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Accurate computations with Wronskian matrices of Bessel and Laguerre polynomials ☆

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This paper provides an accurate method to obtain the bidiagonal factorization of Wronskian matrices of Bessel polynomials and of Laguerre polynomials. This method can be used to compute with high relative accuracy their singular values, the inverse of these matrices, as well as the solution of some related systems of linear equations. Numerical examples illustrating the theoretical results are included.

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1. Introduction

Finding algorithms with high relative accuracy (HRA) for matrix calculations such as obtaining their singular values or inverses is a desirable goal that has been achieved only for a few classes of matrices. Among them, we can mention some subclasses of totally positive matrices for which the bidiagonal factorization can be calculated with HRA. After an adequate parametrization of the matrices, this goal has been achieved for the collocation matrices of some important systems of functions. This was obtained for the collocation matrices of Bessel polynomials (see applications in [5], [11] and references in there) and for the collocation matrices of generalized Laguerre polynomials (see [4]). In both cases, the collocation matrices are totally positive (see Section 2) and a bidiagonal factorization with HRA was obtained for them. This bidiagonal factorization is the start step to apply the algorithms with HRA of [13–15].

It is well known that many fundamental problems in interpolation and approximation require linear algebra computations related to collocation matrices. Wronskian matrices arise when solving Hermite interpolation problems, in particular Taylor interpolation problems. In CAGD, the solution of systems of equations with Wronskian matrices is also important for the definition of bases with good properties in interactive curve design (cf. [3]). Furthermore, in other applications of matrix theory, for example in spectral theory, Wronskian matrices of fundamental solution sets to linear differential equations play a relevant role (cf. [12]).

In [16], the bidiagonal decomposition of the Wronskian matrix of the monomial basis of the space of polynomials of a given degree and the bidiagonal factorization of the Wronskian matrix of the basis of exponential polynomials were obtained. Furthermore, in [17] a procedure to accurately compute the bidiagonal decomposition of collocation and Wronskian matrices of the wide family of Jacobi polynomials is proposed. The obtained results are used to get accurate computations using collocation and Wronskian matrices of well-known types of Jacobi polynomials.

In this paper, we obtain the bidiagonal factorization with HRA for the Wronskian matrices of Bessel polynomials as well as for the Wronskian matrices of generalized Laguerre polynomials, which can be used to calculate with HRA their singular values or inverses.

The layout of the paper is as follows. Section 2 presents basic concepts and results. Section 3 proves the total positivity of the Wronskian matrices of Bessel polynomials defined on positive real numbers and shows that the mentioned algebraic calculations can be performed with HRA. Section 4 deals with the corresponding results for the Wronskian matrices of generalized Laguerre polynomials. Section 5 includes numerical examples illustrating the great accuracy of the presented methods for the computation of all singular values, the inverses of the matrices and the solution of some linear systems.

2. Notations and previous results

In this paper we shall use the following notations. Given an n-times continuously differentiable real function f and $x \in \mathbb{R}$ in its domain, $f'(x)$ denotes the first derivative of f at x. For any $i \leq n$, $f^{(i)}(x)$ denotes the i-th derivative of f at x. Given a basis $(u_0, \ldots, u_n)$ of a space of n-times continuously differentiable functions, defined on a real interval I and $x \in I$, the *Wronskian matrix* at x is

$$W(u_0, \ldots, u_n)(x) := (u_{j-1}^{(i-1)}(x))_{i,j=1,\ldots,n+1}.$$

A matrix is totally positive (TP) if all its minors are nonnegative. Some books with many applications of TP matrices are [1,7,19].

Neville elimination is an alternative procedure to Gaussian elimination and has been used to characterize TP matrices. More details on this elimination method can be found in [8–10].

By Theorem 4.2 and the arguments of p. 116 of [10], a nonsingular TP matrix $A = (a_{i,j})_{1\leq i,j\leq n+1}$ admits a factorization of the form

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n, \tag{1}$$

where F_i and G_i are the TP, lower and upper triangular bidiagonal matrices given by

$$F_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & m_{n+1,n+1-i} & 1 \end{pmatrix},$$

$$G_i^T = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \widetilde{m}_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \widetilde{m}_{n+1,n+1-i} & 1 \end{pmatrix}, \tag{2}$$

and $D = \operatorname{diag}(p_{1,1}, p_{2,2}, \ldots, p_{n+1,n+1})$ has positive diagonal entries. If, in addition, the entries $m_{i,j}$, $\widetilde{m}_{i,j}$ satisfy

$$m_{i,j} = 0 \Rightarrow m_{h,j} = 0, \quad \forall h > i, \quad \text{and} \quad \widetilde{m}_{i,j} = 0 \Rightarrow \widetilde{m}_{i,k} = 0, \quad \forall k > j, \tag{3}$$

then the decomposition (1) is unique. The diagonal entries $p_{i,i}$ of D are the diagonal pivots of the Neville elimination of A and the elements $m_{i,j}$, $\widetilde{m}_{i,j}$ are positive and coincide with the multipliers of the Neville elimination of A and A^T, respectively.

