

1 Quasi-bi-Hamiltonian structures and Superintegrability. Study of
2 a Kepler-related family of systems endowed with generalized
3 Runge-Lenz integrals of motion

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5 **Abstract**

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7 The existence of quasi-bi-Hamiltonian structures for a two-dimensional superintegrable (k_1, k_2, k_3) -
8 dependent Kepler-related problem is studied. We make use of an approach that is related with the
9 existence of some complex functions which satisfy interesting Poisson bracket relations and that was
10 previously applied to the standard Kepler problem as well as to some particular superintegrable systems
11 as the Smorodinsky-Winternitz (SW) system, the Tremblay-Turbiner-Winternitz (TTW) and Post-
12 Winternitz (PW) systems. We prove that these complex functions are important for two reasons: first,
13 they determine the integrals of motion, and second they determine the existence of some geometric
14 structures (in this particular case, quasi-bi-Hamiltonian structures). All the results depend on three
15 parameters (k_1, k_2, k_3) in such a way that in the particular case $k_1 \neq 0$, $k_2 = k_3 = 0$, the properties
16 characterizing the Kepler problem are obtained.

17 This paper can be considered as divided in two parts and every part presents a different approach
18 (different complex functions and different quasi-bi-Hamiltonian structures).

19 *Keywords:* Superintegrability ; Kepler-related Hamiltonians ; Generalized Laplace-Runge-
20 Lenz vectors ; Quasi-bi-Hamiltonian structures.

21 *Running title:* Quasi-bi-Hamiltonian structures of a superintegrable Kepler-related problem.

22 *AMS classification:* 37J35 ; 37J39 ; 70H06 ; 70H33

1 Introduction

It is known that superintegrable systems are very peculiar systems endowed with remarkable properties. In fact these particular systems, and their interesting properties, are very studied in these last years.

The main purpose of this paper is to present a study of some geometric properties of a family of superintegrable Kepler-related systems depending on three parameters k_i , $i = 1, 2, 3$. We will prove that it admits quasi-bi-Hamiltonian structures and, in order to arrive to this result, we will make use of an approach that is related with the existence of some complex functions which satisfy interesting Poisson bracket relations. This formalism was previously applied to the study of, not only the standard Kepler problem [11], but also to other superintegrable two-dimensional systems as the nonlinear isotonic oscillator (SW system) [38] or the Tremblay-Turbiner-Winternitz (TTW) and the Post-Winternitz (PW) systems [39].

So we first recall some basic facts characterizing quasi-bi-Hamiltonian structures and superintegrability.

First, suppose that the phase space of a Hamiltonian system, that is, the $2n$ -dimensional cotangent bundle T^*Q of the configuration space Q endowed with the canonical symplectic structure ω_0 , is equipped with a second symplectic structure $\omega_1 \neq \omega_0$. Then a vector field Γ is said to be bi-Hamiltonian if it is Hamiltonian with respect to both structures, that is,

$$i(\Gamma)\omega_0 = dH_0, \quad \text{and} \quad i(\Gamma)\omega_1 = dH_1. \quad (1)$$

Hence, we have two distinct Hamiltonian formulations for the same dynamical system (we note that in some cases ω_1 can be a closed but nonsymplectic 2-form). A consequence is that the pair (ω_0, ω_1) determines a $(1, 1)$ tensor field R defined as

$$\omega_1(X, Y) = \omega_0(RX, Y), \quad \forall X, Y \in \mathfrak{X}(T^*Q) \quad (2)$$

in such a way that R is Γ -invariant and the eigenfunctions of R are constants of motion. If R has n distinct eigenfunctions and in addition the Nijenhuis tensor N_R of the tensor field R vanishes, then the system is Liouville integrable [13, 16, 18]. Bi-Hamiltonian systems satisfying just (1) are usually called weakly bi-Hamiltonian systems (in opposition to strong structures satisfying the Nijenhuis condition); for example, it is known that systems admitting non-symplectic symmetries [9] or canonoid transformations [10] are bi-Hamiltonian systems and that they can be, in some cases, just weak bi-Hamiltonian.

The point is that bi-Hamiltonian structures are very interesting but, in most of cases, difficult to be obtained. A consequence had been the convenience of introducing the related but weaker concept of quasi-bi-Hamiltonian system [3, 4, 5, 6, 7, 8, 15, 29, 30, 41, 44].

A Hamiltonian vector field Γ on (T^*Q, ω_0) is called quasi-bi-Hamiltonian if, in addition, it is quasi-Hamiltonian with respect to another symplectic structure $\Omega \neq \omega_0$. That is, there exists a (nowhere-vanishing) function μ such that it satisfies the equation $i(\mu\Gamma)\Omega = dh$ for some function h (this function h is a first integral of Γ). So we have

$$i(\Gamma)\omega_0 = dH_0, \quad \text{and} \quad i(\mu\Gamma)\Omega = dh. \quad (3)$$

Second, it is well known that the two more important superintegrable systems are the harmonic oscillator and the Kepler problem and that, associated to them, there are four families of potentials with separability in two different coordinate systems in the Euclidean plane and that they are, therefore, superintegrable with quadratic in the momenta constants of motion (first studied in [20] and then by other authors as e.g. [17, 19, 22, 23, 28, 34, 35, 40, 42]). Two of them are related with the harmonic oscillator and they are not considered in this paper. The other two, that are Kepler-related, are

1 (K1) The following (k_1, k_2, k_3) -dependent Kepler-related potential

$$V_{K1} = \frac{k_1}{\sqrt{x^2 + y^2}} + \frac{k_2}{y^2} + \frac{k_3 x}{y^2 \sqrt{x^2 + y^2}} \quad (4)$$

2 is separable in (i) polar coordinates (r, ϕ) and (ii) parabolic coordinates (τ, σ) .

