G)/K$ in N/K. This correspondence sends conjugacy classes of maximal subgroups of G to conjugacy classes of maximal subgroups of N/K and conjugacy classes of complements of M in G to conjugacy classes of complements of N/K in M/K. From this, it follows that every primitive group of type 2 has maximal subgroups U, such that the projection $\pi_1(U \cap \mathrm{Soc}(G))$ of $U \cap \mathrm{Soc}(G)$ onto S_1 is a non-trivial proper subgroup of S_1. In this case, by Proposition 1.1.44 and Remarks 1.1.46 in [6], G can be regarded as a subgroup of $W \cong Z \wr P_k$, where Z is an almost simple group, P_k is a transitive group of degree $k > 1$, and $U = G \cap (H \wr P_k)$ for a maximal subgroup H of Z.

By Remarks 1.1.46 in [6], if H is a maximal subgroup of Z and $U = G \cap (H \wr P_k)$, then $|G : U| = |Z : H|^k$ and, by Proposition 1.1.44 in [6], U is a core-free maximal subgroup of G. Moreover, all elements in its conjugacy class, consisting of $|G : U|$ elements, are also core-free maximal subgroups of the same index.

Suppose that $S \notin \mathfrak{X} \cup \mathfrak{Y}$. Then, we can consider a core-free maximal subgroup H of Z of index $\mathrm{l}(S)$ and construct $U = G \cap (H \wr P_k)$. Then, the conjugacy class of U in G contains at least $|G : U| = \mathrm{l}(S)^k$ elements.

Suppose that $S \in \mathfrak{X}$. Then, Z contains two non-conjugate core-free maximal subgroups H_1 and H_2 of index $\mathrm{l}(S)$, and so if $U_i = G \cap (H_i \wr P_k)$, $i \in \{1, 2\}$, then $|G : U_1| = |G : U_2| = \mathrm{l}(S)^k$, or Z contains a core-free maximal subgroup H_3 of index v_S, and so if $U_3 = G \cap (H_3 \wr P_k)$, then $|G : U_3| = v_S^k$. By the mentioned result of Gross and Kovács ([15], see also Theorem 1.1.35 in [6]), U_1 and U_2 are in different conjugacy classes in G. It follows that there are at least $2\mathrm{l}(S)^k$ maximal subgroups of G of index $\mathrm{l}(S)^k$ or at least v_S^k maximal subgroups of G of index v_S^k.

Finally, suppose that $S \in \mathfrak{Y}$. Then, Z contain a core-free maximal subgroup H of index v_S, and so if $U = G \cap (H \wr P_k)$, then $|G : U| = v_S^k$. In this case, the conjugacy class of U in G contains v_S^k elements. $\square$

Remark 1. *Consider the O'Nan simple group $S \cong \mathrm{O'N}$, and let $A = \mathrm{Aut}(S)$. According to [11], all the core-free maximal subgroups of A have index greater than its order. Let $W = A \wr C_2$. Let D be the diagonal subgroup of the base group $A \times A$ of W, let $H = DC_2$, and let $G = (\mathrm{Soc}\, W)H$. In the primitive pair (G, H) with simple diagonal action (see Definition 1.1.42 in [6]), we have that $|G : H| = |A|$ is smaller than the index of any maximal subgroup of the form $M \wr C_2$ (see Remarks 1.1.46 in [6]). Hence, the smallest index of a core-free maximal subgroup $\mathrm{l}^*(G)$ of G is smaller than the index $\mathrm{l}^*(A)^2$ corresponding to the product action with the core-free subgroup of smallest index $\mathrm{l}(A)$ of A. What we prove in Lemma 2 is that there are maximal subgroups of indices $\mathrm{l}(S)^k$ and v_S^k.*

Notation 3. *For a group G and a prime p we denote by $\zeta_p(G)$ the number of central p-chief factors of G in a given chief series.*

Notation 4. *Given a natural number $n \geq 2$ and a chief factor A of a group G, we denote by $\mathrm{m}_{n,A}(G)$ the number of maximal subgroups of G of index n for which the socle of the associated primitive group is isomorphic to A.*

Now, we are in a position to establish our lower bound.

Theorem 12. *Let G be a group and let A be a non-Frattini chief factor of G isomorphic to S^k, with k a natural number and S a simple group.*

1. *If $S \in \mathfrak{X}$, then $\mathrm{m}_{x,A}(G) \geq x\lceil (2/3)\mathrm{r}_A(G) \rceil$ for some $x \in \{\mathrm{l}(A), v_S^k\}$ and $v_S^k \leq \mathrm{l}(A)^2$;*
2. *If $S \in \mathfrak{Y}$, then $\mathrm{m}_{x,A}(G) \geq x\mathrm{r}_A(G)$ for $x = v_S^k \leq \mathrm{l}(A)^2$;*
3. *If S is non-abelian and $S \notin \mathfrak{X} \cup \mathfrak{Y}$, then $\mathrm{m}_{n,A}(G) \geq n\mathrm{r}_A(G)$ for $n = \mathrm{l}(A)$;*
4. *If $A \cong C_p^k$, with $k \geq 2$, then $\mathrm{m}_{n,A}(G) \geq n\mathrm{r}_A(G)$ for $n = \mathrm{l}(A) = p^k$;*

5. *If $A \cong C_p$, then $\mathrm{m}_{p,A}(G) \geq p\mathrm{r}_A(G) - p + 1$. Moreover, if $\zeta_p(G) \notin \{1,2\}$, then $\mathrm{m}_{p,A}(G) \geq p\mathrm{r}_A(G)$.*

Proof of Theorem 12. Suppose, first, that A is non-abelian. As in the proof of Corollary 9.3 in [7], there exists a normal section H/N of G that is the direct product of $r = \mathrm{r}_A(G)$ chief factors isomorphic to a direct product $A = A_1 \times \cdots \times A_r$ of r copies of a simple group $S \cong A_i$, $1 \leq i \leq r$, with $\mathrm{l}(A) \leq n$. Consider the groups $C_i = \mathrm{C}_G(A_i)$, $1 \leq i \leq r$. Then, the C_i are different normal subgroups of G and the quotients G/C_i are groups with a unique minimal normal subgroup isomorphic to A.

Suppose that $S \in \mathfrak{X}$. According to Lemma 2, there exists an integer $v_S \leq \mathrm{l}(S)^2$, such that G/C_i has at least $2\mathrm{l}(S)^k$ maximal subgroups of index $\mathrm{l}(S)^k$ or G/C_i has at least v_S maximal subgroups of index v_S. Let b_1 be the number of $i \in \{1, \dots r\}$, such that G/C_i has at least $2\mathrm{l}(S)^k$ maximal subgroups of index $\mathrm{l}(S)^k$ and let b_2 be the number of $i \in \{1, \dots, r\}$, such that G/C_i has at least $\mathrm{l}(S)^k$ maximal subgroups of index $\mathrm{l}(S)^k$. It follows that G has at least $2b_1\mathrm{l}(S)^k$ maximal subgroups of index $\mathrm{l}(S)$ and at least $b_2 v_S^k$ maximal subgroups of index v_S^k. Suppose that $2b_1 \geq b_2$. Then, $b_1 \geq \lceil r/3 \rceil$. Hence, G has at least $2\lceil r/3 \rceil \mathrm{l}(S)^k \geq \lceil 2r/3 \rceil \mathrm{l}(S)^k$ maximal subgroups M of index $\mathrm{l}(S)^k$ with $\mathrm{Soc}(G/M_G) \cong A$. Suppose, now, that $2b_1 < b_2$. Then $b_2 \geq \lceil 2r/3 \rceil$. It follows that G has at least $\lceil 2r/3 \rceil v_S^k$ maximal subgroups M of index v_S^k with $\mathrm{Soc}(G/M_G) \cong A$.

Suppose, now, that $S \in \mathfrak{Y}$. By Lemma 2, there exists an integer $v_S \leq \mathrm{l}(S)^2$ such that G/C_i has at least v_S^k maximal subgroups of index v_S^k for $1 \leq i \leq r$. For $x = v_S^k \leq \mathrm{l}(A)^2$, we have that $\mathrm{m}_{x,A}(G) \geq xr$.

Finally, suppose that S is non-abelian and $S \notin \mathfrak{X} \cap \mathfrak{Y}$. By Lemma 2, G/C_i has at least $\mathrm{l}(S)^k$ maximal subgroups of index $\mathrm{l}(S)^k$ for $1 \leq i \leq r$. Then, for $x = \mathrm{l}(A) = \mathrm{l}(S)^k$, we obtain that $\mathrm{m}_{x,A}(G) \geq xr$.

