NEW EQUILIBRIA OF NON-AUTONOMOUS DISCRETE DYNAMICAL SYSTEMS

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Abstract

In the framework of non-autonomous discrete dynamical systems in metric spaces, we propose new equilibrium points, called quasi-fixed points, and prove that they play a role similar to that of fixed points in autonomous discrete dynamical systems. In this way some sufficient conditions for the convergence of iterative schemes of type $x_{k+1} = T_k x_k$ in metric spaces are presented, where the maps T_k are contractivities with different fixed points. The results include any reordering of the maps, even with repetitions, and forward and backward directions. We also prove generalizations of the Banach fixed point theorems when the self-map is substituted by a sequence of contractivities with different fixed points. The theory presented links the field of dynamical systems with the theory of iterated function systems. We prove that in some cases the set of quasi-fixed points is an invariant fractal set. The hypotheses relax the usual conditions on the underlying space for the existence of invariant sets in countable iterated function systems.

Keywords: Non-autonomous dynamical systems, Discrete-time systems, Fractals, Iterated function systems, Convergence of numerical algorithms.
2000 MSC: 26A18, 28A80, 37B25, 37B55, 37C25

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1. Introduction

This article presents sufficient conditions for the convergence of general iterative schemes of type

$$x_{k+1} = T_k x_k, \tag{1.1}$$

where $x_k \in X$, X is a metric space **and** $T_k : X \to X$ **for all** $k \geq 1$**.** This problem can be approached in the framework of non-autonomous discrete dynamical systems. There is an extensive literature concerning the theory of autonomous dynamical systems, that is to say, iterations of type

$$x_{k+1} = T x_k,$$

where T is a self-map, $T : X \to X$. This model is very general, and it fits a great number of numerical algorithms to solve major problems of the applied mathematics (e.g. Newton-Raphson and fixed point methods for solving nonlinear equations). However, in a rapidly (almost volatile) world, it is unlikely that the parameters of any model remain constant over time. For instance, concerning the ubiquitous growth models (linear and nonlinear) it is very improbable that the growth rates be fixed for long periods of time. For a better understanding of the real-world, it is essential to introduce the temporal variable in the scheme.

In previous papers, the author and team have proposed new fractal functions that arise as fixed points of operators of type

$$T^\alpha : \mathcal{C}^*(I) \to \mathcal{C}^*(I),$$

where $\mathcal{C}^*(I)$ represents a space of functions on a real interval I, and T^α is a contraction (see for instance [1], [14], [15]). The function defined is a limit point of the scheme

$$g_{k+1} = T^\alpha(g_k),$$

and $g_k \in \mathcal{C}^*(I)$. The natural extension of this model is to consider a step (or time) dependent map T^{α_k}, and this has been the main motivation for this paper.

Regarding the system (1.1), there are in the literature classical results of convergence when the cardinal of the collection (T_k) is finite either the underlying space X is compact. In general these outcomes involve only backward orbits (see for instance [6], [10], [19] among others) that is to say, sequences of type:

$$\{T_1 \circ T_2 \circ \ldots \circ T_k(x)\},$$

but these trajectories are not natural in the context of numerical (and non-numerical) algorithms. In the present paper, some hypotheses for the convergence of forward orbits

$$\{T_k \circ T_{k-1} \circ \ldots \circ T_1(x)\}$$

are provided. We consider an infinite sequence of contractions and less restrictive assumptions on the metric space X.

Another major innovation of this paper is the definition of a new equilibrium point that generalizes the classical fixed points, called in this article quasi-fixed points.

In Section 1 we give the first definitions and characterizations of the quasi-fixed points, along with sufficient conditions for their existence. Section 2 reminds the forward invariant sets of a discrete dynamical system. Since the orbits of a non-autonomous model may not be invariant, some generalizations of the orbits are given, referred to as extended orbits and invariant supersets, in order to insure the permanence of the flow. Section 4 proposes some results concerning the stability of quasi-fixed points when (T_k) is a sequence of contractions with different fixed points.

Section 5 presents a code space composed of words of infinite length, whose characters are natural numbers. This is inspired by the ideas of the book of G. Edgar [4] on string spaces, and the reference [8]. Section 6 is focused on the convergence of the forward orbits of (1.1), and Section 7 provides similar results for backward orbits on non-compact metric spaces, proving finally that the set of quasi-fixed points is a fractal.

2. First definitions and results

Let us consider a metric space X with respect to **a** metric d and a sequence of self-maps $T_n : X \to X$, $n \in \mathbb{N}$. A non-autonomous dynamical system is an iterative scheme of type

$$x_{k+1} = T_k x_k, \tag{2.2}$$

for all $k \in \mathbb{N}$ and $x_1 \in X$.
Let us denote

$$\tau_k := T_k \circ T_{k-1} \circ \dots T_1. \tag{2.3}$$

The following concepts are classical in autonomous and non-autonomous dynamical systems.

Definition 2.1. *The orbit of $x \in X$ is the sequence $\gamma(x) := \{\tau_k(x)\}_{k \geq 0}$, where $\tau_0 := Id$.*

Definition 2.2. *An element $x \in X$ is a fixed point of the sequence $(T_k)_{k\geq 1}$ if $\gamma(x) = \{x\}$. That is to say $T_k(x) = x$ for all $k \geq 1$.*

We propose a new equilibrium point for non-autonomous dynamical systems:

Definition 2.3. *An element $\widetilde{x} \in X$ is a quasi-fixed point of the sequence $(T_k)_{k\geq 1}$ (or a quasi-fixed point of the system (1.1)) if $\lim_{k\to\infty} \tau_k(\widetilde{x}) = \widetilde{x}$.*

Remark 2.1. *The fixed points are quasi-fixed, but the converse is not true in general.*

Example 1: Let us consider $X = S^1$, where S^1 is the unit circle, and the discrete system $x_{k+1} = T_k x_k$, where T_k is a rotation of angle θ_k such that $\sum_{k=1}^{\infty} \theta_k = 2\pi$. There are no fixed points but all the elements of the circle are quasi-fixed.

Definition 2.4. *If $y = \lim_{k\to\infty} \tau_k(x)$ the element y is the end-point of x.*

Some elementary results regarding quasi-fixed points are the following.

Proposition 2.1. *A quasi-fixed point is the end-point of itself.*

Proposition 2.2. *If x is a quasi-fixed point its orbit $\gamma(x)$ is a compact. The set $\{(d(x, \tau_k(x))\}_{k\geq 1}$ is bounded and tends to zero as k tends to infinity.*

Definition 2.5. *An element $\overline{x}$ is asymptotically quasi-fixed if there exists a quasi-fixed point $\widetilde{x} \in X$ such that $d(\tau_n(\widetilde{x}), \tau_n(\overline{x}))$ tends to zero as n tends to infinity.*

Definition 2.6. *An element $x \in X$ is a quasi-periodic point if there exists $m > 0$ such that $\tau_i(x) = \lim_{k\to\infty} \tau_{km+i}(x)$, for $i = 1, 2, \dots, m$.*

Definition 2.7. *$x \in X$ is eventually quasi-periodic/quasi-fixed point if there exists $p > 0$ such that $\tau_p(x)$ is quasi-periodic/quasi-fixed.*

In the following we give a characterization theorem for quasi-fixed points. We need the following well known result about Lipschitz maps.