3 (K2) The following (k_1, k_2, k_3) -dependent Kepler-related potential

$$V_{K2} = \frac{k_1}{\sqrt{x^2 + y^2}} + k_2 \frac{[\sqrt{x^2 + y^2} + x]^{1/2}}{\sqrt{x^2 + y^2}} + k_3 \frac{[\sqrt{x^2 + y^2} - x]^{1/2}}{\sqrt{x^2 + y^2}} \quad (5)$$

4 is separable in (i) parabolic coordinates (τ, σ) and (ii) a second system of parabolic coordinates
5 (τ', σ') obtained from (τ, σ) by a rotation.

At this point we recall that the superintegrability of the rational harmonic oscillator (non-central harmonic oscillator with rational ratio of frequencies)

$$H_{mn} = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\alpha_0^2(m^2 x^2 + n^2 y^2),$$

can be proved making use of the complex functions A_x and A_y [25, 32, 9], defined as

$$A_x = p_x + i m \alpha_0 x, \quad A_y = p_y + i n \alpha_0 y,$$

that satisfy

$$\frac{d}{dt} A_x = \{A_x, H_{mn}\} = i m \alpha_0 A_x, \quad \frac{d}{dt} A_y = \{A_y, H_{mn}\} = i n \alpha_0 A_y.$$

6 Then, the function A_{xy} defined as $A_{xy} = (A_x)^n (A_y^*)^m$, is a constant of motion (the two real functions
7 $|A_x|^2$ and $|A_y|^2$ are just the two one-dimensional energies E_x and E_y) and since it is a complex function,
8 it determines not one but two real first integrals, $\text{Re}(A_{xy})$ and $\text{Im}(A_{xy})$ (we have obtained four integrals
9 but, since the system is two dimensional, only three of them can be independent).

10 The important point is that this property (superintegrability related with the existence of some complex
11 functions satisfying certain Poisson brackets properties) is not just an exclusive characteristic of the har-
12 monic oscillator H_{mn} . In fact, it has been recently proved that other superintegrable systems also admit
13 a complex factorization for the additional constants of motion (as the above mentioned SW nonlinear iso-
14 tonic oscillator [38], Tremblay-Turbiner-Winternitz (TTW) and Post-Winternitz (PW) systems [39] and
15 also some particular systems defined in spaces with constant curvature [36, 37]).

16 The aim of this paper is to study the existence of geometric structures associated to the Hamiltonian
17 H_{K2} of the Kepler-related superintegrable potential V_{K2} (the potential V_{K1} , that admits a generalization
18 known as the Post-Winternitz system first studied in [33] and then by other authors, is not studied in this
19 paper). Next we summarize the contents of this paper.

20 First. We will prove the existence of some particular complex functions with interesting Poisson bracket
21 properties and then we will prove that the superintegrability of the H_{K2} system is very related with the
22 properties of these complex functions (we will present two different approaches).

23 Second. We will prove that these complex functions determine the existence of quadratic integrals of
24 motion that can be considered as generalizations of the Laplace-Runge-Lenz vector.

25 Third. We will prove that these complex functions determine the existence of several (complex and real)
26 quasi-bi-Hamiltonian structures.

1 We must clearly advance that we will obtain structures (wedge product of the differentials of complex
2 functions) that do not satisfy the above mentioned Nijenhuis torsion condition (this was also true in the
3 $k_2 = k_3 = 0$ Kepler case [11]); so they are in fact weak quasi-bi-Hamiltonian structures (in opposition
4 to strong structures satisfying the Nijenhuis condition). Nevertheless, the purpose in this paper is not to
5 prove the integrability of a system as consequence of a bi-Hamiltonian structure; in fact, we recall that
6 the multiple separability of V_{K2} was known since [20]. The main idea is that the superintegrable systems
7 are systems endowed with interesting properties deserving to be studied. Now, in this paper, we obtain
8 several new properties all of them related with the above mentioned complex functions.

9 **2 Hamiltonian H_{K2} . Complex functions, Superintegrability, and** 10 **quasi-bi-Hamiltonian structures**

In what follows we will study the second Kepler-related system making use of parabolic coordinates denoted by (τ, σ) and related with the Cartesian coordinates (x, y) by

$$x = \tau^2 - \sigma^2, \quad y = 2\tau\sigma.$$

Then the two linear momenta P_1 and P_2 (corresponding to the Cartesian components p_x and p_y of the linear momentum) and the angular momentum J take the form

$$P_1 = \frac{\tau p_\tau - \sigma p_\sigma}{\tau^2 + \sigma^2}, \quad P_2 = \frac{\tau p_\sigma + \sigma p_\tau}{\tau^2 + \sigma^2}, \quad J = \tau p_\sigma - \sigma p_\tau.$$

A natural Euclidean Hamiltonian $H = T + V$ that takes the form

$$H = \frac{1}{2} \left(\frac{p_\tau^2 + p_\sigma^2}{\tau^2 + \sigma^2} \right) + V(\tau, \sigma), \quad V(\tau, \sigma) = \frac{F(\tau) + G(\sigma)}{\tau^2 + \sigma^2},$$

is Hamilton–Jacobi separable and, therefore, Liouville integrable with the following quadratic function

$$I_2 = (\tau p_\sigma - \sigma p_\tau) \left(\frac{\tau p_\sigma + \sigma p_\tau}{\tau^2 + \sigma^2} \right) + 2 \left(\frac{\tau^2 G(\sigma) - \sigma^2 F(\tau)}{\tau^2 + \sigma^2} \right) = J P_2 + 2 \left(\frac{\tau^2 G(\sigma) - \sigma^2 F(\tau)}{\tau^2 + \sigma^2} \right)$$

11 as the second constant of motion (the first one is the Hamiltonian itself).