Assume that $n = p^k$ is a power of the prime p and that there exist normal subgroups $K \leq H$ of G such that $H/K = A_1 \times \cdots \times A_r$ is a direct product of r non-Frattini chief factors of G isomorphic to an elementary abelian group A of order n. Let $1 \leq i \leq r$ and let M_i be a maximal subgroup supplementing A_i. Consider the primitive group $P_i = G/\mathrm{Core}_G(M_i)$. Then, P_i has a minimal normal subgroup $\tilde{A}_i$, namely the precrown associated with A_i and M, and a maximal subgroup $\tilde{M}_i$ of trivial core.

Assume that $\tilde{M}_i \neq 1$. Then, $\mathrm{N}_{P_i}(\tilde{M}_i) \neq P_i$ and so $\mathrm{N}_G(M_i) \neq G$. By the maximality of M_i, $\mathrm{N}_G(M_i) = M_i$. It follows that the conjugacy class of M_i in G has exactly $|G : M_i| = |A| = n$ elements.

Assume that $\tilde{M}_i = 1$. Then, $P_i \cong A_i$ and A_i is a central chief factor, in particular, $n = p$ is a prime number.

Suppose that H/K has a central G-chief factors and b non-central G chief factors in a given chief series. Note that, in the case that $n \notin \mathbb{P}$, $a = 0$ and $b = r$. For each of the b non-central chief factors, we can obtain with the previous construction n maximal subgroups, they have different core since the core contains the product of all other chief factors. Hence, we obtain at least nb maximal subgroups. Now, suppose that $a > 0$ and consider the a central chief factors, in this case, $n = p$ is prime. By the main result of [8], G has a normal subgroup with elementary abelian quotient of order p^a. This group has $1 + p + \cdots + p^{a-1}$ subgroups of index p, and all of them have in their core the non-central chief factors. It follows that $\mathrm{m}_{p,A}(G) \geq pb + 1 + p + \cdots + p^{a-1} = p(r-a) + 1 + p + \cdots + p^{a-1}$. If $a = 0$, then $\mathrm{m}_{p,A}(G) \geq pr$. If $a \in \{1,2\}$, then $\mathrm{m}_{p,A}(G) \geq pr - p + 1$. If $a \geq 3$, then $1 + p + p^2 + \cdots + p^{a-1} \geq ap$ because $1 + p^2 \geq 2p$, and so $\mathrm{m}_{p,A}(G) \geq pr$. The result follows. □

Remark 2. *The bounds of Theorem 12 for abelian chief factors are attained in groups which are direct products of copies of a primitive group of type 1 with non-cyclic socle or in $(D_{2p})^{r-1} \times C_p$ and $(D_{2p})^{r-2} \times C_p \times C_p$.*

In order to simplify the statement of the main theorem of this section, we propose the following definition.

Definition 7. *For a group G and a chief factor $A \cong S^k$ of G with S a simple group and k natural,*

$$f(G,A) = \begin{cases} \dfrac{\log\lceil (2/3)\mathrm{r}_A(G)\rceil}{2\log \mathrm{l}(A)} & \text{if } S \in \mathfrak{X}, \\ \dfrac{\log \mathrm{r}_A(G)}{2\log \mathrm{l}(A)} & \text{if } S \in \mathfrak{Y}, \\ \dfrac{\log\left(\mathrm{r}_A(G) - 1 + \frac{1}{p}\right)}{\log p} & \text{if } A \cong C_p,\ p \in \mathbb{P} \text{ and } \zeta_p(G) \in \{1,2\}, \\ \dfrac{\log \mathrm{r}_A(G)}{\log \mathrm{l}(A)} & \text{otherwise.} \end{cases}$$

As a consequence of Theorem 12, we obtain a lower bound for $\mathcal{V}(G)$, the inequality of Theorem 4.

Theorem 13. *Let G be a d-generated group. Then,*

$$\mathcal{V}(G) \geq \max\{d, \max_A f(G,A) - 2.5\}$$

where A runs over the non-Frattini chief factors of G.

Proof. Let

$$\mathcal{M}(G) = \max_{n\geq 5} \frac{\log \mathrm{m}_n(G)}{\log n}.$$

By Proposition 1.2 in [2], $\mathcal{M}(G) - 3.5 \leq \mathcal{V}(G)$. Let B be a non-Frattini chief factor of G, such that $\max_A f(G,A) = f(G,B)$. Let $B \cong T^k$ with T a simple group.

Suppose that $B \in \mathfrak{X}$. Then, $\mathrm{m}_x(G) \geq \mathrm{m}_{x,B}(G) \geq x\lceil (2/3)\mathrm{r}_B(G)\rceil$ for $x \in \{\mathrm{l}(B), v_T^k\}$ and $v_T^k \leq \mathrm{l}(B)^2$. In this case,

$$\mathcal{M}(G) \geq \frac{\log \mathrm{m}_x(G)}{\log x} \geq 1 + \frac{\log\lceil (2/3)\mathrm{r}_B(G)\rceil}{2\log \mathrm{l}(B)} = 1 + f(G,B).$$

Suppose that $B \in \mathfrak{Y}$. Then, $\mathrm{m}_x(G) \geq \mathrm{m}_{x,B}(G) \geq x\mathrm{r}_B(G)$ for some $x \leq \mathrm{l}(B)^2$. Hence,

$$\mathcal{M}(G) \geq \frac{\log \mathrm{m}_x(G)}{\log x} \geq 1 + \frac{\log \mathrm{r}_B(G)}{2\log \mathrm{l}(B)} = 1 + f(G,B).$$

Suppose that $B \cong C_p$ for a prime p and that G has exactly one or two central chief factors isomorphic to B in a given chief series. Then, $\mathrm{m}_p(G) \geq \mathrm{m}_{p,B}(G) \geq p\mathrm{r}_B(G) + 1 - p = p\big(\mathrm{r}_B(G) - 1 + (1/p)\big)$ and so

$$\mathcal{M}(G) \geq \frac{\log \mathrm{m}_p(G)}{\log p} \geq 1 + \frac{\log\big(\mathrm{r}_B(G) - 1 + (1/p)\big)}{\log p} = 1 + f(G,B).$$

Suppose that B does not satisfy any of the previous properties. Then, for $n = \mathrm{l}(B)$, $\mathrm{m}_n(G) \geq \mathrm{m}_{n,B}(G) \geq n\mathrm{r}_B(G)$ and so

$$\mathcal{M}(G) \geq \frac{\log \mathrm{m}_n(G)}{\log n} \geq 1 + \frac{\log \mathrm{r}_B(G)}{\log n} = 1 + f(G,B).$$

Consequently,

$$\mathcal{V}(G) \geq \mathcal{M}(G) - 3.5 \geq 1 + f(G,B) - 3.5 \geq \max_A f(G,A) - 2.5.$$

The inequality $\mathcal{V}(G) \geq d$ holds trivially. $\square$

Remark 3. *Since the smallest index of a maximal subgroup of a non-abelian simple group is at least* 5*, if* $B = T^k$ *with* $T \in \mathfrak{X}$*,*

$$\begin{aligned}\mathcal{V}(G) &\geq \frac{\log\lceil(2/3)\mathrm{r}_B(G)\rceil}{2\log \mathrm{l}(B)} - 2.5 \geq \frac{\log \mathrm{r}_B(G)}{2\log \mathrm{l}(B)} + \frac{\log(2/3)}{2\log 5} - 2.5 \\ &\geq \frac{\log \mathrm{r}_B(G)}{2\log \mathrm{l}(B)} - 2.63.\end{aligned}$$

Consequently,

$$\mathcal{V}(G) \geq \max_A \frac{\log \mathrm{r}_A(G)}{2\log \mathrm{l}(A)} - 2.63$$

where A *runs over the set of all non-abelian chief factors in a given chief series of* G*. Moreover, since* $\mathrm{r}_n^{\mathrm{na}}(G) = \max\{\mathrm{r}_A(G) \mid \mathrm{l}(A) \leq n\}$*, we have that, if* N *is such that* $\frac{\log \mathrm{rk}_n(G)}{\log N} = \max_{n\geq 5} \frac{\log \mathrm{rk}_n(G)}{\log n}$ *and* B *is a chief factor of* G *satisfying that* $\mathrm{rk}_n(G) = \mathrm{rk}_n(B)$*, then*

$$\begin{aligned}\max_{n\geq 5} \frac{\log \mathrm{rk}_n(G)}{\log n} &= \frac{\log \mathrm{rk}_n(G)}{\log N} = \frac{\log \mathrm{r}_B(G)}{\log N} \\ &\leq \frac{\log \mathrm{r}_B(G)}{\log \mathrm{l}(B)} \leq \max_A \frac{\log \mathrm{r}_A(G)}{\log \mathrm{l}(A)}\end{aligned}$$

Therefore, this bound improves the bound

$$\mathcal{V}(G) \geq \max\left\{d, \max_{n\geq 5} \frac{\log \mathrm{rk}_n(G)}{c_7 \log n} - 4\right\}$$

given in Theorem 9.5 in [7].

