



Asymptotic and Non-asymptotic Results in the Approximation by Bernstein Polynomials

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Abstract. This paper deals with the approximation of functions by the classical Bernstein polynomials in terms of the Ditzian–Totik modulus of smoothness. Asymptotic and non-asymptotic results are respectively stated for continuous and twice continuously differentiable functions. By using a probabilistic approach, known results are either completed or strengthened.

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1. Introduction and Statements of the Main Results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. As usual, $C[0, 1]$ denotes the space of all real continuous functions defined on $[0, 1]$, and $C^m[0, 1]$, $m \in \mathbb{N}_0$, denotes the subspace of all m -times continuously differentiable functions, with the obvious understanding that $C^0[0, 1] = C[0, 1]$. For $m \in \mathbb{N}$, we denote by $\mathcal{C}^m[0, 1] \supset C^m[0, 1]$ the set of functions $f \in C^{m-1}[0, 1]$ such that $f^{(m-1)}$ is absolutely continuous, i. e.,

$$f^{(m-1)}(y) - f^{(m-1)}(x) = \int_x^y g(u) du, \quad x, y \in [0, 1],$$

for some bounded measurable function g , which can be denoted by $g = f^{(m)}$.

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The indicator function of a set A is denoted by 1_A , and $\mathbb{E}$ stands for mathematical expectation.

Let $f \in C[0, 1]$. The sup-norm of f is simply denoted by $\|f\|$, although, more generally, we use the notation $\|f\|_A = \sup\{|f(x)| : x \in A\}$, $A \subseteq [0, 1]$.

The second order central difference of f is defined by

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h), \quad h \geq 0,$$

whenever $x \pm h \in [0, 1]$. The Ditzian–Totik modulus of smoothness of f with weight function $\varphi(x) = \sqrt{x(1-x)}$ is defined by

$$\omega_2^\varphi(f; \delta) = \sup \left\{ \left| \Delta_{h\varphi(x)}^2 f(x) \right| : 0 \leq h \leq \delta, x \pm h\varphi(x) \in [0, 1] \right\}, \quad \delta \geq 0.$$

The classical first order modulus of continuity is simply denoted by $\omega(f; \delta)$.

In this paper, we will make use of the following important inequality proved by Bustamante [2]:

$$\omega_2^\varphi(f; \lambda\delta) \leq (2 + 3\lambda^2)\omega_2^\varphi(f; \delta), \quad \lambda, \delta \geq 0, \quad \lambda\delta \in [0, 1]. \quad (1)$$

Finally, the n th Bernstein polynomial of f is defined as

$$B_n f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \dots, n.$$

We have the probabilistic representation

$$B_n f(x) = \mathbb{E} f\left(\frac{S_n(x)}{n}\right), \quad (2)$$

where $S_n(x)$ is a random variable having the binomial law with parameters n and x , that is to say,

$$P(S_n(x) = k) = p_{n,k}(x), \quad k = 0, 1, \dots, n. \quad (3)$$

Throughout this paper, whenever we write f , n , x , and y , we are assuming that $f \in C[0, 1]$, $n \in \mathbb{N}$, and $x, y \in [0, 1]$.

Following the works by Ditzian and Ivanov [4] and Totik [9], the rates of uniform convergence for the Bernstein polynomials are characterized by

$$K_1 \omega_2^\varphi\left(f; \frac{1}{\sqrt{n}}\right) \leq \|B_n f - f\| \leq K_2 \omega_2^\varphi\left(f; \frac{1}{\sqrt{n}}\right), \quad (4)$$

for some absolute constants K_1 and K_2 . Whereas no specific values for K_1 have been provided yet, different authors completed statement (4) by showing specific values for the constant K_2 . In this regard, Adell and Sangüesa [1] gave $K_2 = 4$, Gavrea et al. [5] and Bustamante [2] provided $K_2 = 3$, and finally, Păltănea [7] proved the validity of $K_2 = 2.5$, this being the best result up to date and up to our knowledge.

This notwithstanding, if additional smoothness conditions on f are added, then the second inequality in (4) may be valid for values of K_2

smaller than 2.5. In this respect, Bustamante and Quesada [3] and Păltănea [8] obtained the following asymptotic result

$$\lim_{n \rightarrow \infty} \frac{\|B_n f - f\|}{\omega_2^\varphi(f; 1/\sqrt{n})} = \frac{1}{2}, \quad f \in C^2[0, 1], \quad (5)$$

provided that f is not an affine function.