Proposition 2.3. *The composition of Lipschitz maps is Lipschitz whose constant is the product of the Lipschitz constants of the components.*

Theorem 2.1. *If X is complete and T_n are contractivities, then $\widetilde{x}$ is a quasi-fixed point if and only if $\widetilde{x}$ is the limit of the fixed points of the maps $\tau_n = T_n \circ T_{n-1} \ldots \circ T_1$. As a consequence, if a quasi-fixed point exists it is unique.*

Proof. Let us consider the factors k_n of T_n. The Lipschitz constants of τ_n are $K_n := \prod_{i=1}^n k_i \leq K_1 = k_1 < 1$ for any $n \in \mathbb{N}$. Let (x_n) be the sequence of fixed points of τ_n. Then if $\widetilde{x}$ is a quasi-fixed point of the sequence (T_n), as $x_n = \tau_n(x_n)$,

$$d(x_n, \widetilde{x}) \leq d(x_n, \tau_n(\widetilde{x})) + d(\tau_n(\widetilde{x}), \widetilde{x}),$$

$$d(x_n, \widetilde{x}) \leq K_n d(x_n, \widetilde{x}) + d(\tau_n(\widetilde{x}), \widetilde{x}). \tag{2.4}$$

Since $\tau_n(\widetilde{x})$ tends to $\widetilde{x}$, the limit of x_n is $\widetilde{x}$.
For the converse implication, if $\lim_{n\to\infty} x_n = \widetilde{x}$,

$$d(\tau_n(\widetilde{x}), \widetilde{x}) \leq d(\tau_n(\widetilde{x}), \tau_n(x_n)) + d(x_n, \widetilde{x}),$$

and

$$d(\tau_n(\widetilde{x}), \widetilde{x}) \leq (K_n + 1) d(x_n, \widetilde{x}), \tag{2.5}$$

Henceforth $\widetilde{x}$ is a quasi-fixed point.
In general, from (2.4) and (2.5) one has

$$(1 - K_n) d(x_n, \widetilde{x}) \leq d(\tau_n(\widetilde{x}), \widetilde{x}) \leq (K_n + 1) d(x_n, \widetilde{x}),$$

and the rate of convergence of the sequences (x_n) and $(\tau_n(\widetilde{x}))$ is similar. $\square$

Example 2: Let us consider $X = \mathbb{R}$, and $T_n(x) = \left(1 - \frac{1}{n+1}\right)(1-x)$, then $\tau_n(x) = \frac{1}{4(n+1)}(2n+1+(4x-1)(-1)^n)$. The fixed point of τ_n is $x_n = \frac{1}{4}(\frac{2n+1-(-1)^n}{n+1-(-1)^n})$. Consequently, the only quasi-fixed point is 1/2.

Let us prove the expression given for $\tau_n(x)$. **Define**

$$y_n(x) := \frac{1}{4(n+1)}(2n+1+(4x-1)(-1)^n).$$

By induction: for $n = 1$**,**

$$\tau_1(x) = T_1(x) = \frac{1}{2}(1-x) = y_1(x).$$

If it is true that $\tau_k(x) = y_k(x)$ **then**

$$\tau_{k+1}(x) = T_{k+1}(y_k(x)) = \left(\frac{k+1}{k+2}\right)\left(1 - \frac{1}{4(k+1)}(2k+1+(4x-1)(-1)^k)\right)$$

$$\tau_{k+1}(x) = \left(\frac{1}{k+2}\right)\left(\frac{2k+3+(4x-1)(-1)^{k+1}}{4}\right),$$

and the last expression agrees with $y_{k+1}(x)$.

Remark 2.2. *Notice that the result of Theorem 2.1 is true if T_n are contractivities for $n \geq p$* ***for some*** *$p \in \mathbb{N}$, taking the sequence of fixed points for $n \geq p$.*

Corollary 2.1. *If the sequence (T_n) owns a fixed point, this is the only quasi-fixed point.*

Remark 2.3. *If X is a complete normed linear space and T_n are linear and contractive,* ***we have the conditions of the previous corollary, the maps of the sequence share the fixed point zero and consequently the quasi-fixed point is the null element.***

The next theorem is due to Bonsall [2] and provides a sufficient condition for the existence of quasi-fixed points. Other related results can be read in [7] and [13], for instance.

Theorem 2.2. *Let X be a complete metric space and let (f_n) be a sequence of contraction mappings with the same Lipschitz constant $k < 1$ such that the sequence (f_n) converges pointwisely to f. Then for all $n \in \mathbb{N}$, f_n has a unique fixed point x_n and the sequence (x_n) converges* ***to the fixed point $\widetilde{x}$ of f.***

Theorem 2.3. *Let X be a complete metric space and (T_n) be self-contractivities. If the sequence (τ_n) converges pointwisely then there is a unique quasi-fixed point.*

Proof. Let us denote the Lipschitz constants of τ_p as $K_p := \prod_{n=1}^{p} k_n$. This is a decreasing sequence bounded by $K_1 = k_1$. According to Bonsall's Theorem and Theorem 2.1 there exists a unique quasi-fixed point, being the fixed point of the limit of (τ_n). $\square$

Example 3: Let us consider $X = \mathbb{R}^2$, and the discrete system $x_{k+1} = T_k x_k$, where $T_k = \lambda_k A_k$, $\prod_{i=1}^{n} \lambda_i \to \lambda < 1$ as n tends to infinity, and A_k is a rotation of angle θ_k such that $\theta := \sum_{k=1}^{\infty} \theta_k < 2\pi$. The sequence τ_n converges uniformly to $\tau(x) = \lambda A(x)$, where A is a rotation of angle θ. The maps τ_n share the fixed point zero, and this is the only quasi-fixed point of the system.

In Section 6 some sufficient conditions for the convergence of (τ_n) and the existence of quasi-fixed points are given.

3. Invariant sets

Let us consider as before a metric space X and a sequence of self-maps $(T_n)_{n\geq 1}$ The next definition is due to Birkhoff.

Definition 3.1. *An element $y \in X$ is a limit point of $x \in X$ if there is a sequence of natural numbers $(n_j)_{n\geq 1}$ such that $n_j \to \infty$ as $j \to \infty$ and $\tau_{n_j}(x) \to y$. The limit set $\Omega(x)$ (or ω-limit) is the set of limit points of x.*

Definition 3.2. *The ω-limit set of $H \subseteq X$ is the union of the limit points of the elements of H.*

The next proposition is a consequence of the previous definitions.