Example 3. *Let* $S \cong \mathrm{Alt}(5)$ *be the alternating group of degree* 5*. According to a result of Wiegold (Theorem in [16]), since* S *is* 2*-generated and has order* $s = 60$*, we have that* $\mathrm{d}(S^{s^t}) = t + 2$ *for all* $t \geq 0$*. Let* G *be a direct product of* $60^4 = 12{,}960{,}000$ *copies of* S*. We have that* $\mathrm{d}(G) = 6$*. Moreover,* $\max_{n\geq 5} \frac{\log \mathrm{rk}_n(G)}{\log n} - 2.5 > 7.67$*. Hence, the lower bound we obtain is* $\mathcal{V}(G) > 7.67$*, that is,* $\mathcal{V}(G) \geq 8$*. The bound obtained by [7, Theorem 9.5] for this group was just* $\mathcal{V}(G) \geq \mathrm{d}(G) = 6$*, because* $c_7 \geq 2$ *and* $\max_{n\geq 5} \frac{\log \mathrm{rk}_n(G)}{c_7 \log n} - 4 < 1.09$*.*

This example also highlights the fact that the formula for the lower bound only gives interesting values different from $\mathrm{d}(G)$ *in groups with a large number of chief factors isomorphic to a non-abelian characteristically simple group in a given chief series. In fact, in order to obtain a non-trivial value for the lower bound for* $\mathcal{V}(G)$ *with the formula of Theorem 3 in a direct product* S^{60^t} *of* 60^t *copies of* $S \cong \mathrm{Alt}(5)$*, and assuming that* $c_7 = 2$ *by our analysis, we need* $t \geq 26$*. In this case,* $S^{60^{26}}$ *has* 28 *generators (Theorem in [16]) and* $\mathcal{V}(G) \geq \log(60^{26})/\log 5 - 5 > 28.0714$*, that is,* $\mathcal{V}(G) \geq 29$*. Our bound in this case is improved to* $\mathcal{V}(G) \geq \log(60^{26})/\log 5 - 2.5 \geq 63.6429$*, that is,* $\mathcal{V}(G) \geq 64$*. Even the most general bound of Theorem 4 would give* $\mathcal{V}(G) \geq \log(60^{26})/(2\log 5) - 3.085 > 29.9864$*, that is,* $\mathcal{V}(G) \geq 30$*.*

Remark 4. *The effort to show that for every simple group* S *there exists an integer* $v_S \leq \mathrm{l}(S)^2$ *such that every almost simple group with socle* S *has a maximal subgroup of index* $\mathrm{l}(S)$ *or* v_S *is necessary in order to make the lower bounds for* $\mathcal{V}(G)$ *useful. By Theorem 11,* $\mathrm{l}(S)^2 < |S|$ *for all simple groups. In particular,* $\frac{\log|S|}{2\log \mathrm{l}(S)} > 1$ *for all simple groups* S*. Suppose that in the quotient* $\frac{\log \mathrm{rk}_n(G)}{c_7 \log n}$ *the coefficient* c_7 *is greater than* $\log 50\,232\,960/\log 6\,156 \approx 2.0323$*. Then, we see that the lower bound is trivial for the Janko sporadic group* $S \cong J_3$ *of order* $50\,232\,960$ *and with* $\mathrm{l}(S) = 6\,156$*: for the groups of the form* $G \cong S^{|S|^t}$*, with* $t + 2$ *generators by Theorem in [16],* $\mathrm{rk}_{\mathrm{l}(S)}(G) = |S|^t$ *and so* $\frac{\log \mathrm{rk}_{\mathrm{l}(S)}(G)}{c_7 \log \mathrm{l}(S)} < t < \mathrm{d}(G)$*, so in this case we only obtain the trivial bound* $\mathcal{V}(G) \geq \mathrm{d}(G)$*.*

3. Upper Bounds for the Number of Maximal Subgroups of a Given Index in a Group

3.1. Bounds for the Number of Socles of Primitive Groups of Type 2

In Lemma 2.3 in [17], it is shown that the symmetric group $\mathrm{Sym}(n)$ has at most $\mathrm{O}(n)$ isomorphism classes of non-abelian simple subgroups. We will adapt the proof in this paper in order to give precise values to the constants associated with this bound.

Lemma 3. *The number $g(n)$ of isomorphism classes of non-abelian simple subgroups of the symmetric group $\mathrm{Sym}(n)$ for $n \geq 5$ satisfies the inequalities $g(n) \leq g_1(n) \leq g_2(n)$, where*

$$g_1(n) = \frac{9}{2}n + 22 + \frac{7}{2}(\log n)(\sqrt{n} - 1) - \frac{7}{2}\sqrt{n} + 7\log n + 7(\ln\log n)\log n$$

and

$$g_2(n) = 4.8869n + 1088.$$

Proof. The number of subgroups of alternating type or sporadic type is at most $(n-4)+26$. By all results mentioned in the proof of Theorem 11, the minimum degree of a permutation representation of a simple group $T = X_k(q)$ of Lie type of rank k over $\mathbb{F}_q$ is $\mathrm{l}(T) \geq q^k$, with the only exception of $\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$, already considered. Since $\mathrm{l}(T) \leq n$, we obtain that $k \leq \log n$. For each k, the number of possibilities for odd q is at most $\lfloor (n^{1/k} - 1)/2 \rfloor$ (namely $3^k, 5^k, 7^k, \ldots, \lfloor n^{1/k} \rfloor^k$) and for even q it is at most $\lfloor \log n^{1/k} \rfloor$ (namely $2^k, 4^k, 8^k, \ldots, (2^{\lfloor \log n^{1/k} \rfloor})^k$). Once q and k are given, there are at most 7 possibilities for the simple group T (up to isomorphism). Recall that the harmonic sum $H_n = \sum_{j=1}^{n} \frac{1}{j}$ satisfies the inequality $H_n \leq \ln n + 1$, as we can check by using the integral test. Hence, the number $g(n)$ of non-abelian simple subgroups $T \leq \mathrm{Sym}(n)$ is bounded by

$$\begin{aligned}
g(n) &\leq n + 22 + 7\sum_{k=1}^{\lfloor \log n \rfloor} \left(\frac{n^{1/k} - 1}{2} + \frac{1}{k}\log n \right) \\
&\leq n + 22 + \frac{7n}{2} + \frac{7}{2}\sum_{k=2}^{\lfloor \log n \rfloor} n^{1/k} - \frac{7}{2}\lfloor \log n \rfloor + 7(1 + \ln\lfloor \log n \rfloor)\log n \\
&\leq \frac{9}{2}n + 22 + \frac{7}{2}\sum_{k=2}^{\lfloor \log n \rfloor} n^{1/k} - \frac{7}{2}\lfloor \log n \rfloor + 7(1 + \ln\lfloor \log n \rfloor)\log n \\
&\leq \frac{9}{2}n + 22 + \frac{7}{2}(\lfloor \log n \rfloor - 1)\sqrt{n} - \frac{7}{2}\lfloor \log n \rfloor + 7(1 + \ln\lfloor \log n \rfloor)\log n \\
&= \frac{9}{2}n + 22 + \frac{7}{2}\lfloor \log n \rfloor(\sqrt{n} - 1) - \frac{7}{2}\sqrt{n} + 7\log n + 7(\ln\lfloor \log n \rfloor)\log n \\
&\leq \frac{9}{2}n + 22 + \frac{7}{2}(\log n)(\sqrt{n} - 1) - \frac{7}{2}\sqrt{n} + 7\log n + 7(\ln\log n)\log n \\
&= g_1(n).
\end{aligned}$$

Since the second derivative of the function g_1 is negative, the function g_1 is concave. Therefore, the graph of g_1 lies below its tangent on any point, for example, the tangent on $n = 4096$, that is, $g_1(n) \leq 4.8869n + 1088$ as we can compute with Maxima [18]. □

It is also shown in Lemma 2.3 in [17] that the number of almost simple subgroups of $\mathrm{Sym}(n)$ up to isomorphism is at most $\mathrm{O}(n \log^6 n)$. We specify the bound in that lemma by adapting its arguments and show that the exponent 6 can be reduced to 3 by using Theorem 11. We need the following bound for the number of subgroups of the outer automorphism group of a non-abelian simple group.