The contribution of this paper is twofold. In first place, we strength statement (5) by giving a non-asymptotic version of it. In fact, we prove the following result.

Theorem 1. *If $f \in C^2[0, 1]$, then*

$$\left| \|B_n f - f\| - \frac{1}{2} \omega_2^\varphi\left(f; \frac{1}{\sqrt{n}}\right) \right| \leq \frac{1}{4n} \left(\omega\left(f''; \frac{1}{3\sqrt{n}}\right) + \frac{1}{4} \omega_2^\varphi\left(f''; \frac{1}{\sqrt{n}}\right) \right). \quad (6)$$

As a consequence, statement (5) holds true.

In second place, we complete statement (4) in the following asymptotic form.

Theorem 2. *Let $(\tau_n)_{n \geq 1}$ be a sequence of positive real numbers such that*

$$\tau_n \longrightarrow \infty, \quad \frac{\tau_n}{n} \longrightarrow 0, \quad n \rightarrow \infty.$$

If $f \in C[0, 1]$ is not an affine function, then

$$\limsup_{n \rightarrow \infty} \frac{1}{\omega_2^\varphi\left(f; \frac{1}{\sqrt{n}}\right)} \|B_n f - f\|_{[\tau_n/n, 1 - \tau_n/n]} \leq \frac{3}{2}. \quad (7)$$

Moreover, we have in (4),

$$K_2 \geq 1. \quad (8)$$

This result is based upon Theorem 3 in Sect. 3, which gives estimates of the form

$$|B_n f(x) - f(x)| \leq K_2(n, x) \omega_2^\varphi\left(f; \frac{1}{\sqrt{n}}\right),$$

for some explicit constants $K_2(n, x)$ depending on n and x .

The paper is organized as follows. The proof of Theorem 1 is given in Sect. 2. We show Theorem 2 in Sect. 3 with the aid of two kinds of auxiliary results. On the one hand, we define certain smooth approximants $Q_h^a f$ of the function $f \in C[0, 1]$, by antisymmetrizing in an appropriate way the classical Steklov means of f . On the other hand, we estimate the tail probabilities and the truncated variance of the random variable $S_n(x)$ appearing in the probabilistic representation of $B_n f$ given in (2).

2. Proof of Theorem 1

2.1. Preliminaries

The Taylor's formula of order $m \in \mathbb{N}$ for $f \in \mathcal{C}^m[0, 1]$, with remainder in integral form can be written as

$$\begin{aligned} f(y) - \sum_{j=0}^{m-1} \frac{f^{(j)}(x)}{j!} (y-x)^j \\ &= \frac{(y-x)^m}{(m-1)!} \int_0^1 (1-\theta)^{m-1} f^{(m)}(x + (y-x)\theta) d\theta \\ &= \frac{(y-x)^m}{m!} \mathbb{E} f^{(m)}(x + (y-x)\beta_m), \end{aligned} \quad (9)$$

where β_m is a random variable with the beta density $\rho_m(\theta) = m(1-\theta)^{m-1}$, $0 \leq \theta \leq 1$.

Lemma 1. *If $f \in C^2[0, 1]$ and $\delta \geq 0$, then*

$$|\omega_2^\varphi(f; \delta) - \delta^2 \|\varphi^2 f''\|| \leq \frac{\delta^2}{8} \omega_2^\varphi(f''; \delta).$$

Proof. Let $h \geq 0$ with $x \pm h \in [0, 1]$. Using (9) with $m = 2$, we get

$$\begin{aligned} f(x-h) &= f(x) - f'(x)h + \frac{f''(x)}{2}h^2 \\ &\quad + \frac{h^2}{2} \mathbb{E} (f''(x - h\beta_2) - f''(x)), \end{aligned}$$

as well as

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{f''(x)}{2}h^2 \\ &\quad + \frac{h^2}{2} \mathbb{E} (f''(x + h\beta_2) - f''(x)). \end{aligned}$$

Adding these two identities, we obtain

$$\begin{aligned} \Delta_h^2 f(x) &= f''(x)h^2 \\ &\quad + \frac{h^2}{2} \mathbb{E} (f''(x + h\beta_2) - 2f''(x) + f''(x - h\beta_2)). \end{aligned} \quad (10)$$

Replacing in (10) h by $h\varphi(x)$ and applying the reverse triangular inequality, we have

$$\begin{aligned} |\omega_2^\varphi(f; \delta) - \delta^2 \|\varphi^2 f''\|| &\leq \frac{\delta^2}{2} \|\varphi^2\| \omega_2^\varphi(f''; \delta) \\ &= \frac{\delta^2}{8} \omega_2^\varphi(f''; \delta), \end{aligned}$$

thus completing the proof. $\square$

Gonska et al. [6] showed that

$$\left\| B_n f - f - \frac{1}{2n} \varphi^2 f'' \right\| \leq \frac{1}{4n} \omega \left(f''; \frac{1}{3\sqrt{n}} \right). \quad (11)$$

2.2. Proof of Theorem 1

Statement (6) is an immediate consequence of (11), Lemma 1 with $\delta = 1/\sqrt{n}$, and the reverse and direct triangular inequalities. On the other hand, we have from Lemma 1

$$\omega_2^\varphi \left(f; \frac{1}{\sqrt{n}} \right) = \frac{1}{n} \|\varphi^2 f''\| + o \left(\frac{1}{n} \right),$$

since $f \in C^2[0, 1]$. Thus, statement (5) readily follows from (6), and completes the proof.

3. Proof of Theorem 2

3.1. Auxiliary Results

Let $0 < h \leq 1/3$. We consider the Steklov means of f defined as

$$\begin{aligned} P_h f(y) &= \int_{-1}^1 \int_{-1}^1 f \left(y + \frac{h}{2} (v_1 + v_2) \right) dv_1 dv_2 \\ &= \int_{-1}^1 f(y + hv) \rho(v) dv, \quad h \leq y \leq 1 - h, \end{aligned}$$

where

$$\rho(v) = (1+v)1_{[-1,0]} + (1-v)1_{(0,1]}, \quad -1 \leq v \leq 1.$$

In probabilistic terms, the Steklov means of f can be written as follows. Let V_1 and V_2 be independent identically distributed random variables having the uniform distribution on $[-1, 1]$ and set $V = (V_1 + V_2)/2$. Since $\rho(v)$ is the probability density of V , we can write

$$P_h f(y) = \mathbb{E} f(y + hV), \quad h \leq y \leq 1 - h. \quad (12)$$

Lemma 2. Let $0 < h \leq 1/3$ and let $P_h f(y)$ be as in (12). Then,

(a)

$$|P_h f(y) - f(y)| \leq \frac{1}{2} \omega_2^\varphi \left(f; \frac{h}{\varphi(y)} \right).$$

(b)

$$|(P_h f)''(y)| \leq \frac{1}{h^2} \omega_2^\varphi \left(f; \frac{h}{\varphi(y)} \right).$$

Proof. Since V takes values in $[-1, 1]$ and is symmetric (i. e., V and $-V$ have the same law), we see that

$$|P_h f(y) - f(y)| = \frac{1}{2} |\mathbb{E}(f(y + hV) + f(y - hV) - 2f(y))| \leq \frac{1}{2} \omega_2^\varphi \left(f; \frac{h}{\varphi(y)} \right),$$

thus showing (a). On the other hand, it can be checked that

$$P_h f(y) = \frac{1}{h^2} (f_{(2)}(y + h) + f_{(2)}(y - h) - 2f_{(2)}(y)),$$

where $f_{(2)}$ is a second antiderivative of f . This readily implies part (b) and completes the proof. $\square$

We will make use of the approximant $P_h f$, whose domain is the interval $[h, 1 - h]$, to define a further one whose domain is the whole interval $[0, 1]$, keeping at the same time analogous properties to those given in Lemma 2. To this end, we assume that

$$n \geq 3, \quad 0 < a < \frac{\varphi(a/2)}{\sqrt{n}} + a \leq 1. \quad (13)$$

and take

$$h = \frac{\varphi(ax)}{\sqrt{n}}, \quad \frac{1}{a(n+1)} \leq x \leq \frac{1}{2}. \quad (14)$$

