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# On Hardy kernels as reproducing kernels

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*Abstract.* Hardy kernels are a useful tool to define integral operators on Hilbertian spaces like $L^2(\mathbb{R}^+)$ or $H^2(\mathbb{C}^+)$. These kernels entail an algebraic $L^1$-structure which is used in this work to study the range spaces of those operators as reproducing kernel Hilbert spaces. We obtain their reproducing kernels, which in the $L^2(\mathbb{R}^+)$ case turn out to be Hardy kernels as well. In the $H^2(\mathbb{C}^+)$ scenario, the reproducing kernels are given by holomorphic extensions of Hardy kernels. Other results presented here are theorems of Paley–Wiener type, and a connection with one-sided Hilbert transforms.

## 1 Introduction

Let $1 \le p < \infty$, and let $H$ be a Hardy kernel of index $p$, that is, a mapping $H : (0,\infty) \times (0,\infty) \to \mathbb{C}$ which is homogenous of degree $-1$ and satisfies $\int_0^\infty |H(1,s)|s^{-1/p}\,ds < \infty$ (see Definition 2.1). As a straightforward consequence of the celebrated Hardy's inequality [9, Theorem 319], one obtains that $H$ defines an operator $A_H$ given by

$$(A_H f)(r) := \int_0^\infty H(r,s) f(s)\,ds, \quad r > 0,\ f \in L^p(\mathbb{R}^+), \tag{1.1}$$

which is bounded on $L^p(\mathbb{R}^+)$, where $\mathbb{R}^+ := (0,\infty)$. Hardy's inequality also allows us to define a bounded operator $D_H$ on the Hardy spaces on the half plane $H^p(\mathbb{C}^+)$, where $\mathbb{C}^+ := \{z \in \mathbb{C} \,|\, \Re z > 0\}$, by

$$\begin{aligned}(D_H F)(z) &:= \int_0^\infty H(|z|,s) F^\theta(s)\,ds \\ &= \int_0^\infty H(1,s) F(sz)\,ds, \quad z = |z|e^{i\theta} \in \mathbb{C}^+,\ F \in H^p(\mathbb{C}^+),\end{aligned} \tag{1.2}$$

where $F^\theta(r) := F(re^{i\theta})$, for $r > 0$, $\theta \in (-\pi/2, \pi/2)$. Indeed, the last term in (1.2) shows that $D_H F$ is holomorphic (see, for example, [11]), and the boundedness of $D_H F$ follows by an application of Hardy's inequality together with the realization of the norm of $H^p(\mathbb{C}^+)$ given in [19] by

$$\|F\|_{H^p} = \sup_{-\pi/2<\theta<\pi/2} \left( \frac{1}{2\pi} \int_0^\infty |F^\theta(r)|^p\,dr \right)^{\frac{1}{p}}, \qquad F \in H^p(\mathbb{C}^+). \tag{1.3}$$

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We will refer to these families of bounded operators on $L^p(\mathbb{R}^+)$ and $H^p(\mathbb{C}^+)$ as Hardy operators. These families have been actively studied, and are often labeled as Hausdorff operators due to its relation to the Hausdorff summability method through the function $\varphi(t) := H(t,1)$ for $t > 0$ (see the survey articles [4, 15] for more details).

On the other hand, recall that a Hilbert space $X$ of complex-valued functions with domain $\Omega$ is said to be a reproducing kernel Hilbert space (RKHS) if and only if point evaluations $L_x f := f(x)$ are continuous functionals for all $x \in \Omega$. Then, by the Riesz representation theorem, for each $x \in \Omega$, there exists a unique $K_x \in X$ such that $f(x) = L_x f = (f \,|\, K_x)$ for all $f \in X$, where $(\cdot|\cdot)$ denotes the inner product in $X$. Then the reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}$ of $X$ is defined by

$$K(x, y) := K_y(x) = (K_y \,|\, K_x), \quad x, y \in \Omega.$$

The kernel $K$ determines the space $X$. More precisely, $X$ can be recovered from $K$ as the completion of $\operatorname{span}\{K_x \,|\, x \in \Omega\}$ under the norm given by scalar product $(K_y \,|\, K_x) := K(x, y)$ (see the proof of the Moore–Aronszajn theorem [1]).

In this paper, we focus on the range spaces of Hardy operators in the Hilbertian case, that is, for $p = 2$. We show that these spaces are RKHSs and obtain their reproducing kernels. Our work is partly motivated by papers [7, 8], where the range spaces of generalized Cesàro operators $\mathcal{C}_\alpha$ on $L^2(\mathbb{R}^+)$ and $H^2(\mathbb{C}^+)$ are analyzed as RKHSs. In this context, it is more appropriate to deal with Hardy operators using Hardy kernels $H$ rather than one-dimensional functions $\varphi$ associated with Hausdorff operators. Indeed, the set $\mathfrak{H}_p$ of Hardy kernels of index $p$ is naturally endowed with a structure of convolution (see [2, 6, 13]). More precisely, $\mathfrak{H}_p$ is a Banach algebra with multiplication $\bullet$ given by

$$(H \bullet G)(r, s) = \int_0^\infty H(r, t) G(t, s)\, dt \tag{1.4}$$

(see Section 2).