Lemma 4 (Theorem B in [10])**.** *The number of subgroups of the outer automorphism group of a non-abelian simple group S is bounded by $\log^3 \mathrm{l}(S)$.*

Lemma 5. *The number of isomorphism classes of almost simple subgroups of* $\mathrm{Sym}(n)$ *is at most* $g_1(n)\log^3 n \leq g_2(n)\log^3 n$*, where* $g_1(n)$ *and* $g_2(n)$ *are the functions defined in Lemma 3.*

Proof. Let $H \leq \mathrm{Sym}(n)$ be an almost simple subgroup and let $T = \mathrm{Soc}(H)$. There are $g(n)$ possibilities for T. By Lemma 4, we know that the number of subgroups of $\mathrm{Out}\, T$ is bounded by $\log^3 \mathrm{l}(T) \leq \log^3 n$. Therefore, $\mathrm{Sym}(n)$ contains at most $g(n)\log^3 n \leq g_1(n)\log^3(n) \leq g_2(n)\log^3(n)$ almost simple subgroups. The results follow. □

Let us denote by $\mathrm{ex}\, n$ the largest natural number r, such that there exists a natural c, such that $c^r = n$. It is clear that $\mathrm{ex}\, n \leq \log n$, and the equality holds if, and only if, n is a power of 2. Moreover, $\mathrm{ex}\, n$ is the greatest common divisor of the exponents of the different primes appearing in the decomposition of n as a product of prime powers. Note that if $n = n_s^s$, then s divides the exponents of the different primes appearing in the prime power decomposition of n, in particular, $s \mid \mathrm{ex}\, n$.

The following result follows from Theorem 1.1.52 and Proposition 1.1.53 in [6] and a result of Gross and Kovács [15] whose proof can be found in Theorem 1.1.35 in [6].

Proposition 2. *Let* P *be a primitive group with a minimal normal subgroup* $A = S_1 \times \cdots \times S_r$*, where for* $1 \leq i \leq r$*,* S_i *is isomorphic to a given non-abelian simple group* S*. Let* $X = \mathrm{N}_P(S_1)/\mathrm{C}_P(S_1)$ *and let* U *be a maximal subgroup of* P *of a given index* $n = |P : U|$*.*

Exactly one of the following three conditions holds:

Condition 1. $U \cap A = (U \cap S_1) \times \cdots \times (U \cap S_r) \neq 1$*. In this case, there exists a maximal subgroup* $\bar{U}$ *of* X*, such that* $\bar{U} \cap \mathrm{Inn}\, X \cong U \cap S_i$*,* $|U \cap A| = |\bar{U} \cap \mathrm{Inn}\, X|^r$*, and* $n = |X : \bar{U}|^r = m^r$ *for some* m*;*

Condition 2. $|U \cap A| = |S|^a$*, where* $0 < a < r$ *is such that there exists an integer* b *with* $ab = r$*. In this case,* $n = |S|^r/|S|^a = |S|^{a(b-1)} = x^{a(b-1)}$*;*

Condition 3. $|U \cap A| = 1$ *and, in this case,* $n = |S|^r = x^r$*.*

There exists a natural bijection between the conjugacy classes of maximal subgroups of X *of trivial core and the conjugacy classes in* P *of core-free maximal subgroups satisfying the above Condition 1.*

Proposition 3. *The number* $s(n)$ *of isomorphism types of minimal normal subgroups of primitive groups of type 2 with a core-free maximal subgroup of index* n *satisfies the inequalities*

$$\begin{aligned} s(n) &\leq (\mathrm{ex}\, n - 1)g(\sqrt{n}) + g(n) + 2(\mathrm{ex}\, n)^2 + 2\,\mathrm{ex}\, n \\ &\leq (\log n - 1)g(\sqrt{n}) + g(n) + 2\log^2 n + 2\log n. \end{aligned}$$

Proof. Let P be a primitive group with a unique minimal normal subgroup isomorphic to $A = S^m$, where S is a non-abelian simple group, and with a core-free maximal subgroup U of index n.

Let $r = \mathrm{ex}\, n$ and consider $s \mid r$ (the values $r = s = 1$ are valid in this context).

We first look for the possibilities of A corresponding to primitive groups P of type 2 with a core-free maximal subgroup satisfying the Condition 1 of Proposition 2. We must consider simple groups S having an almost simple group with a core-free maximal subgroup of index $n^{1/s}$. By Lemma 3, the number of the possibilities for A is at most $g(n^{1/s}) \leq g(n)$. The Condition 1 can only hold if P is almost simple or $\mathrm{ex}\, n \neq 1$. In the former case, by Lemma 3, the number of possibilities for A is at most $g(n)$. In the latter case, we obtain for each $s \neq 1$ a number of possibilities for A bounded by $g(\sqrt{n})$. Hence, the total number of possibilities for A of this type is at most $(\mathrm{ex}(n) - 1)g(\sqrt{n}) + g(n)$.

Now, we look for the possible socles of P satisfying the Condition 2 of Proposition 2. We must look for simple groups S with order $n^{1/s}$ and, for each divisor a of s, we can have a socle $A = S^{ac}$ where $c = s/a + 1$. Since for each value of $n^{1/s}$ there are at most two simple groups of this order, we find that the number of possibilities for A is less than or equal to $2(\operatorname{ex} n)^2$.

Finally, let us study the possible socles of P satisfying the Condition 3 of Proposition 2. Then, S is a simple group of order $n^{1/r}$ and we have at most two non-abelian simple groups of order $n^{1/r}$. This implies that the number of possibilities for A is less than or equal to $2\operatorname{ex} n$.

It follows that the total number $s(n)$ of possible isomorphism types of the minimal normal subgroups of a primitive group of type 2 with a core-free maximal subgroup of index n satisfies the inequality

$$\begin{aligned} s(n) &\leq \sum_{s \mid \operatorname{ex} n} g(n^{1/s}) + 2(\operatorname{ex} n)^2 + 2\operatorname{ex} n \\ &\leq (\operatorname{ex} n - 1)g(\sqrt{n}) + g(n) + 2(\operatorname{ex} n)^2 + 2\operatorname{ex} n \\ &\leq (\log n - 1)g(\sqrt{n}) + g(n) + 2(\log n)^2 + 2\log n. \quad \square \end{aligned}$$

We isolate in a few technical lemmas some results that will be used to obtain bounds for $s(n)$.

Lemma 6. *Let $g_2(n) = 4.8869n + 1088$ be the function of Lemma 3. Then, for $n \geq 4096$,*

$$(\log n - 1)g_2(\sqrt{n}) + \frac{2}{\log n} + \frac{2}{\log^2 n} \leq 0.187 g_2(n).$$

Proof. Consider the function

$$v(n) = (\log n - 1)g_2(\sqrt{n}) + \frac{2}{\log n} + \frac{2}{\log^2 n} - 0.187 g_2(n).$$

Its derived function is

$$v'(n) = \frac{4.8869 \log n}{2\sqrt{n}} + \frac{(2 - \ln 2) \times 4.8869}{2\sqrt{n}\ln 2} + \frac{1088}{n \ln 2} - \frac{2\ln 2}{n \ln^2 n} - \frac{4\ln^2 2}{n \ln^3(n)} - 4.8869 \times 0.187.$$

Since the functions defined by $(\log n)/\sqrt{n}$, $1/\sqrt{n}$ and $1/n$ are decreasing, for $n \geq 4096$ we find that

$$\begin{aligned} \frac{\log n}{\sqrt{n}} &\leq \frac{\log 4096}{\sqrt{4096}} = \frac{12}{64} = \frac{3}{16}, \\ \frac{1}{\sqrt{n}} &\leq \frac{1}{\sqrt{4096}} \leq \frac{1}{64}, \\ \frac{1}{n} &\leq \frac{1}{4096}. \end{aligned}$$

We conclude that, for $n \geq 4096$,

$$\begin{aligned} v'(n) &\leq \frac{4.8869 \times 3}{2 \times 16} + \frac{(2 - \ln 2) \times 4.8869}{2 \times 64 \times \ln 2} + \frac{1088}{4096 \times \ln 2} - 4.8869 \times 0.187 \\ &\leq 0.9134 - 4.8869 \times 0.187 < 0. \end{aligned}$$

It follows that v is a decreasing function in $[4096, +\infty[$. Therefore, for $n \geq 4096$, $v(n) \leq v(4096) < -7.73 < 0$. Consequently, for $n \geq 4096$,

$$(\log n - 1)g_2(\sqrt{n}) + \frac{2}{\log n} + \frac{2}{\log^2 n} < 0.187g_2(n). \quad \square$$

Lemma 7. *The function*

$$w(n) = 1.187(4.8869n + 1088)$$

satisfies the inequality $w(n) \leq n^{1.218}$ *for all* $n \geq 4096$.

Proof. Consider the function $\ln w(n)/\ln n$, that is a decreasing function because its derivative is negative. Therefore, for $n \geq 4096$, $\ln w(n)/\ln n \leq \ln w(4096)/\ln 4096 \leq 1.218$. Consequently, $w(n) \leq n^{1.218}$. $\square$

We are now in a position to prove Theorem 7.

Proof of Theorem 7. The claim for $n \geq 4096$ follows as a consequence of Proposition 3 and Lemmas 3, 6 and 7. The library of primitive permutation groups of small degree of MAGMA (see [19]) contains all primitive permutation groups of degree at most 4095, that were determined in [20]. From the information in this database, we conclude that the number of primitive groups of degree n, and so the number of isomorphism types of socles of primitive groups of degree n, is bounded by $n^{1.218}$ for $n \leq 4095$. $\square$

3.2. Bounds on the Number of Inequivalent Irreducible G-Modules

We can go further if we specify the values of the constants of this linear combination giving rise to the values of the constant c. The existence of these constants follows from counting arguments in which some terms are known to be $o(1)$, but we have not found any explicit value for them. We begin by estimating the constant c_1 of the following result.

Proposition 4 (Proposition 2.4 in [7])**.** *There exists an absolute constant c_1 such that, for each n, the group* $\mathrm{Sym}(n)$ *has at most c_1^n conjugacy classes of primitive subgroups.*

We will prove the following result.

Proposition 5. *The value of the constant c_1 of Proposition 4 can be taken to be* $c_1 = 2^{42.02} \approx 4.46 \times 10^{12}$.

The proof of Proposition 4 depends on the following result.

Theorem 14 (Theorem I in [21])**.** *The number of conjugacy classes of primitive subgroups of the symmetric group* $\mathrm{Sym}(n)$ *is at most $n^{c\mu(n)}$, where c is some absolute constant and $\mu(n)$ denotes the maximal exponent of a prime in the prime factorisation of the natural number n. Consequently,* $\mathrm{Sym}(n)$ *has at most $n^{c\log n}$ conjugacy classes of primitive subgroups.*

We will obtain a value for this constant c.

Theorem 15. *The constant c of Theorem 14 can be taken to be* $c = 1\,714.95$.

In our arguments, we will replace the term $\mu(n)$ by $\operatorname{ex} n$, the largest number r, such that $n = m^r$ for a natural number m, that is, the greatest common divisor of the exponents in the decomposition of n as a product of prime powers. If $n = p^m$ is a power of a prime, then $\mu(n) = \operatorname{ex} n$, while, in general, $\operatorname{ex} n \leq \mu(n)$.

We use the following result of Pálfy.

Lemma 8 (see Lemma 3.4 (ii) in [22])**.** *The number of conjugacy classes of maximal, irreducible, soluble subgroups of* $\mathrm{GL}_n(p)$ *(p prime) is at most*

$$n^{20\log^3 n+5}.$$

The following two results, of Kovács and Robinson, and Wolf, respectively, concern completely reducible subgroups of $\mathrm{GL}_m(p)$.

Lemma 9 (Theorem in [23]). *If G is a completely reducible subgroup of $\mathrm{GL}_m(p)$, with p a prime, then G can be generated by at most $\lfloor (3/2)m \rfloor$ elements.*

Lemma 10 (Theorem 3.1 in [24]). *A soluble completely reducible subgroup of $\mathrm{GL}_m(p)$ has order at most $24^{-1/3}p^{\alpha m}$, where*

$$\alpha = (3\log 48 + \log 24)/(3\log 9) = 2.24399105059531\ldots.$$

In Lemma 1.5 in [21], it is shown that the number of completely reducible soluble subgroups of $\mathrm{GL}_m(p)$ is at most $p^{(5+\mathrm{o}(1))m^2}$. We specify the term $\mathrm{o}(1)$. In the following results, we will follow the notation and the proofs of [21] and we will indicate only the differences. Hence, these are best followed with [21] at hand.

Lemma 11. *The number of completely reducible soluble subgroups of the linear group $\mathrm{GL}_m(p)$ is at most $p^{(4.366+\varepsilon_1(m,p))m^2}$, where*

$$\varepsilon_1(m,p) = \frac{m-1+20m\log^4 m + 5m\log m}{m^2\log p}.$$

Proof. We follow the proof of Lemma 1.5 in [21]. It is shown at the end of the first paragraph there that M can be chosen in at most

$$2^{m-1}m^{20m\log^3 m+5m}p^{m^2} = p^{1+\varepsilon_1(m,p)}$$

ways. By Lemma 10, once fixed a maximal soluble completely reducible subgroup M of $\mathrm{GL}_m(p)$, by Lemma 10 we find that $|M| \leq 24^{-1/3}p^{\alpha m}$. By Lemma 9, every completely reducible subgroup G of M can be generated by at most $\lfloor (3/2)m \rfloor$ elements. Therefore, the number of completely reducible subgroups G of M is at most $(24^{-1/3}p^{2.244m})^{1.5m} \leq p^{3.366m^2}$. Hence, the number of completely reducible soluble subgroups of $\mathrm{GL}_m(p)$ is bounded by $p^{3.36m^2}p^{(1+\varepsilon_1(m,p))m^2} = p^{(4.366+\varepsilon_1(m,p))m^2}$. □

In Lemma 2.3, in [21], it is shown that the number of subgroups X of $\mathrm{GL}_m(p)$ such that $X = \mathrm{F}^*(G)$ for some irreducible subgroup G of $\mathrm{GL}_m(p)$ is at most $p^{(15+\mathrm{o}(1))m^2}$, where $\mathrm{F}^*(G)$ denotes the generalised Fitting subgroup of G. We specify the term $\mathrm{o}(1)$ in this expression.

Lemma 12. *The number of subgroups X of $\mathrm{GL}_m(p)$, such that $X = \mathrm{F}^*(G)$ or some irreducible subgroup G of $\mathrm{GL}_m(p)$ is at most $p^{(13.098+3\varepsilon_1(m,p))m^2}$, where $\varepsilon_1(m,p)$ is defined in Lemma 11.*

Proof. We can argue as in the proof of Lemma 2.3 in [21], where it is shown that $\mathrm{F}^*(G)$ can be generated by three soluble completely reducible subgroups of $\mathrm{GL}_m(p)$. □

If we replace the term 2.25 by 2.244 in the proof of Lemma 2.7 in [21], we obtain the following result, where the 8 is replaced by 7.78.

Lemma 13. *With the same notation of Lemma 2.7 in [21], if P is a p-subgroup of N/H, then $|P| \leq p^{7.78td}$.*

The following result specifies the bound $p^{(94+\mathrm{o}(1))m^2}$ that appears in Theorem 2.8 in [21].

Theorem 16. *The number of irreducible subgroups G of $\mathrm{GL}_m(p)$ is at most $p^{(87.7724+4\varepsilon_1(m,p))m^2}$, where $\varepsilon_1(m,p)$ is defined in Lemma 11.*

Proof. We can follow the proof of Theorem 2.8 in [21] by replacing the terms $5+\mathrm{o}(1)$ by $4.366+\varepsilon_1(m,p)$, $15+\mathrm{o}(1)$ by $13.098+3\varepsilon_1(m,p)$, and 8 by 7.78 according to Lemmas 11–13, respectively. We obtain that the number of choices for G is at most

$$p^{(4.366+\varepsilon_1(m,p))m^2+m^2+(13.098+3\varepsilon_1(m,p))m^2+m^2+7.78\,\times\,8.78m^2} = p^{(87.7724+4\varepsilon_1(m,p))m^2}. \quad \square$$

The following result specifies the bound $n^{3+\mathrm{o}(1)}$ of Lemma 3.2 in [21].

Lemma 14. *The number of conjugacy classes of subgroups F of $\mathrm{Sym}(n)$, such that $F=\mathrm{Soc}(G)$ for some primitive subgroup G of $\mathrm{Sym}(n)$ is at most*

$$h(n) = 2(\mathrm{ex}\,n)^2 + 2\,\mathrm{ex}\,n + n^2 g_1(n)\log^3 n + \frac{(\mathrm{ex}\,n-1)}{8} n g_1(\sqrt{n})\log^3 n = n^{3+\mathrm{o}(1)},$$

where g_1 is defined in Lemma 3.