It turns out that

$$h \leq \min(ax, 1/3). \quad (15)$$

Now, we define the approximant $Q_h^a f(y)$ by antisymmetrizing $P_h f(y)$ around the axes $y = ax$ and $y = 1 - ax$ as follows

$$Q_h^a f(y) = \begin{cases} 2P_h f(ax) - P_h f(2ax - y), & y \in [0, ax]; \\ P_h f(y), & y \in [ax, 1 - ax]; \\ 2P_h f(1 - ax) - P_h f(2(1 - ax) - y), & y \in (1 - ax, 1]. \end{cases} \quad (16)$$

The fact that $Q_h^a f$ is well defined readily follows from (13) and (14). Also, note that $Q_h^a f$ is twice differentiable except at the points ax and $1 - ax$. In these two points, $Q_h^a f$ only has sided second derivatives. This implies that $Q_h^a f \in \mathcal{C}^2[0, 1]$.

Lemma 3. Let $R_a = [ax, 1 - ax]$. Under assumptions (13) and (14), we have

(a) If $y \in R_a$, then

$$|Q_h^a f(y) - f(y)| \leq \frac{1}{2} \omega_2^\varphi \left(f; \frac{1}{\sqrt{n}} \right), \quad |(Q_h^a f)''(y)| \leq \frac{1}{h^2} \omega_2^\varphi \left(f; \frac{1}{\sqrt{n}} \right).$$

(b) If $y \notin R_a$, then

$$|Q_h^a f(y) - f(y)| \leq \left(\frac{7}{2} + \frac{3\sqrt{anx}}{(1-a)^{3/2}} \right) \omega_2^\varphi \left(f; \frac{h}{\varphi(h)} \right),$$

and

$$|(Q_h^a f)''(y)| \leq \frac{1}{h^2} \omega_2^\varphi \left(f; \frac{h}{\varphi(h)} \right).$$

Proof. (a) If $y \in R_a$, then

$$\frac{h}{\varphi(y)} = \frac{\varphi(ax)}{\varphi(y)} \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}. \quad (17)$$

Thus, the first inequality in part (a) follows from Lemma 2(a) and definition (16), whereas the second one follows from Lemma 2(b).

(b) Suppose that $y \in [0, ax]$. By (16), we can write

$$\begin{aligned} Q_h^a f(y) - f(y) &= 2(P_h f(ax) - f(ax)) \\ &\quad - (P_h f(2ax - y) - f(2ax - y)) \\ &\quad - (f(2ax - y) + f(y) - 2f(ax)). \end{aligned} \quad (18)$$

Since $h \leq ax \leq 2ax - y \leq 1 - h$, we see that

$$\varphi(ax) \geq \varphi(h), \quad \varphi(2ax - y) \geq \varphi(h). \quad (19)$$

We therefore have from Lemma 2(a)

$$\begin{aligned} |Q_h^a f(y) - f(y)| &\leq \frac{3}{2} \omega_2^\varphi \left(f; \frac{h}{\varphi(h)} \right) \\ &\quad + \omega_2^\varphi \left(f; \frac{ax}{\varphi(ax)} \right). \end{aligned} \quad (20)$$

Applying (1) with $\lambda = ax\varphi(h)/(h\varphi(ax))$ and $\delta = h/\varphi(h)$, we obtain

$$\begin{aligned} \omega_2^\varphi \left(f; \frac{ax}{\varphi(ax)} \right) &\leq \left(2 + \frac{3(ax)^2 \varphi^2(h)}{\varphi^2(ax) h^2} \right) \omega_2^\varphi \left(f; \frac{h}{\varphi(h)} \right) \\ &\leq \left(2 + \frac{3\sqrt{anx}}{(1-a)^{3/2}} \right) \omega_2^\varphi \left(f; \frac{h}{\varphi(h)} \right), \end{aligned} \quad (21)$$

as follows from (14) and some simple computations. Hence, the first inequality follows from (20) and (21).

On the other hand, we have from (16), (19), and Lemma 2(b)

$$\begin{aligned} |(Q_h^a f)''(y)| &= |(P_h f)''(2ax - y)| \leq \frac{1}{h^2} \omega_2^\varphi \left(f; \frac{h}{\varphi(2ax - y)} \right) \\ &\leq \frac{1}{h^2} \omega_2^\varphi \left(f; \frac{h}{\varphi(h)} \right). \end{aligned}$$

If $y \in (1 - ax, 1]$, the proof is similar. $\square$

The following estimates concerning the random variable $S_n(x)/n$ will be needed.