Proposition 3.1. *$\widetilde{x}$ is a quasi-fixed point if and only if $\Omega(\widetilde{x}) = \{\widetilde{x}\}$.*

Definition 3.3. *A set $E \subseteq X$ is forward or positively invariant if $T_n(E) \subseteq E$, for any $n \in \mathbb{N}$.*

Remark 3.1. ***For the sake of simplicity, the words forward or positively will be omitted along the text.***

Remark 3.2. *Notice that in a non-autonomous discrete dynamical system, the orbits may not be invariant because, for instance, $T_i(\tau_n(x))$ may not belong to $\gamma(x)$.*

Since the invariant sets of non-autonomous systems are difficult to identify, this section is devoted to their characterization. For it we need some more definitions.

Let us consider the set of words of finite length with alphabet $\mathbb{N}$:

$$\Sigma := \{\alpha = (n_1 n_2 \dots n_p) : n_i \in \mathbb{N}, p \in \mathbb{N}\}, \tag{3.6}$$

and let us denote, for a given sequence of self-maps (T_n):

$$T_\alpha := T_{n_p} \circ T_{n_{p-1}} \circ \dots \circ T_{n_1}. \tag{3.7}$$

Definition 3.4. *A finite itinerary of $y \in X$ is an element of type $T_\alpha(y) = T_{n_p} \circ T_{n_{p-1}} \circ \dots \circ T_{n_1}(y)$ or $T_0(y) := y$. In the first case α is the address of $T_\alpha(y)$.*

Definition 3.5. *The extended orbit* $\mathrm{J}(y)$ *of $y \in X$ is the set of finite itineraries of y.*

Let us consider now the set of words of infinite length with alphabet $\mathbb{N}$:

$$\Sigma^\infty := \{\sigma = (n_1 n_2 \dots) : n_i \in \mathbb{N}\}. \tag{3.8}$$

Definition 3.6. *An infinite itinerary of y with address $\sigma = (n_1 n_2 ... n_p \dots)$ is defined as $T_\sigma(y) :=$* $\lim_{p\to\infty} T_{n_p} \circ T_{n_{p-1}} \circ \dots \circ T_{n_1}(y)$*, if the limit exists.*

Let us denote the set of infinite itineraries of y as

$$J^\infty(y) = \{x \in X : \exists \sigma \in \Sigma^\infty \quad \text{such that} \quad x = T_\sigma(y)\}.$$

It is clear that $J^\infty(y) \subseteq \overline{J(y)}$.

Proposition 3.2. *If $\widetilde{x}$ is a quasi-fixed point then $\widetilde{x} \in \mathrm{J}^\infty(\widetilde{x})$.*

Proposition 3.3. *The extended orbit $J(y)$ of any $y \in X$ owns the following properties:*

- $\gamma(y) \subseteq J(y)$.
- *$J(y)$ is an invariant set.*
- *$J(y)$ is the least invariant set containing y.*
- $\Omega(y) \subseteq \overline{J(y)}$.

The third item means that if E is an invariant set and $y \in E$ then $J(y) \subseteq E$. That is to say,

$$J(y) = \bigcap_{E_y \in \epsilon_y} E_y,$$

where E_y is invariant and $y \in E_y$ (ϵ_y is the family of invariant sets containing y.)

Proof. They are straightforward consequences of the definition of extended orbits. $\square$

Proposition 3.4. *If E is an invariant set then $\overline{E}$ also is.*

Proof. If $x \in \overline{E}$ then exists a sequence x_n whose limit is x, and $x_n \in E$. Since T_i is continuous then $\lim T_i(x_n) = T_i(x)$. Since $T_i(x_n) \in E$ then $T_i(x) \in \overline{E}$ for all $i \in \mathbb{N}$ and consequently $\overline{E}$ is invariant. $\square$

Definition 3.7. *$\overline{J(y)}$ is the invariant closure of y.*

These definitions and results can be extended to a set $M \subseteq X$.

Definition 3.8. *The invariant superset of $M \subseteq X$ is*

$$M^{\Sigma} := \cup_{y \in M} J(y).$$

The invariant closure of M is $\overline{M^{\Sigma}}$.

Proposition 3.5. *The invariant superset of $M \subseteq X$ owns the following properties:*

- $M \subseteq M^{\Sigma}$.

- *M^Σ is invariant.*

- *M^Σ is the least invariant set containing M.*

- $\cup_{y\in M}\gamma(y) \subseteq M^\Sigma$.

- $\Omega(M) \subseteq \overline{M^\Sigma}$.

Remark 3.3. *For any $y \in X$, $\gamma(y)^\Sigma = J(y)$ and $J(y)^\Sigma = J(y)$.*

We give in the following two characterizations of the invariant sets.

Proposition 3.6. *A set $E \subseteq X$ is invariant if and only if there exists $M \subseteq X$ such that $E = M^\Sigma$, that is to say, E is the invariant superset of some $M \subseteq X$.*

Proof. If E is invariant then $E^\Sigma \subseteq E$, but E^Σ is the least invariant set containing E, consequently they agree. The converse implication is obvious. □

Remark 3.4. *If $T_n = T$ for all n then* ***for any*** $y \in X$

- $\gamma(y)^\Sigma = \gamma(y)$.

- $J(y) = \gamma(y)$.

- $M^\Sigma = \cup_{y\in M}\gamma(y)$.

In particular, the orbits are invariant sets.

Proposition 3.7. *A set $E \subseteq X$ is invariant if and only if $E = \cup_{y\in E}J(y)$. That is to say, E is invariant if and only if is the union of extended orbits.*

Proof. It is a straightforward consequence of the definitions given. □

Example 4: Let us consider in $X = \mathbb{R}^2$ the system $x_{k+1} = A_k x_k$ where A_k denotes a rotation of angle θ_k around the origin. All the circles C_r centered at the origin are invariant sets. In this case:

$$C_r = \cup_{x\in C_r}\gamma(x)$$

and the invariant closure of C_r, $\overline{C_r^\Sigma}$, agrees with C_r. **For $r = 0$, $\overline{C_r^\Sigma}$ reduces to zero.**

4. Stability of quasi-fixed points

In the first place let us remind the definition of stable sets.

Notation: $\mathcal{N}(H)$ will represent the set of neighborhoods of a set or an element H.

Definition 4.1. *A point $x \in X$ is Lyapunov stable if $\forall \epsilon > 0$ there exists $\delta > 0$ such that if $y \in B(x, \delta)$ then $\tau_n(y) \in B(\tau_n(x), \epsilon)$ for any $n \in \mathbb{N}$.*

Definition 4.2. *A point x is an attractor if there exists $U \in \mathcal{N}(x)$ such that if $y \in U$ then $\tau_n(y) \to x$ as $n \to \infty$. x is said to be asymptotically stable if it is both, stable and attractor. If $\tau_n(y) \to x$ as $n \to \infty$ for all $y \in X$ then x is said to be globally attracting or a global attractor. If x is a global attractor and stable then it is globally asymptotically stable.*
Unstable means not stable. If x is neither stable nor an attractor it will be strongly unstable.

Definition 4.3. *If x is an attractor, then the basin of attraction of x is the set $\mathcal{B}(x)$ of points $y \in X$ such that $\lim_{n\to\infty} \tau_n(y) = x$.*

The next definition is due to Lasalle [9].