12 Now we consider the Hamiltonian H_{K2} of the Kepler-related superintegrable potential V_{K2} . It takes the
13 following form when written in parabolic coordinates

$$H_{K2} = \frac{1}{2} \left(\frac{p_\tau^2 + p_\sigma^2}{\tau^2 + \sigma^2} \right) + \left[\frac{k_1}{\tau^2 + \sigma^2} + \frac{k_2 \tau}{\tau^2 + \sigma^2} + \frac{k_3 \sigma}{\tau^2 + \sigma^2} \right]. \quad (6)$$

Let us now denote by A and B the complex functions $A = A_1 + i A_2$, $B = B_1 + i B_2$, with A_j and B_j , $j = 1, 2$, given by

$$A_1 = \frac{\tau^2 - \sigma^2}{\tau^2 + \sigma^2}, \quad A_2 = \frac{2\tau\sigma}{\tau^2 + \sigma^2},$$

and

$$B_1 = \frac{(\tau p_\sigma - \sigma p_\tau)^2}{\tau^2 + \sigma^2} + k_1 = \frac{J^2}{\tau^2 + \sigma^2} + k_1, \quad B_2 = \frac{J(\tau p_\tau + \sigma p_\sigma)}{\tau^2 + \sigma^2} + k_3 \tau - k_2 \sigma.$$

(in what follows we always suppose that $J \neq 0$ so that B_1 and B_2 are quadratic functions of the momenta). Then we have the following property : the Poisson bracket with H_{K2} of the function A is proportional to itself and this property is also true for the function B

$$\frac{d}{dt} A = \{A, H_{K2}\} = 2i\lambda A, \quad \frac{d}{dt} B = \{B, H_{K2}\} = 2i\lambda B,$$

where the common factor λ takes the value

$$\lambda = \frac{\tau p_\sigma - \sigma p_\tau}{(\tau^2 + \sigma^2)^2} = \frac{J}{(\tau^2 + \sigma^2)^2}.$$

- 1 (we recall that that $J \neq 0$ and therefore $\lambda \neq 0$). Consequently the Poisson bracket of the complex function
 2 AB^* with the Hamiltonian H_{K2} vanishes

$$\begin{aligned} \{AB^*, H_{K2}\} &= \{A, H_{K2}\} B^* + A \{B^*, H_{K2}\} \\ &= (i 2 \lambda A) B^* + A (-i 2 \lambda B^*) = 0. \end{aligned}$$

- 3 The following proposition summarizes this result.

Proposition 1 *Let us consider the following (k_1, k_2, k_3) -dependent Kepler-related Hamiltonian*

$$H_{K2} = \frac{1}{2} \left(\frac{p_\tau^2 + p_\sigma^2}{\tau^2 + \sigma^2} \right) + \left[\frac{k_1}{\tau^2 + \sigma^2} + \frac{k_2 \tau}{\tau^2 + \sigma^2} + \frac{k_3 \sigma}{\tau^2 + \sigma^2} \right]$$

Then, the complex function J_{34} defined as

$$J_{34} = AB^*$$

- 4 *is a (complex) constant of the motion.*

The complex function J_{34} determines two real first-integrals

$$J_{34} = J_3 + i J_4, \quad \{J_3, H_{K2}\} = 0, \quad \{J_4, H_{K2}\} = 0,$$

- 5 whose coordinate expressions are just the two components (R_x and R_y) of the generalized Laplace-Runge-
 6 Lenz vector

$$\begin{aligned} J_3 &= \text{Re}(J_{34}) = J P_2 + 2 \left[k_1 \frac{1}{2} \frac{\tau^2 - \sigma^2}{\tau^2 + \sigma^2} - k_2 \frac{\tau \sigma^2}{\tau^2 + \sigma^2} + k_3 \frac{\tau^2 \sigma}{\tau^2 + \sigma^2} \right], \\ J_4 &= \text{Im}(J_{34}) = J P_1 - 2 \left[k_1 \frac{\tau \sigma}{\tau^2 + \sigma^2} + \frac{1}{2} k_2 \frac{\sigma (\tau^2 - \sigma^2)}{\tau^2 + \sigma^2} + \frac{1}{2} k_3 \frac{\tau (\sigma^2 - \tau^2)}{\tau^2 + \sigma^2} \right]. \end{aligned}$$

- 7 As it is well known the existence of this conserved vector is one of the main characteristics of the Kepler
 8 problem (the standard Laplace-Runge-Lenz vector corresponds to $k_1 \neq 0, k_2 = k_3 = 0$) and the importance
 9 of this fact has led many authors to the study of Kepler-related systems admitting generalizations of the
 10 Laplace-Runge-Lenz vector [1, 2, 12, 24, 26, 27, 31, 43]. Now we have arrived to a new property: it can
 11 also be obtained as a consequence of this complex formalism.

Summarizing, (i) The superintegrability of the Kepler-related Hamiltonian H_{K2} is directly related with the existence of two complex functions, A and B , whose Poisson brackets with the Hamiltonian are proportional, with a common complex factor $2i\lambda$, to themselves, and (ii), The two components of the (k_1, k_2, k_3) -dependent Laplace-Runge-Lenz vector, J_3 and J_4 , appear as the real and imaginary parts of the complex first-integral of motion. Remark that A is a complex function of constant modulus one, while the modulus of B is a polynomial of degree four in the momenta given by

$$B B^* = J_3^2 + J_4^2 = 2J^2 H_{K2} + 2J(k_3 p_\tau - k_2 p_\sigma) + k_1^2 + (k_2 \sigma - k_3 \tau)^2.$$

Let us now denote by Y_{34} the (complex) Hamiltonian vector field of J_{34}

$$i(Y_{34}) \omega_0 = dJ_{34},$$

that obviously satisfies $Y_{34}(H_{K2}) = \{H_{K2}, J_{34}\} = 0$, and by Y_A and Y_B the Hamiltonian vector fields of A and B :

$$i(Y_A)\omega_0 = dA, \quad i(Y_B)\omega_0 = dB,$$

that is

$$Y_A = \left(\frac{\partial A}{\partial p_\tau}\right) \frac{\partial}{\partial \tau} + \left(\frac{\partial A}{\partial p_\sigma}\right) \frac{\partial}{\partial \sigma} - \left(\frac{\partial A}{\partial \tau}\right) \frac{\partial}{\partial p_\tau} - \left(\frac{\partial A}{\partial \sigma}\right) \frac{\partial}{\partial p_\sigma}$$

$$Y_B = \left(\frac{\partial B}{\partial p_\tau}\right) \frac{\partial}{\partial \tau} + \left(\frac{\partial B}{\partial p_\sigma}\right) \frac{\partial}{\partial \sigma} - \left(\frac{\partial B}{\partial \tau}\right) \frac{\partial}{\partial p_\tau} - \left(\frac{\partial B}{\partial \sigma}\right) \frac{\partial}{\partial p_\sigma}$$