Lemma 4. *In the setting of Lemma 3, denote by $r = 1 - a$. Then,*

(a)

$$P\left(\frac{S_n(x)}{n} \notin R_a\right) \leq e^{-n x r^2/2} + 3e^{-n x r^2/(2e)} =: \epsilon_n(x).$$

(b)

$$\begin{aligned} & \frac{1}{h^2} \mathbb{E} \left(\frac{S_n(x)}{n} - x \right)^2 1_{\left\{ \frac{S_n(x)}{n} \notin R_a \right\}} \\ & \leq \frac{n x}{a(1-ax)} \left(e^{-n x r^2/2} + 6e^{-(n-2)x r^2/(2e)} \right) =: \delta_n(x). \end{aligned}$$

Proof. (a) As follows from (3), we have

$$\mathbb{E} e^{\theta S_n(x)} = (1 + x(e^\theta - 1))^n, \quad \theta \in \mathbb{R}. \quad (22)$$

Let $\theta \geq 0$. By (22) and Chebyshev's inequality, we have

$$\begin{aligned} P(S_n(x) < a n x) &= P(e^{-\theta S_n(x)} > e^{-\theta a n x}) \leq \mathbb{E} e^{-\theta S_n(x) + \theta a n x} \\ &= e^{-n(-\log(1+x(1-e^{-\theta}))) - \theta a x} \leq e^{-n x((1-e^{-\theta}) - a\theta)} \leq e^{-n x(r\theta - \theta^2/2)}, \end{aligned} \quad (23)$$

where we have used the inequalities

$$-\log(1-u) \geq u, \quad u \geq 0, \quad 1 - e^{-\theta} \geq \theta - \frac{\theta^2}{2}, \quad \theta \geq 0.$$

Choosing $\theta = r$ in (23) (the value minimizing the exponent), we get

$$P(S_n(x) < a n x) \leq e^{-n x r^2/2}. \quad (24)$$

On the other hand, we claim that

$$P(S_n(x) > n(1-ax)) \leq P(S_n(x) > n(1-ax) - 1) \leq 3e^{-n x r^2/(2e)}. \quad (25)$$

Indeed, let $0 \leq \theta \leq 1$. Using the inequalities

$$\log(1+u) \leq u, \quad u \geq 0, \quad e^\theta - 1 \leq \theta + \frac{e\theta^2}{2}, \quad 0 \leq \theta \leq 1,$$

we have, as in the proof of (24),

$$\begin{aligned} P(S_n(x) > n(1-ax) - 1) &= P(e^{\theta S_n(x)} > e^{\theta n(1-ax) - \theta}) \\ &\leq \mathbb{E} e^{\theta S_n(x) - n\theta(1-ax) + \theta} \leq 3\mathbb{E} e^{\theta S_n(x) - n\theta(1-ax)} \\ &= 3e^{n(\log(1+x(e^\theta - 1))) - \theta(1-ax)} \leq 3e^{n(x\theta + e x \theta^2/2 - \theta(1-ax))} \\ &= 3e^{n\theta(2x-1)} e^{n x(e\theta^2/2 - r\theta)} \leq 3e^{n x(e\theta^2/2 - r\theta)}, \end{aligned} \quad (26)$$

since $x \leq 1/2$. Thus, claim (25) follows by choosing $\theta = r/e$ in (26). Hence, part (a) follows from (24) and (25).

(b) From (24), we see that

$$\begin{aligned} & \mathbb{E} \left(\frac{S_n(x)}{n} - x \right)^2 1_{\{S_n(x) < anx\}} \\ & \leq x^2 P(S_n(x) < anx) \leq x^2 e^{-n x r^2 / 2}. \end{aligned} \quad (27)$$

On the other hand, since $1 - ax \geq 1/2$, we have

$$\frac{k}{k-1} \leq \frac{n/2}{n/2-1} = \frac{n}{n-2}, \quad k > n(1-ax).$$

We therefore have

$$\begin{aligned} & \mathbb{E} \left(\frac{S_n(x)}{n} - x \right)^2 1_{\{S_n(x) > n(1-ax)\}} \leq \frac{1}{n^2} \mathbb{E} S_n(x)^2 1_{\{S_n(x) > n(1-ax)\}} \\ & = \frac{1}{n^2} \sum_{k > n(1-ax)} \binom{n}{k} k^2 x^k (1-x)^{n-k} \\ & = \frac{n-1}{n} x^2 \sum_{k > n(1-ax)} \binom{n-2}{k-2} \frac{k}{k-1} x^{k-2} (1-x)^{n-k} \\ & \leq \frac{n-1}{n-2} x^2 P(S_{n-2}(x) > n(1-ax) - 2) \\ & \leq 2x^2 P(S_{n-2}(x) > n(1-ax) - 2), \end{aligned} \quad (28)$$

since $n \geq 3$. Observe that

$$n(1-ax) - 2 = (n-2)(1-ax) - 2ax \geq (n-2)(1-ax) - 1,$$

as follows from assumptions (13) and (14). By (25), the right-hand side in (28) can be bounded above by

$$2x^2 P(S_{n-2}(x) > (n-2)(1-ax) - 1) \leq 6x^2 e^{-(n-2)xr^2/(2e)}.$$

This, together with (27) and (28), shows part (b) and completes the proof. $\square$

We are in a position to give the following local estimate.

Theorem 3. *In the setting of Lemma 4, we have*

$$|B_n f(x) - f(x)| \leq \left(1 + \frac{1}{2} \frac{\varphi^2(x)}{\varphi^2(ax)} \right) \omega_2^\varphi \left(f; \frac{1}{\sqrt{n}} \right) + \nu_n(x) \omega_2^\varphi \left(f; \frac{h}{\varphi(h)} \right),$$

where

$$\nu_n(x) = \left(\frac{7}{2} + \frac{3\sqrt{anx}}{(1-a)^{3/2}} \right) \epsilon_n(x) + \frac{1}{2} \delta_n(x). \quad (29)$$

Proof. We use the notation $Qf(y) = Q_h^a f(y)$ and write

$$\begin{aligned} B_n f(x) - f(x) &= (Qf(x) - f(x)) + (B_n f(x) - B_n(Qf)(x)) \\ &\quad + (B_n(Qf)(x) - Qf(x)) \\ &=: I + II + III. \end{aligned} \quad (30)$$

By Lemma 3(a), we have

$$|I| \leq \frac{1}{2} \omega_2^\varphi \left(f; \frac{1}{\sqrt{n}} \right). \quad (31)$$

By (2) and Lemma 3(a) and (b), we see that

$$\begin{aligned} |II| &= \left| \mathbb{E} Qf \left(\frac{S_n(x)}{n} \right) - \mathbb{E} f \left(\frac{S_n(x)}{n} \right) \right| \\ &\leq \frac{1}{2} \omega_2^\varphi \left(f; \frac{1}{\sqrt{n}} \right) \\ &\quad + \left(\frac{7}{2} + \frac{3\sqrt{anx}}{(1-a)^{3/2}} \right) P \left(\frac{S_n(x)}{n} \notin R_a \right) \omega_2^\varphi \left(f; \frac{h}{\varphi(h)} \right). \end{aligned} \quad (32)$$

Finally, denote by $\xi_n(x) = x + (S_n(x)/n - x)\beta_2$. Applying (9) with $m = 2$ and Lemma 3, we get

$$\begin{aligned} |III| &= \frac{1}{2} \left| \mathbb{E} (Qf)''(\xi_n(x)) \left(\frac{S_n(x)}{n} - x \right)^2 \right| \\ &\leq \frac{1}{2h^2} \omega_2^\varphi \left(f; \frac{1}{\sqrt{n}} \right) \mathbb{E} \left(\frac{S_n(x)}{n} - x \right)^2 1_{\{S_n(x)/n \in R_a\}} \\ &\quad + \frac{1}{2h^2} \omega_2^\varphi \left(f; \frac{h}{\varphi(h)} \right) \mathbb{E} \left(\frac{S_n(x)}{n} - x \right)^2 1_{\{S_n(x)/n \notin R_a\}} \\ &\leq \frac{1}{2} \frac{\varphi^2(x)}{\varphi^2(ax)} \omega_2^\varphi \left(f; \frac{1}{\sqrt{n}} \right) \\ &\quad + \frac{1}{2h^2} \omega_2^\varphi \left(f; \frac{h}{\varphi(h)} \right) \mathbb{E} \left(\frac{S_n(x)}{n} - x \right)^2 1_{\{S_n(x)/n \notin R_a\}}, \end{aligned} \quad (33)$$

where we have used (14), the inequality $1/\sqrt{n} \leq h/\varphi(h)$, and the well known fact that

$$\mathbb{E} \left(\frac{S_n(x)}{n} - x \right)^2 = \frac{\varphi^2(x)}{n}.$$

The result follows from (30)–(33) and Lemma 4. $\square$

3.2. Proof of Theorem 2

Since the random variables $S_n(x)$ and $n - S_n(1 - x)$ have the same law, we have

$$B_n f(1 - x) - f(1 - x) = \mathbb{E} f \left(1 - \frac{S_n(x)}{n} \right) - f(1 - x).$$