In the setting of Hardy operators on $L^2(\mathbb{R}^+)$, our main result is that, for a Hardy kernel $H$ of index 2, the range space $\mathcal{A}(H) = A_H(L^2(\mathbb{R}^+))$ becomes an RKHS (continuously included in $L^2(\mathbb{R}^+)$) if and only if $H$ belongs to a certain ideal of $\mathfrak{H}_2$ (see Theorem 3.3). In this case, the reproducing kernel $K_H$ of $\mathcal{A}(H)$ is itself another Hardy kernel, given by

$$K_H = H \bullet H^*, \tag{1.5}$$

where $H^*$ is the adjoint kernel of $H$ (see Definition 2.2).

In the setting of Hardy spaces on the half plane, we prove in Theorem 4.3 that, for a given Hardy kernel $H$, the range space $\mathcal{D}(H)$ of a Hardy operator $D_H$ is an RKHS, continuously included in $H^2(\mathbb{C}^+)$, with reproducing kernel given by

$$\mathcal{K}_H = (H \bullet \mathcal{S} \bullet H^*)^{hol}.$$

Here, $\mathcal{S}$ is the Stieltjes kernel and $(\cdot)^{hol}$ denotes the extension to $\mathbb{C}^+ \times \mathbb{C}^+$, which is holomorphic in the first variable and anti-holomorphic in the second one, whenever such an extension exists (Theorem 4.3).

Next, we establish Paley–Wiener-type results in Section 5. We show that the Laplace transform $\mathcal{L}$ provides an isometric isomorphism between $\mathcal{A}(H)$ and $\mathcal{D}(H^\top)$

(see Definition 2.2 for $H^{\top}$), and between $\mathcal{A}(H)$ and $\mathcal{D}(H)$ under additional requirements on $H$.

In Section 6, we apply the results of preceding sections to several examples of range spaces of Hardy operators, such as generalized Cesàro and generalized Poisson operators. In particular, we retrieve results concerning generalized Cesàro operators in [8], with simpler proofs.

## 2 Banach algebras of Hardy kernels

In this section, we are concerned with arbitrary $p \in [1, \infty)$.

***Definition 2.1*** Let $1 \le p < \infty$, and let $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ be a measurable map. $H$ is said to be a Hardy kernel of index $p$ if the following conditions hold.

(i) $H$ is homogeneous of degree $-1$; that is, for all $\lambda > 0$, $H(\lambda r, \lambda s) = \lambda^{-1} H(r,s)$ for all $r, s > 0$.

(ii) $\int_0^\infty |H(1,s)| s^{-1/p} ds < \infty$, which is equivalent to $\int_0^\infty |H(r,1)| r^{-1/p'} dr < \infty$, where $p'$ is such that $1/p + 1/p' = 1$ (with $p' = \infty$ if $p = 1$ as usual).

Hardy kernels are useful tools to construct bounded operators on the Lebesgue spaces $L^p(\mathbb{R}^+)$ through (1.1). This is a well-known result of Hardy, Littlewood, and Pólya (see [9, Theorem 319]), and it is part of folklore that such operators can be described as convolution operators by identifying a Hardy kernel $H$ with the function $g_H \in L^1(\mathbb{R})$ given by

$$g_H(t) := H(1, e^{-t}) e^{-t/p'}, \ t \in \mathbb{R}$$

(see, for example, [2, 6]). If one wants $H \mapsto g_H$ to be a bijection, one must consider the following equivalence classes in the set of Hardy kernels of index $p$. We set that two Hardy kernels $H, G$ of index $p$ are equivalent, $H \sim G$, if and only if $H(r,1) = G(r,1)$ for a.e. $r > 0$. From now on, $\mathfrak{H}_p$ will denote this set of equivalence classes of Hardy kernels of index $p$, and we will refer to $H \in \mathfrak{H}_p$ as a Hardy kernel rather than an equivalence class of Hardy kernels, so we identify an equivalence class by any of its elements.

As a consequence, the mapping $\Phi_p : \mathfrak{H}_p \to L^1(\mathbb{R})$ defined by $\Phi_p(H) := g_H$ is a bijection, with inverse given by $(\Phi_p^{-1} g)(r,s) = r^{-1/p} s^{-1/p'} g\left(\log \frac{r}{s}\right)$, for a.e. $r, s > 0$, $g \in L^1(\mathbb{R})$.