Their local coordinate expressions are, respectively, given by

$$Y_A = \frac{2}{(\tau^2 + \sigma^2)^2} \left(-2\tau\sigma + i(\tau^2 - \sigma^2) \right) \left(\sigma \frac{\partial}{\partial p_\tau} - \tau \frac{\partial}{\partial p_\sigma} \right)$$

and

$$Y_B = Y_{Bh} - Y_{Bv}$$

$$Y_{Bh} = \frac{1}{(\tau^2 + \sigma^2)} \left(2J \left(\tau \frac{\partial}{\partial \sigma} - \sigma \frac{\partial}{\partial \tau} \right) + i \left(J \left(\tau \frac{\partial}{\partial \tau} + \sigma \frac{\partial}{\partial \sigma} \right) + (\tau p_\tau + \sigma p_\sigma) \left(\tau \frac{\partial}{\partial \sigma} - \sigma \frac{\partial}{\partial \tau} \right) \right) \right)$$

$$Y_{Bv} = \frac{1}{(\tau^2 + \sigma^2)^2} W \left(\sigma \frac{\partial}{\partial p_\tau} - \tau \frac{\partial}{\partial p_\sigma} \right) + i \left(k_3 \frac{\partial}{\partial p_\sigma} - k_2 \frac{\partial}{\partial p_\tau} \right)$$

where W denotes the following complex function

$$W = 2 \left((\tau^2 - \sigma^2) p_\tau p_\sigma - \tau \sigma (p_\tau^2 - p_\sigma^2) \right) + i \left((\tau^2 - \sigma^2) (p_\tau^2 - p_\sigma^2) + 4\tau \sigma p_\tau p_\sigma \right) = i(\tau + i\sigma)^2 (p_\tau - ip_\sigma)^2$$

Then, the vector field Y_{34} appears as a linear combination of Y_A and Y_B^* ; more specifically we have

$$Y_{34} = B^* Y_A + A Y_B^* = Y + Y', \quad Y = B^* Y_A, \quad Y' = A Y_B^*.$$

The vector field Y_{34} is certainly a symmetry of the Hamiltonian system (T^*Q, ω_0, H_{K2}) , but the two vector fields, Y and Y' , are neither symmetries of the symplectic form ω_0 (that is, $\mathcal{L}_Y \omega_0 \neq 0$ and $\mathcal{L}_{Y'} \omega_0 \neq 0$) nor symmetries of the Hamiltonian (that is, $\mathcal{L}_Y H_{K2} \neq 0$ and $\mathcal{L}_{Y'} H_{K2} \neq 0$). Moreover, remark that they are not symmetries of the dynamics, because

$$[Y, \Gamma_{K2}] \neq 0, \quad [Y', \Gamma_{K2}] \neq 0, \quad i(\Gamma_{K2})\omega_0 = dH_{K2}.$$

Then it can be proved (by direct computation) that the Lie bracket of the dynamical vector field Γ_{K2} with Y is given by

$$[\Gamma_{K2}, Y] = i J_{34} X_\lambda,$$

- 1 where X_λ is the Hamiltonian vector field of the function λ . The vector field X_λ on the right hand side
- 2 represents an obstruction for Y to be a dynamical symmetry. Only when λ is a numerical constant the
- 3 vector field Y (and also Y') is a dynamical symmetry of Γ_K .

Let us denote by Ω the complex 2-form defined as the wedge product of the differentials of the two complex functions

$$\Omega = dA \wedge dB^*,$$

and by ω_Y and ω'_Y the two complex 2-forms obtained by Lie derivative of ω_0 , that is,

$$\mathcal{L}_Y \omega_0 = \omega_Y, \quad \mathcal{L}_{Y'} \omega_0 = \omega'_Y.$$

Then we have the following property

$$\mathcal{L}_Y \omega_0 = i_Y(d\omega_0) + d(i_Y \omega_0) = d(i_Y \omega_0) = d(B^* dA) = -\Omega$$

$$\mathcal{L}_{Y'} \omega_0 = i_{Y'}(d\omega_0) + d(i_{Y'} \omega_0) = d(i_{Y'} \omega_0) = d(A dB^*) = \Omega$$

1 Using the preceding results we can prove:

2 **Proposition 2** *The Hamiltonian vector field Γ_{K2} of the (k_1, k_2, k_3) -dependent Kepler-related problem H_{K2}*
 3 *is a quasi-Hamiltonian system with respect to the complex 2-form Ω .*

Proof.- The contraction of the vector field Γ_{K2} with the complex 2-form Ω gives:

$$i(\Gamma_{K2}) \Omega = \Gamma_{K2}(A) dB^* - \Gamma_{K2}(B^*) dA,$$

and recalling that

$$\Gamma_{K2}(A) = \{A, H_{K2}\} = i \lambda A, \quad \Gamma_{K2}(B^*) = \{B^*, H_{K2}\} = -i \lambda B^*,$$

we arrive to

$$i(\Gamma_{K2}) \Omega = (i \lambda A) dB^* + (i \lambda B^*) dA = i \lambda d(AB^*).$$

4

□

The complex 2-form Ω can be written as

$$\Omega = \Omega_1 + i \Omega_2$$

5 where the two real 2-forms, $\Omega_1 = \text{Re}(\Omega)$ and $\Omega_2 = \text{Im}(\Omega)$, take the form

$$\begin{aligned} \Omega_1 &= dA_1 \wedge dB_1 + dA_2 \wedge dB_2 \\ &= \alpha_{12} d\tau \wedge d\sigma + \alpha_{13} d\tau \wedge dp_\tau + \alpha_{14} d\tau \wedge dp_\sigma + \alpha_{23} d\sigma \wedge dp_\tau + \alpha_{24} d\sigma \wedge dp_\sigma \\ \Omega_2 &= -dA_1 \wedge dB_2 + dA_2 \wedge dB_1 \\ &= \beta_{12} d\tau \wedge d\sigma + \beta_{13} d\tau \wedge dp_\tau + \beta_{14} d\tau \wedge dp_\sigma + \beta_{23} d\sigma \wedge dp_\tau + \beta_{24} d\sigma \wedge dp_\sigma \end{aligned}$$

with α_{ij} and β_{ij} functions of the coordinates τ and σ . Then we have

$$i(\Gamma_{K2}) \Omega_1 = -\lambda dJ_4, \quad i(\Gamma_{K2}) \Omega_2 = \lambda dJ_3,$$

6 which means that Γ_{K2} is also quasi-bi-Hamiltonian with respect to the two real 2-forms (ω_0, Ω_1) and
 7 (ω_0, Ω_2) .

8 Therefore, the two complex functions, A and B , that determine the existence of superintegrability (exis-
 9 tence of additional constants of motion) are also directly related with the existence of quasi-bi-Hamiltonian
 10 structures; first complex (ω_0, Ω) and then real $(\omega_0, \Omega_1, \Omega_2)$.

Remark that the complex 2-form Ω is closed but it is not symplectic. In fact, we have verified that $\Omega_1 \wedge \Omega_1 = 0$, $\Omega_2 \wedge \Omega_2 = 0$, and $\Omega_1 \wedge \Omega_2 = 0$, and therefore we obtain $\Omega \wedge \Omega = 0$. The linear combinations (pencil of 2-forms) $\omega_0 - \mu \Omega_1$ and $\omega_0 - \mu \Omega_2$ satisfy

$$(\omega_0 - \mu \Omega_1)^{\wedge 2} = 2(1 - \mu(\alpha_{24} + \alpha_{13})) V,$$

$$(\omega_0 - \mu \Omega_2)^{\wedge 2} = 2(1 - \mu(\beta_{24} + \beta_{13})) V,$$

11 where V denotes the standard volume $V = d\tau \wedge dp_\tau \wedge d\sigma \wedge dp_\sigma$.

The distribution defined by the kernel of Ω_1 , that is two-dimensional, is given by

$$\text{Ker } \Omega_1 = \{ f_1 X_{11} + f_2 X_{12} \mid f_1, f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{C} \},$$

where the vector fields X_{11} and X_{12} are

$$\begin{aligned} X_{11} &= \left(2\tau\sigma p_\sigma - (\tau^2 + \sigma^2)p_\tau \right) \frac{\partial}{\partial p_\tau} + \left(2\tau\sigma p_\sigma - (\tau^2 + \sigma^2)p_\tau \right) \frac{\partial}{\partial p_\sigma}, \\ X_{12} &= \tau \frac{\partial}{\partial p_\tau} + \sigma \frac{\partial}{\partial p_\sigma} - \frac{(\tau^2 - \sigma^2)(\tau k_3 - \sigma k_2)}{(\tau^2 + \sigma^2)p_\tau - 2\tau\sigma p_\tau} \frac{\partial}{\partial p_\sigma}. \end{aligned}$$

In a similar way the kernel of Ω_2 is given by

$$\text{Ker } \Omega_2 = \{ f_1 X_{21} + f_2 X_{22} \mid f_1, f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{C} \},$$

where the vector fields X_{21} and X_{22} are

$$X_{21} = \tau^2 p_\sigma \frac{\partial}{\partial p_\tau} + \sigma^2 p_\tau \frac{\partial}{\partial p_\sigma}, \quad X_{22} = \tau \frac{\partial}{\partial p_\tau} + \sigma \frac{\partial}{\partial p_\sigma} - \frac{\sigma(\tau k_3 - \sigma k_2)}{\tau p_\sigma} \frac{\partial}{\partial p_\sigma}.$$

We have

$$[\text{Ker } \Omega_1, \text{Ker } \Omega_1] \subset \text{Ker } \Omega_1, \quad [\text{Ker } \Omega_2, \text{Ker } \Omega_2] \subset \text{Ker } \Omega_2.$$

If Y_3 and Y_4 are the Hamiltonian vector fields (with respect to the canonical symplectic form ω_0) of the first integrals J_3 and J_4 , then the dynamical vector field Γ_{K2} is orthogonal to Y_4 with respect to the structure Ω_1 and it is also orthogonal to Y_3 with respect to the structure Ω_2 , that is,

$$i(\Gamma_{K2}) i(Y_4) \Omega_1 = 0, \quad i(\Gamma_{K2}) i(Y_3) \Omega_2 = 0.$$

The bi-Hamiltonian structure (ω_0, Ω) determines a complex recursion operator R defined as

$$\Omega(X, Y) = \omega_0(RX, Y), \quad \forall X, Y \in \mathfrak{X}(T^*Q).$$

But as Ω and R are complex, we can introduce two real recursion operator R_1 and R_2 defined as

$$\Omega_1(X, Y) = \omega_0(R_1 X, Y), \quad \Omega_2(X, Y) = \omega_0(R_2 X, Y).$$

We recall that $\widehat{\omega}_0$ is the map $\widehat{\omega}_0 : \mathfrak{X}(T^*Q) \rightarrow \wedge^1(T^*Q)$ given by contraction, that is $\widehat{\omega}_0(X) = i(X)\omega_0$, and then the nondegenerate character of ω_0 means that the map $\widehat{\omega}_0$ is a bijection. Using this notation we can write the two operators R_1 and R_2 as follows

$$R_1 = \widehat{\omega}_0^{-1} \circ \widehat{\Omega}_1, \quad R_2 = \widehat{\omega}_0^{-1} \circ \widehat{\Omega}_2.$$

1 Then we have the following properties

2 (i) The coordinates expressions of R_1 and R_2 are

$$\begin{aligned} R_1 &= \left[\alpha_{13} \frac{\partial}{\partial \tau} + \alpha_{14} \frac{\partial}{\partial \sigma} - \alpha_{12} \frac{\partial}{\partial p_\sigma} \right] \otimes d\tau + \left[\alpha_{23} \frac{\partial}{\partial \tau} + \alpha_{24} \frac{\partial}{\partial \sigma} + \alpha_{12} \frac{\partial}{\partial p_\tau} \right] \otimes d\sigma \\ &+ \left[\alpha_{13} \frac{\partial}{\partial p_\tau} + \alpha_{23} \frac{\partial}{\partial p_\sigma} \right] \otimes dp_\tau + \left[\alpha_{14} \frac{\partial}{\partial p_\tau} + \alpha_{24} \frac{\partial}{\partial p_\sigma} \right] \otimes dp_\sigma \end{aligned}$$

3 and

$$\begin{aligned} R_2 &= \left[\beta_{13} \frac{\partial}{\partial \tau} + \beta_{14} \frac{\partial}{\partial \sigma} - \beta_{12} \frac{\partial}{\partial p_\sigma} \right] \otimes d\tau + \left[\beta_{23} \frac{\partial}{\partial \tau} + \beta_{24} \frac{\partial}{\partial \sigma} + \beta_{12} \frac{\partial}{\partial p_\tau} \right] \otimes d\sigma \\ &+ \left[\beta_{13} \frac{\partial}{\partial p_\tau} + \beta_{23} \frac{\partial}{\partial p_\sigma} \right] \otimes dp_\tau + \left[\beta_{14} \frac{\partial}{\partial p_\tau} + \beta_{24} \frac{\partial}{\partial p_\sigma} \right] \otimes dp_\sigma \end{aligned}$$

- (ii) R_1 and R_2 have two different eigenvalues doubly degenerate and one of them is null (that is, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 \neq 0$). Therefore we have

$$\det[R_1] = \det[R_2] = 0,$$

1 which is a consequence of the singular character of Ω_1 and Ω_2 .

2 **3 Hamiltonian H_{K2} . New complex functions and new quasi-bi-** 3 **Hamiltonian structures**

4 The expressions of the two complex functions A and B (studied in the previous section 2) have a rather
5 different form (lack of symmetry between these functions). Now, in this new section we present a new
6 approach that makes use of two new complex functions (to be denoted by M_a and M_b) that are quite
7 similar one to the other ; that is, it is a more symmetric approach.

Let us now consider a second set of complex functions $M_a = M_{a1} + i M_{a2}$, $M_b = M_{b1} + i M_{b2}$, with M_{aj} and M_{bj} , $j = 1, 2$, defined by:

$$M_{a1} = \frac{1}{\sqrt{\tau^2 + \sigma^2}} \left(Jp_\tau - (k_2\sigma - k_3\tau)\tau \right), \quad M_{a2} = \frac{1}{\sqrt{\tau^2 + \sigma^2}} \left(-Jp_\sigma - 2k_1\tau + (k_2\sigma - k_3\tau)\sigma \right),$$

and

$$M_{b1} = \frac{1}{\sqrt{\tau^2 + \sigma^2}} \left(Jp_\sigma - (k_2\sigma - k_3\tau)\sigma \right), \quad M_{b2} = \frac{1}{\sqrt{\tau^2 + \sigma^2}} \left(Jp_\tau - 2k_1\sigma - (k_2\sigma - k_3\tau)\tau \right).$$

Then we have the following property

$$\{M_a, H_{K2}\} = i \lambda M_a, \quad \{M_b, H_{K2}\} = i \lambda M_b.$$

Proposition 3 *The complex function K_{34} defined as*

$$K_{34} = M_a M_b^*$$

8 *is a (complex) constant of the motion for the dynamics of the (k_1, k_2, k_3) -dependent Kepler-related system*
9 *described by the Hamiltonian H_{K2} .*

10 The proof is quite similar to the proof of the previous Proposition 1.

11 Note that the modulus of the complex functions M_a and M_b , that are constants of motion, are given by

$$\begin{aligned} M_a M_a^* &= 2 \left(J^2 H_{K2} + k_1 R_x + J(k_3 p_\tau - k_2 p_\sigma) \right) + (k_2\sigma - k_3\tau) + 2k_1^2, \\ M_b M_b^* &= 2 \left(J^2 H_{K2} - k_1 R_x + J(k_3 p_\tau - k_2 p_\sigma) \right) + (k_2\sigma - k_3\tau) + 2k_1^2. \end{aligned}$$

The complex function K_{34} determines two real functions that are first integrals for the H_{K2} :

$$K_{34} = K_3 + i K_4, \quad \{K_3, H_{K2}\} = 0, \quad \{K_4, H_{K2}\} = 0,$$

12 with K_3 and K_4 given by

$$K_3 = \text{Re}(K_{34}) = M_{a1}M_{b1} + M_{a2}M_{b2} = JP_1 - \left(\frac{2k_1\tau\sigma}{\tau^2 + \sigma^2} + \frac{(k_2\sigma - k_3\tau)(\tau^2 - \sigma^2)}{(\tau^2 + \sigma^2)} \right),$$

$$K_4 = \text{Im}(K_{34}) = M_{a2}M_{b1} - M_{a1}M_{b2} = 2J^2 H_{K2} + 2J(k_3 p_\tau - k_2 p_\sigma) + (k_2 \sigma - k_3 \tau)^2.$$

The function K_3 is the component R_y of the generalized Laplace-Runge-Lenz constant, K_4 is a fourth order in the momenta polynomial and $M_a M_a^* - M_b M_b^*$ is just the other component R_x of the above mentioned vector

$$M_a M_a^* - M_b M_b^* = 4k_1 \left[JP_2 + 2 \left(k_1 \frac{1}{2} \frac{\tau^2 - \sigma^2}{\tau^2 + \sigma^2} - k_2 \frac{\tau \sigma^2}{\tau^2 + \sigma^2} + k_3 \frac{\tau^2 \sigma}{\tau^2 + \sigma^2} \right) \right].$$

Let us now denote by Z_{34} the Hamiltonian vector field of the function K_{34} , i.e. $i(Z_{34})\omega_0 = dK_{34}$, such that $Z_{34}(H_{K2}) = 0$, and by Z_a and Z_b the Hamiltonian vector fields of the complex functions M_a and M_b , that is,

$$i(Z_a)\omega_0 = dM_a, \quad i(Z_b)\omega_0 = dM_b.$$

Their coordinate expressions are given by

$$Z_a = \left(\frac{\partial M_a}{\partial p_\tau} \right) \frac{\partial}{\partial \tau} + \left(\frac{\partial M_a}{\partial p_\sigma} \right) \frac{\partial}{\partial \sigma} - \left(\frac{\partial M_a}{\partial \tau} \right) \frac{\partial}{\partial p_\tau} - \left(\frac{\partial M_a}{\partial \sigma} \right) \frac{\partial}{\partial p_\sigma} = Z_{a0} + k_1 Z_{a1} + k_2 Z_{a2} + k_3 Z_{a3}$$

with Z_{a0} and Z_{ai} , $i = 1, 2, 3$, given by

$$Z_{a0} = \frac{1}{\sqrt{\tau^2 + \sigma^2}} \left(((\tau p_\sigma - 2\sigma p_\tau + i\sigma p_\sigma) \frac{\partial}{\partial \tau} + (\tau p_\tau + i(\sigma p_\tau - 2\tau p_\sigma)) \frac{\partial}{\partial \sigma} - \frac{(\tau p_\tau + \sigma p_\sigma)}{\tau^2 + \sigma^2} ((p_\tau - i p_\sigma) (\sigma \frac{\partial}{\partial p_\tau} - \tau \frac{\partial}{\partial p_\sigma}))) \right),$$

1

$$\begin{aligned} Z_{a1} &= \frac{2i\sigma}{(\tau^2 + \sigma^2)^{3/2}} \left(\sigma \frac{\partial}{\partial p_\tau} - \tau \frac{\partial}{\partial p_\sigma} \right), \\ Z_{a2} &= \frac{1}{(\tau^2 + \sigma^2)^{3/2}} \left[(\sigma^3 \frac{\partial}{\partial p_\tau} + \tau^3 \frac{\partial}{\partial p_\sigma}) + i\sigma \left(\tau \sigma \frac{\partial}{\partial p_\tau} - (2\tau^2 + \sigma^2) \frac{\partial}{\partial p_\sigma} \right) \right], \\ Z_{a3} &= \frac{1}{(\tau^2 + \sigma^2)^{3/2}} \left[\tau \left(-(\tau^2 + 2\sigma^2) \frac{\partial}{\partial p_\tau} + \tau \sigma \frac{\partial}{\partial p_\sigma} \right) + i \left(\sigma^3 \frac{\partial}{\partial p_\tau} + \tau^3 \frac{\partial}{\partial p_\sigma} \right) \right], \end{aligned}$$

and

$$Z_b = \left(\frac{\partial M_b}{\partial p_\tau} \right) \frac{\partial}{\partial \tau} + \left(\frac{\partial M_b}{\partial p_\sigma} \right) \frac{\partial}{\partial \sigma} - \left(\frac{\partial M_b}{\partial \tau} \right) \frac{\partial}{\partial p_\tau} - \left(\frac{\partial M_b}{\partial \sigma} \right) \frac{\partial}{\partial p_\sigma} = Z_{b0} + k_1 Z_{b1} + k_2 Z_{b2} + k_3 Z_{b3}$$

with Z_{b0} and Z_{bi} , $i = 1, 2, 3$, given by

$$Z_{b0} = \frac{1}{\sqrt{\tau^2 + \sigma^2}} \left((-\sigma p_\sigma + i(\tau p_\sigma - 2\sigma p_\tau)) \frac{\partial}{\partial \tau} + (2\tau p_\sigma - \sigma p_\tau + i\tau p_\tau) \frac{\partial}{\partial \sigma} - \frac{(\tau p_\tau + \sigma p_\sigma)}{\tau^2 + \sigma^2} ((p_\sigma + i p_\tau) (\sigma \frac{\partial}{\partial p_\tau} - \tau \frac{\partial}{\partial p_\sigma})) \right),$$

2

$$\begin{aligned} Z_{b1} &= -\frac{2i\tau}{(\tau^2 + \sigma^2)^{3/2}} \left(\sigma \frac{\partial}{\partial p_\tau} - \tau \frac{\partial}{\partial p_\sigma} \right), \\ Z_{b2} &= \frac{1}{(\tau^2 + \sigma^2)^{3/2}} \left[\sigma \left(-\tau \sigma \frac{\partial}{\partial p_\tau} + (2\tau^2 + \sigma^2) \frac{\partial}{\partial p_\sigma} \right) + i \left(\sigma^3 \frac{\partial}{\partial p_\tau} + \tau^3 \frac{\partial}{\partial p_\sigma} \right) \right], \\ Z_{b3} &= -\frac{1}{(\tau^2 + \sigma^2)^{3/2}} \left[(\sigma^3 \frac{\partial}{\partial p_\tau} + \tau^3 \frac{\partial}{\partial p_\sigma}) + i\tau \left((\tau^2 + 2\sigma^2) \frac{\partial}{\partial p_\tau} - \tau \sigma \frac{\partial}{\partial p_\sigma} \right) \right]. \end{aligned}$$

Now recalling that

$$dK_{34} = d(M_a M_b^*) = M_b^* d(M_a) + M_a d(M_b^*),$$

we obtain

$$K_{34} = M_b^* Z_a + M_a Z_b^* = Z + Z', \quad \text{where} \quad Z = M_b^* Z_a, \quad Z' = M_a Z_b^*.$$

In the following we will denote by Ω_M the complex 2-form defined as $\Omega_M = dM_a \wedge dM_b^*$. Then the two 2-forms ω_Z and $\omega_{Z'}$ obtained by Lie derivation of ω_0 with respect to Z and Z' are given by

$$\mathcal{L}_Z \omega_0 = \omega_Z = -\Omega_M, \quad \mathcal{L}_{Z'} \omega_0 = \omega_{Z'} = \Omega_M.$$

Proposition 4 *The Hamiltonian vector field Γ_{K2} of the (k_1, k_2, k_3) -dependent Kepler-related problem H_{K2} is a quasi-Hamiltonian system with respect to the complex 2-form Ω_M .*

Proof.- This can be proved by a direct computation:

$$\begin{aligned} i(\Gamma_{K2}) \Omega_M &= \Gamma_{K2}(M_a) dM_b^* - \Gamma_{K2}(M_b^*) dM_a \\ &= (i \lambda M_a) dM_b^* + (i \lambda M_b^*) dM_a = i \lambda d(M_a M_b^*). \end{aligned}$$

3

□

The complex 2-form Ω_M can be decomposed as

$$\Omega_M = \Omega_{M1} + i \Omega_{M2},$$

where the two real 2-forms, $\Omega_{M1} = \text{Re}(\Omega_M)$ and $\Omega_{M2} = \text{Im}(\Omega_M)$, take the form

$$\Omega_{M1} = dM_{a1} \wedge dM_{b1} + dM_{a2} \wedge dM_{b2}, \quad \Omega_{M2} = -dM_{a1} \wedge dM_{b2} + dM_{a2} \wedge dM_{b1},$$

and then considering the real and imaginary parts of the equation

$$i(\Gamma_{K2}) \Omega_M = i \lambda d(K_3 + i K_4).$$

we obtain:

$$i(\Gamma_{K2}) \Omega_{M1} = -\lambda dK_4, \quad i(\Gamma_{K2}) \Omega_{M2} = \lambda dK_3,$$

4 The following proposition summarizes this result

Proposition 5 *The Hamiltonian vector field Γ_{K2} of the (k_1, k_2, k_3) -dependent Kepler-related problem H_{K2} is also quasi-bi-Hamiltonian with respect to the two pairs of real 2-forms (ω_0, Ω_{M1}) and (ω_0, Ω_{M2}) .*

5 We close this section with the following three items that generalize to the (k_1, k_2, k_3) -dependent system
6 some previous properties of the k_1 -dependent Kepler problem [11]

9 (i) The two real 2-forms are closed but not symplectic. In fact we have verified that $\Omega_{M1} \wedge \Omega_{M1} = 0$,
10 $\Omega_{M2} \wedge \Omega_{M2} = 0$, and also $\Omega_{M1} \wedge \Omega_{M2} = 0$.

(ii) These two 2-forms, Ω_{M1} and Ω_{M2} , determine two recursion operators ((1, 1) tensor fields) R'_1 and R'_2 defined as

$$\Omega_{M1}(X, Y) = \omega_0(R'_1 X, Y), \quad \Omega_{M2}(X, Y) = \omega_0(R'_2 X, Y),$$

or in an equivalent way

$$R'_1 = \widehat{\omega_0}^{-1} \circ \widehat{\Omega_{M1}}, \quad R'_2 = \widehat{\omega_0}^{-1} \circ \widehat{\Omega_{M2}}.$$

As in section 2, a consequence of the singular character of Ω_{M1} and Ω_{M2} is that

$$\det[R'_1] = \det[R'_2] = 0.$$

(iii) If we denote by Z_3 and Z_4 the Hamiltonian vector fields (with respect to the canonical symplectic form ω_0) of the integrals K_3 and K_4 , then the dynamical vector field Γ_{K2} is orthogonal to Z_4 with respect to the structure Ω_{M1} and it is also orthogonal to Z_3 with respect to the structure Ω_{M2} , that is,

$$i(\Gamma_{K2}) i(Z_4) \Omega_{M1} = 0, \quad i(\Gamma_{K2}) i(Z_3) \Omega_{M2} = 0.$$

4 Final comments

As observed in the introduction superintegrable systems are systems endowed with very interesting properties directly related with the existence of associated geometric structures. Now, in this paper we have analyzed the second Kepler-related family of superintegrable systems and we have obtained several different quasi-bi-Hamiltonian structures. A remarkable property is that they arise as a consequence of the properties of certain particular complex functions.

We close this paper pointing out some open questions.

First, we have two different ways of obtaining quasi-bi-Hamiltonian structures for the same system, generated by different couples of complex functions. The second way looks more fundamental since M_a and M_b are quite similar one to the other and the formalism is more symmetric than the first one. In any case it is natural to ask what are the relations between the two structures. Second, the complex functions method presented in this paper (as well in some other previous papers mentioned in the Introduction) is restricted to the two dimensional case; it is convenient to study the generalization to the three-dimensional case (the multiple separability of three-dimensional systems was first studied in [17]) and also to constant curvature spaces (the superintegrability of some particular systems was studied in [14, 21, 36, 37] making use of curvature-dependent coordinates); the generalization of the system studied in this paper must be done making use of curvature-dependent parabolic coordinates. Third, the complex functions (A, B) or (M_a, M_b) are important for two reasons since they determine the integrals of motion (AB^* or $M_a M_b^*$) and also the geometric structures; probably there are some additional properties hidden behind these functions deserving to be studied making use of tools of complex differential geometry.

Two more questions. Quasi-bi-Hamiltonian structures are not very well studied yet (in contrast to the bi-Hamiltonian systems) so that their relation with superintegrability can be a good motivation to undertake a better study of these structures. Finally, classical superintegrability is interesting not only by itself but also as a first step for the study of the corresponding quantum versions (quantum superintegrability is related with the degeneracy of the energy levels). Thus, quantum version of the properties presented in this paper can also be considered as a matter to be studied.

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