Definition 4.4. *An orbit $\gamma(x)$ is said to be stable in the sense of Lagrange if it is bounded.*

Example 5: Let us consider $X = \mathbb{R}^2$ and the system $x_{k+1} = A_k x_k$ for $k \geq 1$, where A_k is a rotation around the origin of angle θ_k such that $\sum_{k=1}^{\infty} \theta_k = 2\pi$. Then for any $x \in \mathbb{R}^2$, x is a Lyapunov stable quasi-fixed point. The orbits are stable in the sense of Lagrange.

Example 6: Let us resume Example 2, considering the system

$$x_{k+1} = (1 - \frac{1}{k+1})(1 - x_k).$$

$1/2$ is a quasi-fixed point globally asymptotically stable.

Example 7: For $X = \mathbb{R}^2$ and the system $x_{k+1} = \lambda_k A_k x_k$ for $k \geq 1$, where $|\lambda_k| < 1$ such that $\prod_{k=1}^{n} |\lambda_k| \to 0$ as k tends to infinity, and A_k is a rotation around the origin of angle θ_k then $(0, 0)$ is a quasi-fixed point and a global attractor. The orbits are stable in the sense of Lagrange.

Example 8: For $X = \mathbb{R}^+ \times \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of real numbers greater or equal than zero, and the system

$$x_{k+1} = y_k^{\alpha_k}$$

$$y_{k+1} = x_k^{\beta_k},$$

where $0 < \alpha_k, \beta_k < c < 1$, the origin is a strongly unstable quasi-fixed point.

Theorem 4.1. *If $\widetilde{x} \in X$ is a quasi-fixed point of a sequence (T_n) of Lipschitz maps with constants k_i such that $\prod_{i=1}^{n} k_i \to 0$ as $n \to \infty$ then $\widetilde{x}$ is globally asymptotically stable.*

Proof. For $x \in X$, and $\widetilde{x}$ quasi-fixed point, defining $x_n := \tau_n(x)$,

$$d(\tau_n(x), \tau_n(\widetilde{x})) \leq k_n d(x_{n-1}, \widetilde{x}_{n-1}) \leq (\prod_{i=1}^{n} k_i) d(x, \widetilde{x}). \tag{4.9}$$

Since $\tau_n(\widetilde{x})$ tends to $\widetilde{x}$ then $\tau_n(x)$ tends also to $\widetilde{x}$, and $\widetilde{x}$ is attracting. The inequality (4.9) implies that $\widetilde{x}$ is also stable. □

Corollary 4.1. *With the hypothesis given on the Lipschitz constants k_n, the quasi-fixed point $\widetilde{x}$ is unique.*

Corollary 4.2. *As a particular case, with the hypotheses of Theorem 4.1, if $\widetilde{x}$ is a fixed point then is globally asymptotically stable.*

Corollary 4.3. *If X is a normed space, and (T_n) are self-maps linear and bounded such that $\prod_{i=1}^{n} ||T_i|| \to 0$ as $n \to \infty$ then the origin is globally asymptotically stable and it is the only quasi-fixed point.*

Remark 4.1. *Let us notice that, due to Theorem 2.1, $\widetilde{x}$ is the limit of the fixed points of the maps τ_n, but at the moment its existence is only guaranteed in the case of convergence of either τ_n or their fixed points. This question will be addressed again in the last sections of the paper.*

Remark 4.2. *The arguments of Theorem 4.1 are also valid for trajectories of type $T_1 \circ T_2 \circ \ldots \circ T_n(x)$, defining a quasi-fixed point through the equality*

$$\widetilde{x} = \lim_{n\to\infty} T_1 \circ T_2 \circ \ldots \circ T_n(\widetilde{x}).$$

The previous theorem can be generalized to the following result.

Proposition 4.1. *If $k_n \leq k < 1$ for any $n \in \mathbb{N}$, then if a sequence (T_{n_k}) has a quasi-fixed point, it is a global attractor for the system $x_{k+1} = T_{n_k} x_k$.*

Remark 4.3. *The previous result is very general because the sequence (T_{n_k}) need not be a subsequence of (T_n). It must be composed of elements of it, but it may have infinite repeated elements, for instance.*

All over the article (T_{n_j}) means any sequence taken from (T_n), and it may have repeated elements.

A local result of stability for fixed points of specific discrete systems can be found in [3], with an application in [16].

5. An ultrametric space associated with a sequence of contractivities

Let (T_n) be a sequence of contractions with contractivity factors (k_n), and let $k := \sup_n k_n < 1$, and $k_n > 0$ for any $n \in \mathbb{N}$.
Let Σ^∞ be the set of words of infinite length and alphabet $\mathbb{N}$. That is to say, $\sigma \in \Sigma^\infty$ if it has the form

$$\sigma = (n_1 n_2 n_3 \ldots) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} ...$$

Σ^∞ can be endowed with the product topology, but we will consider here a structure of metric space associated with the sequence (T_n).

Definition 5.1. *A word of finite length $\alpha \in \Sigma$, $\alpha = (n_1 n_2 n_3 \ldots n_p)$ is a prefix of $\sigma \in \Sigma^\infty$ if $\sigma = (n_1 n_2 n_3 \ldots n_p \ldots)$. The number $p \in \mathbb{N}$ is the length of α, denoted as $p = card(\alpha) = |\alpha|$.*

Definition 5.2. *The words $\sigma, \sigma' \in \Sigma^\infty$ have a maximal common prefix $\alpha = (n_1 n_2 n_3 \ldots n_p)$ if $\sigma = (n_1 n_2 n_3 \ldots n_p n_{p+1} \ldots)$, $\sigma' = (n_1 n_2 n_3 \ldots n_p n'_{p+1} \ldots)$ and $n_{p+1} \neq n'_{p+1}$.*

We define a metric d_T in Σ^∞ associated with the sequence (T_n) in the following way:

- If $\sigma, \sigma' \in \Sigma^\infty$, $\sigma = (n_1 n_2 \ldots)$ and $\sigma' = (n'_1 n'_2 \ldots)$ are such that $n_1 \neq n'_1$ then $d_T(\sigma, \sigma') = 1$.

- If $\sigma = \sigma'$ then $d_T(\sigma, \sigma') = 0$.

- If σ, σ' have a maximal common prefix $\alpha = (n_1 n_2 n_3 \ldots n_p)$ then $d_T(\sigma, \sigma') = k_{n_1} k_{n_2} \ldots k_{n_p}$, where k_{n_j} is the contractivity factor of T_{n_j}.

We will prove that (X, d_T) is an ultrametric space, that is to say, the following inequality holds for any $\sigma, \sigma', \sigma''$:

$$d_T(\sigma, \sigma') \leq \max\{d_T(\sigma, \sigma''), d_T(\sigma'', \sigma')\} \leq d_T(\sigma, \sigma'') + d_T(\sigma'', \sigma').$$

An ultrametric space is a particular case of metric space and the distance must satisfy the classical conditions of positivity and simmetry. For the third property of the ultrametric conditions, let us consider $\sigma = (n_1 n_2 \ldots n_p n_{p+1} \ldots)$ and $\sigma' = (n_1 n_2 \ldots n_p n'_{p+1} \ldots)$ such that $n_{p+1} \neq n'_{p+1}$. According to the distance definition $d_T(\sigma, \sigma') = k_{n_1} k_{n_2} \ldots k_{n_p}$. Let σ'' be any other word, and let us assume that $d_T(\sigma, \sigma'') \geq d_T(\sigma', \sigma'')$ for instance. If $\sigma'' = (n_1 n_2 \ldots n_q n''_{q+1} n''_{q+2} \ldots)$, and $n''_{q+1} \neq n_{q+1}$ then $d_T(\sigma, \sigma'') = k_{n_1} k_{n_2} \ldots k_{n_q}$.