In [13], the bidiagonal factorization (1) of an $(n+1) \times (n+1)$ nonsingular and TP matrix A is represented by defining a matrix $BD(A) = (BD(A)_{i,j})_{1 \le i,j \le n+1}$ such that

$$BD(A)_{i,j} := \begin{cases} m_{i,j}, & \text{if } i > j, \\ p_{i,i}, & \text{if } i = j, \\ \widetilde{m}_{j,i}, & \text{if } i < j. \end{cases} \tag{4}$$

The bidiagonal factorization (1) can be used to represent more classes of matrices. In fact, if we consider the factorization given by (1), (2) and (3) without any further requirement than the nonsingularity of D then, by Proposition 2.2 of [2], the uniqueness of (1) holds. From now on, $BD(A)$ will denote the bidiagonal decomposition of a matrix that satisfies these hypotheses.

Let us observe that if a matrix A is nonsingular and TP, then A^T is also a nonsingular and TP matrix. Moreover, the bidiagonal decomposition of A^T can be computed as

$$A^T = G_n^T G_{n-1}^T \cdots G_1^T D F_1^T \cdots F_{n-1}^T F_n^T, \tag{5}$$

where F_i and G_i, $i = 1, \ldots, n$, are the lower and upper triangular bidiagonal matrices in (1). In fact, $BD(A^T) = (BD(A))^T$ (see Section 4 of [13]).

We say that a real x is computed with high relative accuracy (HRA) whenever the computed value $\widehat{x}$ satisfies

$$\frac{\|x - \widehat{x}\|}{\|x\|} < Ku,$$

where u is the unit round-off and $K > 0$ is a constant independent of the arithmetic precision. Clearly, HRA implies that the relative errors in the computations have the same order as the machine precision. It is well known that a sufficient condition to assure that an algorithm can be computed with HRA is the non inaccurate cancellation (NIC) condition and it is satisfied if it only evaluates products, quotients, sums of numbers of the same sign, subtractions of numbers of opposite sign or subtraction of initial data (cf. [6], [13]).

If the bidiagonal factorization (1) of a nonsingular TP matrix A is computed with HRA then, using the algorithms in [14], we can also compute with HRA its singular values, the matrix A^{-1} and even the solution of $Ax = b$ for vectors b with alternating signs. In the following sections we shall obtain the bidiagonal factorization (1) of Wronskian matrices associated to Bessel and Laguerre polynomials, analyzing whether it can be computed with HRA.

3. Total positivity and factorization of Wronskian matrices of Bessel polynomials

Let us denote by $\mathbf{P}^n$ the space of polynomials of degree less than or equal to n and $(p_0, \ldots, p_n)$ the monomial basis of $\mathbf{P}^n$ such that

$$p_i(x) := x^i, \quad i = 0, \ldots, n. \tag{6}$$

The following result restates Corollary 1 of [16] providing the bidiagonal factorization (1) of the Wronskian matrix $W(p_0, \ldots, p_n)(x)$, $x \in \mathbb{R}$.

Proposition 1. *Let $(p_0, \ldots, p_n)$ be the monomial basis given in (6). For any $x \in \mathbb{R}$, the Wronskian matrix $W(p_0, \ldots, p_n)(x)$ is nonsingular and can be factorized as follows,*

$$W(p_0, \ldots, p_n)(x) = DG_{1,n} \cdots G_{n-1,n}G_{n,n}, \tag{7}$$

where $D = diag\{0!, 1!, \ldots, n!\}$ and $G_{i,n}$, $i = 1, \ldots, n$, are the upper triangular bidiagonal matrices in (2) with

$$\widetilde{m}_{k,k-i} = x, \quad i + 1 \le k \le n + 1. \tag{8}$$

Moreover, if $x > 0$ then $W(p_0, \ldots, p_n)(x)$ is nonsingular and TP, its bidiagonal decomposition (1) is given by (7) and (8) and it can be computed with HRA.

Let us recall that the Bessel basis of $\mathbf{P}^n$ is the polynomial system $(B_0, \ldots, B_n)$ with

$$B_i(x) := \sum_{k=0}^{i} \frac{(i+k)!}{2^k (i-k)!k!} x^k, \quad i = 0, \ldots, n. \tag{9}$$

In [5], the total positivity of the matrix of change of basis between the Bessel polynomial basis $(B_0, \ldots, B_n)$ and the monomials $(p_0, \ldots, p_n)$ is proved. As a consequence, accurate computations when considering collocation matrices $(B_{j-1}(x_{i-1}))_{1\le i,j\le n+1}$ with $(0 <)x_0 < x_1 < \cdots < x_n$ are derived.

Now, let $W(B_0, \ldots, B_n)(x)$ be the Wronskian matrix at $x \in \mathbb{R}$ of the basis (9) of Bessel polynomials. The following result extends the results in [5] to $W(B_0, \ldots, B_n)(x)$ at $x > 0$ and establishes the total positivity of this Wronskian matrix.

Theorem 2. *Let $(B_0, \ldots, B_n)$ be the Bessel polynomial basis defined in (9). For any $x > 0$, the Wronskian matrix $W := W(B_0, \ldots, B_n)(x)$ is nonsingular TP and its bidiagonal decomposition (1) can be computed with HRA. Furthermore, the computation of its singular values, the inverse of W, as well as the solution of the linear systems $Wx = b$, where $b = (b_0, \ldots, b_n)^T$ has alternating signs, can be performed with HRA.*

Proof. It can be checked that

$$(B_0, \dots, B_n)^T = A(p_0, \dots, p_n)^T, \tag{10}$$

where $(p_0, \dots, p_n)$ is the monomial basis given in (6) and the change of basis matrix $A = (a_{ij})_{1 \le i,j \le n+1}$ is lower triangular and satisfies

$$a_{i,j} := \frac{(i+j-2)!}{2^{j-1}(i-j)!(j-1)!}, \quad i \ge j. \tag{11}$$

Using formula (10), it can be checked that

$$W(B_0, \dots, B_n)(x) = W(p_0, \dots, p_n)(x)A^T, \tag{12}$$

where $W(p_0, \dots, p_n)(x)$ is the Wronskian matrix of the monomial basis $(p_0, \dots, p_n)$ and A is the lower triangular matrix described by (11).

By Proposition 1, $W(p_0, \dots, p_n)(x)$, $x > 0$, is nonsingular and TP and its bidiagonal factorization (1) can be computed with HRA. Furthermore, by Theorem 3 of [5], A is a nonsingular TP matrix and admits a factorization of the form

$$A = F_n F_{n-1} \cdots F_1 D, \tag{13}$$

where F_i, $i = 1, \dots, n$, are the lower triangular bidiagonal matrices described in (2) and $D = \operatorname{diag}(p_{1,1}, \dots, p_{n+1,n+1})$. The entries $m_{i,j}$, $1 \le j < i \le n+1$, and $p_{i,i}$, $1 \le i \le n+1$, are given by

$$m_{i,j} = \frac{(2i-2)(2i-3)}{(2i-j-1)(2i-j-2)}, \quad p_{i,i} = (2i-3)!!, \tag{14}$$

with the following double factorial notation for a positive integer k,

$$k!! := \prod_{j=0}^{\lfloor (k-1)/2 \rfloor} (k-2j),$$

where $\lfloor (k-1)/2 \rfloor$ is the greatest integer less than or equal to $(k-1)/2$. Clearly, $m_{i,j}$, and $p_{i,i}$ are positive and can be obtained with HRA. The bidiagonal factorization (1) of A^T is given by $A^T = DF_1^T \cdots F_{n-1}^T F_n^T$.

On the other hand, $W(B_0, \dots, B_n)(x)$, $x > 0$, is nonsingular and TP since, by (12), it can be expressed as the product of two nonsingular TP matrices (see Theorem 3.1 of [1]).

Using Algorithm 5.1 of [14], if the bidiagonal decomposition (1) of two nonsingular TP matrices is provided with HRA, then the corresponding bidiagonal decomposition (1) of the product is computed with HRA. Consequently, the bidiagonal decomposition

(1) of $W = W(B_0, \dots, B_n)(x)$, $x > 0$, can be computed with HRA. This fact guarantees that algebraic problems such as the computation of all the singular values, the inverse matrix of W, and the solution of the linear systems $Wx = b$, where $b = (b_0, \dots, b_n)^T$ has alternating signs, can be performed with HRA (see Section 3 of [6]). $\square$

Let us recall that the basis of reverse Bessel polynomials in $\mathbf{P}^n$ is $(R_0, \dots, R_n)$ with

$$R_i(x) := \sum_{k=0}^{i} \frac{(i+k)!}{2^k (i-k)! k!} x^{i-k}, \quad i = 0, \dots, n. \tag{15}$$

Let us observe that this basis is obtained when reversing the order of the coefficients of the Bessel polynomials $B_i(x)$, $i = 0, \dots, n$, in (9).

In [5] it is proved that the matrix of change of basis between the reverse Bessel polynomials $(R_0, \dots, R_n)$ and the monomials $(p_0, \dots, p_n)$ is TP. Therefore, accurate computations with collocation matrices $(R_{j-1}(x_{i-1}))_{1 \le i,j \le n+1}$ where $(0 <)x_0 < x_1 < \cdots < x_n$ are provided.

Given $x \in \mathbb{R}$, $W(R_0, \dots, R_n)(x)$ denotes the Wronskian matrix at x of the basis (15) of reverse Bessel polynomials. The following result extends the results in [5] to $W(R_0, \dots, R_n)(x)$ at $x > 0$ and establishes the total positivity of this Wronskian matrix.

Theorem 3. *Let $(R_0, \dots, R_n)$ be the reverse Bessel polynomials basis given in (15). For any $x > 0$, the Wronskian matrix $W_R := W(R_0, \dots, R_n)(x)$ is nonsingular TP and its bidiagonal decomposition (1) can be computed with HRA. Furthermore, the computation of its singular values, the inverse of W_R, as well as the solution of the linear systems $W_R x = b$, where $b = (b_0, \dots, b_n)^T$ has alternating signs, can be performed with HRA.*

Proof. It can be checked that

$$(R_0, \dots, R_n)^T = C(p_0, \dots, p_n)^T, \tag{16}$$

where $(p_0, \dots, p_n)$ is the monomial basis given in (6) and $C = (c_{ij})_{1 \le i,j \le n+1}$ is the lower triangular change of basis matrix such that

$$c_{i,j} = \frac{(2i-j-1)!}{2^{i-j}(j-1)!(i-j)!}, \quad i \ge j. \tag{17}$$

By formula (16) we can write

$$W(R_0, \dots, R_n)(x) = W(p_0, \dots, p_n)(x) C^T, \tag{18}$$

where $W(p_0, \dots, p_n)(x)$ is the Wronskian matrix of $(p_0, \dots, p_n)$ at x and C is the lower triangular matrix described by (17).

