CONCERNING THE VECTOR-VALUED FRACTAL INTERPOLATION FUNCTIONS ON THE SIERPIŃSKI GASKET

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Abstract. The present paper is concerned with the study of vector-valued interpolation functions on the Sierpiński gasket by certain classes of fractal functions. This extends the known results on the real-valued and vector-valued fractal interpolation functions on a compact interval in $\mathbb{R}$ and the real-valued fractal interpolation on the Sierpiński gasket. We study the smoothness property of the vector-valued fractal interpolants on the Sierpiński gasket. A few elementary properties of the fractal approximants and the fractal operator that emerge in connection with the vector-valued fractal interpolation on the Sierpiński gasket are indicated. Some constrained approximation aspects of the vector-valued fractal interpolation function on the Sierpiński gasket are pointed out.

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1. INTRODUCTION

For approximating naturally occurring functions which display some kind of self-similarity under magnification, Barnsley [1] introduced univariate real-valued interpolation functions defined on a compact interval in $\mathbb{R}$. These functions are known as Fractal Interpolation Functions (FIFs for short), and their construction is rooted in the theory of Iterated Function System (IFS) [12]. The pioneering work on fractal interpolation [1] has received a good deal of attention in the literature, and it continues to flourish. Various extensions and generalizations of the notion of FIF, for instance, the hidden variable FIF [4], coalescence hidden variable FIF [8], vector-valued FIF [20], bilinear FIF [3], spline FIF [7], Hermite FIF [21], rational FIF [30], local FIF [19], FIF with partial self-similarity [16] have been considered. Researches reporting many important properties of FIFs, including the box dimension and Hausdorff dimension of their graphs, smoothness, stability, sensitivity, and constrained approximation properties are scattered in the literature. For expository reading materials on fractal approximation theory, we cite the well-known treatises [17, 18]. Navascués [22, 23] scrutinized a subclass of FIFs which transpired the existence of a parameterized family of fractal functions, called α-fractal functions, associated with a prescribed continuous function defined on a compact interval. Considerable interest is now being evinced in the study of fractal operator inherent with the notion of α-fractal function; see, for instance, [30, 31]. In all these approaches, FIFs are constructed using the concepts of IFS via the Banach fixed point theory. Recently, in reference [24], the author constructed a new nonlinear

2000 *Mathematics Subject Classification.* Primary 28A80; Secondary 41A30.

Key words and phrases. Fractal interpolation function, Sierpiński gasket, Hölder continuity, Hausdorff dimension, Fractal operator.

FIF using the Rakotch and the Matkowski fixed point theorems thereby initiating the application of various fixed point theorems in the construction of FIF.

The existing literature on FIF deals primarily on segments in the univariate case, and triangles and rectangles in the multivariate case. That is, the domains of the aforementioned extensively studied FIFs are some classical geometric objects, but not fractal sets. Smooth functions and fractal functions defined on the fractal domains have received recent attention. For instance, an explicit construction of a topological basis in the space of Whitney functions on the Cantor-type set is given in [10]. In [6], Celik et al. extended the concept of fractal interpolation function to include interpolation of a data set on a well-known self-similar fractal domain, namely, the Sierpiński Gasket (SG). Following this paper, Ruan [26] defined FIFs on post critically finite self-similar sets, which are introduced and analyzed by Kigami [15], and proved that FIFs have finite energy under certain conditions. In [25], Ri and Ruan established some basic properties of a class of FIF, which they call uniform FIFs, on SG. The present paper is also meant to be a modest contribution to the theory of fractal functions on a fractal domain, but does so in the context of vector-valued functions on SG.

Motivated by the real-valued FIF on SG studied in [6] and the vector-valued FIF on a real compact interval introduced in [20], in the present paper we seek vector-valued FIFs on SG. In fact, we will work in a more general setting in the sense that an arbitrary "base function" is considered in the place of a harmonic function in [6]. Thus, part of our investigations herein may be viewed as a systematic extension of the observation made in [6] to a broad class of functions, but in the setting of vector-valued functions. The transition from real-valued FIF on SG [6] or vector-valued FIF on a compact interval [20] to the vector-valued FIF on SG studied in this paper calls for the pursuance of a different, although analogous, tack.

In the subsequent parts of this paper we shall restrict to a subclass of the constructed vector-valued FIFs, which enables us to define a parameterized family of fractal functions associated with a prescribed continuous vector-valued function defined on SG. The reason to deal with this restricted class of the vector-valued FIFs on SG is that the approximation theoretic aspects of these fractal functions are more tractable than those in the general class. However, our methods are sufficiently general in scope to allow the readers to carefully carry out similar analysis to the general setting of the vector-valued FIFs on SG constructed herein. The vector-valued FIFs on SG to which the emphasis is given in this paper will be referred to as the *vector-valued α-fractal functions on SG*. Not unexpectedly, this class constitutes a counterpart to the so-called α-fractal functions in the context of the real-valued fractal interpolation on a compact interval in $\mathbb{R}$.

We discuss a few basic properties of these vector-valued α-fractal functions on SG such as the continuous dependence on the parameters, box counting and Hausdorff dimension of their graphs, and energy. Let us denote the fractal set SG under consideration by Δ. The vector-valued α-fractal function corresponding to a prescribed germ function on SG leads us, quite naturally, to the study of a fractal operator defined on $\mathcal{C}(\Delta, \mathbb{R}^N)$, the space of all continuous vector-valued functions defined on SG. Some constrained approximation aspects of the vector-valued α-fractal functions on SG are considered. It is our belief that the basic principles of our arguments for vector-valued FIFs on SG carry over to the case of FIFs on post critically finite self-similar sets. Thus, our paper could serve as a preliminary step towards the eventual treatment of approximation aspects of fractal functions on post critically finite self-similar sets and could find potential applications, for instance, in the theory of analysis on fractals, in particular, differential equations on fractals.

2. Vector-valued fractal interpolation functions on SG

We commence with a brief discussion of the pertinent definitions and preliminaries on the Sierpiński Gasket (SG). For details, the reader may refer [6, 15, 29].

Let us recall that SG, denoted by Δ, is the attractor of the IFS in the plane consisting of three functions $\{u_1, u_2, u_3\}$ such that

$$u_i : \mathbb{R}^2 \to \mathbb{R}^2,\ u_i(t) = \frac{1}{2}(t + p_i),\ i = 1, 2, 3,$$

where p_1, p_2 and p_3 are non-collinear points in $\mathbb{R}^2$. That is, SG is the unique non-empty compact set satisfying

$$\Delta = u_1(\Delta) \cup u_2(\Delta) \cup u_3(\Delta).$$

The sets $u_i(\Delta)$ are referred to as the cells of order 1. Since $t \in \mathbb{R}^2$, we felt that $\boldsymbol{t}$ would have been better notation, but for notational simplicity, we may just write t. Consider the alphabet $I = \{1, 2, 3\}$. Let us denote n-th times Cartesian product of I with itself by I^n. Fix a number $n \in \mathbb{N}$. The chain $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_n) = \omega_1\omega_2 \dots \omega_n$, where $\omega_j \in I$, is called a word of length n, denoted by $|\boldsymbol{\omega}| = n$. Consider the iterations $u_{\boldsymbol{\omega}} = u_{\omega_1} \circ u_{\omega_2} \circ \dots \circ u_{\omega_n}$ corresponding to a word $\boldsymbol{\omega}$ of length n. Define $V_0 = \{p_1, p_2, p_3\}$, the set of vertices of SG. The union of images of V_0 under these iterations $u_{\boldsymbol{\omega}}$ with $|\boldsymbol{\omega}| = n$ constitutes the set of n-th stage vertices V_n of Δ. Further, define $V_* = \cup_{n=1}^{\infty} V_n$. The set $u_{\boldsymbol{\omega}}(\Delta) = u_{\omega_1} \circ u_{\omega_2} \circ \dots \circ u_{\omega_n}(\Delta)$ is called an n-cell.

Let Γ_0 be the complete graph on the vertex set V_0. We construct the graph Γ_m recursively as follows. Let the graph Γ_{m-1} with the vertex set V_{m-1} for some $m \geq 1$, be constructed. Now to define the graph Γ_m on V_m, assume that for any $t, z \in V_m$, the edge relation $t \sim_m z$ holds if and only if $t = u_i(t'), z = u_i(z')$ with $t' \sim_{m-1} z'$ and $i \in I$. Equivalently, $t \sim_m z$ if and only if there exists $\boldsymbol{\omega} \in I^m$ such that $t, z \in u_{\boldsymbol{\omega}}(V_0)$.

Definition 2.1. For $m = 0, 1, 2, \dots$, the graph energy E_m on Γ_m is defined by

$$E_m(f) := \left(\frac{5}{3}\right)^m \sum_{t \sim_m z} (f(t) - f(z))^2.$$

The graph energy sequence $(E_m)_{m=0}^{\infty}$ satisfies $E_{m-1}(f) = \min E_m(\tilde{f})$, where the minimum is taken over all $\tilde{f}$ satisfying $\tilde{f}|_{V_{m-1}} = f$ for any $f : V_* \to \mathbb{R}$ and for any $m \geq 1$. Note that for each function f on V_*, the sequence $\big(E_m(f)\big)_{m=0}^{\infty}$ is increasing. The limit

$$E(f) := \lim_{m \to \infty} E_m(f)$$

is termed the energy of f on V_*. Furthermore, we say f has finite energy if $E(f) < +\infty$.

It is well known that a function f with $E(f) < +\infty$ is uniformly continuous on V_*. Since V_* is dense in SG, it follows at once that f can be uniquely extended to a continuous function on SG.

Definition 2.2. If f is continuous and $E_{m-1}(f) = E_m(f)$ for all $m \geq 1$, then f is called a harmonic function on SG.

Definition 2.3. The space $\mathcal{S}(H_0, V_m, \mathbb{R})$ of piecewise harmonic functions of level m is defined to be the space of continuous functions such that $f \circ u_{\boldsymbol{\omega}}$ is harmonic for all $\boldsymbol{\omega} \in I^m$. Further, define dom(E) $:= \{f \in \mathcal{C}(\triangle, \mathbb{R}) : E(f) < \infty\}$.

Remark 2.4. Note that $\mathcal{S}(H_0, V_m, \mathbb{R})$ is contained in dom(E) and it is a finite dimensional linear space of dimension equal to the cardinality of V_m.