Case I: $p \geq q$. In this case

$$d_T(\sigma, \sigma'') = k_{n_1} k_{n_2} \ldots k_{n_q} \geq k_{n_1} k_{n_2} \ldots k_{n_p} = d_T(\sigma, \sigma'),$$

and thus

$$\max\{d_T(\sigma, \sigma''), d_T(\sigma', \sigma'')\} \geq d_T(\sigma, \sigma').$$

Case II: $p < q$. These hypotheses contradict the definition of $d_T(\sigma, \sigma')$ since as

$$\sigma = (n_1 n_2 \ldots n_p n_{p+1} \ldots)$$

$$\sigma' = (n_1 n_2 \ldots n_p n'_{p+1} \ldots)$$

$$\sigma'' = (n_1 n_2 \ldots n_p \ldots n_q \ldots)$$

then

$$d_T(\sigma', \sigma'') \leq d_T(\sigma, \sigma'') < k_{n_1} \ldots k_{n_p}.$$

The words σ, σ'' must have in common the letter in the $(p+1)$-th position, n_{p+1}, and σ', σ'' would agree at the $(p+1)$-th character, $(n'_{p+1} = n_{p+1})$. Consequently σ, σ' should have the $(p+1)$-th letter in common.

Notation: If $\sigma = (n_1 n_2 \ldots)$ and $p \in \mathbb{N}$ we will use the following expression:

$$\sigma|p := (n_1 n_2 \ldots n_p) \in \Sigma. \tag{5.10}$$

Proposition 5.1. *Let us consider $\sigma, \sigma' \in \Sigma^\infty, \sigma = (n_1 n_2 \ldots n_p \ldots)$. Then $\sigma|p = \sigma'|p$ if and only if $d_T(\sigma, \sigma') \leq k_{n_1} k_{n_2} \ldots k_{n_p}$.*

Proof. The direct implication is evident given that the contractivity factors are lower than 1. For the converse one, let us assume that $d_T(\sigma, \sigma') \leq k_{n_1} k_{n_2} \ldots k_{n_p}$ and $n_i \neq n'_i$ for some $i \leq p$. Then $d_T(\sigma, \sigma') = k_{n_1} k_{n_2} \ldots k_{n_r}$ for $r < i \leq p$ and thus we would have $k_{n_1} k_{n_2} \ldots k_{n_r} \leq k_{n_1} k_{n_2} \ldots k_{n_p}$ for $r < p$, and this is impossible. Consequently $n_i = n'_i$ for all $i \leq p$. □

Definition 5.3. *A cylinder of prefix $\alpha \in \Sigma$ is the set of words of infinite length with prefix α. It can be expressed as*

$$C(\alpha) = \{\sigma = \alpha\sigma' : \sigma' \in \Sigma^\infty\} = \alpha \times \Sigma^\infty$$

or

$$C(\alpha) = \{\sigma \in \Sigma^\infty : \sigma|p = \alpha\},$$

where $p = |\alpha|$.

Proposition 5.2. *$C(\alpha)$ is a clopen set, that is to say, is open and closed.*

Proof. Let $p = |\alpha|$ and $\sigma \in C(\alpha)$. Let α_i be the letters of α and $\sigma = (\alpha_1\alpha_2 \ldots \alpha_p n_{p+1} n_{p+2} \ldots)$. Let us choose $R < \prod_{i=1}^{p} k_{\alpha_i}$ and let σ' be such that $d_T(\sigma, \sigma') < R$. The previous result implies that $\sigma|p = \sigma'|p$ and consequently $\sigma' \in C(\alpha)$. Then $B(\sigma, R) \subseteq C(\alpha)$.
Bearing in mind that $(C(\alpha))^c = \cup_{\alpha' \neq \alpha} C(\alpha')$, we have that the cylinder is closed.

□

In fact the cylinders compose a base for the open sets in the metric topology defined.

Let us define the shift map $s : \Sigma^\infty \to \Sigma^\infty$ defined for $\sigma = (n_1 n_2 n_3 \ldots)$ as $s(\sigma) = (n_2 n_3 \ldots)$.

Proposition 5.3. *If $k := \sup_n k_n < 1$, s is continuous with respect to the metric d_T.*

Proof. Let $\epsilon > 0$ and $\sigma = (n_1 n_2 n_3 \ldots) \in \Sigma^\infty$. Let us see that s is continuous at σ. Let $p \in \mathbb{N}$ be such that $k^p < \epsilon$, and let us choose $\delta = k_{n_1} k_{n_2} \ldots k_{n_p} k_{n_{p+1}}$.
If $d_T(\sigma, \sigma') < \delta$ then, by Proposition 5.1, $\sigma|(p+1) = \sigma'|(p+1)$ and thus $s(\sigma)|p = s(\sigma')|p$. Then

$$d_T(s(\sigma), s(\sigma')) \leq k_{n_2} k_{n_3} \ldots k_{n_p} k_{n_{p+1}} \leq k^p < \epsilon.$$

□

Definition 5.4. *Let (T_n) be a sequence of contractivities on a metric space X, with contractivity factors k_n. If there exists a solution $s \in \mathbb{R}$ for the equation*

$$\sum_{n=1}^{\infty} k_n^s = 1,$$

then s is the similarity dimension of the sequence (T_n).

The previous definition is a generalization of the homonymous dimension in the finite case (see for instance [4]). There are numerous results concerning the relation between similarity, box and Hausdorff dimensions in the finite case (see for instance [5] and references therein). One classical result is due to Moran (1946) [12]:

Theorem 5.1. *Let $\mathcal{F}$ be an Euclidean Iterated Function System $\mathcal{F} = (f_1, f_2, \ldots, f_n)$ composed of similarities whose attractor is K. Let k_i be the similarity ratio associated with f_i, and assume that $\mathcal{F}$ satisfies the open set condition. If s is the solution of the equation*

$$\sum_{i=1}^{n} k_i^s = 1,$$

then $s = dim_B(K) = dim_H(K)$, where $dim_B(K)$ and $dim_H(K)$ are the box and Hausdorff dimensions of K respectively.

If there exists a similarity dimension for (T_n) then Σ^∞ can be endowed with a measure μ defined for the basic sets as

$$\mu(C(\alpha)) = (k_{n_1} k_{n_2} \ldots k_{n_p})^s$$

if $\alpha = (n_1 n_2 \ldots n_p)$. Since $\Sigma^\infty = \cup_{n=1}^{\infty} C(\alpha_n)$ **where** $\alpha_n = (n)$ **for all** n, then $\mu(\Sigma^\infty) = \sum_{n=1}^{\infty} k_n^s = 1$.