Let us recall that, by Proposition 1, $W(p_0,\dots,p_n)(x)$, $x>0$, is nonsingular and TP and its bidiagonal factorization (1) can be computed with HRA. On the other hand, by Theorem 5 of [5], the matrix C is nonsingular and TP and admits a factorization

$$C = F_n F_{n-1} \cdots F_1 D, \tag{19}$$

where F_i, $i=1,\dots,n$, are the lower triangular bidiagonal matrices described in (2) and $D = \operatorname{diag}(p_{1,1},\dots,p_{n+1,n+1})$. The entries $m_{i,j}$ and $p_{i,i}$ are given by

$$m_{i,j} = 2i-2j-1,\ 1\le j<i\le n+1, \qquad p_{i,i}=1,\ 1\le i\le n+1, \tag{20}$$

and, clearly, can be obtained with HRA. The bidiagonal factorization (1) of C^T is given by $C^T = DF_1^T\cdots F_{n-1}^T F_n^T$.

Since $W(R_0,\dots,R_n)(x)$, $x>0$, is the product of two nonsingular TP matrices, by (18), we deduce that it is nonsingular and TP (see Theorem 3.1 of [1]).

Using Algorithm 5.1 of [14], if the bidiagonal decomposition (1) of two nonsingular TP matrices is provided with HRA, then the corresponding bidiagonal decomposition (1) of the product is computed with HRA. Consequently, the bidiagonal decomposition (1) of $W(R_0,\dots,R_n)(x)$, $x>0$, can be computed with HRA and so, its inverse matrix, its singular values and the solutions of the mentioned linear systems (see Section 3 of [6]). $\square$

Section 5 shows accurate results obtained when solving the above algebraic problems using the bidiagonal factorization (1) and the algorithms presented in [14] and [15].

4. Total positivity and factorization of Wronskian matrices of Laguerre polynomials

Given $\alpha > -1$, the generalized Laguerre basis of $\mathbf{P}^n$ is the polynomial system $(L_0^{(\alpha)},\dots,L_n^{(\alpha)})$ described by

$$L_i^{(\alpha)}(x) := \sum_{k=0}^{i} (-1)^k \binom{i+\alpha}{i-k} \frac{x^k}{k!}, \quad i=0,\dots,n. \tag{21}$$

It is well known that this polynomial basis is orthogonal on the interval $[0,\infty)$ with respect to the weight function $x^\alpha e^{-x}$.

In [4] accurate computations when considering collocation matrices of Laguerre polynomials $\left(L_{j-1}^{(\alpha)}(x_{i-1})\right)_{1\le i,j\le n+1}$, with $(0>)x_0>x_1>\cdots>x_n$ are provided.

The following result analyzes the total positivity of Laguerre Wronskian matrices and provides a factorization that allows to solve with HRA some algebraic problems.

Theorem 4. *Let $(L_0^{(\alpha)},\dots,L_n^{(\alpha)})$ be the Laguerre basis defined in (21) and J the diagonal matrix $J := diag((-1)^{i-1})_{1\le i\le n+1}$. Then, for any $x<0$, the matrix*

$$W_J := JW(L_0^{(\alpha)}, \ldots, L_n^{(\alpha)})(x) \tag{22}$$

is a nonsingular TP matrix and its bidiagonal decomposition (1) can be computed with HRA. Furthermore, the computation of all the singular values, the inverse of W_J, as well as the solution of the linear systems $W_J x = b$, where $b = (b_0, \ldots, b_n)^T$ has alternating signs, can be performed with HRA.

Proof. In Theorem 2 of [4] it is shown that the matrix A of the change of basis between the generalized Laguerre basis (21) and the monomial basis (6) such that

$$(L_0^{(\alpha)}, \ldots, L_n^{(\alpha)}) = (p_0, \ldots, p_n)A, \tag{23}$$

satisfies

$$A = JS_\alpha^{-1} P_U S_0^{-1} S_\alpha, \tag{24}$$

where $S_\alpha := \operatorname{diag}((\alpha + i - 1)^{i-1})_{1 \le i \le n+1}$ and $P_U \in \mathbb{R}^n$ is an upper triangular Pascal matrix, that is, the $(n+1) \times (n+1)$ upper triangular matrix with $\binom{j-1}{i-1}$ as (i,j)-entry for $j \ge i$.

Let $(\ell_0, \ldots, \ell_n)$ such that $\ell_i(x) = (-x)^i$, $i = 0, \ldots, n$. Since $(p_0, \ldots, p_n) = (\ell_0, \ldots, \ell_n)J$, taking into account identities (23) and (24), we can write

$$(L_0^{(\alpha)}, \ldots, L_n^{(\alpha)}) = (\ell_0, \ldots, \ell_n) S_\alpha^{-1} P_U S_0^{-1} S_\alpha. \tag{25}$$

Let us observe that the upper triangular Pascal matrix P_U is nonsingular and TP (see [7]) and so are the positive diagonal matrices S_α^{-1}, S_0^{-1} and S_α. Then, we can deduce that $S_\alpha^{-1} P_U S_0^{-1} S_\alpha$ is also nonsingular and TP since it is a product of nonsingular and TP matrices.