Let $n \in \mathbb{N}$ and $B : V_n \to \mathbb{R}^N$ be a given vector-valued function. Following the fractal interplation theorem of Barnsley [1], we construct an IFS whose attractor is the graph of a continuous function $f : \Delta \to \mathbb{R}^N$ such that $f|_{V_n} = B$. Further, we note that this function f can be obtained as the unique fixed point of the RB-operator. To this end, define maps $W_{\boldsymbol{\omega}} : \Delta \times \mathbb{R}^N \to \Delta \times \mathbb{R}^N$ by

$$W_{\boldsymbol{\omega}}(t, x) = \Big(u_{\boldsymbol{\omega}}(t), F_{\boldsymbol{\omega}}(t, x)\Big), \ \boldsymbol{\omega} \in I^n,$$

where $F_{\boldsymbol{\omega}}(t, x) : \Delta \times \mathbb{R}^N \to \mathbb{R}^N$ are so chosen that the following conditions are satisfied.

$$\big\|F_{\boldsymbol{\omega}}(., x) - F_{\boldsymbol{\omega}}(., x')\big\| \leq c_{\boldsymbol{\omega}}\|x - x'\|, \quad F_{\boldsymbol{\omega}}\big(p_j, B(p_j)\big) = B\big(u_{\boldsymbol{\omega}}(p_j)\big),$$

for every $\boldsymbol{\omega} \in I^n, j \in I$, where $0 < c_{\boldsymbol{\omega}} < 1$. For our part, we consider

$$F_{\boldsymbol{\omega}}(t, x) = \alpha_{\boldsymbol{\omega}}(t)x + q_{\boldsymbol{\omega}}(t),$$

where $\alpha_{\boldsymbol{\omega}} : \Delta \to \mathbb{R}$ and $q_{\boldsymbol{\omega}} : \Delta \to \mathbb{R}^N$ are continuous functions with $\|\alpha\|_\infty := \{\|\alpha_{\boldsymbol{\omega}}\|_\infty : \boldsymbol{\omega} \in I^n\} < 1$. Let us consider the IFS $\{K; W_{\boldsymbol{\omega}} : \boldsymbol{\omega} \in I^n\}$. The following theorem is a counterpart to [1, Theorem 1]. Proof follows on similar lines and hence omitted.

Theorem 2.5. *Let $n \in \mathbb{N}$ and $B : V_n \to \mathbb{R}^N$ be the given function.*

(1) *The IFS $\{K; W_{\boldsymbol{\omega}}, \boldsymbol{\omega} \in I^n\}$ defined above has a unique attractor G_g. The set G_g is the graph of a continuous function $g : \Delta \to \mathbb{R}^N$ which satisfies $g|_{V_n} = B$, referred to as the vector-valued FIF on SG.*
(2) *Let $\mathcal{C}(\Delta, \mathbb{R}^N)$ denote the Banach space of all vector-valued continuous functions $h : \Delta \to \mathbb{R}^N$ with norm $\|h\|_\infty = \max\big\{\|h(t)\|_2 : t \in \Delta\big\}$. Let*

$$\mathcal{C}^*(\Delta, \mathbb{R}^N) = \Big\{h \in \mathcal{C}(\Delta, \mathbb{R}^N) : h(p_1) = B(p_1), \ h(p_2) = B(p_2), \ h(p_3) = B(p_3)\Big\}.$$

Define a map $T : \mathcal{C}^*(\Delta, \mathbb{R}^N) \to \mathcal{C}^*(\Delta, \mathbb{R}^N)$ by

$$(Th)(t) = F_{\boldsymbol{\omega}}\big(u_{\boldsymbol{\omega}}^{-1}(t), h(u_{\boldsymbol{\omega}}^{-1}(t))\big), \ \forall \ t \in u_{\boldsymbol{\omega}}(\Delta), \ \boldsymbol{\omega} \in I^n.$$

Then T is a contraction and its unique fixed point is the function g in the previous item. In particular,

$$g(t) = F_{\boldsymbol{\omega}}\big(u_{\boldsymbol{\omega}}^{-1}(t), g(u_{\boldsymbol{\omega}}^{-1}(t))\big), \ \forall \ t \in u_{\boldsymbol{\omega}}(\Delta), \ \boldsymbol{\omega} \in I^n.$$

Remark 2.6. In the above theorem crucial fact is that T is well-defined, which can be seen as follows. From the very construction of SG, for any $x \in u_{\boldsymbol{\omega}}(\Delta) \cap u_{\boldsymbol{\tau}}(\Delta)$, we have the following relation: $\boldsymbol{\omega} = \omega_1\omega_2 \dots \omega_{n-1}i$, $\boldsymbol{\tau} = \omega_1\omega_2 \dots \omega_{n-1}j$ and $x =$

$u_{\boldsymbol{\omega}}(p_j) = u_{\boldsymbol{\tau}}(p_i)$ for some $i, j \in I$. Now, using the so-called join-up conditions $F_{\boldsymbol{\omega}}\big(p_j, B(p_j)\big) = B\big(u_{\boldsymbol{\omega}}(p_j)\big)$, we have

$$\begin{aligned}(Th)(x) &= F_{\boldsymbol{\omega}}\big(u_{\boldsymbol{\omega}}^{-1}(x), h(u_{\boldsymbol{\omega}}^{-1}(x))\big)\\ &= F_{\boldsymbol{\omega}}\big(p_j, h(p_j)\big)\\ &= F_{\boldsymbol{\omega}}\big(p_j, B(p_j)\big)\\ &= B\big(u_{\boldsymbol{\omega}}(p_j)\big)\\ &= B(x),\end{aligned}$$

and

$$\begin{aligned}(Th)(x) &= F_{\boldsymbol{\tau}}\big(u_{\boldsymbol{\tau}}^{-1}(x), h(u_{\boldsymbol{\tau}}^{-1}(x))\big)\\ &= F_{\boldsymbol{\tau}}\big(p_i, h(p_i)\big)\\ &= F_{\boldsymbol{\tau}}\big(p_i, B(p_i)\big)\\ &= B\big(u_{\boldsymbol{\tau}}(p_i)\big)\\ &= B(x).\end{aligned}$$

Again, using the join-up conditions,

$$(Th)(p_i) = F_{\boldsymbol{\omega}}\big(u_{\boldsymbol{\omega}}^{-1}(p_i), h(u_{\boldsymbol{\omega}}^{-1}(p_i))\big) = F_{\boldsymbol{\omega}}\big(p_i, h(p_i)\big) = F_{\boldsymbol{\omega}}\big(p_i, B(p_i)\big) = B(p_i),$$

for each $i \in I$ and $\boldsymbol{\omega} = ii\dots i$ (n-times). Further, since $F_{\boldsymbol{\omega}}$ is continuous for each $\boldsymbol{\omega} \in I^n$, Th is continuous, and $Th \in \mathcal{C}^*(\Delta, \mathbb{R}^N)$. Hence, T is well-defined.

Remark 2.7. Note that in the construction of the real-valued FIF on SG given in [6], the emphasis is given only in the RB-operator approach and it is not evident if the graph of the FIF on SG is the attractor of a suitable IFS. Interpreting the polynomials of degree 1 as the classical harmonic functions on an interval, in [6] the author replaces polynomials of degree 1 by harmonic functions of fractal analysis. Thus, [6] provides SG counterpart to the affine FIF on an interval, but not to a general (non-affine) FIF. Our approach here is a realization that the classical Barnsley theory of the real-valued FIF on an interval, in its most general form as appeared in [1], can be extended almost verbatim to the SG setting.

Definition 2.8. Let $\Phi = \{\phi_1, \phi_2, \dots, \phi_n\}$ be an IFS on $\mathbb{R}^k$ and A be the attractor.

(1) Φ satisfies the Strong Separation Condition (SSC) if $\phi_i(A) \cap \phi_j(A) = \emptyset$ for distinct $i, j \in \{1, 2, \dots, n\}$.
(2) Φ is said to satisfy the Open Set Condition (OSC) if there exists a non-empty open set $U \subset \mathbb{R}^k$ such that
$$\cup_{i=1}^n \phi_i(U) \subset U; \quad \phi_i(U) \cap \phi_j(U) = \emptyset, \text{ for } i \neq j.$$
(3) If the above open set U satisfies $U \cap A \neq \emptyset$, then we say that Φ satisfies the Strong Open Set Condition (SOSC).

Proposition 2.9. Let $\phi_1, \phi_2, \dots, \phi_n$ be maps on $\mathbb{R}^k$ such that

$$r_i\|x - y\|_2 \le \|\phi_i(x) - \phi_i(y)\|_2 \le R_i\|x - y\|_2 \ , \forall\ x, y \in \mathbb{R}^k,$$

where $0 < r_i \le R_i < 1 \ \forall\ i \in \{1, 2, \dots, n\}$. Further assume that the maps ϕ_i satisfy SOSC. Let A be the unique compact set satisfying

$$A = \cup_{i=1}^n \phi_i(A).$$

Then $s_* \le \dim_H(A) \le \overline{\dim}_B(A) \le s^*$, where s_* and s^* satisfy $\sum_{i=1}^n r_i^{s_*} = 1$ and $\sum_{i=1}^n R_i^{s^*} = 1$ respectively.

Proof. In view of Proposition 9.6 in [9] we obtain the required upper bound for the upper box dimension of A. For the desired lower bound of the Hausdorff dimension of A we proceed as follows. Let U be an open set produced in the SOSC, $J := \{1, 2, \dots, n\}$, $A_{\boldsymbol{\omega}} := \phi_{\boldsymbol{\omega}}(A) := \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_m}(A)$ for $\boldsymbol{\omega} \in J^m$ and $J^* := \cup_{m \in \mathbb{N}} J^m$.

Since $U \cap A \neq \emptyset$, we have an index $\boldsymbol{\tau} \in J^*$ with $A_{\boldsymbol{\tau}} \subset U$. It can be seen that for any m and $\boldsymbol{\omega} \in J^m$, the sets $A_{\boldsymbol{\omega\tau}}$ are disjoint. Furthermore, the IFS $\{\phi_{\boldsymbol{\omega\tau}} : \boldsymbol{\omega} \in J^m\}$ satisfies the assumptions of Proposition 9.7 in [9]. Therefore, with the notation $r_{\boldsymbol{\omega}} = r_{\omega_1} r_{\omega_2} \dots r_{\omega_m}$, we have $s_m \leq \dim_H(A^*)$, where A^* is an attractor of the aforesaid IFS and s_m is such that $\sum_{\boldsymbol{\omega} \in J^m} r_{\boldsymbol{\omega\tau}}^{s_m} = 1$. Since $A^* \subset A$, we get $s_m \leq \dim_H(A^*) \leq \dim_H(A)$.