Proof. We follow the proof of Lemma 3.2 in [21]. If F acts regularly on Ω, we have at most $2\,\mathrm{ex}\,n$ choices for F up to conjugacy. If F is not regular and has diagonal action, then there are at most $2(\mathrm{ex}\,n)^2$ choices for F up to conjugacy. Suppose that F is not regular on Ω and has a wreath product action. Then there are at most $\mathrm{ex}\,n$ choices for n_0. By Lemma 5, $\mathrm{Sym}(n)$ has at most $g_1(n_0)\log^3 n_0$ almost simple subgroups up to isomorphism and so G_0 can be chosen in at most $g_1(n_0)\log^3 n_0$ as an abstract group. Once G_0 is fixed up to isomorphism, G_0 has at most n_0^2 core-free maximal subgroups of index n_0 by Theorem 1.3 in [2]. Hence there are $n_0^2 g_1(n_0)\log^3 n_0$ possibilities for the conjugacy class of G_0 in $\mathrm{Sym}(\Omega_0)$. We can distinguish the case $n_0=n$ and the rest of the cases, corresponding to $n_0 \leq \sqrt{n}$. Consequently, the number is bounded by

$$2(\mathrm{ex}\,n)^2 + 2\,\mathrm{ex}\,n + n^2 g_1(n)\log^3 n + \frac{(\mathrm{ex}\,n-1)}{8} n g_1(\sqrt{n})\log^3 n. \quad \square$$

The following result specifies the bound $24^{(1/6+\mathrm{o}(1))n^2}$ on the number of subgroups of $\mathrm{Sym}(n)$ given in Corollary 3.3 in [22]. It is based on the proof of this last result.

Theorem 17. *The number of subgroups of $\mathrm{Sym}(n)$ is at most*

$$24^{(n^2-1)/6}(n!)^2 2^{17n}.$$

Now we can prove Theorem 15.

Proof of Theorem 15. We follow the proof in Proof of Theorem I in [21].

Assume that G has abelian socle and $n=p^m$, we obtain at most

$$v_a(m,p) = p^{(87.7724+4\varepsilon_1(m,p))m^2} = n^{(87.7724+4\varepsilon_1(m,p))m}$$

choices for G up to conjugacy.

Suppose now that G has a non-abelian socle $F=L^r$, where L is a simple group. By Lemma 14, F can be chosen in at most $h(n)$ ways in $\mathrm{Sym}(n)$ up to conjugacy. The element $\tilde{g}$ can be chosen in at most $n^2 r!$ ways. The total number of subgroups of $\mathrm{Sym}(r)$ is at most $24^{(r^2-1)/6+17r}(r!)^2$. Given $|\tilde{O}\tilde{S}|$, there are at most n^{12r} choices for $\tilde{S}$. It follows that the number of choices for $\tilde{S}$ is at most n^{12r}. Let $u = \operatorname{ex} n$ and $n = s^u$. Since $r \leq u$, we obtain that G can be chosen in at most

$$\begin{aligned} v_n(u,s) &= h(n)(u!)n^2 24^{(u^2-1)/6+17u}(u!)^2 n^{12u} \\ &= h(s^u)(u!)^3 s^{2u+12u^2} 24^{(u^2-1)/6+17u} \end{aligned}$$

ways.

In order to obtain a bound for the number of conjugacy classes of primitive groups of degree n, it will be enough to find a bound on $v(u,s) = v_a(u,s) + v_n(u,s)$.

Since we must obtain a bound for $v(u,s)$ of the form $n^{cu} = s^{cu^2}$, it will be enough to maximise $\left(\log v(u,s)\right)/(u^2 \log s)$. We can check, with the help of a computer algebra system, such as Maxima [18], that $\left(\log v_a(u,s)\right)/(u^2 \log s)$ is bounded by 1714.94 (the bound is attained if $s = 54$ and $u = 2$), while the bound for $\left(\log v_n(u,s)\right)/(u^2 \log s)$ is bounded by 47.569 (the bound is attained for $u = 5$, $s = 1$). Therefore, since $1 + n^{-1\,667.371u} \leq n^{0.01u}$ for $n \geq 2$ and $u \geq 1$, we obtain that

$$\begin{aligned} v(u,s) &\leq n^{1\,714.94u} + n^{47.569u} = n^{1\,714.94u}(1 + n^{-1\,667.371u}) \\ &\leq n^{1\,714.94u} n^{0.01u} = n^{1\,714.95u}. \end{aligned}$$

Consequently Theorem 14 holds with $c = 1\,714.95$. □

Finally, we can prove Proposition 5.

Proof of Proposition 5. To obtain the constant c_1 of Proposition 4, in which we need a bound of the type $c_1^n = c_1^{s^u}$, we must find a bound for $\left(\ln v(u,s)\right)/s^u$, that will correspond to $\ln c_1$. With the help of Maxima [18], we show that $v_a(s,u) \leq c_a^{s^u}$, where $\log c_a = 439.662$, with the maximum attained in $(u,s) = (5,2)$, and $v_n(s,u) \leq c_n^{s^u}$, where $\log c_n = 22.0903$, with the maximum attained in $(u,s) = (1,5)$. Since $c_n/c_a = 2^{-417.5717}$, we have that

$$v(u,s) \leq c_a^{s^u} + c_n^{s^u} = c_a^{s^u}\left(1 + \left(\frac{c_n}{c_a}\right)^{s^u}\right) \leq \left(c_a(1 + c_n/c_a)\right)^{s^u} \leq c_0^n,$$

where $c_0 = 439.663$.

However, we see that for $s^u \geq 4096$, the value of $\log \hat{c}_a = 42.019$, corresponding to $(u,s) = (12,2)$, satisfies that $v_a(u,s) \leq \hat{c}_a^{s^u}$, and $\log \hat{c}_n = 1.402$, also corresponding to $(u,s) = (12,2)$, satisfies that $v_n(u,s) \leq \hat{c}_n^{s^u}$. As above, for $s^u \geq 4096$, we have that

$$v(u,s) \leq \hat{c}_a^{s^u} + \hat{c}_n^{s^u} = \hat{c}_a^{s^u}\left(1 + \left(\frac{\hat{c}_n}{\hat{c}_a}\right)^{s^u}\right) \leq \left(\hat{c}_a(1 + \hat{c}_n/\hat{c}_a)\right)^{s^u} \leq c_1^n,$$

where $c_1 = 42.02$. This bound also holds for all values of $n \leq 4095$ as we can check with MAGMA (see [19]). □

Our next step will be to estimate the constant associated to the number of conjugacy classes of transitive subgroups of the symmetric group $\mathrm{Sym}(n)$ of degree n. In the next results we will follow the notation and the arguments of [7]. Hence, it will be convenient for the reader to have that paper at hand.

In order to avoid confusion between the constants in [7] and our constants, we will use capital letters to refer to the constants of [7] and reserve the lowercase letters for our constants when they are different. In Theorem 3.1 in [7], it was shown that the number of conjugacy classes of transitive d-generated subgroups of the symmetric group $\mathrm{Sym}(n)$ of degree n is at most C_t^{nd}, where $C_t = (4c_1)^3$ and c_1 was the constant of Proposition 5 whose value can be taken to be equal to $2^{42.02}$ and so C_t takes the value $2^{132.06}$. In this subsection, we will show that the value of the constant c_t can be reduced to $2^{77.034} \approx 1.5472 \times 10^{23}$.

Theorem 18. *The number of conjugacy classes of transitive d-generated subgroups of the symmetric group* $\mathrm{Sym}(n)$ *of degree n is at most c_t^{nd}, where $c_t = 2^{77.034}$.*

Proof. It is enough to follow the proof of [7, Theorem 3.1]. We use the notation of this result. We will argue by induction on n and assume that $n \geq 5$. Given a d-generated transitive subgroup of $\mathrm{Sym}(n)$ and a system $\{B_1, \dots, B_s\}$ of blocks for T, such that $b = |B_1| = b > 1$ and H_1 the stabiliser of B_1, such that H_1 acts primitively on B_1, then T can be regarded as a subgroup of $\mathrm{Sym}(b) \wr \mathrm{Sym}(s)$. Let P be the image of H_1 in the symmetric group on B_1, isomorphic to $\mathrm{Sym}(b)$, and let $\tilde{K}$ be the kernel of the action of T on the blocks. Then, $T/\tilde{K}$ can be naturally embedded into $\mathrm{Sym}(s)$ and T into $P \wr (T/\tilde{K})$. We can divide the d-generated transitive subgroups of $\mathrm{Sym}(n)$ into three families:

1. The first family corresponds to the case in which P does not contain the alternating group $\mathrm{Alt}(b)$ or $b \leq 4$. We have that the number of conjugacy classes of d-generated groups in this family is bounded by
$$N_1 = c_t^{d(n/2+1)} 4^{nd} c_1^n. \tag{1}$$
2. The second family corresponds to the case in which $b \geq 5$, P contains $\mathrm{Alt}(5)$, and $\tilde{K} \neq 1$. The number of d-generated groups in this family is bounded by
$$N_2 = c_t^{d(n/5+1)} 2^{nd}. \tag{2}$$
3. The third family corresponds to the case in which $b \geq 5$, P contains $\mathrm{Alt}(5)$, and $\tilde{K} = 1$. The number of d-generated groups in this family is bounded by
$$N_3 = c_t^{d(n/5+1)} 2^{nd} n^2. \tag{3}$$