On the other hand, if $g(y) = f(1 - y)$, we obviously have

$$\omega_2^\varphi(g; \delta) = \omega_2^\varphi(f; \delta), \quad \delta \geq 0.$$

Thus, without loss of generality, we can assume that $0 < x \leq 1/2$.

In the setting of Lemma 4, we claim that

$$\begin{aligned} \omega_2^\varphi\left(f; \frac{h}{\varphi(h)}\right) \\ \leq \left(2 + 3\sqrt{\frac{anx}{1-a}}\right) \omega_2^\varphi\left(f; \frac{1}{\sqrt{n}}\right). \end{aligned} \quad (34)$$

Actually, choose $\lambda = h\sqrt{n}/\varphi(h)$ and $\delta = 1/\sqrt{n}$. By definition (14) and the fact that $h \leq ax$, we see that

$$\lambda^2 = \frac{h^2 n}{\varphi^2(h)} = \frac{\varphi^2(ax)}{\varphi^2(h)} = \frac{ax(1-ax)}{h(1-h)} \leq \frac{ax}{h} = \frac{ax\sqrt{n}}{\varphi(ax)} = \frac{\sqrt{anx}}{\sqrt{1-ax}} \leq \sqrt{\frac{anx}{1-a}}.$$

This, in conjunction with (1), shows claim (34).

From Theorem 3 and (34), we have

$$\frac{|B_n f(x) - f(x)|}{\omega_2^\varphi(f; 1/\sqrt{n})} \leq 1 + \frac{1}{2} \frac{\varphi^2(x)}{\varphi^2(ax)} + \left(2 + 3\sqrt{\frac{anx}{1-a}}\right) \nu_n(x), \quad (35)$$

where $\nu_n(x)$ is defined in (29). Recalling the definitions of $\epsilon_n(x)$ and $\delta_n(x)$ given in Lemma 4, we see that

$$\left(2 + 3\sqrt{\frac{anx}{1-a}}\right) \nu_n(x) \leq P_3(\sqrt{nx}) e^{-cnx}, \quad (36)$$

where $P_3(\cdot)$ is a polynomial of degree three and c is a positive constant not depending on n and x . Observe that, whenever $t_n \rightarrow \infty$, as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \sup_{u \geq t_n} P_3(\sqrt{u}) e^{-cu} = 0. \quad (37)$$

Let τ_n be as in Theorem 2. From (35) and (36), we get

$$\frac{1}{\omega_2^\varphi(f; 1/\sqrt{n})} \|B_n f - f\|_{[\tau_n/n, 1/2]} \leq 1 + \frac{1}{2a} + \sup_{u \geq \tau_n} P_3(\sqrt{u}) e^{-cu}.$$

By (37) and the fact that $\tau_n \rightarrow \infty$, as $n \rightarrow \infty$, this implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{\omega_2^\varphi(f; 1/\sqrt{n})} \|B_n f - f\|_{[\tau_n/n, 1/2]} \leq 1 + \frac{1}{2a},$$

which shows (7), since $0 < a < 1$ is arbitrary.

On the other hand, let $x \in (0, 1/n)$. Consider the function

$$f_x(y) = \left(1 - \frac{y}{x}\right) 1_{[0, x]}(y).$$

Observe that $\omega_2^\varphi(f_x; 1/\sqrt{n}) = 1$, as well as

$$B_n f_x(x) - f_x(x) = \mathbb{E}_{f_x} \left(\frac{S_n(x)}{n} \right) = P(S_n(x) = 0) = (1 - x)^n,$$

thus implying that $K_2 \geq (1 - x)^n$. Therefore, letting $x \rightarrow 0$, we see that $K_2 \geq 1$. This shows (8) and completes the proof.

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