Suppose, on the contrary, that $\dim_H(A) < s_*$, where $\sum_{i=1}^n r_i^{s_*} = 1$. We have $s_m < s_*$. We obtain

$$\begin{aligned} r_{\boldsymbol{\tau}}^{-s_m} &= \sum_{\boldsymbol{\omega} \in J^m} r_{\boldsymbol{\omega}}^{s_m} \\ &\geq \sum_{\boldsymbol{\omega} \in J^m} r_{\boldsymbol{\omega}}^{\dim_H(A)} \\ &= \sum_{\boldsymbol{\omega} \in J^m} r_{\boldsymbol{\omega}}^{s_*} r_{\boldsymbol{\omega}}^{\dim_H(A) - s_*} \\ &\geq \sum_{\boldsymbol{\omega} \in J^m} r_{\boldsymbol{\omega}}^{s_*} r_{\max}^{m(\dim_H(A) - s_*)} \\ &= r_{\max}^{m(\dim_H(A) - s_*)}. \end{aligned}$$

Since $r_{\max} < 1$ and the term on the left side in the above expression is bounded, the above inequality provides a contradiction as m tends to infinity. Thus our assumption is wrong, which implies that $\dim_H(A) \geq s_*$, and hence the required result. □

Remark 2.10. The above proposition improves [9, Proposition 9.7] in the following sense. Proposition 9.7 in [9] has been proved with the SSC. However, we could replace this assumption with a relatively weaker assumption, namely, the SOSC.

Remark 2.11. Let us remark in passing that the IFS used in the construction of the real-valued FIF on a compact interval appeared in [1] satisfies the SOSC. Consequently, with the aid of the above theorem we can obtain bounds for the Hausdorff dimension of the graph of the FIF in [1]. More interestingly, the lower bound for the Hausdorff dimension of the graph of the FIF provided in [1, Theorem 4] can be obtained without the additional condition imposed thereat, namely,

$$t_1.t_N \leq (\min\{a_1, a_N\}) \Big(\sum_{n=1}^N t_n^l \Big)^{2/l}.$$

Theorem 2.12. *Let n be a fixed natural number. If the IFS $\{\Delta \times \mathbb{R}^N; W_{\boldsymbol{\omega}},\ \boldsymbol{\omega} \in I^n\}$ defined earlier in the construction of the vector-valued FIF f on SG satisfies the condition*

$$r_{\boldsymbol{\omega}} \|(t, x) - (z, y)\|_2 \leq \|W_{\boldsymbol{\omega}}(t, x) - W_{\boldsymbol{\omega}}(z, y)\|_2 \leq R_{\boldsymbol{\omega}} \|(t, x) - (z, y)\|_2,$$

for all $(t, x), (z, y) \in \Delta \times \mathbb{R}^N$, where $0 < r_{\boldsymbol{\omega}} \leq R_{\boldsymbol{\omega}} < 1\ \forall\ \boldsymbol{\omega} \in I^n$. Then

$$s_* \leq \dim_H(G_f) \leq \overline{\dim}_B(G_f) \leq s^*,$$

where s_* and s^* are such that $\sum_{\boldsymbol{\omega}\in I^n} r_{\boldsymbol{\omega}}^{s_*} = 1$ *and* $\sum_{\boldsymbol{\omega}\in I^n} R_{\boldsymbol{\omega}}^{s^*} = 1$.

Proof. Let Conv(Δ) be the convex hull of Δ and define $H = \big(\mathrm{Conv}(\Delta)\big)^0 \times \mathbb{R}^N$. It is plain to see that

$$u_i\Big(\big(\mathrm{Conv}(\Delta)\big)^0\Big) \cap u_j\Big(\big(\mathrm{Conv}(\Delta)\big)^0\Big) = \emptyset,$$

for every $i, j \in I$ with $i \neq j$. This immediately yields

$$W_{\boldsymbol{\omega}}(H) \cap W_{\boldsymbol{\tau}}(H) = \emptyset,$$

for every $\boldsymbol{\omega}, \boldsymbol{\tau} \in I^n$ satisfying $\boldsymbol{\omega} \neq \boldsymbol{\tau}$. Using $H \cap G_f \neq \emptyset$, one deduces that the IFS satisfies the SOSC. Hence the result follows from Theorem 2.9. □

3. On a Parameterized Family of Vector-valued Fractal Approximants on SG

In this section, we consider a subclass of the vector-valued FIFs on SG constructed in the previous section. This enable us to obtain a parameterized family of continuous vector-valued fractal approximants associated with a continuous function $f : \Delta \to \mathbb{R}^N$, called the source function. The research works on α-fractal functions by the first author [22, 23] motivated this part of the current paper.

3.1. **Vector-valued α-fractal Functions on SG.** Let $f : \Delta \to \mathbb{R}^N$ be a prescribed continuous function. Consider $B = f|_{V_n}$, where V_n is defined at the beginning of Section 2. In the IFS

$$F_{\boldsymbol{\omega}}(t, x) = \alpha_{\boldsymbol{\omega}}(t)x + q_{\boldsymbol{\omega}}(t),$$

considered in the previous section, we let

$$q_{\boldsymbol{\omega}}(t) = f(u_{\boldsymbol{\omega}}(t)) - \alpha_{\boldsymbol{\omega}}(t)b(t),$$

where $w \in I^n$, $b : \Delta \to \mathbb{R}^N$ is a fixed continuous map that fulfills the condition $b(p_1) = f(p_1)$, $b(p_2) = f(p_2)$ and $b(p_3) = f(p_3)$. To reduce some effort, we may represent by $b|_{V_0} = f|_{V_0}$ the conditions $b(p_1) = f(p_1)$, $b(p_2) = f(p_2)$ and $b(p_3) = f(p_3)$. Theorem 2.5 ensures the existence of a vector-valued fractal function, which we call vector-valued α-fractal function associated to the source function $f : \Delta \to \mathbb{R}^N$ and denote it by $f^{\alpha}_{n,b}$. The reader is encouraged to see [23] for an analogous construction of real-valued function on a compact interval. Furthermore,

$$f^{\alpha}_{n,b}(t) = f(t) + \alpha_{\boldsymbol{\omega}}\big(u_{\boldsymbol{\omega}}^{-1}(t)\big)\Big[f^{\alpha}_{n,b}\big(u_{\boldsymbol{\omega}}^{-1}(t)\big) - b\big(u_{\boldsymbol{\omega}}^{-1}(t)\big)\Big],\ \forall\, t \in u_{\boldsymbol{\omega}}(\Delta),\ \boldsymbol{\omega} \in I^n. \tag{3.1}$$

For notational convenience, we shall denote the function $f^{\alpha}_{n,b}$ by f^{α}. In fact, we obtain a class of fractal functions $\{f^{\alpha}_{n,b}\}$ for various choices of the set of n-th stage vertices V_n, and the maps α and b.

3.2. **Hölder Continuity.** This subsection is targeted to discuss the smoothness property of f^{α}, which is a fractal analogue of the given continuous function $f : \Delta \to \mathbb{R}^N$ obtained through the IFS:

$$u_{\boldsymbol{\omega}}(t) = \frac{1}{2^n}t + \sum_{k=1}^{n} \frac{1}{2^k}p_{\omega_k}; \quad F_{\boldsymbol{\omega}}(t, x) = \alpha_{\boldsymbol{\omega}}(t)x + f(u_{\boldsymbol{\omega}}(t)) - \alpha_{\boldsymbol{\omega}}(t)b(t),\ \boldsymbol{\omega} \in I^n.$$

To simply the presentation, we introduce the notation $a = \frac{1}{2^n}$ and $a_{\boldsymbol{\omega}} = \sum_{k=1}^{n} \frac{1}{2^k}p_{\omega_k}$ t. Then we have $u_{\boldsymbol{\omega}}(t) = at + a_{\boldsymbol{\omega}}$ for $\boldsymbol{\omega} \in I^n$. In this subsection, for computational convenience we consider the Sierpiński gasket with vertices $p_1 = (0,0),\ p_2 = (1,0)$ and $p_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Then we get $\|t - t'\|_2 \le 1$ for all $t, t' \in \Delta$.

We prove that for suitable choices of the parameters, the fractal function f^α preserves the Hölder continuity of the source function f. For $t \in \Delta$ and $\boldsymbol{\omega}_j \in I^n$, let

$$u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m}(t) = u_{\boldsymbol{\omega}_1} \circ u_{\boldsymbol{\omega}_2} \circ \cdots \circ u_{\boldsymbol{\omega}_m}(t),\ u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m}(\Delta) = u_{\boldsymbol{\omega}_1} \circ u_{\boldsymbol{\omega}_2} \circ \cdots \circ u_{\boldsymbol{\omega}_m}(\Delta).$$

Define a shift operator σ by

$$\sigma(\boldsymbol{\omega}_1\boldsymbol{\omega}_2 \ldots \boldsymbol{\omega}_m) = (\boldsymbol{\omega}_2\boldsymbol{\omega}_3 \ldots \boldsymbol{\omega}_m).$$

Let σ^k denote the $k-$fold autocomposition of σ such that for $1 \le k \le m-1$, $u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(t) = u_{\boldsymbol{\omega}_{k+1}\boldsymbol{\omega}_{k+2}\ldots\boldsymbol{\omega}_m}(t)$ otherwise $u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(t) = t$. Using the successive iteration and induction, we can prove the following lemma; see also [32].

Lemma 3.1. *Let f^α be an α-fractal function corresponding to f. For any $t \in \Delta$ and $\boldsymbol{\omega}_j \in I^n$,*

$$u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m}(t) = a^m t + \sum_{k=1}^{m} a^{k-1} a_{\boldsymbol{\omega}_k}$$

and

$$\begin{aligned} f^\alpha\big(u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m}(t)\big) =& \Big(\prod_{k=1}^{m} \alpha_{\boldsymbol{\omega}_k}(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(t))\Big) f^\alpha(t) + \sum_{r=1}^{m} \Big(\prod_{k=1}^{r-1} \alpha_{\boldsymbol{\omega}_k}(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(t))\Big) \\ & .f\big(u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(t)\big) - \sum_{r=1}^{m} \Big(\prod_{k=1}^{r} \alpha_{\boldsymbol{\omega}_k}(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(t))\Big) \\ & .b\big(u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_n)}(t)\big), \end{aligned}$$

where

$$u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(t) = a^{m-k} t + \sum_{l=1}^{m-k} a^{l-1} a_{\boldsymbol{\omega}_{k+l}}.$$

The following lemma can be consulted in [32].

Lemma 3.2. *Let $r_i, q_i,\ i = 1, 2, \ldots, m$, be given real numbers. Then*

$$\prod_{i=1}^{m} r_i - \prod_{i=1}^{m} q_i = \sum_{i=1}^{m} \Big(\prod_{k=1}^{m-1} c_k^{(i)}\Big)(r_i - q_i),$$

where, for all $k = 1, 2, \ldots, m-1$, each $c_k^{(i)}, i = 1, 2, \ldots m$, is a real number from the set

$$\{r_1, r_2, \ldots r_{i-1}, r_{i+1}, \ldots, r_m, q_1, q_2, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m\}.$$

Theorem 3.3. *Let the source function f, the parameter maps b, $\alpha_{\boldsymbol{\omega}}$ ($\boldsymbol{\omega} \in I^n$) be Hölder continuous functions with constants K_f, K_b, $K_{\alpha_{\boldsymbol{\omega}}}$ and Hölder exponents s_f, s_b, $s_{\alpha_{\boldsymbol{\omega}}}$ respectively. Define $\|\alpha\|_\infty = \max\{\|\alpha_{\boldsymbol{\omega}}\|_\infty : \boldsymbol{\omega} \in I^n\}$, $\delta = \frac{\|\alpha\|_\infty}{a}$ and $s_\alpha = \min\{s_{\alpha_{\boldsymbol{\omega}}} : \boldsymbol{\omega} \in I^n\}$. Then the following hold:*

1. *If $\|\alpha\|_\infty < \frac{1}{2^n}$, then f^α is a Hölder continuous function with exponent s, where $s = \min\{s_f, s_b, s_\alpha\}$.*
2. *If $\|\alpha\|_\infty = \frac{1}{2^n}$, then f^α is a Hölder continuous function with exponent $s - \mu$ for some $0 < \mu < 1$.*

(3) *If* $\|\alpha\|_\infty > \frac{1}{2^n}$, *then* f^α *is a Hölder continuous function with exponent* λ, *where* $0 < \lambda \leq s - 1 + \frac{\ln \|\alpha\|_\infty}{\ln a} < 1$.

Proof. For any $t, t' \in \Delta$, it is possible to find $m \geq 0$ such that $t \in u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m}(\Delta)$ and

$$a^{m+1} \leq \|t - t'\|_2 \leq a^m.$$

If $m = 0$, we set $u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m}(\Delta) = \Delta$. Let $t, t' \in u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m}(\Delta)$. Since $t \in u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m}(\Delta)$, there exists a $\bar{t} \in \Delta$ such that the following conditions hold due to Lemma 3.1,

$$t = u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m}(\bar{t}) = a^m\bar{t} + \sum_{h=1}^{m} a^{h-1}a_{\boldsymbol{\omega}_h} \tag{3.2}$$

and

$$\begin{aligned} f^\alpha(t) =& f^\alpha(u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots i\boldsymbol{\omega}_m}(\bar{t}))\\ =& \Big(\prod_{k=1}^{m} \alpha_{\boldsymbol{\omega}_k}(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(t))\Big) f^\alpha(\bar{t}) + \sum_{r=1}^{m}\Big(\prod_{k=1}^{r-1} \alpha_{\boldsymbol{\omega}_k}(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t}))\Big) f\big(u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big)\\ &- \sum_{r=1}^{m}\Big(\prod_{k=1}^{r} \alpha_{\boldsymbol{\omega}_k}(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t}))\Big) b\big(u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big). \end{aligned}$$

From (3.2), $\bar{t}$ can be written as

$$\bar{t} = a^{-m}\Big[t - \sum_{h=1}^{m} a^{h-1}a_{\boldsymbol{\omega}_h}\Big].$$

Using Lemma 3.1 and the above equation, we get

$$u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t}) = a^{-m}\Big[t - \sum_{h=1}^{m} a^{h-1}a_{\boldsymbol{\omega}_h}\Big] + \sum_{l=1}^{m-k} a^{l-1}a_{\boldsymbol{\omega}_{k+l}}.$$

Similarly, since $t' \in u_{\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m}(\Delta)$, there exists $\overline{t'} \in \triangle$ such that t' and $f^\alpha(t')$ have expressions similar to the above. Consequently,

$$\begin{aligned}
&\big\|f^\alpha(t) - f^\alpha(t')\big\|_2 \\
&\le \sum_{r=1}^{m}\Bigg\|\Big(\prod_{k=1}^{r-1}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big)\Big)\ f\big(u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big) \\
&- \Big(\prod_{k=1}^{r-1}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{x'})\big)\Big) f\big(L_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{x'})\big)\Bigg\|_2 \\
&+ \Bigg\|\Big(\prod_{k=1}^{m}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big)\Big)\ f^\alpha(\bar{t}) - \Big(\prod_{k=1}^{m}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\Big)\ f^\alpha(\overline{t'})\Bigg\|_2 \\
&+ \sum_{r=1}^{m}\Bigg\|\Big(\prod_{k=1}^{r}\alpha_{i_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big)\Big)\ b\big(u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big) \\
&- \Big(\prod_{k=1}^{r}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\Big)\ b\big(u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\Bigg\|_2 \\
\le &\sum_{r=1}^{m}\Big|\prod_{k=1}^{r-1}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big)\Big|\ \Big\|f\big(u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big) - f\big(u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\Big\|_2 \\
&+ \sum_{r=1}^{m}\Big|\prod_{k=1}^{r-1}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big) - \prod_{k=1}^{r-1}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\Big|\big\|f\big(u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\big\|_2 \\
&+ \sum_{r=1}^{m}\Big|\prod_{k=1}^{r}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big)\Big|\ \Big\|b\big(u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big) - b\big(u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\Big\|_2 \\
&+ \sum_{r=1}^{m}\Big|\prod_{k=1}^{r}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big) - \prod_{k=1}^{r}\alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\Big|\big\|b\big(u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\big\|_2 \\
&+ 2(\|\alpha\|_\infty)^m\|f^\alpha\|_\infty.
\end{aligned}$$

Let $s = \min\big\{s_f, s_b, s_\alpha\big\}$. The equality $\|u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t}) - u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\|_2 = a^{-r}\|t-t'\|_2$ together with the fact that f is Hölder continuous function with exponent s_f yields

$$\begin{aligned}
&\Big\|f\big(u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big) - f\big(u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\Big\|_2 \\
&\le K_f\Big\|u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t}) - u_{\sigma^{r-1}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\Big\|_2^{s_f} \\
&= K_f\Big[a^{1-r}\|t-t'\|_2\Big]^{s_f} \\
&\le K_f a^{1-r}\|t-t'\|_2^{s}.
\end{aligned}$$

On similar lines, we have

$$\begin{aligned}
\Big\|b\big(u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big) - b\big(u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\big)\Big\|_2 &\le K_b\Big\|u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t}) - u_{\sigma^r(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\overline{t'})\Big\|_2^{s_b} \\
&= K_b\Big[a^{-r}\|t-t'\|_2\Big]^{s_b} \\
&\le K_b a^{-r}\|t-t'\|_2^{s}.
\end{aligned}$$

By hypotheses, each $\alpha_{\boldsymbol{\omega}}$ is Hölder continuous with exponent $s_{\alpha_{\boldsymbol{\omega}}}$ and Hölder constant $K_{\alpha_{\boldsymbol{\omega}}}$. Let $K_\alpha = \max\{K_{\alpha_{\boldsymbol{\omega}}} : \boldsymbol{\omega} \in I^n\}$. By Lemma 3.2,

$$\begin{aligned}
&\Big|\prod_{k=1}^{r-1} \alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t})\big) - \prod_{k=1}^{r-1} \alpha_{\boldsymbol{\omega}_k}\big(u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t'})\big)\Big| \\
&\le \sum_{k=1}^{r-1} (\|\alpha\|_\infty)^{r-2} K_\alpha \Big\| u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t}) - u_{\sigma^k(\boldsymbol{\omega}_1\boldsymbol{\omega}_2\ldots\boldsymbol{\omega}_m)}(\bar{t'})\Big\|_2^{s_\alpha} \\
&= \sum_{k=1}^{r-1} (\|\alpha\|_\infty)^{r-2} K_\alpha \Big[a^{-k}\|t-t'\|_2\Big]^{s_\alpha} \\
&\le K_\alpha (\|\alpha\|_\infty)^{r-2} \|t-t'\|_2^s \sum_{k=1}^{r-1} a^{-k} \\
&\le K_\alpha (\|\alpha\|_\infty)^{r-2} \|t-t'\|_2^s \frac{a^{1-r}}{1-a} \\
&= \frac{K_\alpha \delta^{r-2}}{a(1-a)} \|t-t'\|_2^s.
\end{aligned}$$

Combining these estimates, we obtain

$$\begin{aligned}
&\big\|f^\alpha(t) - f^\alpha(t')\big\|_2 \\
&\le \sum_{r=1}^{m} \big(\|\alpha\|_\infty\big)^{r-1} K_f a^{1-r} \|t-t'\|_2^s + \sum_{r=1}^{m} \big(\|\alpha\|_\infty\big)^{r} K_b a^{-r} \|t-t'\|_2^s \\
&\quad + \sum_{r=2}^{m} \frac{K_\alpha \|f\|_\infty \delta^{r-2}}{a(1-a)} \|t-t'\|_2^s + \sum_{r=1}^{m} \frac{K_\alpha \|b\|_\infty \delta^{r-1}}{a(1-a)} \|t-t'\|_2^s + 2\big(\|\alpha\|_\infty\big)^m \|f^\alpha\|_\infty \\
&= \sum_{r=1}^{m} \big(\|\alpha\|_\infty\big)^{r-1} K_f a^{r-1} \|t-t'\|_2^s + \sum_{r=1}^{m} \big(\|\alpha\|_\infty\big)^{r} K_b a^{-r} \|t-t'\|_2^s \\
&\quad + \sum_{r=2}^{m} \frac{K_\alpha \|f\|_\infty \delta^{r-2}}{a(1-a)} \|t-t'\|_2^s + \sum_{r=1}^{m} \frac{K_\alpha \|b\|_\infty \delta^{r-1}}{a(1-a)} \|t-t'\|_2^s + 2\|f^\alpha\|_\infty \delta^m a^m.
\end{aligned}$$

Since $a^{m+1} \le \|t-t'\|_2$, for a suitable constant H we have

$$\begin{aligned}
&\big\|f^\alpha(t) - f^\alpha(t')\big\|_2 \\
&\le \sum_{r=1}^{m} \delta^{r-1} K_f \|x-t'\|_2^s + \sum_{r=1}^{m} \delta^r K_b \|t-t'\|_2^s \\
&\quad + \sum_{r=2}^{m} \frac{K_\alpha \|f\|_\infty \delta^{r-2}}{a(1-a)} \|t-t'\|_2^s + \sum_{r=1}^{m} \frac{K_\alpha \|b\|_\infty \delta^{r-1}}{a(1-a)} \|t-t'\|_2^s + \frac{2\|f^\alpha\|_\infty \delta^m}{a} \|t-t'\|_2 \\
&\le \left[\sum_{r=1}^{m} \delta^{r-1} K_f + \sum_{r=1}^{m} \delta^r K_b + \sum_{r=2}^{m} \frac{K_\alpha \|f\|_\infty \delta^{r-2}}{a(1-a)} + \sum_{r=1}^{m} \frac{K_\alpha \|b\|_\infty \delta^{r-1}}{a(1-a)} + \frac{2\|f^\alpha\|_\infty \delta^m}{a}\right] \|t-t'\|_2^s \\
&\le 4K \|t-t'\|_2^s \sum_{r=1}^{m+1} \delta^{r-1}.
\end{aligned}$$