Note that $N_1 + N_2 + N_3 \leq \max\{2N_1, 2(N_2 + N_3)\}$. In order to obtain a value of c_t, such that $N_1 + N_2 + N_3 \leq c_t^{nd}$, it will be enough to obtain a value of c_t, such that $2N_1 \leq c_t^{nd}$ and $2(N_2 + N_3) \leq c_t^{nd}$. The condition $2N_1 \leq c_t^{nd}$ is equivalent to $2 \times 4^{nd} \times c_1^n \leq c_t^{(n/2-1)d}$. By taking logarithms, it is equivalent to
$$\frac{1 + 2nd + n \log c_1}{(\frac{n}{2} - 1)d} \leq \log c_t.$$

We maximise the left hand side with the help of Maxima [18] and using that $\log c_1 = 42.02$ and $d \geq 2$. We obtain that the maximum of this expression is less than 77.034 and so the value of $c_t = 2^{77.034}$ satisfies this inequality. We have to show that $2(N_2 + N_3) \leq c_t^{nd}$. Since $2(N_1 + N_2) = 2c_t^{d(n/5+1)} 2^{nd}(1 + n^2)$, the condition $2(N_2 + N_3) \leq c_t^{nd}$ is equivalent to
$$2 \times 2^{nd}(1 + n^2) \leq c_t^{(4n/5-1)d}.$$

By taking logarithms, it is equivalent to
$$\frac{1 + \log(1 + n^2) + nd}{\left(\frac{4n}{5} - 1\right)d} \leq \log c_t.$$

We also use Maxima [18] to check that the maximum of the first expression is $2.6167 < 77.034$. □

We recall the constant associated to the number of epimorphisms from a d-generated group onto a transitive group of degree n.

Proposition 6 (Proposition 4.1 and Remark 4.2 in [7])**.** *Let G be a d-generated group and T a transitive group of degree n. Then, there are at most $|T|c_r^{dn}\max\{1, \mathrm{rk}_n(G)\}$ epimorphisms from G onto T, where $c_r = 16$.*

The following result is Proposition 5.6 in [7], with the precise value of the constant c_3.

Proposition 7. *There exists a constant c_3, such that if H is a quasisimple group and U is an absolutely irreducible FH-module (where F is a finite field), such that $|H| > |U|^{c_3}$, then one of the following holds:*

1. *$H = \mathrm{Alt}(m)$ and W is the natural $\mathrm{Alt}(m)$-module;*
2. *$H = \mathrm{Cl}_d(K)$, a classical group over $K \leq F$, and $U = F \otimes_K U_0$, where U_0 is the natural module for $\mathrm{Cl}_d(K)$.*

In fact, the constant $c_3 = 7$ satisfies these conditions.

Proof. This follows as a consequence of [25–27]. □

This gives a value of $c_4 = c_2 + 1 + \max\{3, c_3\} = 23$ to the constant defined before Proposition 5.7 in [7].

In Proposition 5.9 in [7], it is shown that the number of conjugacy classes of primitive d-generated groups P of $\mathrm{GL}_{\mathbb{F}_p}(W)$ is at most $|W|^{C_5 d}$, where $C_5 = 6c_4 + 31 + c_2 = 184$. A slight variation of the same arguments can be used to give a lower bound for this number.

Proposition 8. *The number of conjugacy classes of primitive d-generated subgroups P of $\mathrm{GL}_{\mathbb{F}_p}(W)$ is at most $|W|^{c_5 d + k_5}$, where $c_5 = 2c_4 + 10 = 56$, $k_5 = 2c_4 + 26 = 72$.*

Proof. We follow the same arguments of Proposition 5.9 in [7] and we will show how to modify that argument to obtain our bound. We divide the primitive groups P into several families.

The family 1 corresponds to $|P| > |W|^{c_4}$. There are, at most,

$$N_1 = |W|^{9d+9}$$

conjugacy classes of primitive d-generated groups in this family.

The family 2 corresponds to $|P| \leq |W|^{c_4}$ and P almost fixing a non-trivial tensor product decomposition $U' \otimes_F U$ of W. The induction argument shows that there are at most $|U|^{c_5(d+1)+k_5}$ choices for Y up to conjugacy in $\mathrm{GL}_{\mathbb{F}_p}(U)$, and so the number of conjugacy classes of primitive d-generated groups in this family is bounded by

$$n|W|^{2(d+1)}|U|^{c_5(d+1)+k_5}|W|^{d(c_4+3)} \leq N_2 = |W|^{2(d+1)+\frac{c_5}{2}(d+1)+\frac{k_5}{2}+1+d(c_4+3)}.$$

The family 3 consists of the groups in which $|P| \leq |W|^{c_4}$ and P does not almost fix any non-trivial tensor product decomposition $U' \otimes_F U$ of W. The subfamily 3.1 corresponds to the case in which $\mathrm{F}^*(P)$ is the product of a q-group T of symplectic type and a cyclic group C of order coprime to p and q. We obtain that the number of conjugacy classes of primitive d-generated groups in this subfamily is bounded by

$$N_3 = |W|^{c_2 d+8}.$$

The subfamily 3.2 corresponds to the case in which $\mathrm{F}^*(P)$ is a central product of k copies of a quasisimple group S and a cyclic group C. The number of conjugacy classes of primitive d-generated groups in this family is bounded by

$$N_4 = |W|^{4d+2c_4+2}.$$

Now, since we can assume that $|W| \geq 4$, we obtain that

$$\begin{aligned} N_1 + N_2 + N_3 + N_4 &\leq 4 \max\{N_1, N_2, N_3, N_4\} \\ &\leq |W| \max\{N_1, N_2, N_3, N_4\} \\ &\leq |W|^{\max\{9d+10, 2(d+1)+\frac{c_5}{2}(d+1)+\frac{k_5}{2}+2+d(c_4+3), c_2 d+9, 4d+2c_4+3\}}. \end{aligned}$$

The inequality

$$2(d+1) + \frac{c_5}{2}(d+1) + \frac{k_5}{2} + 2 + d(c_4+3) \leq c_5 d + k_5$$

is equivalent to

$$2(d+1) + 2 + d(c_4+3) \leq \frac{c_5}{2}d - \frac{c_5}{2} + \frac{k_5}{2},$$

which is, in turn, equivalent to

$$2(d+1) + (d-1)(c_4+3) + c_4 + 9 \leq \frac{c_5}{2}(d-1) + \frac{k_5}{2}$$

and is satisfied for $c_5 = 2c_4 + 10$, $k_5 = 2c_4 + 26$. Since all other terms involved the maximum are less than or equal to the second one, the equality holds. □

In Proposition 6.1 in [7], it is shown that the number of conjugacy classes of d-generated irreducible subgroups of $\mathrm{GL}_{\mathbb{F}_p}(V)$ is at most $|V|^{C_i d}$, where $C_i = 7 + c_4 + C_5 + \log C_t = 7 + 23 + 184 + 132.06 = 346.06$. In Proposition 7.1 in [7], it is proved that if G is a d-generated group and T is an irreducible linear subgroup of $\mathrm{GL}_{\mathbb{F}_p}(V)$, then there are at most $\max\{1, \mathrm{r}_{|V|}(G)\}|T||V|^{dc_l}$ epimorphisms from G onto T, where

$$c_l = 4 + \max\{c_4, 4\} = 27.$$

Given an irreducible subgroup of $\mathrm{GL}_{\mathbb{F}_p}(V)$, we have that the number of T-conjugacy classes of epimorphisms from a group G onto T is at most $|V||\mathrm{Epi}(G,T)|/|T|$ by Lemma 7.2 in [7]. As a consequence, we can obtain the following result (compare with Corollary 7.3 in [7]).

Corollary 2. *Let G be a d-generated group. There exists a constant C_6, such that the number of irreducible G-modules of size n is at most*

$$\max\{1, \mathrm{rk}_n(V)\} n^{C_6 d}.$$

The value of the constant C_6 is not presented in [7], but it seems clear that the arguments in this paper that the constant $C_6 = c_i + c_l + 1 = 346.06 + 27 + 1 = 374.06$ satisfies the condition.

3.3. Determination of the Constants

We are now in a position to complete the proof of Theorem 6.