On the other hand, since $\ell_j^{(i)}(x) = (-1)^i p_j^{(i)}(-x)$, $0 \le i \le j \le n$, the following matrix equality can be easily deduced

$$JW(\ell_0, \ldots, \ell_n)(x) = W(p_0, \ldots, p_n)(-x), \quad x \in \mathbb{R}. \tag{26}$$

Then, using equality (26), we can deduce that the scaled Wronskian matrix $JW(L_0^{(\alpha)}, \ldots, L_n^{(\alpha)})(x)$, $x \in \mathbb{R}$, satisfies

$$\begin{aligned} JW(L_0^{(\alpha)}, \ldots, L_n^{(\alpha)})(x) &= JW(\ell_0, \ldots, \ell_n)(x) S_\alpha^{-1} P_U S_0^{-1} S_\alpha \\ &= W(p_0, \ldots, p_n)(-x) S_\alpha^{-1} P_U S_0^{-1} S_\alpha. \end{aligned}$$

Moreover, from Proposition 1, $W(p_0, \ldots, p_n)(-x)$, $x < 0$, is nonsingular TP and so is the matrix $JW(L_0^{(\alpha)}, \ldots, L_n^{(\alpha)})(x)$, since it is the product of nonsingular TP matrices.

The bidiagonal factorization (1) of $JW(\ell_0, \ldots, \ell_n)(x) = W(p_0, \ldots, p_n)(-x)$, $x < 0$, is described by (7) and (8). Clearly, it can be computed with HRA. On the other hand, in Theorem 2 of [4] it is shown that the bidiagonal factorization (1) of the matrix $S_\alpha^{-1} P_U S_0^{-1} S_\alpha$ is

$$S_\alpha^{-1} P_U S_0^{-1} S_\alpha = S_0^{-1} G_1 \cdots G_n,$$

where G_k, $k = 1, \ldots, n$, is the bidiagonal upper triangular matrix with unit diagonal whose $(i, i+1)$ entry is

$$\widetilde{m}_{i,i-k} = \frac{i+\alpha}{i}, \quad k < i.$$

Again, this factorization can be computed with HRA. Finally, following Section 5.2 of [14], the bidiagonal factorization (1) of $JW(L_0^{(\alpha)}, \ldots, L_n^{(\alpha)})(x)$, $x < 0$, can be computed with HRA using the subtraction-free Algorithm 5.1 in [14], and the bidiagonal factorizations (1) of $JW(\ell_0, \ldots, \ell_n)(x)$ and $S_\alpha^{-1} P_U S_0^{-1} S_\alpha$, which can be provided with HRA.

This fact guarantees that algebraic problems such that the computation of all the singular values, the inverse matrix of W_J, and the solution of the linear systems $W_J x = b$, where $b = (b_0, \ldots, b_n)^T$ has alternating signs, can be performed with HRA (see Section 3 of the [6]). $\square$

Let us observe that by, Theorem 4, we can clearly see that the Wronskian matrix W of the Laguerre basis $(L_0^{(\alpha)}, \ldots, L_n^{(\alpha)})$ is not TP at any $x < 0$. Consequently, for the accurate solution of algebraic problems with W, such as the computation of W^{-1}, the algorithms of [13–15] can not be used directly, using a bidiagonal decomposition of W. Nevertheless, as it is shown in the next result, the solution of some algebraic problems related with W is closely related to that corresponding to the TP matrix $W_J = JW$, whose bidiagonal decomposition can be computed with HRA. Then, using the algorithms of [13–15] with the bidiagonal decomposition (1) of W_J, the solution with HRA of the considered algebraic problems can be achieved and then, from the computed results, the solutions corresponding to the matrix W can also be obtained with HRA.

Corollary 5. *Let $W := W(L_0^{(\alpha)}, \ldots, L_n^{(\alpha)})(x)$ be the Wronskian matrix of the Laguerre basis $(L_0^{(\alpha)}, \ldots, L_n^{(\alpha)})$ defined in (21). Then, for any $x < 0$, the bidiagonal factorization (1) of W can be computed with HRA. Moreover, the computation of all its singular values, the inverse of W, as well as the solution of the linear systems $Wx = b$, where the elements of $b = (b_0 \ldots, b_n)^T$ have the same sign, can be performed with HRA.*

Proof. Let $J := \operatorname{diag}((-1)^{i-1})_{1 \le i \le n+1}$. By Theorem 4, the bidiagonal decomposition (1) of $W_J := JW$ can be computed with HRA. By multiplying this factorization by $J = J^{-1}$, we can derive with HRA the corresponding bidiagonal factorization of W.

On the other hand, let us observe that, since J is a unitary matrix, the singular values of W coincide with those of W_J and then, from Theorem 4, their computation for $x < 0$ can be performed with HRA. Similarly, taking into account that

$$W^{-1} = W_J^{-1} J,$$

Theorem 4 also guarantees the accurate computation of W^{-1}. Finally, if we have a linear system of equations $Wx = b$, where the elements of $b = (b_i \dots, b_n)^T$ have the same sign, from Theorem 4, we will be able to solve with HRA the equivalent system $JWx = Jb$, since Jb has alternating signs. □

Let us now consider the polynomial basis $(\bar{L}_0^{(\alpha)}, \dots, \bar{L}_n^{(\alpha)})$ obtained by changing the variable in the Laguerre basis as follows:

$$\bar{L}_i^{(\alpha)}(x) := L_i^{(\alpha)}(-x) = \sum_{k=0}^{i} \binom{i+\alpha}{i-k} \frac{x^k}{k!}, \quad i = 0, \dots, n. \tag{27}$$

As in Theorem 4, using the results in this paper, the analysis of the total positivity of the Wronskian matrix $W(\bar{L}_0^{(\alpha)}, \dots, \bar{L}_n^{(\alpha)})(x)$, $x \in \mathbb{R}$, can also be performed.