Case 1: If $\delta < 1$, then we have

$$\sum_{r=1}^{m+1} \delta^{r-1} \leq \frac{1}{1-\delta}$$

and hence we obtain

$$|f^{\alpha}(t) - f^{\alpha}(t')| \leq \frac{4K}{1-\delta} \|t - t'\|_2^s.$$

That is, f^{α} is Hölder continuous with exponent s. Let t and t' do not belong to the same cell, but the conditions hold. In this case, t and t' must belong to two adjacent cells. Let $\tilde{t}$ be the common boundary point of the adjacent cells. Then one gets

$$\|f(t) - f(t')\|_2 \leq \|f(t) - f(\tilde{t})\|_2 + \|f(\tilde{t}) - f(t')\|_2.$$

Therefore, we have

$$\|f(t) - f(t')\|_2 \leq \frac{8H}{1-\delta} \|t - t'\|_2^s.$$

Case 2: If $\delta = 1$, then

$$\|f(t) - f(t')\|_2 \leq 4K(m+1)\|t - t'\|_2^s.$$

Since $\|t - t'\|_2 \leq a^m < 1$, we have $m \leq \ln(\|t - t'\|_2)/\ln(a)$. Using a known inequality

$$0 < -x^{\mu} \ln x \leq \frac{1}{\mu e}, \text{ for } 0 < x \leq 1 \text{ and } 0 < \mu < 1,$$

choose some $0 < \mu < 1$ so that

$$\begin{aligned}
(m+1)\|t - t'\|_2^s &\leq \Big(1 + \frac{\ln(\|t - t'\|_2)}{\ln a}\Big)\|t - t'\|_2^s \\
&= \|t - t'\|_2^s + \frac{-\|t - t'\|_2^{\mu} \ \ln(\|t - t'\|_2)}{|\ln a|}\|t - t'\|_2^{s-\mu} \\
&\leq \|t - t'\|_2^s + \frac{1}{\mu e |\ln a|}\|t - t'\|_2^{s-\mu} \\
&\leq \Big(1 + \frac{1}{\mu e \ |\ln a|}\Big)\|t - t'\|_2^{s-\mu}.
\end{aligned}$$

Therefore, we obtain

$$\|f(t) - f(t')\|_2 \leq 4K\Big(1 + \frac{1}{\mu e |\ln a|}\Big)\|t - t'\|_2^{s-\mu}.$$

Case 3: If $\delta > 1$, then

$$\|f(t) - f(t')\|_2 \leq 4K\frac{\delta^{m+1}}{\delta - 1} \ \|t - t'\|_2^s.$$

We take a positive number λ with $0 < \lambda < 1$ such that

$$\delta^{m+1}\|t - t'\|_2^s \leq \|t - t'\|_2^{\lambda}.$$

Further we get

$$\lambda \leq s + \frac{(m+1)\ln \delta}{\ln(\|t - t'\|_2)}.$$

Since $a^{m+1} \leq \|t-t'\|_2$, we obtain $\dfrac{1}{\ln(\|t-t'\|_2)} \leq \dfrac{1}{(m+1)\ln(a)}$. This gives $\lambda \leq s + \dfrac{\ln \delta}{\ln a} = s - 1 + \frac{\ln \|\alpha\|_\infty}{\ln a} < 1$. Hence, we have

$$\|f(t) - f(t')\|_2 \leq \frac{4K}{\delta - 1} \|t-t'\|_2^{\lambda}.$$

This completes the proof. □

3.3. **Continuous Dependence on Parameters.** Let $f \in \mathcal{C}(\Delta, \mathbb{R}^N)$ be a fixed source function. Let $b : \Delta \to \mathbb{R}^N$ be fixed and consider the vector-valued fractal function $f^{\alpha}_{n,b}$ corresponding to f (see the self-referential equation in (3.1)). Define

$$S = \big\{\alpha \in (\mathcal{C}(\Delta, \mathbb{R}))^{3^n} : \|\alpha\|_\infty \leq r < 1 \text{ and } r \text{ is a fixed number}\big\}$$

and consider the map $\Psi : S \to \mathcal{C}(\Delta, \mathbb{R}^N)$ defined by

$$\Psi(\alpha) = f^{\alpha}_{n,b}.$$

Theorem 3.4. *The map $\Psi : S \to \mathcal{C}(\Delta, \mathbb{R}^N)$ is continuous.*

Proof. For $\alpha, \beta \in S$, from the functional equation for the vector-valued α-fractal function, we have

$$\Psi(\alpha)(t) = f^{\alpha}_{n,b}(t) = f(t) + \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))\ (f^{\alpha}_{n,b} - b) \circ u^{-1}_{\boldsymbol{\omega}}(t),\ \forall\ t \in u_{\boldsymbol{\omega}}(\Delta),\ \boldsymbol{\omega} \in I^n$$

and

$$\Psi(\beta)(t) = f^{\beta}_{n,b}(t) = f(t) + \beta_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(f^{\beta}_{n,b} - b) \circ u^{-1}_{\boldsymbol{\omega}}(t),\ \forall\ t \in u_{\boldsymbol{\omega}}(\Delta),\ \boldsymbol{\omega} \in I^n.$$

Consequently,

$$\begin{aligned} f^{\alpha}_{n,b}(t) - f^{\beta}_{n,b}(t) =& \big[f(t) + \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(f^{\alpha}_{n,b} - b) \circ u^{-1}_{\boldsymbol{\omega}}(t)\big] - \big[f(t) + \beta_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(f^{\beta}_{n,b} - b) \circ u^{-1}_{\boldsymbol{\omega}}(t)\big] \\ =& \big[\alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))f^{\alpha}_{n,b}(u^{-1}_{\boldsymbol{\omega}}(t)) - \beta_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))f^{\beta}_{n,b}(u^{-1}_{\boldsymbol{\omega}}(t))\big] + \big(\beta_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t)) - \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))\big)b\big(u^{-1}_{\boldsymbol{\omega}}(t)\big) \\ =& \big(\alpha_{\boldsymbol{\omega}} f^{\alpha}_{n,b} - \beta_{\boldsymbol{\omega}} f^{\alpha}_{n,b} + \beta_{\boldsymbol{\omega}} f^{\alpha}_{n,b} - \beta_{\boldsymbol{\omega}} f^{\beta}_{n,b}\big)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big) + \big((\beta_{\boldsymbol{\omega}} - \alpha_{\boldsymbol{\omega}})b\big)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big) \\ =& \big((\alpha_{\boldsymbol{\omega}} - \beta_{\boldsymbol{\omega}})f^{\alpha}_{n,b} + \beta_{\boldsymbol{\omega}}(f^{\alpha}_{n,b} - f^{\beta}_{n,b})\big)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big) + \big((\beta_{\boldsymbol{\omega}} - \alpha_{\boldsymbol{\omega}})b\big)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big). \end{aligned}$$

Using the triangle inequality and the definition of the uniform norm, we have

$$\begin{aligned} \big|f^{\alpha}_{n,b}(t) - f^{\beta}_{n,b}(t)\big| \leq& \Big|\big((\alpha_{\boldsymbol{\omega}} - \beta_{\boldsymbol{\omega}})f^{\alpha}_{n,b} + \beta_{\boldsymbol{\omega}}(f^{\alpha}_{n,b} - f^{\beta}_{n,b})\big)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big)\Big| + \Big|\big((\beta_{\boldsymbol{\omega}} - \alpha_{\boldsymbol{\omega}})b\big)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big)\Big| \\ \leq& \|\alpha - \beta\|_\infty \|f^{\alpha}_{n,b}\|_\infty + \|\beta\|_\infty \|f^{\alpha}_{n,b} - f^{\beta}_{n,b}\|_\infty + \|\beta - \alpha\|_\infty \|b\|_\infty \\ =& \|\alpha - \beta\|_\infty(\|f^{\alpha}_{n,b}\|_\infty + \|b\|_\infty) + \|\beta\|_\infty \|f^{\alpha}_{n,b} - f^{\beta}_{n,b}\|_\infty. \end{aligned}$$

This implies that

$$\|f^{\alpha}_{n,b} - f^{\beta}_{n,b}\|_\infty \leq \|\alpha - \beta\|_\infty(\|f^{\alpha}_{n,b}\|_\infty + \|b\|_\infty) + \|\beta\|_\infty \|f^{\alpha}_{n,b} - f^{\beta}_{n,b}\|_\infty.$$

Using $1 - \|\beta\|_\infty \geq 1 - r$, finally we have,

$$\|\Psi(\alpha) - \Psi(\beta)\|_\infty = \|f^{\alpha}_{n,b} - f^{\beta}_{n,b}\|_\infty \leq \frac{\|\alpha - \beta\|_\infty}{1 - r}(\|f^{\alpha}_{n,b}\|_\infty + \|b\|_\infty).$$

The previous inequality reveals that Ψ is continuous at α. Since α is arbitrary, Ψ is continuous on S, establishing the assertion. □

Theorem 3.5. *Let $f \in \mathcal{C}(\Delta, \mathbb{R}^N)$ and $X_f = \big\{b \in C(\triangle, \mathbb{R}^n) : b|_{V_0} = f|_{V_0}\big\}$. Let $\alpha \in \big(\mathcal{C}(\Delta, \mathbb{R})\big)^{3^n}$ with $\|\alpha\|_\infty < 1$. Then the map $\Phi : X_f \to \mathcal{C}(\Delta, \mathbb{R}^N)$ defined by $\Phi(b) = f^{\alpha}_{n,b}$ is Lipschitz continuous.*

Proof. Let $b, c \in X_f$. From the functional equation for the vector-valued α-fractal function on SG, we have

$$\Phi(b)(t) = f^{\alpha}_{n,b}(t) = f(t) + \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))\ (f^{\alpha}_{n,b} - b) \circ u^{-1}_{\boldsymbol{\omega}}(t),\ \forall\ t \in u_{\boldsymbol{\omega}}(\Delta),\ \boldsymbol{\omega} \in I^n$$

and

$$\Phi(c)(t) = f^{\alpha}_{n,c}(t) = f(t) + \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(f^{\alpha}_{n,c} - c) \circ u^{-1}_{\boldsymbol{\omega}}(t),\ \forall\ t \in u_{\boldsymbol{\omega}}(\Delta),\ \boldsymbol{\omega} \in I^n.$$

From these two equations, for $t \in u_{\boldsymbol{\omega}}(\Delta)$ we get

$$\begin{aligned} f^{\alpha}_{n,b}(t) - f^{\alpha}_{n,c}(t) =& [f(t) + \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(f^{\alpha}_{n,b} - b)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big)] - [f(t) + \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(f^{\alpha}_{n,c} - c)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big)] \\ =& \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(f^{\alpha}_{n,b} - f^{\alpha}_{n,c})\big(u^{-1}_{\boldsymbol{\omega}}(t)\big) + \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))\ (c - b)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big). \end{aligned}$$

The above equation yields

$$\begin{aligned} |f^{\alpha}_{n,b}(t) - f^{\alpha}_{n,c}(t)| =& \Big|\alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(f^{\alpha}_{n,b} - f^{\alpha}_{n,c})\big(u^{-1}_{\boldsymbol{\omega}}(t)\big) + \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(c - b)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big)\Big|. \\ \le& \Big|\alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(f^{\alpha}_{n,b} - f^{\alpha}_{n,c})\big(u^{-1}_{\boldsymbol{\omega}}(t)\big)\Big| + \Big|\alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(c - b)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big)\Big|. \\ \le& \|\alpha\|_\infty \|f^{\alpha}_{n,b} - f^{\alpha}_{n,c}\|_\infty + \|\alpha\|_\infty \|c - b\|_\infty. \end{aligned}$$

The above inequality holds for all $t \in u_{\boldsymbol{\omega}}(\Delta)$, and therefore we have

$$\|f^{\alpha}_{n,b} - f^{\alpha}_{n,c}\|_\infty \le \|\alpha\|_\infty \|f^{\alpha}_{n,b} - f^{\alpha}_{n,c}\|_\infty + \|\alpha\|_\infty \|c - b\|_\infty.$$

This can be recasted as,

$$\|f^{\alpha}_{n,b} - f^{\alpha}_{n,c}\|_\infty \le \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|b - c\|_\infty.$$

That is, $\|\Phi(b) - \Phi(c)\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|b - c\|_\infty$, showing that Φ is a Lipschitz continuous map with a Lipschitz constant $\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}$. □

3.4. **Energy.** Here we derive conditions on the parameters so that $f^{\alpha}_{n,b}$ has finite energy, whenever so has the source function $f : \Delta \to \mathbb{R}^N$.

Recall the notion of energy of a function $f : \Delta \to \mathbb{R}$ given in Definition 2.1. For a function $f \in \mathcal{C}(\Delta, \mathbb{R}^N)$, $f = (f_1, f_2, \dots, f_N)$, where $f_j : \Delta \to \mathbb{R}$, we define the energy of f as follows.

$$E(f, \mathbb{R}^N) = \max_{1 \le j \le N} E(f_j).$$

Similarly, we define the energy $E(\alpha)$ for $\alpha \in (\mathcal{C}(\triangle, \mathbb{R}))^k$ by

$$E(\alpha, \mathbb{R}^k) = \max_{1 \le j \le k} E(\alpha_j).$$

Remark 3.6. Alternatively, we may define energy for a vector-valued function $f : \Delta \to \mathbb{R}^N$ as follows. First we define the graph energy $E^*_m(f, \mathbb{R}^N)$ on Γ_m by

$$E^*_m(f, \mathbb{R}^N) := \Big(\frac{5}{3}\Big)^m \sum_{t \sim_m z} \|f(t) - f(z)\|_2^2.$$

The energy $E^*(f, \mathbb{R}^N)$ of $f : \Delta \to \mathbb{R}^N$ can now be defined as

$$E^*(f, \mathbb{R}^N) = \lim_{m \to \infty} E^*_m(f, \mathbb{R}^N).$$

Using $\|x\|_\infty \le \|x\|_2 \le \sqrt{N}\ \|x\|_\infty$, we infer that

$$E^*(f, \mathbb{R}^N) < \infty \text{ if and only if } E(f, \mathbb{R}^N) < \infty.$$

We define $\mathrm{dom}(E,\mathbb{R}^N)=\big\{f\in\mathcal{C}(\Delta,\mathbb{R}^N): E(f,\mathbb{R}^N)<\infty\big\}$.

Theorem 3.7. *Let $n\in\mathbb{N}$. Consider the source function $f\in dom(E,\mathbb{R}^N)$ and the parameter map $b\in dom(E,\mathbb{R}^N)$ with $b|_{V_0}=f|_{V_0}$. If $\alpha\in(dom(E,\mathbb{R}))^{3^n}$ satisfies $\|\alpha\|_\infty<\frac{1}{\sqrt{5^n\ 4}}$, then the corresponding vector-valued α-fractal function $f^\alpha_{n,b}$ denoted by f^α belongs to $dom(E,\mathbb{R}^N)$. Furthermore, we have*

$$E(f^\alpha,\mathbb{R}^N)\le\frac{3E(f,\mathbb{R}^N)+5^n8\|\alpha\|^2_\infty E(b,\mathbb{R}^N)+5^n4(\|f^\alpha\|^2_\infty+2\|b\|^2_\infty)E(\alpha,\mathbb{R}^{3^n})}{1-5^n4\|\alpha\|^2_\infty}.$$

Proof. Using the functional equation for f^α, the following estimate for the coordinate functions of f^α can be easily obtained.

$$\begin{aligned}
\big|f^\alpha_j(t)-f^\alpha_j(z)\big|^2=&\Big|f_j(t)-f_j(z)+\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(t))f^\alpha_j(u^{-1}_{\boldsymbol\omega}(t))-\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(z))f^\alpha_j(u^{-1}_{\boldsymbol\omega}(z))\\
&+\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(z))b(u^{-1}_{\boldsymbol\omega}(z))-\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(t))b(u^{-1}_{\boldsymbol\omega}(t))\Big|^2\\
\le&\ 4\big|f_j(t)-f_j(z)\big|^2+4|\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(t))\big|^2\big|f^\alpha_j(u^{-1}_{\boldsymbol\omega}(t))-f^\alpha_j(u^{-1}_{\boldsymbol\omega}(z))\big|^2\\
&+4\big|f^\alpha_j(u^{-1}_{\boldsymbol\omega}(z))\big|^2\big|\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(t))-\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(z))\big|^2\\
&+8\big|\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(z))\big|^2\big|b_j(u^{-1}_{\boldsymbol\omega}(t))-b_j(u^{-1}_{\boldsymbol\omega}(z))\big|^2\\
&+8\big|b^\alpha_j(u^{-1}_{\boldsymbol\omega}(t))\big|^2\big|\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(t))-\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(z))\big|^2\\
\le&\ 4\big|f_j(t)-f_j(z)\big|^2+4\|\alpha\|^2_\infty\big|f^\alpha_j(u^{-1}_{\boldsymbol\omega}(t))-f^\alpha_j(u^{-1}_{\boldsymbol\omega}(z))\big|^2\\
&+4\|f^\alpha\|^2_\infty\big|\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(t))-\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(z))\big|^2\\
&+8\|\alpha\|^2_\infty\big|b_j(u^{-1}_{\boldsymbol\omega}(t))-b_j(u^{-1}_{\boldsymbol\omega}(z))\big|^2\\
&+8\|b\|^2_\infty\big|\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(t))-\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(z))\big|^2\\
\le&\ 4\big|f_j(t)-f_j(z)\big|^2+4\|\alpha\|^2_\infty\big|f^\alpha_j(u^{-1}_{\boldsymbol\omega}(t))-f^\alpha_j(u^{-1}_{\boldsymbol\omega}(z))\big|^2\\
&+4\big(\|f^\alpha\|^2_\infty+2\|b\|^2_\infty\big)\big|\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(t))-\alpha_{\boldsymbol\omega}(u^{-1}_{\boldsymbol\omega}(z))\big|^2\\
&+8\|\alpha\|^2_\infty\big|b_j(u^{-1}_{\boldsymbol\omega}(t))-b_j(u^{-1}_{\boldsymbol\omega}(z))\big|^2
\end{aligned}$$

On calculating the energy at the m-th level we obtain

$$\begin{aligned}
E_m(f^\alpha_j)\le&\ 4E_m(f_j)+\big(\frac{5}{3}\big)^n3^n4\|\alpha\|^2_\infty E_{m-n}(f^\alpha_j)\\
&+\big(\frac{5}{3}\big)^n3^n8\|\alpha\|^2_\infty E_{m-n}(b_j)+\big(\frac{5}{3}\big)^n3^n4\big(\|f^\alpha\|^2_\infty+2\|b\|^2_\infty\big)E_{m-n}(\alpha_{\boldsymbol\omega}).
\end{aligned}$$

Taking limit as $m\to\infty$, the above inequality yields

$$\begin{aligned}
E(f^\alpha_j)-5^n4\|\alpha\|^2_\infty E(f^\alpha_j)\le&\ 4E(f_j)+5^n8\|\alpha\|^2_\infty E(b_j)\\
&+5^n4\big(\|f^\alpha\|^2_\infty+2\|b\|^2_\infty\big)E(\alpha_{\boldsymbol\omega}).
\end{aligned}$$

Using the hypotesis $\|\alpha\|_\infty<\frac{1}{\sqrt{5^n\ 4}}$ and the definition of $E(b),E(f)$ and $E(\alpha)$ we have

$$\begin{aligned}
\Big(1-5^n4\|\alpha\|^2_\infty\Big)E(f^\alpha_j)\le&\ 4E(f,\mathbb{R}^N)+5^n8\|\alpha\|^2_\infty E(b,\mathbb{R}^N)\\
&+5^n4\big(\|f^\alpha\|^2_\infty+2\|b\|^2_\infty\big)E(\alpha,\mathbb{R}^{3^n}).
\end{aligned}$$

This proves the assertion. □

Remark 3.8. If α is a constant function, then we could get a weaker condition, namely, $\|\alpha\|_\infty < \frac{1}{\sqrt{5^n\ 2}}$ for the above theorem to hold. This improves the bound obtained in the setting of the real-valued α-fractal function on SG in [28].

Remark 3.9. The above theorem can be compared with Theorem 4.1 of [25] wherein the authors calculated the energy of the uniform (real-valued) fractal interpolation function. Theorem 4.1 of [25] deals with $n = 1,\ N = 1$ and the IFS $F_i(t, x) = dx + q_i(t)$, where $d \in (-1, 1)$, $q_i(t) = h_i(t)$ for all $t \in \Delta$ with $h_i(p_j) = B(q_{ij}) - d_j B(p_j),\ i, j \in I$, where $B : V_1 \to \mathbb{R}$ is a given function.