In the following we give sufficient conditions on the contractivity factors to ensure the existence of similarity dimension for the sequence (T_n).

Proposition 5.4. *If the contractiviy factors k_n satisfy the inequality*

$$\frac{1}{n^2} \leq k_n \leq \frac{c}{n^{1+\epsilon}},$$

for some $c < 1$ and $\epsilon > 0$, then there is $s \in [1, +\infty)$ such that

$$\sum_{n=1}^{\infty} k_n^s = 1.$$

Proof. Let us consider the function $\Phi : [1, +\infty) \to (0, +\infty)$ defined as

$$\Phi(x) = \sum_{n=1}^{\infty} k_n^x.$$

Let us find a bound for $\Phi(x)$:

$$\Phi(x) \leq \sum_{n=1}^{\infty} \frac{c^x}{n^x} \leq \sum_{n=1}^{\infty} c \frac{1}{n^x} = c\xi(x),$$

where ξ is the Riemann zeta function and $x > 1$. Let us prove now that Φ is continuous. With the hypotheses given

$$\Phi(x) \leq \sum_{n=1}^{\infty} k_n \leq \sum_{n=1}^{\infty} c \frac{1}{n^{1+\epsilon}}$$

for $1 \leq x < +\infty$. Using the Weierstrass criterion, the series $\Phi(x)$ converges uniformly, and consequently is continuous. Now $\Phi(1) \geq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} > 1$, and

$$\lim_{x\to\infty} \Phi(x) \leq c \lim_{x\to\infty} \xi(x) = c < 1.$$

Applying Darboux's theorem there exists $s \geq 1$ such that $\Phi(s) = 1$. $\square$

Proposition 5.5. *If the contractiviy factors k_n satisfy the inequality*

$$\left(\frac{1}{2}\right)^n \leq k_n \leq r^n,$$

for some $r < 1$ then there is a real $s \in [1, +\infty)$ such that

$$\sum_{n=1}^{\infty} k_n^s = 1.$$

Proof. Let us define Φ as before. This function is continuous applying the Weierstrass criterion of uniform convergence. With the hypotheses given, for $x \geq 1$,

$$\Phi(x) \leq \sum_{n=1}^{\infty} r^{nx} = \frac{r^x}{1 - r^x},$$

and $\Phi(1) \geq 1$. If $x \to \infty$ then $\frac{r^x}{1-r^x} \to 0$. Then there exists $s \geq 1$ such that $\Phi(s) = 1$. $\square$

Remark 5.1. *If $k_n = r^n$ where $0 < r < 1$ the similarity dimension can be explicitly computed as $s = \frac{\log(1/2)}{log(r)}$*, ***since the equation for s reduces to***

$$\frac{r^s}{1 - r^s} = 1$$

and the expression for s is easily calculated from it.

6. Convergence of non-autonomous iterative schemes

In this section we consider as before a metric space X, and a sequence of self-contractivities (T_n), and we give sufficient conditions for the convergence of iterative schemes of type

$$x_{k+1} = T_{n_k} x_k,$$

for $x_k \in X, n_k \in \mathbb{N}$.
Given a word $\sigma \in \Sigma^\infty$, $\sigma = (n_1 n_2 \ldots)$ and $p \in \mathbb{N}$ we will use the following notations:

$$\sigma|p := (n_1 n_2 \ldots n_p) \tag{6.11}$$

$$T_{\sigma|p} := T_{n_p} \circ T_{n_{p-1}} \circ \ldots \circ T_{n_1} : X \to X, \tag{6.12}$$

and for a set $M \subseteq X$,

$$M_p := T_{\sigma|p}(M). \tag{6.13}$$

In the first place, we define a new type of (forward) invariance.

Definition 6.1. *Let $\sigma \in \Sigma^\infty$, $\sigma = (n_1 n_2 \ldots)$ and $M \subseteq X$. The set M is σ-invariant if $T_{n_j}(M) \subseteq M$ for all $j \in \mathbb{N}$.*

If M is σ-invariant $T_{\sigma|p}(M) \subseteq M$ and the flow remains in M.

For a σ-invariant set M we will assume in this section that $\cap_{p=1}^{\infty} M_p \neq \emptyset$.

We consider the set sequence $(T_{\sigma|p}(M))_{p\in\mathbb{N}}$ and let us notice that even if M is σ-invariant, it is not a nested decreasing sequence, that is to say, $T_{\sigma|p}(M)$ may not contain $T_{\sigma|(p+1)}(M)$. For instance $T_{n_1}(M)$ may not contain $T_{n_2} \circ T_{n_1}(M)$. As usual, k_n is the contractivity factor of T_n.

Theorem 6.1. *If X is complete, $M \subseteq X$ is σ-invariant and bounded, and $K_p := \prod_{j=1}^{p} k_{n_j} \to 0$ as p tends to infinity, then there exists $\widetilde{x}_\sigma \in \overline{M}$ such that*

$$\lim_{p\to\infty} T_{\sigma|p}(x) = \widetilde{x}_\sigma,$$

for any $x \in M$. Further if M is closed, then there exists a unique quasi-fixed point of (T_{n_j}), $\widetilde{x}_\sigma \in M$, globally attracting in M, and the sequence of self-maps $(T_{\sigma|p})_{p\in M}$ converges uniformly in M to the constant map T_σ defined as $T_\sigma(x) = \widetilde{x}_\sigma$, for all $x \in M$.

Proof. Let us consider $M_p = T_{\sigma|p}(M)$, and $N_p := \cap_{j=1}^{p}\overline{M_j}$. As M is σ-invariant, the sequence (N_p) is a nested sequence of closed sets in $\overline{M}$ (complete):

$$\overline{M} \supseteq \overline{M_1} \supseteq (\overline{M_1} \cap \overline{M_2}) \supseteq \ldots \supseteq (\cap_{j=1}^{p}\overline{M_j}) \supseteq \ldots$$

and

$$diam(N_p) \leq diam(\overline{M_p}) = diam(M_p) \leq (\prod_{j=1}^{p} k_{n_j})diam(M) \to 0,$$

as $p \to \infty$. Consequently, applying the Cantor's intersection theorem:

$$\cap_{p=1}^{\infty} N_p = \{\widetilde{x}_\sigma\}.$$

where $\widetilde{x}_\sigma \in \overline{M}$. If $x \in M$, let us denote

$$x_p := T_{\sigma|p}(x) \in M_p.$$

$\widetilde{x}_\sigma \in \overline{M_p}$ since

$$\widetilde{x}_\sigma \in N_p = (\cap_{j=1}^{p}\overline{M_j}) \subseteq \overline{M_p}.$$

Then we have

$$d(x_p, \widetilde{x}_\sigma) \leq diam(\overline{M_p}) = diam(M_p) \leq K_p diam(M), \tag{6.14}$$

and consequently $\lim_{p\to\infty} x_p = \widetilde{x}_\sigma$.
If M is closed, $\widetilde{x}_\sigma \in M$ and $\lim x_p = \widetilde{x}_\sigma$, for all $x \in M$. Hence $\widetilde{x}_\sigma$ is a globally attracting quasi-fixed point in M. For the convergence of the maps $T_{\sigma|p}$ to T_σ defined as $T_\sigma(x) = \widetilde{x}_\sigma$ for all $x \in M$, let us think that

$$d(T_{\sigma|p}(x), T_\sigma(x)) = d(x_p, \widetilde{x}_\sigma) \leq K_p diam(M),$$

and $T_{\sigma|p} \to T_\sigma$ uniformly when p tends to infinity. □

As a consequence of the previous result, we have the following proposition.

Proposition 6.1. *If X is a complete metric space, $M \subseteq X$ is σ-invariant and compact, and K_p tends to zero, there exists a quasi-fixed point $\widetilde{x}_\sigma \in M$ globally attracting in M. The sequence $(T_{\sigma|p})$ converges uniformly to T_σ, and $\lim_{p\to\infty} N_p = \{\widetilde{x}_\sigma\}$ in the Hausdorff metric.*

Proof. It is a particular case of the previous proposition. In this case (N_p) is a nested and decreasing sequence of compacts, whose intersection is a singleton. A decreasing sequence of compacts is Cauchy. As the space $\mathcal{K}(X)$ of compacts of X is complete, the sequence is convergent to its intersection. $\square$

Theorem 6.2. *If X is complete and bounded and the contractivity factors k_n satisfy the inequality $k := \sup k_n < 1$ then $\forall \sigma \in \Sigma^\infty$, $\sigma = (n_1 n_2 \ldots)$, the sequence of self-maps (T_{n_j}) owns a quasi-fixed point $\widetilde{x}_\sigma \in X$ globally asymptotically stable, that is to say, the iterative scheme*

$$x_{k+1} = T_{n_k} x_k \tag{6.15}$$

for all $k \geq 1$ is convergent to $\widetilde{x}_\sigma \in X$, and this limit does not depend on the starting point $x_1 \in X$. In particular,

$$\lim_{n \to \infty} \tau_n(x) = \widetilde{x}_{\sigma^*},$$

for all $x \in X$ and $\sigma^* = (123\ldots)$.
For any $\sigma \in \Sigma^\infty$, the sequence of maps $(T_{\sigma|p})_{p \in \mathbb{N}}$ converges uniformly to T_σ defined as $T_\sigma(x) = \widetilde{x}_\sigma \; \forall x \in X$.

Proof. The existence, attraction and stability are straightforward consequences of Theorems 6.1 and 4.1. $\square$

Corollary 6.1. *If X is compact, any iterative scheme of type (6.15) is convergent with the hypothesis given on the contractivity factors.*

On the hypotheses of Theorem 6.2, let us consider now the set Q of quasi-fixed points of the sequences (T_{n_j}) for any $\sigma = (n_1 n_2 \ldots)$, and the map

$$\pi : \Sigma^\infty \to Q$$

defined as $\pi(\sigma) = \widetilde{x}_\sigma$.

Definition 6.2. *σ is called an address for $\widetilde{x}_\sigma$.*

We have the following result.

Theorem 6.3. *If X is complete and bounded and the contractivity factors k_n satisfy the inequality $k := \sup k_n < 1$, the map π defined above is continuous and surjective.*

Proof. The surjectivity is obvious from the definition of Q. Let us define now the map $\mathcal{T} : \Sigma^\infty \to \mathcal{C}(X)$, where $\mathcal{C}(X)$ is the space of continuous (and bounded since X is), defined as $\mathcal{T}(\sigma) = T_\sigma$, where T_σ is defined as in Theorem 6.2.
$\mathcal{C}(X)$ is complete with respect to the uniform metric since X is. Let us define $\mathcal{T}_p : \Sigma^\infty \to \mathcal{C}(X)$, defined as $\mathcal{T}_p(\sigma) = T_{\sigma|p} = T_{n_p} \circ T_{n_{p-1}} \circ T_{n_1}$.
First claim: $\mathcal{T}_p$ is continuous.
Given $\epsilon > 0$, $\sigma \in \Sigma^\infty$, $\sigma = (n_1 n_2 \ldots)$. Take $\delta < K_p = \prod_{j=1}^{p} k_{n_j}$. If $d_T(\sigma, \sigma') < \delta$, according to Proposition 5.1 $\sigma|p = \sigma'|p$ and $d_{sup}(\mathcal{T}_p(\sigma), \mathcal{T}_p(\sigma')) = d_{sup}(T_{\sigma|p}, T_{\sigma'|p}) < \epsilon$. Hence $\mathcal{T}_p$ is continuous.
Second claim: $(\mathcal{T}_p)$ converges uniformly to $\mathcal{T}$.

$$d^*(\mathcal{T}_p, \mathcal{T}) := \sup_{\sigma \in \Sigma^\infty} d_{sup}(\mathcal{T}_p(\sigma), \mathcal{T}(\sigma)) = \sup_{\sigma \in \Sigma^\infty} \sup_{x \in X} d(T_{\sigma|p}(x), T_\sigma(x))$$

then, defining $x_p := T_{\sigma|p}(x)$,

$$d^*(\mathcal{T}_p, \mathcal{T}) = \sup_{\sigma \in \Sigma^\infty} \sup_{x \in X} d(x_p, \widetilde{x}_\sigma) \leq K_p diam(X) \leq k^p diam(X),$$

using the inequality (6.14) for $M = X$. Consequently $d^*(\mathcal{T}_p, \mathcal{T}) \to 0$ as p tends to infinity, and $\mathcal{T}$ is continuous.
The continuity of π is equivalent to that of $\mathcal{T}$ since:

$$d(\pi(\sigma), \pi(\sigma')) = d(\widetilde{x}_\sigma, \widetilde{x}_{\sigma'}) = \sup_{x \in X}(T_\sigma(x), T_{\sigma'}(x)) = d_{sup}(\mathcal{T}(\sigma), \mathcal{T}(\sigma')).$$

□

Definition 6.3. *A sequence of self-maps (T_n) is a Picard sequence if there exists $\widetilde{x} \in X$ such that $\lim T_n \circ T_{n-1} \circ \ldots \circ T_1(x) = \widetilde{x}\ \forall x \in X$.*

Proposition 6.2. *On the hypotheses of Theorem 6.3, any sequence (T_{n_j}) is a Picard sequence.*

Remark 6.1. *Let us notice that by Theorem 2.1 we have also proved that the sequence of fixed points of $(T_{\sigma|p})$ converges to a quasi-fixed point for any $\sigma \in \Sigma^\infty$ with the hypotheses given.*

7. The fractal set of quasi-fixed points

In this section we consider backward orbits of type:

$$\{T_{n_1} \circ T_{n_2} \circ \ldots T_{n_p}(x)\}$$

and quasi-fixed points defined for $\sigma = (n_1 n_2 \ldots)$ as

$$\widetilde{x}_\sigma = \lim_{p\to\infty} T_{n_1} \circ T_{n_2} \circ \ldots T_{n_p}(\widetilde{x}_\sigma).$$

For the sake of simplicity we will use the same notation for the backward orbits:

$$T_{\sigma|p} := T_{n_1} \circ T_{n_2} \circ \ldots \circ T_{n_p},$$

for $p \in \mathbb{N}$ and $\sigma \in \Sigma^\infty$. Let us define a σ-invariant set M as in the previous section, that is to say, satisfying $T_{n_j}(M) \subseteq M$ if $\sigma = (n_1 n_2 \ldots)$. If M is σ-invariant then $T_{\sigma|p}(M) \subseteq M$ for all $p \in \mathbb{N}$. Also we have, if M is bounded,

$$diam(T_{\sigma|p}(M)) \leq (\prod_{j=1}^{p} k_{n_j}) diam(M). \tag{7.16}$$

Unlike the case of the previous section, if M is σ-invariant the sequence $(T_{\sigma|p}(M))$ is a nested decreasing sequence, that is to say,

$$M \supseteq T_{\sigma|1}(M) \supseteq T_{\sigma|2}(M) \supseteq \ldots \supseteq T_{\sigma|p}(M) \supseteq \ldots$$

since for instance

$$T_{n_1} \circ T_{n_2}(M) \subseteq T_{n_1}(M) \subseteq M.$$

Proposition 7.1. *If X is complete, M is σ-invariant and bounded, and for $\sigma = (n_1, n_2 \ldots)$, $K_p := \prod_{j=1}^{p} k_{n_j} \to 0$ as $p \to \infty$, then there exists $\widetilde{x}_\sigma \in \overline{M}$ such that for all $x \in M$*

$$\lim_{p\to\infty} T_{n_1} \circ T_{n_2} \circ \ldots \circ T_{n_p}(x) = \widetilde{x}_\sigma.$$

Further if M is σ-invariant and closed, there exists a unique quasi-fixed point $\widetilde{x}_\sigma \in M$ globally attracting in M, and the sequence of maps $(T_{\sigma|p})$ converges uniformly in M to T_σ defined as $T_\sigma(x) = \widetilde{x}_\sigma$ for all $x \in M$.

Proof. Let us define $M_p := T_{\sigma|p}(M)$. We have then a nested sequence of closed sets

$$\overline{M} \supseteq \overline{M_1} \supseteq \overline{M_2} \supseteq \ldots \supseteq \overline{M_p} \supseteq \ldots$$

whose diameter tends to zero since

$$diam(\overline{M_p}) = diam(M_p) \leq K_p diam(M).$$

By the Cantor's intersection theorem, there exists $\widetilde{x}_\sigma \in \overline{M}$ such that

$$\cap_{p\in\mathbb{N}} \overline{M_p} = \{\widetilde{x}_\sigma\}.$$

If $x \in M$, let us define as usual $x_p = T_{\sigma|p}(x) \in M_p$. Since $x_p, \widetilde{x}_\sigma \in \overline{M_p}$ then $d(x_p, \widetilde{x}_\sigma) \leq diam(\overline{M_p}) = diam(M_p) \to 0$, and thus $\lim_{p\to\infty} x_p = \widetilde{x}_\sigma$.
If M is closed, $\widetilde{x}_\sigma$ is a globally attracting quasi-fixed point in M. Let us see that $(T_{\sigma|p})$ converges uniformly in M to $T_\sigma(x) = \widetilde{x}_\sigma$ for $x \in M$. It comes from the inequality

$$d_{sup}(T_{\sigma|p}, T_\sigma) = \sup_{x\in M} d(x_p, \widetilde{x}_\sigma) \leq K_p diam(M).$$

□

Proposition 7.2. *If X is a complete metric space, $M \subseteq X$ is σ-invariant and compact, and $K_p \to 0$, there exists a quasi-fixed point $\widetilde{x}_\sigma \in M$ globally attracting in M. The sequence $(T_{\sigma|p}(M))$ converges to $\{\widetilde{x}_\sigma\}$ in the Hausdorff metric.*

Proof. It is a particular case of the previous proposition. The sequence $(T_{\sigma|p}(M))$ is a nested decreasing sequence of compact sets and consequently there exists $\widetilde{x}_\sigma \in M$ such that

$$\{\widetilde{x}_\sigma\} = \cap_{p\in\mathbb{N}} T_{\sigma|p}(M).$$

The sequence of the sets M_p converges to their intersection in the Hausdorff metric d_H. □

Let us define, as in the previous section, the set Q of quasi-fixed points associated to the sequence (T_n), and let us see that it is a fractal set.

Theorem 7.1. *If X is a complete and bounded metric space, and the contractivity factors are such that $k := \sup k_n < 1$ then $\forall \sigma \in \Sigma^\infty$, $\sigma = (n_1 n_2 \dots)$, the sequence of self-maps (T_{n_j}) owns a global attractor $\widetilde{x}_\sigma \in X$, and the sequence $(T_{\sigma|p})$ converges uniformly to the constant map defined as $T_\sigma(x) = \widetilde{x}_\sigma$ for all $x \in X$. The transformation*

$$\pi : \Sigma^\infty \to Q,$$

defined as $\pi(\sigma) = \widetilde{x}_\sigma$ is surjective and continuous. Further Q is a fractal set.

Proof. The first part is analogous to that of Theorem 6.3. Let us prove now that Q is a fractal set. Q is defined in this case as

$$Q = \cup_{\sigma \in \Sigma^\infty} (\cap_{p \in \mathbb{N}} T_{\sigma|p}(X)).$$

If s is the shift map, defined as $s(\sigma) = (n_2 n_3 \dots)$,

$$T_{n_1}(\widetilde{x}_{s(\sigma)}) = \lim_{p \to \infty} T_{n_1} \circ T_{n_2} \circ \dots \circ T_{n_p}(x) = \pi(\sigma),$$

and consequently $T_{n_1}(\pi(s(\sigma))) = \pi(\sigma)$. Consequently Q is an invariant set of the countable iterated function system $\{X, (T_n)\}$, that is to say,

$$Q = \cup_{n \in \mathbb{N}} T_n(Q).$$

As Q is the union of ”copies” of itself, it is a fractal set. □

Let us notice that the concept of invariance here is more restrictive than in Section 3, where only forward invariant sets are considered.

Remark 7.1. *This result proves that quasi-fixed points are the ”atoms” of a fractal set linked to the countable iterated function system $\{X, (T_n)\}$.*

Remark 7.2. *The set Q may not be a compact but if the cardinal of (T_n) is finite then Q is compact (see for instance [6], [10], [19]).*

If the space X is compact, the sequence $(T_{\sigma|p}(B))$ is composed of compact sets if B is compact. According to the theory of countable iterated function systems, defining $\forall B \in \mathcal{K}(X)$ the transformation

$$T(B) = \overline{\cup_{n=1}^{\infty} T_n(B)} \in \mathcal{K}(X),$$

One has $\lim_{m\to\infty} T^m(B) = A$ for some compact $A \subseteq X$ (see for instance [11], [17], [18]).

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