Proof of Theorem 6. The arguments of the proof of Corollary 9.1 in [7] show that the constant c_p appearing there is essentially the same as the constant C_6 of Corollary 2. The constant c such that the number of maximal subgroups of index n of G is bounded by $n^{cd} \max\{1, \mathrm{rk}_n(G)\}$ can be taken as $c = C_6 + 1 = 375.06$. This is the constant that appears in Theorem 3. □

We can present the arguments of Sections 6 and 7 of [7] in a different way to improve the bound for the number of irreducible G-modules of order n.

Theorem 19. *Let G be a d-generated group. The number of irreducible non-equivalent G-modules of size n is at most $n^{c_6 d+k_6}\max\{1,\mathrm{rk}_n(G)\}$, where $c_6 = 2c_4 + c_5 + \log c_t + 4 = 183.034$ and $k_6 = k_5 + 2 = 74$.*

Proof. As in Propositions 6.1 and 7.1 in [7], let T be an irreducible d-generated subgroup of $\mathrm{GL}_{\mathbb{F}_p}(V)$ and let H be a subgroup of T, such that the representation of T is induced from a primitive representation of H. Let W be a primitive H-module such that $V = \mathbb{F}_p T \oplus_{\mathbb{F}_p H} W$. Let P be the image of H in $\mathrm{GL}_{\mathbb{F}_p}(W)$ and $b = \dim_{\mathbb{F}_p} W$. Set $\tilde{K} = H_T$. Then, $T/\tilde{K}$ is a transitive group of degree $s = m/b$, where $m = \dim_{\mathbb{F}_p} V$, and T is a subgroup of $P \wr (T/\tilde{K})$.

We divide the d-generated irreducible subgroups of $\mathrm{GL}_{\mathbb{F}_p}(V)$ into two families, as in Proposition 6.1 in [7].

In the first family, $|P| \leq |W|^{c_4}$. Note that

$$|T| \leq |P|^s |T/\tilde{K}| \leq |W|^{c_4 s}|T|/|\tilde{K}|$$

and so $|\tilde{K}| \leq |W|^{sc_4} = |V|^{c_4}$, that is, we are in Case 1 in the proof of Proposition 7.1. The argument in the proof of Proposition 6.1 in [7], with the replacement of $|W|^{c_5\,\mathrm{d}(H)}$ by $|W|^{c_5\,\mathrm{d}(H)+k_5}$ and $|V|^{c_5 d}$ by $|V|^{c_5 d+k_5}$ shows that the number of conjugacy classes of d-generated irreducible subgroups in the first family is bounded by

$$n|V|^{c_5 d+k_5} c_t^{sd} |V|^{c_4 d} \leq |V|^{(c_5+c_4+\log c_t)d+k_5+1}.$$

Now, we obtain a bound for $|\mathrm{Epi}(G,T)|$. The argument in Case 1 of Proposition 7.1 in [7] shows that

$$|\mathrm{Epi}(G,T)| \leq |T|\max\{1,\mathrm{r}_{|V|}(G)\}|V|^{d(c_4+4)}.$$

By Lemma 7.2 in [7], we conclude that the number of irreducible $\mathbb{F}_p G$-modules of size n obtained starting from a primitive linear group P with $|P| \leq |W|^{c_4}$ is bounded by

$$|V|^{(c_5+2c_4+\log c_t+c_4)d+k_5+2}\max\{1,\mathrm{r}_{|V|}(G)\}.$$

In the second family, $|P| > |W|^{c_4}$. This corresponds to Case 2 in Proposition 7.1 in [7]. The number of choices of conjugacy classes of primitive d-generated groups in this family, according to the proof of Proposition 8, family 1, is bounded by $|W|^{9d+9}$. As in the proof of Proposition 6.1 in [7], we can consider two subfamilies. In the subfamily 2.1, there are at most

$$|V|^{3d+4} c_t^{dm}$$

conjugacy classes of d-generated irreducible subgroups, while in the subfamily 2.2, there are at most

$$|V|^{3d+5} c_t^{dm}$$

conjugacy classes of d-generated irreducible subgroups. In Case 2 of Proposition 7.1 in [7], we see that

$$|\mathrm{Epi}(G,T)| \leq \max\{1,\mathrm{r}_{|V|}(G)\}|T||V|^{4d}.$$

By Lemma 7.2 in [7], we conclude that the number of irreducible $\mathbb{F}_p G$-modules of size n obtained starting from a primitive linear group P with $|P| \leq |W|^{c_4}$ is bounded by

$$\begin{aligned}(|V|^{3d+4} + |V|^{3d+5})c_t^{dm}\max\{1,\mathrm{r}_{|V|}(G)\}|V|^{4d}|V| \\ \leq |V|^{7d+7}c_t^{dm}\max\{1,\mathrm{r}_{|V|}(G)\} \leq |V|^{(7+\log c_t)d+7}\max\{1,\mathrm{r}_{|V|}(G)\},\end{aligned}$$

because $c_t^{dm} = 2^{md\log c_t} \leq |V|^{d\log c_t}$.

Putting everything together, we obtain that the number of irreducible G-modules of size n is bounded by

$$(|V|^{(7+\log c_t)d+7} + |V|^{(2c_4+c_5+\log c_t+4)+k_5+2}) \max\{1, \mathrm{rk}_n(G)\} \\ \leq |V|^{(2c_4+c_5+\log c_t+4)d+k_5+2} \max\{1, \mathrm{rk}_n(G)\}. \quad \square$$

Proof of Theorem 10. To determine the values of the constants, we can follow the results needed to prove Theorem 19 avoiding all arguments with insoluble groups. We will only state the differences.

First of all, we can use the bound of Lemma 11 as a bound for the number of irreducible soluble subgroups of $\mathrm{GL}_m(p)$. We can replace the bound of Theorem 16 by this value. We use the arguments of the proof of Proposition 4, but taking into account that now only the term $v_a(m, p)$ appears, to obtain a bound of the form $\hat{c}_1^n$ for the number of conjugacy classes of primitive subgroups of $\mathrm{Sym}(n)$. We obtain that the corresponding value of $\hat{c}_1$ is $\hat{c}_1 = 2^{9.886}$.

The result corresponding to Theorem 18 says that the number of conjugacy classes of soluble transitive d-generated subgroups of the symmetric group $\mathrm{Sym}(n)$ is at most $\hat{c}_t^{nd}$. We can use the same arguments than in the proof of Theorem 18, but here only the first family should be taken into account. By induction, we obtain that $N_1 \leq \hat{c}_t^{nd}$ is obtained for $\hat{c}_t = 2^{27.772}$.

The corresponding version of Theorem 19, that is, the number of irreducible G-modules of size n for a d-generated soluble group G is at most $n^{\hat{c}_6 d+\hat{k}_6}$, holds then for $\hat{c}_6 = 2c_4 + c_5 + \log \hat{c}_t + 4 = 133.772$ and $\hat{k}_6 = k_5 + 2 = 66$. $\square$

Note that for soluble groups, the exponent of n in the bound can be reduced by $49.262d$ with respect to Theorem 19.

The fact that the number of crowns associated to chief factors of order n is obviously bounded by the number of representations gives an immediate bound for the number of irreducible G-modules of dimension r over a field of p elements that is useful for modules of small dimension.

Proposition 9. *Let G be a d-generated group. The number of G-modules of size $n = p^r$, where p is a prime and $r \in \mathbb{N}$, is at most $(n-1)^{dr}$.*

Proof. Let $G = \langle x_1, \ldots, x_d \rangle$, let V be a G-module over G and let $\{v_1, \ldots, v_r\}$ be a basis of V as a vector space over $\mathbb{F}_p$. Then, the action of G on V is completely determined by the action of each of the x_i on the v_j, say $v_j^{x_i}$. Each of these images can take at most $n-1$ values. This gives the result. $\square$

Proposition 9 makes clear that the bound of Theorem 19 is only useful for prime-power values $n = p^r$ with r big, say

$$r > c_6 d + k_6 + \log_n \max\{1, \mathrm{rk}_n(G)\} = 183.034d + 74 + \log_n \max\{1, \mathrm{rk}_n(G)\},$$

otherwise Proposition 9 gives better bounds for the number of irreducible G-modules of size $n = p^r$. For example, if G is a d-generated group, then Proposition 9 gives better bounds for the number of irreducible G-modules of size $n = p^r$ with p a prime.

Putting together Theorem 19 and Proposition 9, we obtain the following result.

Theorem 20. *The number of non-equivalent irreducible G-modules of size $n = p^r$, where p is a prime and $r \in \mathbb{N}$, is at most*

$$n^{\min\{c_6 d+k_6+\log_n \max\{1,\mathrm{rk}_n(G)\}, dr\}},$$

where c_6 and k_6 are the constants of Theorem 19.

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