4. A Fractal Operator on $\mathcal{C}(\Delta, \mathbb{R}^N)$

Let $f \in \mathcal{C}(\Delta, \mathbb{R}^N)$ be arbitrary, but fixed. In the construction of vector-valued α-fractal function corresponding to f given in the previous section we take

$$b = Lf,$$

where $L : \mathcal{C}(\Delta, \mathbb{R}^N) \to \mathcal{C}(\Delta, \mathbb{R}^N)$ is an operator such that $(Lf)|_{V_0} = f|_{V_0}$. The corresponding fractal function $f^\alpha_{n,L}$ satisfies the functional equation:

$$f^\alpha_{n,L}(t) = f(t) + \alpha_{\boldsymbol{\omega}}(u^{-1}_{\boldsymbol{\omega}}(t))(f^\alpha_{n,L} - Lf)\big(u^{-1}_{\boldsymbol{\omega}}(t)\big),\ \forall\ t \in u_{\boldsymbol{\omega}}(\Delta),\ \boldsymbol{\omega} \in I^n.$$

Definition 4.1. The rule that assigns the vector-valued α-fractal function $f^\alpha_{n,L}$ to a given continuous function f is called the α-fractal operator denoted by $\mathcal{F}^\alpha_{n,L}$. That is, $\mathcal{F}^\alpha_{n,L} : \mathcal{C}(\Delta, \mathbb{R}^N) \to \mathcal{C}(\Delta, \mathbb{R}^N)$ is defined by

$$\mathcal{F}^\alpha_{n,L}(f) = f^\alpha_{n,L}.$$

For our convenience, we write $f^\alpha_{n,L}$ as f^α and the operator as $\mathcal{F}^\alpha$ instead of $\mathcal{F}^\alpha_{n,L}$.

Remark 4.2. Recall that by the definition of $f^\alpha_{n,L}$, the operator $\mathcal{F}^\alpha_{n,L}$ satisfies the following interpolatory property

$$\mathcal{F}^\alpha_{n,L}(f)(x) = f(x),\ \forall\ x \in V_n.$$

In the upcoming theorem we gather some elementary properties of the fractal operator $\mathcal{F}^\alpha$ defined on $\mathcal{C}(\Delta, \mathbb{R}^N)$, which is endowed with the uniform norm. For $J \subset \mathbb{R}$, similar results for the fractal operator $\mathcal{F}^\alpha$ defined on the space $\mathcal{C}(J, \mathbb{R})$ are well-known; see, for instance, [22, 23]. Hence the proof of the following theorem is withheld. For basic definitions from functional analysis and perturbation theory of operators needed in the sequel, the reader may refer the books [14, 27].

Theorem 4.3. *Let $\mathcal{F}^\alpha : \mathcal{C}(\Delta, \mathbb{R}^N) \to \mathcal{C}(\Delta, \mathbb{R}^N)$ be the fractal defined above and Id be the identity operator on $\mathcal{C}(\Delta, \mathbb{R}^N)$. The following hold.*

(a) *For any $f \in \mathcal{C}(\Delta, \mathbb{R}^N)$, the perturbation error satisfies*

$$\|\mathcal{F}^\alpha(f) - f\|_\infty \le \|\alpha\|_\infty \|\mathcal{F}^\alpha(f) - Lf\|_\infty.$$

Consequently,

$$\big\|\mathcal{F}^\alpha(f) - f\big\|_\infty \le \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|f - Lf\|_\infty.$$

This implies that the fractal operator $\mathcal{F}^\alpha$ is relatively bounded with respect to L. Further, if $\|\alpha\|_\infty = 0$ or $L = Id$, then $\mathcal{F}^\alpha = Id$.

(b) *If L is a bounded linear operator, then so is the fractal operator $\mathcal{F}^\alpha$. Further, the operator norm satisfies*
$$\|\mathcal{F}^\alpha\| \le 1 + \frac{\|\alpha\|_\infty \|Id - L\|}{1 - \|\alpha\|_\infty}.$$
(c) *If L is a bounded linear operator and $\|\alpha\|_\infty < \|L\|^{-1}$, then $\mathcal{F}^\alpha$ is bounded below and, in particular, injective. Consequently, in this case, $\mathcal{F}^\alpha$ is not compact.*
(d) *If $f_L \in \mathcal{C}(\Delta, \mathbb{R}^N)$ is a fixed point of L, then f_L is a fixed point of $\mathcal{F}^\alpha$ also.*
(e) *If α be such that $\alpha_j : \Delta \to \mathbb{R}$ is non-null for some $j \in I^n$, then the fixed points of L and the fixed points of the fractal operator $\mathcal{F}^\alpha$ are same.*

As mentioned earlier, the aforementioned properties of the fractal operator are well-known in the setting of $\mathcal{C}(J, \mathbb{R})$, where J is a compact interval in $\mathbb{R}$. In what follows, we provide a pair of results which forms an addendum to the aforementioned properties of the fractal operator.

Theorem 4.4. *There exists a non-trivial closed invariant subspace for the bounded linear fractal operator $\mathcal{F}^\alpha : \mathcal{C}(\Delta, \mathbb{R}^N) \to \mathcal{C}(\Delta, \mathbb{R}^N)$.*

Proof. Choose a non-zero function $f \in \mathcal{C}(\Delta, \mathbb{R}^N)$ such that $f(x) = 0$ for all $x \in V_n$. Let $\big(\mathcal{F}^\alpha\big)^r$ denote the composition of $\mathcal{F}^\alpha$ with itself r-times and $(\mathcal{F}^\alpha)^0(f) = f$. Consider the non-zero subspace
$$W_f = \operatorname{span}\big\{f, \mathcal{F}^\alpha(f), (\mathcal{F}^\alpha)^2(f), \dots\big\}.$$
Note that $\mathcal{F}^\alpha(W_f) \subseteq W_f$, that is, W_f is an invariant subspace of $\mathcal{F}^\alpha$. Next we see that if $g \in W_f$, then $g(x) = 0$ for all $x \in V_n$.

By the definition of W_f, there exist constants $\beta_i \in \mathbb{R}$ such that
$$g = \beta_1 (\mathcal{F}^\alpha)^{r_1}(f) + \beta_2 (\mathcal{F}^\alpha)^{r_2}(f) + \dots + \beta_m (\mathcal{F}^\alpha)^{r_m}(f),$$
where $r_i \in \mathbb{N} \cup \{0\}$. By the interpolatory property of the fractal operator, we have
$$\big(\mathcal{F}^\alpha(f)\big)(x) = f(x), \ \forall\, x \in V_n.$$
Consequently, for all $x \in V_n$, we have $g(x) = 0$.

Take $W = \overline{W_f}$. Then $\mathcal{F}^\alpha(W) \subseteq W$, and what remains is to show that $W \neq \mathcal{C}(\Delta, \mathbb{R}^N)$. For $h \in W$, there exists a sequence $(h_m)_{m \in \mathbb{N}} \subset W_f$ such that $h_m \to h$ uniformly. Since $h_m(x) = 0$ for all m and $x \in V_n$, it follows that $h(x) = 0$ for all $x \in V_n$. Therefore a function $g \in \mathcal{C}(\Delta, \mathbb{R}^N)$ such that $g(x) \neq 0$ for some $x \in V_n$. cannot be an element in W, that is, $W \neq \mathcal{C}(\Delta, \mathbb{R}^N)$, completing the proof. □

The next result is a generalization of [11, Theorem 1], which was proved in the setting of an operator on a Hilbert space.

Lemma 4.5. [5, Lemma 1]. *Let $(X, \|.\|)$ be a Banach space, $A : X \to X$ be a linear operator. Suppose there exist constants $\lambda_1, \lambda_2 \in [0, 1)$ such that*
$$\|Ax - x\| \le \lambda_1 \|x\| + \lambda_2 \|Ax\|, \quad \forall\, x \in X.$$
Then A is a topological isomorphism, and
$$\frac{1 - \lambda_2}{1 + \lambda_1}\|x\| \le \|A^{-1}x\| \le \frac{1 + \lambda_2}{1 - \lambda_1}\|x\|, \quad \forall\, x \in X.$$

Theorem 4.6. *Let L be a bounded linear operator and $\|\alpha\|_\infty < \|L\|^{-1}$. Then the fractal operator $\mathcal{F}^\alpha : \mathcal{C}(\Delta, \mathbb{R}^N) \to \mathcal{C}(\Delta, \mathbb{R}^N)$ is a topological isomorphism.*

Proof. In view of Theorem 4.3 we have

$$\begin{aligned}\big\|f-\mathcal{F}^\alpha(f)\big\|_\infty &\le \|\alpha\|_\infty \big\|\mathcal{F}^\alpha(f)-Lf\big\|_\infty\\ &\le \|\alpha\|_\infty\Big[\big\|\mathcal{F}^\alpha(f)\big\|_\infty+\|Lf\|_\infty\Big]\\ &= \|\alpha\|_\infty\Big[\big\|\mathcal{F}^\alpha(f)\big\|_\infty+\|L\|\|f\|_\infty\Big].\end{aligned}$$

Recall that the standing assumption on α is that $\|\alpha\|_\infty < 1$. In addition, by the hypothesis, we have $\|\alpha\|_\infty < \|L\|^{-1}$. From the previous lemma, it follows that the fractal operator $\mathcal{F}^\alpha$ is a topological isomorphism. □

Remark 4.7. Let $J \subset \mathbb{R}$ be a compact interval. It is known that [22] for $\|\alpha\|_\infty < \big(1+\|Id-L\|\big)^{-1}$, the fractal operator $\mathcal{F}^\alpha : \mathcal{C}(J,\mathbb{R}) \to \mathcal{C}(J,\mathbb{R})$ is a topological isomorphism. Note that

$$\|L\| = \|L-Id+Id\| \le \|Id-L\|+1,$$

Therefore, thanks to [5, Lemma 1], our approach in the previous theorem illustrates that the fractal operator $\mathcal{F}^\alpha$ is a topological isomorphism with a weaker condition on α.

5. Some Approximation Aspects of Vector-valued α-fractal functions on SG

In view of the fact that a topological isomorphism preserves Schauder bases, the following result is an immediate consequence of Theorem 4.6. However, for the sake of exposition, we provide a slightly expanded details of the arguments.

Theorem 5.1. *The space $\mathcal{C}(\Delta,\mathbb{R})$ of all real-valued continuous functions on SG endowed with the supnorm possesses a Schauder basis consisting of α-fractal functions.*

Proof. Following the notation in [13], let $\{\psi_\xi : \xi \in \mathcal{V}\}$ be a Schauder-basis for $\mathcal{C}(\Delta,\mathbb{R})$ established in [13, Proposition 4.2], which can be viewed as an analogue on SG of the classical Faber-Schauder system. Choose an appropriate bounded linear operator L and consider the parameter map α such that $\|\alpha\|_\infty < \|L\|^{-1}$. The corresponding fractal operator $\mathcal{F}^\alpha : \mathcal{C}(\Delta,\mathbb{R}) \to \mathcal{C}(\Delta,\mathbb{R})$ is a topological isomorphism. Let $f \in \mathcal{C}(\Delta,\mathbb{R})$ so that $(\mathcal{F}^\alpha)^{-1}(f) \in \mathcal{C}(\Delta,\mathbb{R})$. By [13, Proposition 4.2] we have

$$(\mathcal{F}^\alpha)^{-1}(f) = \sum_{i=0}^{\infty}\sum_{\xi\in\mathcal{V}_i} c_\xi\big((\mathcal{F}^\alpha)^{-1}(f)\big)\ \psi_\xi,$$

where $\cup_{i=0}^{\infty}\mathcal{V}_i = \mathcal{V}$ (The reader should consult [13] for these notation). Since $\mathcal{F}^\alpha$ is a bounded linear map we get

$$f = \sum_{i=0}^{\infty}\sum_{\xi\in\mathcal{V}_i} c_\xi\big((\mathcal{F}^\alpha)^{-1}(f)\big)\ \psi_\xi^\alpha,$$

where $\psi_\xi^\alpha := \mathcal{F}^\alpha(\psi_\xi)$. To prove the uniqueness of the representation, let $f = \sum_{i=0}^{\infty}\sum_{\xi\in\mathcal{V}_i} d_\xi\ \psi_\xi^\alpha$. By the continuity of the map $(\mathcal{F}^\alpha)^{-1}(f)$ we have

$$(\mathcal{F}^\alpha)^{-1}(f) = \sum_{i=0}^{\infty}\sum_{\xi\in\mathcal{V}_i} d_\xi\ \psi_\xi,$$

from which we deduce that $d_\xi = c_\xi\big((\mathcal{F}^\alpha)^{-1}(f)\big)$. Therefore, $\{\psi_\xi^\alpha : \xi \in \mathcal{V}\}$ is a Schauder basis consisting of α-fractal functions for the space $\mathcal{C}(\Delta, \mathbb{R})$. □

The following definition is a counterpart to Definition 2.3.

Definition 5.2. The space $\mathcal{S}(H_0, V_m, \mathbb{R}^N)$ of vector-valued piecewise harmonic functions of level m is defined to be the space of continuous functions f such that $f \circ u_{\boldsymbol{\omega}}$ is harmonic for all $\boldsymbol{\omega} \in I^m$, where $u_{\boldsymbol{\omega}}(t) = \frac{1}{2^m}t + \sum_{k=1}^m \frac{1}{2^k} p_{w_k}$.

The following theorem about denseness of vector-valued piecewise harmonic functions can be easily proved; see, for instance, [29, Theorem 1.4.4].

Theorem 5.3. *Any function f in $\mathcal{C}(\Delta, \mathbb{R}^N)$ can be approximated uniformly by a sequence (h_m) in $\mathcal{S}(H_0, V_m, \mathbb{R}^N)$ with $h_m|_{V_m} = f|_{V_m}$.*

Definition 5.4. Consider the space $\mathcal{S}(H_0, V_m, \mathbb{R}^N)$ of vector-valued piecewise harmonic functions of level m and the bounded linear operator $\mathcal{F}^\alpha : \mathcal{C}(\Delta, \mathbb{R}^N) \to \mathcal{C}(\Delta, \mathbb{R}^N)$. The space $\mathcal{F}^\alpha\big(\mathcal{S}(H_0, V_m, \mathbb{R}^N)\big)$ will be referred to as the space of all (vector-valued) fractal piecewise harmonic functions. That is, a continuous function $h^\alpha : \Delta \to \mathbb{R}^N$ is a fractal piecewise harmonic function if $h^\alpha = \mathcal{F}^\alpha(h)$ for some piecewise harmonic function $h \in \mathcal{S}(H_0, V_m, \mathbb{R}^N)$.

In the following theorem we show that a given $f \in \mathcal{C}(\triangle, \mathbb{R}^N)$ can be uniformly well-approximated by a fractal piecewise harmonic function.

Theorem 5.5. *Let $f \in \mathcal{C}(\Delta, \mathbb{R}^N)$ and $\epsilon > 0$. Let $n \in \mathbb{N}$ and consider a bounded linear operator $L : \mathcal{C}(\Delta, \mathbb{R}^N) \to \mathcal{C}(\Delta, \mathbb{R}^N)$, $L \neq Id$ satisfying $(Lf)|_{V_0} = f|_{V_0}$. Then there exist a scale vector $\alpha = \alpha(\epsilon, L)$ in $(\mathcal{C}(\Delta, \mathbb{R}))^{3^n}$, $\alpha \neq 0$ and a fractal piecewise harmonic function $h_{n,L}^\alpha \in \mathcal{C}(\triangle, \mathbb{R}^N)$ such that*

$$\|f - h_{n,L}^\alpha\|_\infty < \epsilon.$$

Proof. Let $\epsilon > 0$ be given. By Theorem 5.3, there exists a piecewise harmonic function $h \in \mathcal{C}(\Delta, \mathbb{R}^N)$ such that

$$\|f - h\|_\infty < \frac{\epsilon}{2}.$$

For a natural number n and for a bounded linear operator $L : \mathcal{C}(\triangle, \mathbb{R}^N) \to \mathcal{C}(\triangle, \mathbb{R}^N)$, $L \neq Id$ satisfying $(Lf)|_{V_0} = f|_{V_0}$, select $\alpha \in (\mathcal{C}(\triangle, \mathbb{R}))^{3^n}$, $\alpha \neq 0$ such that

$$\|\alpha\|_\infty < \frac{\frac{\epsilon}{2}}{\frac{\epsilon}{2} + \|Id - L\|\|h\|_\infty}.$$

Consider $h_{n,L}^\alpha = \mathcal{F}_{n,L}^\alpha(h)$ with the aforementioned choice of α. Then, using item (a) in Theorem 4.3, we have

$$\begin{aligned}
\|f - h_{n,L}^\alpha\|_\infty &\leq \|f - h\|_\infty + \|h - h_{n,L}^\alpha\|_\infty \\
&\leq \|f - h\|_\infty + \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|Id - L\|\|h\|_\infty \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

completing the proof. □

Theorem 5.6. *Let $f \in \mathcal{C}(\Delta, \mathbb{R}^N)$ and $\epsilon > 0$. Let $n \in \mathbb{N}$ and $\alpha \in (\mathcal{C}(\Delta, \mathbb{R}))^{3^n}$ be arbitrary but fixed. Then, there exist a bounded linear operator $L : \mathcal{C}(\triangle, \mathbb{R}^N) \to \mathcal{C}(\triangle, \mathbb{R}^N)$, $L \neq Id$ satisfying $(Lf)|_{V_0} = f|_{V_0}$ and an α-fractal piecewise harmonic function $h^\alpha_{n,L} = \mathcal{F}^\alpha_{n,L}(h)$ such that*

$$\|f - h^\alpha_{n,L}\|_\infty < \epsilon.$$

Proof. Let $\epsilon > 0$ be given. By Theorem 5.3, there exists a piecewise harmonic function $h \in \mathcal{C}(\Delta, \mathbb{R}^N)$ such that

$$\|f - h\|_\infty < \frac{\epsilon}{2}.$$

Choose a natural number n and a scale vector $0 \neq \alpha \in (\mathcal{C}(\Delta, \mathbb{R}))^{3^n}$ satisfying $\|\alpha\|_\infty < 1$, but otherwise arbitrary. Now let us consider a bounded linear operator $L : \mathcal{C}(\Delta, \mathbb{R}^N) \to \mathcal{C}(\Delta, \mathbb{R}^N)$, $L \neq Id$ satisfying $(Lf)|_{V_0} = f|_{V_0}$, such that

$$\|Id - L\| < \frac{1 - \|\alpha\|_\infty}{\|\alpha\|_\infty \|h\|_\infty} \frac{\epsilon}{2}.$$

With the above choice of L in the construction of the fractal operator $\mathcal{F}^\alpha_{n,L}$, consider $h^\alpha_{n,L} = \mathcal{F}^\alpha_{n,L}(h)$. We have

$$\begin{aligned}
\|f - h^\alpha_{n,L}\|_\infty &\leq \|f - h\|_\infty + \|h - h^\alpha_{n,L}\|_\infty \\
&\leq \|f - h\|_\infty + \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|Id - L\| \|h\|_\infty \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

and this completes the proof. □

Remark 5.7. Let $f \in \mathcal{C}(\triangle, \mathbb{R}^N)$. The above theorems, in particular, assert the following.

(1) For $m \in \mathbb{N}$, let $\alpha^m \in (\mathcal{C}(\Delta, \mathbb{R}))^{3^n}$, $\|\alpha^m\|_\infty < 1$ and $\alpha^m \to 0$ as $m \to \infty$. Then there exists a sequence of fractal piecewise harmonic functions $\big(h^{\alpha^m}_{n,L}\big)_{m \in \mathbb{N}}$ which converges to f uniformly as $m \to \infty$.
(2) Let $(L_m)_{m \in \mathbb{N}}$ be a sequence of bounded linear operators on $\mathcal{C}(\Delta, \mathbb{R}^N)$ satisfying $L_m(g) \to g$ for each $g \in \mathcal{C}(\Delta, \mathbb{R}^N)$. Then there exists a sequence of fractal piecewise harmonic functions $\big(h^\alpha_{n,L_m}\big)_{m \in \mathbb{N}}$ which converges to f uniformly. In particular, the piecewise harmonic operators corresponding to f serve our purpose.

The following theorem is a direct consequence of the above remark.

Theorem 5.8. *Let $\mathcal{F}^\alpha_{n,P_k}\big(\mathcal{S}(H_0, V_m, \mathbb{R}^N)\big)$ be the class of all fractal piecewise harmonic functions with a fixed choice of the α and piecewise harmonic operator P_k. The set $\bigcup_{k \in \mathbb{N}} \mathcal{F}^\alpha_{n,P_k}\big(\mathcal{S}(H_0, V_m, \mathbb{R}^N)\big)$ is dense in $\mathcal{C}(\triangle, \mathbb{R}^N)$.*

Acknowledgements. The first author thanks the University Grants Commission (UGC), India, for financial support in the form of a Junior Research Fellowship.

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