Theorem 6. *Let $(\bar{L}_0^{(\alpha)}, \dots, \bar{L}_n^{(\alpha)})$ be the polynomial basis defined in (27). Then, for any $x > 0$, the Wronskian matrix $W := W(\bar{L}_0^{(\alpha)}, \dots, \bar{L}_n^{(\alpha)})(x)$, $x > 0$, is nonsingular and TP and its bidiagonal decomposition (1) can be computed with HRA. Furthermore, the computation of all the singular values, the inverse of W, as well as the solution of the linear systems $Wx = b$, where $b = (b_0, \dots, b_n)^T$ has alternating signs, can be performed with HRA.*

Proof. It can be easily checked that the matrix A of change of basis between the basis (27) and the monomial basis (6), such that $(\bar{L}_0^{(\alpha)}, \dots, \bar{L}_n^{(\alpha)}) = (p_0, \dots, p_n)A$, satisfies

$$A = S_\alpha^{-1} P_U S_0^{-1} S_\alpha,$$

where $S_\alpha := \operatorname{diag}((\alpha+i-1)^{i-1})_{1 \le i \le n+1}$ and P_U is the $(n+1) \times (n+1)$ upper triangular Pascal matrix (see [4]). Consequently, $W(\bar{L}_0^{(\alpha)}, \dots, \bar{L}_n^{(\alpha)})(x)$ satisfies

$$W(\bar{L}_0^{(\alpha)}, \dots, \bar{L}_n^{(\alpha)})(x) = W(p_0, \dots, p_n)(x) S_\alpha^{-1} P_U S_0^{-1} S_\alpha, \quad x \in \mathbb{R},$$

where $S_\alpha^{-1} P_U S_0^{-1} S_\alpha$ is nonsingular and TP because it is a product of nonsingular and TP matrices. As in the proof of Theorem 4, from Proposition 1 and taking into account that the product of nonsingular and TP matrices is nonsingular and TP, it can be deduced that $W = W(\bar{L}_0^{(\alpha)}, \dots, \bar{L}_n^{(\alpha)})(x)$ is nonsingular TP and its bidiagonal factorization (1) can be provided with HRA, which guarantees that its inverse matrix, its singular values, as well as the solution of the linear systems $Wx = b$, where $b = (b_0, \dots, b_n)^T$ has alternating signs, can be performed with HRA. □

Section 5 shows accurate results obtained when solving the above algebraic problems using the bidiagonal factorization (1) and the algorithms presented in [14] and [15].

5. Numerical experiments

Let us suppose that A is an $(n+1)\times(n+1)$ nonsingular, TP matrix, whose bidiagonal decomposition (1) is represented by means of the matrix $BD(A)$ given in (4). If $BD(A)$ can be computed with HRA, then the Matlab functions `TNSingularValues`, `TNInverseExpand` and `TNSolve` of the library TNTool in [15] take as input argument $BD(A)$ and compute with HRA the singular values of A, its inverse matrix A^{-1} (using the algorithm presented in [18]) and the solution of systems of linear equations $Ax = b$, for vectors b whose entries have alternating signs. The function `TNProduct` is also available in the mentioned library. If the bidiagonal decomposition (1) of two nonsingular and TP matrices A and B can be computed with HRA, `TNProduct` computes with HRA the bidiagonal decomposition (1) of AB. The computational cost of the functions `TNSolve` and `TNInverseExpand` is $O(n^2)$ elementary operations, whereas the computational cost of the other mentioned functions is $O(n^3)$.

For the Bessel polynomials basis $(B_0,\dots,B_n)$, $n\in\mathbb{N}$, using Theorem 2 and function `TNProduct`, we have implemented a Matlab function that computes $BD(W)$ for its Wronskian matrix W. Furthermore, for the reverse Bessel polynomials $(R_0,\dots,R_n)$, $n\in\mathbb{N}$, considering Theorem 3 and the function `TNProduct` we have implemented a Matlab function that computes $BD(W)$ for its Wronskian matrix W.

Given $\alpha > -1$, for the generalized Laguerre polynomials basis $(L_0^{(\alpha)},\dots,L_n^{(\alpha)})$, considering Theorem 4 and the function `TNProduct`, we have also implemented a Matlab function, which computes $BD(W_J)$ for the matrix $W_J := JW$, obtained from its Wronskian matrix W at $x<0$ (see (22)). At last, for the polynomial basis $(\bar{L}_0^{(\alpha)},\dots,\bar{L}_n^{(\alpha)})$ defined in (27), using Theorem 6 and the function `TNProduct`, we have implemented a Matlab function for the computation of $BD(W)$ for its Wronskian matrix W at $x>0$.

In the numerical experimentation, we have considered Wronskian matrices corresponding to different $(n+1)$-dimensional bases proposed in this paper. The numerical results illustrate the accuracy of the computations for dimensions $n+1 = 10, 15, 20, 25$. The authors will provide upon request the software with the implementation of the above mentioned routines.

The 2-norm condition number of the considered Wronskian matrices has been obtained with the Mathematica command `Norm[A,2]`$\cdot$ `Norm[Inverse[A],2]` and is shown in Table 1. We can clearly observe that the condition numbers significantly increase with the dimension of the matrices. This explains that traditional methods do not obtain accurate solutions when solving the aforementioned algebraic problems. In contrast, the numerical results will illustrate the high accuracy obtained when using the bidiagonal decompositions deduced in this paper with the Matlab functions available in [15].

In our first numerical example we have computed the singular values of the considered matrices with the following algorithms:

- Matlab's function `TNSingularValues` taking as argument the matrix representation (4) of the corresponding deduced bidiagonal decomposition (1).

Table 1
From left to right, condition number of Wronskian matrices of the Bessel polynomials bases at $x_0 = 2$ and $x_0 = 50$, reverse Bessel polynomials bases at $x_0 = 0.3$ and $x_0 = 50$, generalized Laguerre polynomials at $x_0 = -5$ (with $\alpha = 2$) and, finally, polynomial basis defined in (27) at $x_0 = 2$ (with $\alpha = 0$).

$n+1$	$\kappa_2(W)$	$\kappa_2(W)$	$\kappa_2(W)$	$\kappa_2(W)$	$\kappa_2(W)$	$\kappa_2(W)$
10	4.2×10^{14}	3.4×10^{31}	2.5×10^{8}	2.5×10^{25}	7.3×10^{8}	1.1×10^{6}
15	3.4×10^{26}	3.5×10^{49}	1.7×10^{15}	1.2×10^{37}	1.0×10^{12}	9.6×10^{8}
20	3.5×10^{39}	4.7×10^{67}	1.1×10^{23}	1.6×10^{49}	1.1×10^{15}	8.7×10^{11}
25	2.6×10^{54}	6.4×10^{86}	2.8×10^{32}	8.6×10^{54}	1.0×10^{18}	8.0×10^{14}

Table 2
Relative errors when computing the lowest singular value of Wronskian matrices of Bessel polynomials bases at $x_0 = 2$ (left) and reverse Bessel polynomials bases at $x_0 = 0.3$ (right).

$n+1$	`svd`(W)	`TNSV`$(BD(W))$	`svd`(W)	`TNSV`$(BD(W))$
10	3.0×10^{-4}	2.1×10^{-16}	1.4×10^{-9}	3.9×10^{-15}
15	7.6×10^{-1}	5.7×10^{-16}	5.3×10^{-3}	2.4×10^{-15}
20	7.9	3.9×10^{-16}	3.7×10^{-1}	6.8×10^{-15}
25	8.0	1.6×10^{-16}	9.6×10^{-1}	5.9×10^{-15}

- Matlab's command svd.
- Mathematica's routine `Singularvalues` with a 100-digit arithmetic for computing the singular values of the considered Wronskian matrices.

The values provided by Mathematica have been considered as the exact solution of the algebraic problem and the relative error e of each approximation has been computed as $e := |a - \widehat{a}|/|a|$, where a denotes the singular value computed with Mathematica and $\widehat{a}$ the singular value computed with Matlab. Let us observe that, for the computation of the singular values of Wronskian matrices W of Laguerre bases, we have considered instead the TP matrices $W_J = JW$ whose singular values coincide with those of W (see Corollary 5).

In Tables 2 and 3, the relative errors of the approximation to the lowest singular value of the considered matrices are shown. We can observe that our methods provide very accurate results in contrast to the non accurate results provided by the Matlab commands `svd`.

On the other hand, in our second experiment, we have computed the inverse matrix of the considered Wronskian matrices with the following algorithms:

- Matlab's function `TNInverseExpand` with the corresponding matrix representation (4) of the bidiagonal decomposition (1) as argument.
- Matlab's routine `inv`.
- Mathematica's routine `Inverse` in 100-digit arithmetic for computing the inverse of the considered Wronskian matrices.

Table 3
Relative errors when computing the lowest singular value of Wronskian matrices of generalized Laguerre polynomials at $x_0 = -5$ with $\alpha = 2$ (left) and Wronskian matrices of the polynomials bases defined in (27) at $x_0 = 2$ with $\alpha = 0$ (right).

$n+1$	svd(W)	TNSV($BD(W_J)$)	svd(W)	TNSV($BD(W)$)
10	3.6×10^{-11}	2.2×10^{-16}	1.6×10^{-13}	8.3×10^{-16}
15	1.6×10^{-9}	1.2×10^{-15}	5.6×10^{-12}	1.3×10^{-17}
20	2.4×10^{-9}	4.7×10^{-15}	1.3×10^{-9}	3.0×10^{-15}
25	4.8×10^{-6}	2.6×10^{-15}	1.1×10^{-7}	1.4×10^{-15}

Table 4
Relative errors when computing the inverse of the Wronskian matrices of Bessel polynomials bases at $x_0 = 50$ (left) and reverse Bessel polynomials bases at $x_0 = 50$ (right).

$n+1$	inv(W)	TNIE($BD(W)$)	inv(W)	TNIE($BD(W)$)
10	2.0×10^{-14}	1.8×10^{-16}	6.1×10^{-15}	5.2×10^{-17}
15	3.7×10^{-12}	1.1×10^{-16}	6.6×10^{-11}	1.8×10^{-16}
20	3.5×10^{-9}	4.8×10^{-17}	1.0×10^{-7}	4.6×10^{-16}
25	2.4×10^{-6}	2.4×10^{-16}	5.0×10^{-5}	3.0×10^{-16}

To look over the errors we have compared both Matlab approximations with the inverse matrix A^{-1} computed by Mathematica using 100-digit arithmetic, taking into account the formula $e = \|A^{-1} - \widehat{A}^{-1}\|_2/\|A^{-1}\|_2$ for the corresponding relative error, where A^{-1} denotes the inverse matrix computed with Mathematica and $\widehat{A}^{-1}$ the inverse matrix computed with Matlab. For the computation of the inverse of the Wronskian matrices W of Laguerre bases, we have considered instead the TP matrices $W_J = JW$. Once W_J^{-1} is computed, the inverse of W can be accurately obtained, taking into account that

$$W^{-1} = W_J^{-1} J,$$

by means of a suitable change of sign of the accurate computed entries of W_J^{-1} (see Corollary 5).

The obtained relative errors are shown in Tables 4 and 5. Observe that the relative errors achieved through the bidiagonal decompositions obtained in this paper are much smaller than those obtained with the Matlab command `inv`.

At last, in our third experiment, given random nonnegative integer values d_i, $i = 1, \ldots, n+1$, we have computed the solutions of the linear systems $Wc = d$ where, in the case of Bessel polynomials bases, reverse Bessel polynomials bases and the polynomials bases defined in (27), $d = ((-1)^{i+1} d_i)^T_{1 \le i \le n+1}$ and, in the case of the generalized Laguerre bases, $d = (d_i)^T_{1 \le i \le n+1}$ (see Corollary 5).

We have computed the solution of these systems of linear equations associated to the considered Wronskian matrices with the next algorithms:

Table 5
Relative errors when computing the inverse of the Wronskian matrices of generalized Laguerre polynomials at $x_0 = -5$ with $\alpha = 2$ (left) and polynomials bases defined in (27) at $x_0 = 2$ with $\alpha = 0$ (right).

$n+1$	`inv`(W)	`TNIE`$(BD(W_J))$	`inv`(W)	`TNIE`$(BD(W))$
10	7.4×10^{-14}	1.8×10^{-16}	1.9×10^{-14}	5.7×10^{-17}
15	2.7×10^{-11}	2.1×10^{-16}	8.8×10^{-13}	2.9×10^{-16}
20	4.7×10^{-10}	4.8×10^{-15}	4.3×10^{-11}	3.6×10^{-15}
25	1.5×10^{-8}	1.6×10^{-15}	4.1×10^{-10}	1.6×10^{-15}

Table 6
Relative errors when solving $\mathbf{Wc} = \mathbf{d}$ with Wronskian matrices of Bessel polynomials bases at $x_0 = 50$ (left) and reverse Bessel polynomials bases at $x_0 = 50$ (right).

$n+1$	$W \setminus d$	`TNSolve`$(BD(W), d)$	$W \setminus d$	`TNSolve`$(BD(W), d)$
10	1.4×10^{-13}	2.8×10^{-17}	3.2×10^{-14}	2.8×10^{-16}
15	1.4×10^{-11}	3.5×10^{-16}	2.7×10^{-11}	1.3×10^{-16}
20	5.1×10^{-9}	3.1×10^{-16}	6.1×10^{-8}	3.7×10^{-16}
25	1.4×10^{-6}	3.4×10^{-16}	2.0×10^{-5}	2.5×10^{-16}

Table 7
Relative errors when solving $\mathbf{Wc} = \mathbf{d}$ with Wronskian matrices of generalized Laguerre polynomials at $x_0 = -5$ with $\alpha = 2$ (left) and polynomials bases defined in (27) at $x_0 = 2$ with $\alpha = 0$ (right).

$n+1$	$W \setminus d$	`TNSolve`$(BD(W_J), Jd)$	$W \setminus d$	`TNSolve`$(BD(W), d)$
10	6.7×10^{-14}	7.2×10^{-17}	2.4×10^{-14}	7.2×10^{-17}
15	3.1×10^{-11}	1.9×10^{-16}	6.5×10^{-13}	3.3×10^{-16}
20	1.9×10^{-10}	3.8×10^{-15}	1.6×10^{-11}	2.6×10^{-15}
25	1.3×10^{-8}	1.5×10^{-15}	1.8×10^{-10}	6.6×10^{-15}

- Matlab's function `TNSolve` by using the matrix representation (4) of the proposed bidiagonal decompositions (1).
- Matlab's command \.
- Mathematica's routine `LinearSolve` in 100-digit arithmetic.

The vector provided by Mathematica has been considered as the exact solution c. Then, we have computed in Mathematica the relative error of the computed approximation with Matlab $\widehat{c}$, taking into account the formula $e = \|c - \widehat{c}\|_2 / \|c\|_2$. When considering the Wronskian matrices W of Laguerre bases, we have considered instead the equivalent system $JWc = Jd$ (see Corollary 5).

In Tables 6 and 7, the relative errors when solving the aforementioned linear systems for different values of n are shown. Notice that the proposed methods preserve the accuracy, which does not considerably decrease with the dimension of the system in contrast with the results obtained with the Matlab command \.

Declaration of competing interest

This study does not have any conflicts to disclose.

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