Next, we endow the linear space $\mathfrak{H}_p$ with the norm and product given, respectively, by $\|H\|_{\mathfrak{H}_p} := \|\Phi_p(H)\|_{L^1(\mathbb{R})}$, $H \bullet G := \Phi_p^{-1}((\Phi_p H) * (\Phi_p G))$ for all $H, G \in \mathfrak{H}_p$, where $*$ stands for the usual convolution of two elements of $L^1(\mathbb{R})$. We will denote the Banach algebra of bounded linear operators on a Banach space $X$ by $\mathcal{B}(X)$.

***Proposition 2.1*** *Let $1 \le p < \infty$. The space $\mathfrak{H}_p$ is a commutative Banach algebra if provided with the norm and product*

$$\|H\|_{\mathfrak{H}_p} = \int_0^\infty |H(1,s)| s^{-1/p} \, ds,$$

$$(H \bullet G)(r,s) = \int_0^\infty H(r,t) G(t,s) \, dt, \quad r, s > 0.$$

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*Moreover, the mappings* $A_p : \mathfrak{H}_p \to \mathcal{B}(L^p(\mathbb{R}^+))$, $D_p : \mathfrak{H}_p \to \mathcal{B}(H^p(\mathbb{C}^+))$, *given by* $A_p(H) := A_H$ *and* $D_p(H) := D_H$, *are bounded Banach algebra homomorphisms.*

**Proof** It is readily seen that $\|H\|_{\mathfrak{H}_p} = \|g_H\|_{L^1(\mathbb{R})} = \int_0^\infty |H(1,s)|s^{-1/p}\,ds$. Let us prove the product identity. It follows that for $H, G \in \mathfrak{H}_p$,

$$\begin{aligned}(H \bullet G)(r,s) &= \Phi_p^{-1}\left(\Phi_p(H) * \Phi_p(G)\right)(r,s) = r^{-1/p}s^{-1/p'}\left(g_H * g_G\right)\left(\log\frac{r}{s}\right)\\ &= r^{-1/p}s^{-1/p'}\int_{-\infty}^{\infty} G(1,e^{-t})e^{-t/p'}H\left(1,e^{t-\log(r/s)}\right)e^{(t-\log(r/s))/p'}\,dt\\ &= \frac{1}{r}\int_0^\infty G(1,u^{-1})H\left(1,u\frac{s}{r}\right)\frac{du}{u} = \frac{1}{r}\int_0^\infty H\left(1,\frac{v}{r}\right)G\left(1,\frac{s}{v}\right)\frac{dv}{v}\\ &= \int_0^\infty H(r,v)\,G(v,s)\,dv, \qquad r,s>0.\end{aligned}$$

Next, it follows by Hardy's inequality [9, Theorem 319] that $\|A_H\|_{\mathcal{B}(L^p)} \le \|H\|_{\mathfrak{H}_p}$. Moreover, one has that

$$\begin{aligned}(A_{H\bullet G}f)(r) &= \int_0^\infty (H\bullet G)(r,s)f(s)\,ds = \int_0^\infty H(r,t)\int_0^\infty G(t,s)f(s)\,ds\,dt\\ &= (A_H A_G f)(r), \qquad f \in L^p(\mathbb{R}^+),\ \text{a.e. } r>0.\end{aligned}$$

Note that $(D_H F)^\theta = A_H F^\theta$. Thus, by (1.3) and what we have just proved,

$$\|D_H F\|_{H^p} \le \|A_H\|_{\mathcal{B}(L^p)} \sup_{-\pi/2<\theta<\pi/2} \frac{1}{2\pi}\|F^\theta\|_{L^p} \le \|H\|_{\mathfrak{H}_p}\|F\|_{H^p}, \quad F \in H^p(\mathbb{C}^+).$$

Similarly, $(D_{H\bullet G}F)^\theta = A_{H\bullet G}F^\theta = A_H A_G F^\theta = (D_H D_G F)^\theta$ for any $F \in H^p(\mathbb{C}^+)$ and $\theta \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, and thus $D_{H\bullet G} = D_H D_G$. ∎

Next, we give a few definitions and properties regarding Hardy kernels that will be needed later. Let us denote by $\overline{z}$ the conjugate of $z \in \mathbb{C}$.

***Definition 2.2*** Let $1 < p < \infty$, and let $H \in \mathfrak{H}_p$. Set $H^\top(r,s) := H(s,r)$ for all $r,s>0$. Similarly, set $H^*(r,s) := \overline{H(s,r)}$ for all $r,s>0$.

***Remark 2.2*** Let $1 < p < \infty$, and let $H, G \in \mathfrak{H}_p$. One has that $H^\top, H^* \in \mathfrak{H}_{p'}$, that $(H \bullet G)^\top = H^\top \bullet G^\top$, $(H\bullet G)^* = H^* \bullet G^*$, and that $(H^\top)^* = (H^*)^\top$.

***Definition 2.3*** