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Explicit estimates for Comtet numbers of the first kind

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ABSTRACT

We give explicit upper and lower bounds for a large subset of Comtet numbers $s_{\boldsymbol{\alpha}}(n, m)$ of the first kind, including the r-Stirling numbers of the first kind, among others. In many occasions, such estimates are asymptotically sharp. The form of the bounds varies according to m lying in the central or non-central regions of $\{1, \ldots, n\}$. Depending on this fact, we use different probabilistic representations of $s_{\boldsymbol{\alpha}}(n, m)$ in terms of well known random variables.

1. Introduction

Given a sequence $\boldsymbol{\alpha} = (\alpha_j)_{j\geq 0}$ of real numbers, the Comtet numbers of the first kind associated to $\boldsymbol{\alpha}$, denoted by $s_{\boldsymbol{\alpha}}(n, m)$, are defined by

$$\prod_{j=0}^{n-1}(z - \alpha_j) = \sum_{m=0}^{n} s_{\boldsymbol{\alpha}}(n, m) z^m. \tag{1}$$

These numbers were introduced by Comtet [1] and subsequently generalized by El-Desouky and Cakić [2] and El-Desouky et al. [3], among others. Indeed, the authors in [3] consider a new family of Whitney numbers which includes, as particular cases, the r-Whitney numbers considered by Mező [4] and the r-Stirling numbers introduced by Broder [5] (different properties and applications of such numbers can be found in Komatsu and Mező [6], Bényi et al. [7], and the references therein). More specifically, the r-Stirling numbers of the first kind are given by

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = (-1)^{n-m} s_{\boldsymbol{\alpha}}(n, m), \tag{2}$$

when we choose in (1) $\alpha_j = 0$, $0 \leq j \leq r-1$, and $\alpha_j = j$, $j \geq r$ for some $r = 1, 2, \ldots$.

Since the pioneering work of Moser and Wyman [8], there are many papers in the literature devoted to obtain asymptotic expansions for the classical Stirling numbers of the first kind (see, for instance, Temme [9], Hwang [10], Tsylova [11], Chelluri et al. [12], Louchard [13], and the references therein), as well as explicit estimates (cf. Moser and Wyman [8] and Arratia and De Salvo [14]). Asymptotic formulas for r-Stirling numbers of the first kind can be found in Corcino et al. [15], and Corcino, C.B. and Corcino, R.B. [16].

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The aim of this paper is to give explicit estimates for the Comtet numbers of the first kind under the assumption that

$$\alpha_j = 0, \quad j = 0, 1, \dots, r-1, \qquad \alpha_j > 0, \quad j = r, r+1, \dots, \tag{3}$$

for some $r = 1, 2, \dots$. In such a case, we have from (1)

$$\prod_{j=0}^{n-1}(z - \alpha_{r+j}) = \sum_{m=0}^{n} s_{\boldsymbol{\alpha}}(n+r, m+r) z^m. \tag{4}$$

Three features should be emphasized. In first place, our results always provide explicit upper and lower bounds for the numbers $s_{\boldsymbol{\alpha}}(n, m)$ under assumption (3). In many occasions, such estimates are asymptotically sharp. In second place, the form of such estimates varies according to m lying in the central or non-central regions of $\{1, \dots, n\}$. We find this feature too in the aforementioned papers devoted to obtain estimates for the classical Stirling numbers of the first kind. And thirdly, we use a probabilistic approach based on different representations of $s_{\boldsymbol{\alpha}}(n, m)$ in terms of expectations involving well known random variables. In fact, we give various probabilistic representations of $s_{\boldsymbol{\alpha}}(n, m)$ whose usefulness depends on the value of m in $\{1, \dots, n\}$.

This paper is organized as follows. In Section 2, we give probabilistic representations of $s_{\boldsymbol{\alpha}}(n, m)$ involving sums of independent random variables having the Bernoulli, the uniform, and the exponential distributions. Such representations, which may have an independent interest, suggest the way of estimating the Comtet numbers of the first kind. The main results are stated in Section 3 and illustrated in Section 4, by considering a variety of examples including the r-Stirling and some special cases of the Jacobi–Stirling numbers of the first kind, among others. Some brief comments are also made if assumption (3) is dropped. Sections 5–7 are devoted to prove our main results. The proofs use different techniques, namely, binomial approximation, normal approximation, and moment estimates depending on the probabilistic representations given in Section 2.

2. Probabilistic representations

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout this paper, we assume that $n \in \mathbb{N}$ and $z \in \mathbb{C}$. Denote by $\boldsymbol{P}_n$ the set of finite sequences $\boldsymbol{p}_n = (p_j)_{0 \le j \le n-1}$ such that $0 \le p_j \le 1$, $0 \le j \le n-1$. Each probability p_j may depend on n, but this dependence is not explicitly written in order to avoid a cumbersome notation.

Associated to each $\boldsymbol{p}_n \in \boldsymbol{P}_n$, we consider a finite sequence $(X_j)_{0 \le j \le n-1}$ of independent random variables such that X_j has the Bernoulli distribution with success probability p_j, i.e.,

$$P(X_j = 1) = p_j = 1 - P(X_j = 0). \tag{5}$$

The probability distribution of the random variable

$$W_n = X_0 + \cdots + X_{n-1} \tag{6}$$

is rather involved and known in the literature as the Poisson-binomial distribution (see, for instance, Ehm [17] and Roos [18]). The mean and the variance of W_n are respectively given by

$$\mu_n = \mathbb{E}W_n = \sum_{j=0}^{n-1} p_j, \quad \sigma_n^2 = Var(W_n) = \mathbb{E}(W_n - \mu_n)^2 = \sum_{j=0}^{n-1} p_j(1 - p_j), \tag{7}$$

where $\mathbb{E}$ stands for mathematical expectation. If $p_j = p$, $0 \le j \le n-1$, the probability distribution of $S_n(p) := X_0 + \cdots + X_{n-1}$ is the binomial distribution with parameters n and p, that is,

$$P(S_n(p) = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \dots, n. \tag{8}$$

In this case, we obviously have

$$\mathbb{E}S_n(p) = np, \qquad Var(S_n(p)) = \mathbb{E}(S_n(p) - np)^2 = np(1-p). \tag{9}$$

Given $\boldsymbol{p}_n \in \boldsymbol{P}_n$ and $t > 0$, we consider the success probabilities

$$p_j(t) = \frac{p_j t}{p_j t + 1}, \quad j = 0, 1, \dots, n-1, \tag{10}$$

together with their associated random variable $W_n(t)$ as in (6). The mean and the variance of $W_n(t)$ are respectively denoted by

$$\mu_n(t) = \sum_{j=0}^{n-1} p_j(t), \qquad \sigma_n^2(t) = \sum_{j=0}^{n-1} p_j(t)(1 - p_j(t)). \tag{11}$$

In the following result, we give two different probabilistic representations of the Comtet numbers $s_{\boldsymbol{\alpha}}(n+r, m+r)$ in terms of W_n and $W_n(t)$. To this end, denote

$$\beta = \min_{0\le j\le n-1} \alpha_{r+j}, \qquad Q = \prod_{j=0}^{n-1} \alpha_{r+j}, \qquad Q(t) = \prod_{j=0}^{n-1} (\alpha_{r+j} + \beta t). \tag{12}$$

To simplify the notation, the dependence on n of the three parameters in (12) is dropped. However, this dependence must be taken into account when considering their asymptotic behavior as $n \to \infty$ (see the examples in Section 4).

Proposition 2.1. *Let $t > 0$ and let $\boldsymbol{p}_n \in \boldsymbol{P}_n$ be given by*

$$p_j = \frac{\beta}{\alpha_{r+j}}, \quad j = 0, 1, \ldots, n-1. \tag{13}$$

For any $m = 0, 1, \ldots, n$, we have

$$\begin{aligned}(-1)^{n-m} s_{\boldsymbol{\alpha}}(n+r, m+r) &= \frac{Q}{\beta^m} \mathbb{E}\binom{W_n}{m} \\ &= \frac{Q(t)}{(\beta t)^m} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{E} e^{i\theta(W_n(t)-m)}\, d\theta.\end{aligned} \tag{14}$$

Proof. From (5) and (6), we have

$$\mathbb{E}(1-z)^{W_n} = \prod_{j=0}^{n-1} \mathbb{E}(1-z)^{X_j} = \prod_{j=0}^{n-1} (1 - p_j z). \tag{15}$$

This, together with (4), (12), and (13), implies that

$$\begin{aligned}\sum_{m=0}^{n} s_{\boldsymbol{\alpha}}(n+r, m+r)(\beta z)^m &= \prod_{j=0}^{n-1} (\beta z - \alpha_{r+j}) = (-1)^n Q \prod_{j=0}^{n-1} (1 - p_j z) \\ &= (-1)^n Q\, \mathbb{E}(1-z)^{W_n} = (-1)^n Q \sum_{m=0}^{n} \mathbb{E}\binom{W_n}{m} (-z)^m,\end{aligned}$$

which shows the first equality in (14). On the other hand, we claim that

$$\mathbb{E}(1+tz)^{W_n} = \mathbb{E}(1+t)^{W_n} \mathbb{E} z^{W_n(t)}, \qquad t > 0. \tag{16}$$

In fact, it follows from (10) and (15) that

$$\begin{aligned}\mathbb{E}(1+tz)^{W_n} &= \prod_{j=0}^{n-1} (1 + p_j t z) = \prod_{j=0}^{n-1} (1 + p_j t) \prod_{j=0}^{n-1} \left(1 - \frac{p_j t}{p_j t + 1}(1-z)\right) \\ &= \mathbb{E}(1+t)^{W_n} \mathbb{E}(1-(1-z))^{W_n(t)},\end{aligned}$$

thus showing claim (16). It turns out that

$$\begin{aligned}\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{E}(1 + te^{i\theta})^{W_n} e^{-im\theta}\, d\theta &= \sum_{j=0}^{n} t^j \mathbb{E}\binom{W_n}{j} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta(j-m)}\, d\theta \\ &= t^m \mathbb{E}\binom{W_n}{m}.\end{aligned} \tag{17}$$

By (15) and (16), the first integral in (17) can be written as

$$\mathbb{E}(1+t)^{W_n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{E} e^{i\theta(W_n(t)-m)}\, d\theta = \prod_{j=0}^{n-1} (1 + p_j t) \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{E} e^{i\theta(W_n(t)-m)}\, d\theta.$$

Thus, the second equality in (14) follows from the first one and (12) and (13). □

The generating functions of the r-Stirling numbers of the first kind are given by (cf. Broder [5, Th. 15])

$$\sum_{n=m}^{\infty} \begin{bmatrix} n+r \\ m+r \end{bmatrix}_r \frac{z^n}{n!} = \frac{1}{m!} \left(\frac{1}{1-z}\right)^r (-\log(1-z))^m, \quad m \in \mathbb{N}_0, \tag{18}$$

for $|z|<1$. In order to give probabilistic representations for such numbers, let $(U_j)_{j\geq 1}$, $(Y_j)_{j\geq 1}$, and $(\widetilde{Y}_j)_{j\geq 1}$ be three sequences of independent identically distributed random variables such that U_1 has the uniform distribution on $[0,1]$, and Y_1 and $\widetilde{Y}_1$ have the exponential density $\rho(\theta)=e^{-\theta},\ \theta\geq 0$. We assume that the three sequences are mutually independent and denote

$$T_n=U_1Y_1+\cdots+U_nY_n,\qquad \widetilde{T}_n=\widetilde{Y}_1+\cdots+\widetilde{Y}_n,\quad (T_0=\widetilde{T}_0=0). \tag{19}$$

Proposition 2.2. *For any $m=0,1,\ldots,n$, we have*

$$\begin{bmatrix} n+r \\ m+r \end{bmatrix}_r=\binom{n}{m}\mathbb{E}\left(\widetilde{T}_r+T_m\right)^{n-m}.$$

Proof. Let $|z|<1$. Observe that

$$\mathbb{E}e^{zU_1T_1}=\mathbb{E}\left(\frac{1}{1-zU_1}\right)=-\frac{\log(1-z)}{z}.$$

Hence, using the independence of the random variables involved, we have from (19)

$$\mathbb{E}e^{z(\widetilde{T}_r+T_m)}=\left(\mathbb{E}e^{z\widetilde{Y}_1}\right)^r\left(\mathbb{E}e^{zU_1T_1}\right)^m=\frac{1}{(1-z)^r}\left(\frac{-\log(1-z)}{z}\right)^m. \tag{20}$$

We can therefore rewrite the generating function in (18) as

$$\sum_{n=m}^{\infty}\begin{bmatrix} n+r \\ m+r \end{bmatrix}_r\frac{z^n}{n!}=\frac{z^m}{m!}\mathbb{E}e^{z(\widetilde{T}_r+T_m)}$$
$$=\frac{z^m}{m!}\sum_{k=0}^{\infty}\mathbb{E}\left(\widetilde{T}_r+T_m\right)^k\frac{z^k}{k!}=\sum_{n=m}^{\infty}\binom{n}{m}\mathbb{E}\left(\widetilde{T}_r+T_m\right)^{n-m}\frac{z^n}{n!}.$$

This completes the proof. □

An analogous probabilistic representation for the classical Stirling numbers of the first kind was obtained by Sun and Wang [19] (see also [20]).

3. Main results

The probabilistic representations given in Section 2 suggest the way of estimating the Comtet numbers $s_{\boldsymbol{\alpha}}(n+r,m+r)$. Assume from now on that $\boldsymbol{p}_n\in\boldsymbol{P}_n$ is given by (13). It is well known (see, for instance, Ehm [17] and Roos [18]) that W_n can be approximated by $S_n(p)$, whenever both random variables have similar means and variances. Under such circumstances, we can approximate the binomial moments of W_n in formula (14) by the corresponding binomial moments of $S_n(p)$, which are easy to compute, since

$$\mathbb{E}\binom{S_n(p)}{m}=\binom{n}{m}p^m,\quad m=0,1,\ldots,n, \tag{21}$$

as follows from (8). More specifically, choose p in such a way that

$$np=\mu_n, \tag{22}$$

so that the random variables W_n and $S_n(p)$ have the same mean, as follows from (7) and (9). Denote

$$\gamma_n^2=\sum_{j=0}^{n-1}(p_j-p)^2. \tag{23}$$

It can be checked that

$$Var(S_n(p))-Var(W_n)=np(1-p)-\sum_{j=0}^{n-1}p_j(1-p_j)=\gamma_n^2. \tag{24}$$

With the preceding notations, we give our first main estimate.

Theorem 3.1. *Let $\boldsymbol{p}_n$ be as in (13) and assume (22). For any $m=0,\ldots,n$, with*

$$m\leq C\frac{\mu_n}{\gamma_n}, \tag{25}$$

for some positive constant C, we have

$$\left|\frac{(-1)^{n-m}\beta^m}{Q}s_\alpha(n+r,m+r)-\binom{n}{m}p^m\left(1-\frac{n}{2(n-1)}\frac{m(m-1)\gamma_n^2}{\mu_n^2}\right)\right|$$
$$\leq K_C\binom{n}{m}p^m\left(\frac{m\gamma_n}{\mu_n}\right)^3,$$

where

$$K_C=\frac{1}{C^3}\sum_{k=3}^{\infty}\left(\frac{Ce^{1/2}}{\sqrt{k}}\right)^k<\infty. \tag{26}$$

The expectation function $\mu_n(t)$ defined in (11) is strictly increasing and satisfies $\mu_n(0)=0$ and $\mu_n(\infty)=n$. Denote by $\tau=\tau(m,n)>0$ the unique solution of the equation

$$\mu_n(\tau)=m,\qquad m=1,\dots,n-1. \tag{27}$$

Theorem 3.2. *Let $\tau=\tau(m,n)>0$ be as in (27), $m=1,\dots,n-1$. Then,*

$$\left|\frac{2\pi(-1)^{n-m}(\beta\tau)^m}{Q(\tau)}s_\alpha(n+r,m+r)-\frac{1}{\sqrt{2\pi}\sigma_n(\tau)}\right|$$
$$\leq\frac{2}{\sigma_n^3(\tau)\sqrt{2\pi}}+\frac{3e^{-2\sigma_n^2(\tau)\sin^2(1/4)}}{\pi}+\frac{2e^{-\sigma_n^2(\tau)/8}}{\pi\sigma_n^2(\tau)}.$$

Theorems 3.1 and 3.2 always give us upper and lower bounds for $s_\alpha(n+r,m+r)$ for appropriate values of m. However, their accuracy depends on the range of m. Indeed, Theorem 3.1 holds for $m\leq C\mu_n/\gamma_n$, but it is asymptotically sharp if $m=o(\mu_n/\gamma_n)$. On the other hand, Theorem 3.2 is asymptotically sharp if $\sigma_n(\tau)\to\infty$, as $n\to\infty$. These ideas are illustrated in the following section by considering various different examples.

From a technical point of view, we mention that Eq. (27) extends to the Comtet numbers of the first kind the approach used by Moser and Wyman [8], Temme [9], Chelluri et al. [12], and Louchard [13] to deal with the classical Stirling numbers of the first kind. A similar approach to the r-Stirling numbers of the first kind can be found in Corcino et al. [15] and Corcino, C.B. and Corcino R.B. [16].

Let $\widetilde{T}_r$ and T_m be the random variables defined in (19). Note that

$$\nu_m:=\mathbb{E}\left(\widetilde{T}_r+T_m\right)=r\mathbb{E}\widetilde{Y}_1+m\mathbb{E}U_1Y_1=r+\frac{m}{2}, \tag{28}$$

and

$$\tau_m^2:=Var\left(\widetilde{T}_r+T_m\right)=rVar\left(\widetilde{Y}_1\right)+mVar(U_1Y_1)=r+\frac{5}{12}m. \tag{29}$$

With these notations, we state the following estimate for the r-Stirling numbers of the first kind.

Theorem 3.3. *Let $m\in\mathbb{N}$ with $n-m\geq 2$. If*

$$\frac{(n-m)}{\sqrt{m}}\leq\frac{1}{2\sqrt{2}}, \tag{30}$$

then

$$\left|\binom{n}{m}^{-1}\begin{bmatrix}n+r\\m+r\end{bmatrix}_r-\nu_m^{n-m}\left(1+\binom{n-m}{2}\left(\frac{\tau_m}{\nu_m}\right)^2\right)\right|$$
$$\leq 2^{r+11/2}e^{2/5-r/2}\nu_m^{n-m}\left(\frac{n-m}{\sqrt{m}}\right)^3.$$

Using Stein–Chen Poisson approximation, Arratia and DeSalvo [14] obtained sharp upper and lower bounds for the classical Stirling numbers of the first kind in the same range for m as in Theorem 3.3. In view of (28) and (29), this result gives us asymptotically sharp estimates for the r-Stirling numbers of the first kind in the range $n-m=o(\sqrt{m})$ or, equivalently, $m=n-o(\sqrt{n})$.

4. Illustrative examples

Given two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ of real numbers, we set $x_n\sim y_n$ if $x_n/y_n\to 1$, as $n\to\infty$. The generalized harmonic numbers and the Riemann zeta function are respectively defined by

$$H_n(a)=\sum_{j=0}^{n-1}\frac{1}{(j+1)^a},\quad a\in\mathbb{R},\qquad\zeta(r)=\sum_{j=0}^{\infty}\frac{1}{(j+1)^r},\quad r>1.$$

Table 1
Asymptotic behavior of μ_n/γ_n, according to the values of $a \neq 0$.

$a > 1$	$a = 1$	$\frac{1}{2} < a < 1$	$a = \frac{1}{2}$	$0 < a < \frac{1}{2}$	$a < 0$
$\frac{\zeta(a)}{\sqrt{\zeta(2a)}}$	$\frac{H_n}{\sqrt{\zeta(2)}}$	$\frac{n^{1-a}}{(1-a)\sqrt{\zeta(2a)}}$	$\frac{2\sqrt{n}}{\sqrt{H_n}}$	$\frac{\sqrt{1-2a}}{a}\sqrt{n}$	$\frac{\sqrt{1-2a}}{-a}\sqrt{n}$

We simply denote by $H_n = H_n(1)$. Observe that

$$H_n(a) \sim \frac{n^{1-a}}{1-a}, \quad a < 1. \tag{31}$$

To illustrate Theorems 3.1 and 3.2, consider the case in which

$$\alpha_{r+j} = (j+1)^a, \quad j = 0, 1, \ldots, n-1, \quad a \in \mathbb{R}. \tag{32}$$

If $r = 1$ and $a = 2$, then $s_{\boldsymbol{\alpha}}(n+1, m+1)$ are a particular case of the Jacobi–Stirling numbers of the first kind introduced by Everitt et al. [21].

The case $a = 0$ is trivial, since formula (4) directly gives us

$$s_{\boldsymbol{\alpha}}(n+r, m+r) = (-1)^{n-m}\binom{n}{m}, \quad m = 0, 1, \ldots, n.$$

The asymptotic accuracy of Theorems 3.1 and 3.2 depends on the asymptotic behavior of the quantities μ_n/γ_n and $\sigma_n(\tau)$, respectively. To describe both quantities, we distinguish the following two cases.

Case 1: $a > 0$. From (12), (13), and (32), we have in this case

$$\beta = 1, \quad Q = (n!)^a, \quad p_j = \frac{1}{(j+1)^a}, \quad j = 0, 1, \ldots, n-1. \tag{33}$$

By (7), (22), (23), and (33), we also have

$$\mu_n = H_n(a), \quad \gamma_n^2 = H_n(2a) - \frac{H_n^2(a)}{n}. \tag{34}$$

For large values of n, the quotient μ_n/γ_n has the order of magnitude given in Table 1, depending on the values of a. Actually, the resulting expressions in this table follow from (31) and (34).

Case 2: $a < 0$. Set $b = -a$. As in Case 1, we have

$$\beta = \frac{1}{n^b}, \quad Q = \frac{1}{(n!)^b}, \quad p_j = \left(\frac{j+1}{n}\right)^b, \quad j = 0, 1, \ldots, n-1. \tag{35}$$

Using (31), we have from (7) and (23)

$$\frac{\mu_n}{\gamma_n} \sim \frac{\sqrt{1+2b}}{b}\sqrt{n},$$

which is the last column of Table 1.

To describe the asymptotic behavior of $\sigma_n(\tau)$, we distinguish the cases $a > 0$ and $a < 0$. In this respect, the following two auxiliary results will be useful.

Lemma 4.1. *Let $t > 0$ and $0 < q < 1/2$. Assume that $p_j = 1/(j+1)^a$, $j = 0, 1, \ldots, n-1$, for some $a > 0$. Then,*

$$\sigma_n^2(t) \geq q(1-q)\left\lfloor \left(\left(\frac{1-q}{q}\right)^{1/a} - \left(\frac{q}{1-q}\right)^{1/a}\right) t^{1/a} \right\rfloor,$$

where $\lfloor x \rfloor$ stands for the integer part of x.

Proof. Fix $0 < q < 1/2$. Denote by $|A|$ the cardinality of $A \subseteq \mathbb{N}_0$. By (11), it is clear that

$$\sigma_n^2(t) \geq q(1-q)|A_q| \tag{36}$$

where

$$\begin{aligned} A_q &= \left\{0 \leq j \leq n-1 : \; q \leq p_j(t) \leq 1-q\right\} \\ &= \left\{0 \leq j \leq n-1 : \; \frac{q}{(1-q)t} \leq p_j \leq \frac{1-q}{qt}\right\}. \end{aligned}$$

as follows from (10). Since $p_j = 1/(j+1)^a$, we see that

$$A_q = \left\{ 0 \le j \le n-1 : \left(\frac{q}{1-q} \right)^{1/a} t^{1/a} \le j+1 \le \left(\frac{1-q}{q} \right)^{1/a} t^{1/a} \right\}. \tag{37}$$

On the other hand, it is easily checked that $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x+y \rfloor$, $x, y \in \mathbb{R}$. This implies that

$$|[r,s] \cap \mathbb{N}_0| \ge \lfloor s \rfloor - \lfloor r \rfloor \ge \lfloor s-r \rfloor, \quad 0 \le r \le s.$$

This, together with (36) and (37), shows the result. □

Lemma 4.2. *Let $(f_n(y))_{n\ge1}$ be a sequence of nonnegative, nonincreasing, and uniformly bounded functions defined on $[0,\infty)$. Let $s_n = f_n(1) + \cdots + f_n(n)$. If $s_n \to \infty$, as $n \to \infty$, then*

$$s_n \sim \int_0^n f_n(y)\, dy.$$

Proof. Since $f_n(y)$ is nonnegative and nonincreasing, we have

$$\int_0^n f_n(y)\,dy \ge s_n \ge \int_1^{n+1} f_n(y)\,dy$$
$$\ge \int_0^n f_n(y)\,dy - f_n(0) \ge \int_0^n f_n(y)\,dy - \sup_{n\ge1} f_n(0).$$

This shows the result, since the sequence $(f_n(0))_{n\ge1}$ is bounded. □

Lemma 4.3. *Let $t > 0$. Assume that $p_j = (j+1)^b/n^b$, $j = 0, 1, \ldots, n-1$, for some $b > 0$. Then,*

Proof. Applying Lemma 4.2 with $f_n(y) = n^b/(ty^b + n^b)$, $y \ge 0$, and recalling (10) and (11), we have

$$\begin{aligned} \mu_n(t) &= \sum_{j=0}^{n-1} \frac{(j+1)^b t}{(j+1)^b t + n^b} = n - \sum_{j=0}^{n-1} \frac{n^b}{(j+1)^b t + n^b} \\ &\sim n - n^b \int_0^n \frac{dy}{ty^b + n^b} = n \left(1 - \int_0^1 \frac{dx}{tx^b + 1} \right). \end{aligned} \tag{38}$$

Applying again Lemma 4.2 with $f_n(y) = n^{2b}/(ty^b + n^b)^2$, $y \ge 0$, we have from (11) and (38)

$$\begin{aligned} \sigma_n^2(t) &= \sum_{j=0}^{n-1} \frac{p_j t}{(p_j t + 1)^2} = n^b \sum_{j=0}^{n-1} \frac{(j+1)^b t}{((j+1)^b t + n^b)^2} \\ &= \sum_{j=0}^{n-1} \frac{n^b}{(j+1)^b t + n^b} - \sum_{j=0}^{n-1} \frac{n^{2b}}{((j+1)^b t + n^b)^2} \\ &\sim n - \mu_n(t) - n^{2b} \int_0^n \frac{dy}{(ty^b + n^b)^2} = n - \mu_n(t) - n \int_0^1 \frac{dx}{(tx^b+1)^2}. \end{aligned} \tag{39}$$

On the other hand, using integration by parts, we get

$$\begin{aligned} \int_0^1 \frac{dx}{tx^b + 1} &= \frac{1}{t+1} + tb \int_0^1 \frac{x^b}{(tx^b+1)^2}\,dx \\ &= \frac{1}{t+1} + b \int_0^1 \frac{dx}{tx^b + 1} - b \int_0^1 \frac{dx}{(tx^b+1)^2}. \end{aligned} \tag{40}$$

The result follows from (38)–(40) and some simple computations. □

Having in mind Table 1 and Lemmas 4.1 and 4.3, we discuss how to apply Theorems 3.1 and 3.2 when $n \to \infty$. In this regard, recall that if $m \to \infty$, then $\tau \to \infty$, as follows from (27). Also, we denote by C an absolute positive constant whose value may change from line to line.

In first place, suppose that $a = 1$. If

$$m = o\left(\frac{\mu_n}{\gamma_n} \right) = o(H_n) = o(\log n), \tag{41}$$

then we use Theorem 3.1 to obtain sharp estimates for

$$\frac{(-1)^{n-m}\beta^m}{Q} s_{\boldsymbol{\alpha}}(n+r,m+r) = \frac{(-1)^{n-m}}{n!} s_{\boldsymbol{\alpha}}(n+r,m+r),$$

where the equality follows from (33). If, on the contrary, $m \to \infty$, but m does not satisfy (41), then we use Theorem 3.2 to estimate

$$\frac{2\pi(-1)^{n-m}(\beta\tau)^m}{Q(\tau)} s_{\boldsymbol{\alpha}}(n+r,m+r) = \frac{2\pi(-1)^{n-m}\tau^m}{\prod_{j=0}^{n-1}(j+1+\tau)} s_{\boldsymbol{\alpha}}(n+r,m+r),$$

where the equality follows from (12), (32), and (33). This estimate is again sharp since, by virtue of Lemma 4.1, $\sigma_n^2(\tau) \geq C\tau$.

If $0 < a < 1$, the discussion is similar and we omit it. If $a < 0$, we use Theorem 3.1 to obtain sharp estimates, whenever

$$m = o(\sqrt{n}). \tag{42}$$

If $m \to \infty$, but m does not satisfy (42), we have from (27) and Lemma 4.3

$$\sigma_n^2(\tau) \sim -\frac{1}{a}\left(\frac{\tau}{\tau+1}n - m\right).$$

Hence, Theorem 3.2 gives us sharp estimates provided that $m \leq rn$, for some $0 < r < 1$. If, on the contrary, $m \sim n$, then Theorem 3.2 only gives us upper and lower bounds.

Finally, for $a > 1$, we use Theorem 3.1 to obtain upper and lower bounds when $0 \leq m \leq C$, whereas if $m \to \infty$, we apply Theorem 3.2 to get sharp estimates, since $\sigma_n^2(\tau) \geq C\tau^{1/a}$, as follows from Lemma 4.1.

To conclude this section, let us say some words if assumption (3) is dropped. If the sequence $(\alpha_j)_{j\geq 0}$ satisfies $\alpha_j = 0$, $0 \leq j \leq r-1$, and $\alpha_j < 0$, $j \geq r$, for some $r \in \mathbb{N}$, we can apply the results in Section 3 to the sequence $-\boldsymbol{\alpha} = (-\alpha_j)_{j\geq 0}$, by taking into account that

$$s_{\boldsymbol{\alpha}}(n+r,m+r) = (-1)^{n-m} s_{-\boldsymbol{\alpha}}(n+r,m+r),$$

as follows from (4). However, for an arbitrary sequence $(\alpha_j)_{j\geq 0}$, we do not know how to obtain estimates for $s_{\boldsymbol{\alpha}}(n,m)$ being sharp for any $m = 0, 1, \ldots, n$, particularly when the numbers α_j are almost symmetric around the origin. In fact, if $\alpha_j = (-1)^{j-1}\alpha$, $j \in \mathbb{N}$, for some $\alpha \in \mathbb{R}$, then Eq. (4) becomes

$$\sum_{m=0}^{2n} s_{\boldsymbol{\alpha}}(2n+1,m+1)z^m = (z-\alpha)^n(z+\alpha)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-m}\alpha^{2(n-k)}z^{2k},$$

thus implying that the even coefficients have large oscillations, whereas the odd coefficients are zero.

5. Binomial approximation

Having in mind the first probabilistic representation in (14), we will approximate the binomial moments of W_n by those of the random variable $S_n(p)$ having the same mean. To this end, recall that the usual kth forward difference of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$\Delta^k f(x) = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}f(x+j), \qquad k \in \mathbb{N}_0, \qquad x \in \mathbb{R}.$$

We start with the following identity for any function $\phi : \{0, 1, \ldots, n\} \to \mathbb{R}$

$$\mathbb{E}\phi(W_n) = \sum_{k=0}^{n} a_k(n)\mathbb{E}\Delta^k\phi(S_{n-k}(p)), \tag{43}$$

where p is defined in (22), $a_0(n) = 1$ and

$$a_k(n) = \sum_{0\leq i_1<\cdots<i_k\leq n-1}(p_{i_1}-p)\cdots(p_{i_k}-p), \qquad k = 1, \ldots, n.$$

Formula (43) was shown by Roos [18] for $\phi = 1_{\{m\}}$, $m = 0, \ldots, n$, where 1_A stands for the indicator function of the set A, and can be easily extended for any function ϕ, since

$$\phi = \sum_{m=0}^{n}\phi(m)1_{\{m\}}.$$

An alternative proof of (43) can be derived from [22, Corollary 4.1]. It is easily checked that

$$a_1(n) = 0, \qquad a_2(n) = -\gamma_n^2/2, \tag{44}$$

where γ_n^2 is defined in (23). On the other hand,

$$|a_k(n)| \le \left(\frac{\gamma_n}{\sqrt{k}}\right)^k \left(\frac{n}{n-k}\right)^{(n-k)/2} \le \left(\frac{e^{1/2}\gamma_n}{\sqrt{k}}\right)^k, \qquad k = 3, \dots, n. \tag{45}$$

The first inequality in (45) was shown by Roos [18, formula (30)], whereas the second one readily follows from the inequality $1 + x \le e^x$, $x \ge 0$.

We will need the following special case of identity (43).

Lemma 5.1. *For any $m = 0, 1, \dots, n$, we have*

$$\mathbb{E}\binom{W_n}{m} = \sum_{k=0}^{m} a_k(n)\binom{n-k}{m-k}p^{m-k}.$$

Proof. Consider the function $\phi(l) = (l)_m/m!$, $l = 0, 1, \dots, n$. It can be shown by induction on k that

$$\Delta^k\phi(l) = \binom{l}{m-k}, \qquad l = 0, 1, \dots, n, \qquad k = 0, 1, \dots, m. \tag{46}$$

In (46), we replace l by the random variable $S_{n-k}(p)$ and then take expectations to obtain

$$\mathbb{E}\Delta^k\phi(S_{n-k}(p)) = \mathbb{E}\binom{S_{n-k}(p)}{m-k} = \binom{n-k}{m-k}p^{m-k}, \qquad k = 0, 1, \dots, m.$$

where in the last equality we have used (21). Therefore, the result follows from (43). □

Proof of Theorem 3.1. In view of (22), (44), and Lemma 5.1, we can write

$$\mathbb{E}\binom{W_n}{m} = \binom{n}{m}p^m\left(1 - \frac{n}{2(n-1)}\frac{m(m-1)\gamma_n^2}{\mu_n^2} + \sum_{k=3}^{m} a_k(n)\frac{(m)_k}{(n)_k}\frac{n^k}{\mu_n^k}\right). \tag{47}$$

Observe that $(m-j)/(n-j) \le m/n$, $j = 0, \dots, m$. By (25) and (45), the modulus of the sum in (47) is bounded above by

$$\sum_{k=3}^{m}\left(\frac{Ce^{1/2}}{\sqrt{k}}\right)^k\left(\frac{m\gamma_n}{C\mu_n}\right)^k \le \left(\frac{m\gamma_n}{C\mu_n}\right)^3\sum_{k=3}^{m}\left(\frac{Ce^{1/2}}{\sqrt{k}}\right)^k \le K_C\left(\frac{m\gamma_n}{\mu_n}\right)^3,$$

where K_C is defined in (26). This, (47), and the first equality in (14) show the result. □

6. Normal approximation

The proof of Theorem 3.2 will be based on the second representation of $s_\alpha(n+r, m+r)$ given in (14) and some auxiliary results. In the following lemma, we will use the inequality

$$|\log(1+z) - z| \le |z|^2, \qquad |z| < 1/2. \tag{48}$$

Lemma 6.1. *Let X be a random variable having the Bernoulli distribution with success probability p. If $-\pi \le \theta \le \pi$, then*

(a) $\left|\mathbb{E}e^{i\theta(X-p)}\right| \le e^{-2p(1-p)\sin^2(\theta/2)}$.

(b) If $|\theta| < 2$, then

$$\mathbb{E}e^{i\theta(X-p)} = \exp\left(-\frac{p(1-p)}{2}\theta^2\left(1 + \frac{(1-2p)i}{3}\theta + \frac{\theta^2}{4}t_p(\theta)\right)\right),$$

where

$$|t_p(\theta)| < 1. \tag{49}$$

Proof. (a) By (5), we have

$$\mathbb{E}e^{i\theta(X-p)} = e^{-ip\theta}(pe^{i\theta} + 1 - p),$$

thus implying that

$$\left|\mathbb{E}e^{i\theta(X-p)}\right|^2 = 1 - 4p(1-p)\sin^2(\theta/2).$$

This, together with the inequality $1 - x \le e^{-x}$, $0 \le x \le 1$, shows part (a).

(b) Observe that

$$\mathbb{E}(X-p)^k = p(1-p)((1-p)^{k-1} + (-1)^k p^{k-1}), \qquad k \in \mathbb{N}. \tag{50}$$

By Taylor's formula, we have

$$\begin{aligned} z_p(\theta) &:= \mathbb{E}e^{i\theta(X-p)} - 1 \\ &= \sum_{k=2}^{2r-1} \frac{(i\theta)^k}{k!}\mathbb{E}(X-p)^k + \frac{(i\theta)^{2r}}{(2r)!}\mathbb{E}(X-p)^{2r} e^{i\zeta(\theta)(X-p)}, \end{aligned} \tag{51}$$

where $\zeta(\theta)$ is a point between 0 and θ and $r \in \mathbb{N}$.

From (50) and (51) with $r = 2$, we see that

$$\begin{aligned} &\log(1 + z_p(\theta)) = z_p(\theta) + \log(1 + z_p(\theta)) - z_p(\theta) \\ &= -\frac{p(1-p)}{2}\theta^2 \left(1 + \frac{(1-2p)i}{3}\theta + \frac{\theta^2}{4} t_p(\theta)\right), \end{aligned} \tag{52}$$

where

$$t_p(\theta) = -\frac{\mathbb{E}(X-p)^4 e^{i\zeta(\theta)(X-p)}}{3p(1-p)} - \frac{8}{p(1-p)\theta^4}(\log(1 + z_p(\theta)) - z_p(\theta)). \tag{53}$$

By (51) and (52), part (b) will follow as soon as we show (49). To this end, the modulus of the first term on the right-hand side in (53) is bounded above by

$$\frac{(1-p)^3 + p^3}{3} \le \frac{1}{3}, \tag{54}$$

as follows from (50). On the other hand, formula (51) with $r = 1$ gives us

$$|z_p(\theta)| \le \frac{p(1-p)\theta^2}{2} \le \frac{\theta^2}{8} < \frac{1}{2}. \tag{55}$$

Therefore, by inequality (48), the modulus of the second term on the right-hand side in (53) is bounded by

$$\frac{8}{p(1-p)\theta^4}|z_p(\theta)|^2 \le 2p(1-p) \le \frac{1}{2}. \tag{56}$$

This, together with (54), shows (49) and completes the proof. □

Let Z be a random variable having the standard normal density

$$\rho(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \qquad u \in \mathbb{R}. \tag{57}$$

It is well known that

$$\mathbb{E}Z^{2m} = \frac{(2\,m)!}{m!2^m}, \qquad m \in \mathbb{N}_0. \tag{58}$$

In the following crucial lemma, we will use the inequality

$$|e^z - 1 - z| \le \frac{|z|^2}{2} e^{|z|}. \tag{59}$$

Lemma 6.2. *Let W_n be as in (6), whose mean μ_n and variance σ_n^2 are given in (7). Then,*

$$\left| \int_{-\pi}^{\pi} \mathbb{E}e^{i\theta(W_n-\mu_n)}\,d\theta - \frac{\sqrt{2\pi}}{\sigma_n} \right| \le \frac{2\sqrt{2\pi}}{\sigma_n^3} + 6e^{-2\sigma_n^2 \sin^2(1/4)} + \frac{4e^{-\sigma_n^2/8}}{\sigma_n^2}.$$

Proof. As in (15), we have

$$\mathbb{E}e^{i\theta(W_n-\mu_n)} = \prod_{j=1}^{n} \mathbb{E}e^{i\theta(X_j-p_j)}. \tag{60}$$

This implies, by virtue of Lemma 6.1(a), that

$$\left| \int_{1/2 \le |\theta| \le \pi} \mathbb{E} e^{i\theta(W_n - \mu_n)} \, d\theta \right| \le 2 \int_{1/2}^{\pi} e^{-2\sigma_n^2 \sin^2(\theta/2)} \, d\theta \le 6 e^{-2\sigma_n^2 \sin^2(1/4)}. \tag{61}$$

In the rest of the proof, we assume that $|\theta| < 1/2$. From (7), (60), and Lemma 6.1(b), we can write

$$\mathbb{E} e^{i\theta(W_n - \mu_n)} = e^{-\sigma_n^2 \theta^2/2 - v_n(\theta)}, \tag{62}$$

where

$$v_n(\theta) = \frac{i}{6} a_n \theta^3 + \frac{1}{8} b_n(\theta) \theta^4 \tag{63}$$

and

$$a_n = \sum_{j=0}^{n-1} p_j(1-p_j)(1-2p_j), \qquad b_n(\theta) = \sum_{j=0}^{n-1} p_j(1-p_j) t_{p_j}(\theta). \tag{64}$$

By (49), we have the upper bounds

$$|a_n| \le \sigma_n^2, \quad |b_n(\theta)| \le \sigma_n^2, \quad |v_n(\theta)| \le \left(\frac{1}{6} + \frac{1}{16} \right) \sigma_n^2 |\theta|^3 < \sigma_n^2 |\theta|^3/4. \tag{65}$$

By (57) and (62), we have the identity

$$\begin{aligned} &\int_{|\theta|<1/2} \mathbb{E} e^{i\theta(W_n-\mu_n)} \, d\theta - \frac{\sqrt{2\pi}}{\sigma_n} = \int_{|\theta|<1/2} e^{-\sigma_n^2\theta^2/2 - v_n(\theta)} \, d\theta - \int_{-\infty}^{\infty} e^{-\sigma_n^2\theta^2/2} \, d\theta \\ &= -\int_{|\theta|\ge 1/2} e^{-\sigma_n^2\theta^2/2} \, d\theta - \int_{|\theta|<1/2} v_n(\theta) e^{-\sigma_n^2\theta^2/2} \, d\theta \\ &+ \int_{|\theta|<1/2} e^{-\sigma_n^2\theta^2/2} \left(e^{-v_n(\theta)} - 1 + v_n(\theta) \right) \, d\theta =: I + II + III. \end{aligned} \tag{66}$$

Observe that

$$|I| = \int_{|\theta|\ge 1/2} e^{-\sigma_n^2\theta^2/2} \, d\theta \le 2 \int_{|\theta|\ge 1/2} |\theta| e^{-\sigma_n^2\theta^2/2} \, d\theta = \frac{4 e^{-\sigma_n^2/8}}{\sigma_n^2}. \tag{67}$$

On the other hand, using the second bound in (65) and the antisymmetry of the function $\theta^3 \exp(-\sigma_n^2\theta^2/2)$, we have from (63)

$$\begin{aligned} |II| &\le \frac{\sigma_n^2}{8} \int_{|\theta|<1/2} \theta^4 e^{-\sigma_n^2\theta^2/2} \, d\theta = \frac{1}{8\sigma_n^3} \int_{|u|<\sigma_n/2} u^4 e^{-u^2/2} \, du \\ &\le \frac{\sqrt{2\pi}}{8\sigma_n^3} \mathbb{E} Z^4 = \frac{3\sqrt{2\pi}}{8\sigma_n^3}, \end{aligned} \tag{68}$$

as follows from (57) and (58).

Finally, by inequality (59) and the third upper bound in (65), we get after making the change $\sigma_n \theta \sqrt{3} = 2u$

$$\begin{aligned} |III| &\le \frac{\sigma_n^4}{32} \int_{|\theta|<1/2} \theta^6 e^{-\sigma_n^2\theta^2/2} e^{\sigma_n^2\theta^2/8} \, d\theta \le \frac{4}{27\sqrt{3}\sigma_n^3} \int_{-\infty}^{\infty} u^6 e^{-u^2/2} \, du \\ &\le \frac{4\sqrt{2\pi}}{27\sqrt{3}\sigma_n^3} \mathbb{E} Z^6 = \frac{20\sqrt{2\pi}}{9\sqrt{3}\sigma_n^3}, \end{aligned} \tag{69}$$

where the last equality follows from (58). The result follows from (61) and (67)–(69). □

Proof of Theorem 3.2. Let τ be as in (27). From the second representation in (14), we have

$$\frac{(-1)^{n-m}(\beta\tau)^m}{Q(\tau)} s_\alpha(n+r, m+r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{E} e^{i\theta(W_n(\tau) - \mu_n(\tau))} \, d\theta.$$

Therefore, the result follows from Lemma 6.2. □

7. Moment approximation

As seen in (28) and (29), the mean and the variance of the random variable $\widetilde{T}_r + T_m$ are, respectively,

$$\nu_m = r + \frac{m}{2}, \qquad \tau_m^2 = r + \frac{5}{12} m. \tag{70}$$

On the other hand, we write the expansion

$$-\log(1-z)=\sum_{k=1}^{\infty}\frac{z^k}{k},\qquad |z|\le 1/2,$$

as

$$-\log(1-z)=z+\frac{z^2}{2}+\frac{2}{3}z^3s(z),\quad s(z)=\frac{3}{2}\sum_{k=3}^{\infty}\frac{z^{k-3}}{k}.\quad |z|\le 1/2. \tag{71}$$

Note that

$$|s(z)|\le\frac{1}{2}\sum_{k=3}^{\infty}|z|^{k-3}=\frac{1}{2}\frac{1}{1-|z|}\le 1,\quad |z|\le 1/2. \tag{72}$$

In view of Proposition 2.2, Theorem 3.3 will be shown once we prove the following result on moment approximation.

Lemma 7.1. *Let $m\in\mathbb{N}$ with $n-m\ge 2$. If*

$$\frac{n-m}{\sqrt{m}}\le\frac{1}{2\sqrt{2}}, \tag{73}$$

then

$$\left|\mathbb{E}\left(\widetilde{T}_r+T_m\right)^{n-m}-\nu_m^{n-m}\left(1+\binom{n-m}{2}\left(\frac{\tau_m}{\nu_m}\right)^2\right)\right|$$
$$\le 2^{r+11/2}e^{2/5-r/2}\nu_m^{n-m}\left(\frac{n-m}{\sqrt{m}}\right)^3.$$

Proof. Consider the standardized random variable

$$Z_m=\frac{\widetilde{T}_r+T_m-\nu_m}{\tau_m},$$

which obviously satisfies $\mathbb{E}Z_m=0$ and $Var(Z_m)=\mathbb{E}Z_m^2=1$. We write

$$\begin{aligned}\mathbb{E}\left(\widetilde{T}_r+T_m\right)^{n-m}&=\mathbb{E}\left(\tau_m Z_m+\nu_m\right)^{n-m}\\&=\sum_{l=0}^{n-m}\binom{n-m}{l}\nu_m^{n-m-l}\tau_m^l\mathbb{E}Z_m^l\\&=\nu_m^{n-m}\left(1+\binom{n-m}{2}\left(\frac{\tau_m}{\nu_m}\right)^2+\sum_{l=3}^{n-m}\binom{n-m}{l}\left(\frac{\tau_m}{\nu_m}\right)^l\mathbb{E}Z_m^l\right).\end{aligned} \tag{74}$$

The modulus of the sum on the right-hand side in (74) is bounded above by

$$\sum_{l=3}^{n-m}\left(\frac{2(n-m)\tau_m}{\nu_m}\right)^l\frac{\mathbb{E}(|Z_m|/2)^l}{l!}. \tag{75}$$

By (70), we see that $\tau_m/\nu_m\le\sqrt{2/m}$, thus implying, by virtue of assumption (73), that

$$\frac{2(n-m)\tau_m}{\nu_m}\le 1.$$

Therefore, the sum in (75) is bounded above by

$$\left(\frac{2(n-m)\tau_m}{\nu_m}\right)^3\mathbb{E}e^{|Z_m|/2}\le 2^{9/2}\left(\frac{n-m}{\sqrt{m}}\right)^3\mathbb{E}e^{|Z_m|/2}.$$

This, together with (74) and (75), will imply the result, as soon as we show that

$$\mathbb{E}e^{|Z_m|/2}\le\mathbb{E}e^{Z_m/2}+\mathbb{E}e^{-Z_m/2}\le 2\left(\frac{2}{\sqrt{e}}\right)^r e^{2/5}, \tag{76}$$

To prove (76), let $u = 1/(2\tau_m)$ or $u = -1/(2\tau_m)$ and observe that $|u| \le 1/2$, as follows from (70). Thus, we have from (20) and (70)–(72)

$$\begin{aligned}
&\mathbb{E}e^{u(\widetilde{T}_r + T_m - \nu_m)} = \mathbb{E}e^{u(\widetilde{T}_r - r)}\mathbb{E}e^{u(T_m - m/2)} \\
&= \left(\mathbb{E}e^{u(\widetilde{Y}_1 - 1)}\right)^r \left(\mathbb{E}e^{u(U_1 Y_1 - 1/2)}\right)^m \\
&= \left(\frac{e^{-u}}{1-u}\right)^r \left(-\frac{e^{-u/2}\log(1-u)}{u}\right)^m \\
&= \left(\frac{e^{-u}}{1-u}\right)^r e^{-mu/2}\left(1 + \frac{u}{2} + \frac{2}{3}u^2 s(u)\right)^m .
\end{aligned} \tag{77}$$

Using the inequality $1 + x \le e^x$, $x \ge -1$, we finally obtain from (70)–(72) and (77)

$$\begin{aligned}
&\mathbb{E}e^{u(\widetilde{T}_r + T_m - \nu_m)} \le \left(\frac{e^{-u}}{1-u}\right)^r e^{2mu^2|s(u)|/3} \\
&\le \left(\frac{2}{\sqrt{e}}\right)^r e^{m/6\tau_m^2} \le \left(\frac{2}{\sqrt{e}}\right)^r e^{2/5}.
\end{aligned}$$

This shows claim (76) and completes the proof. □

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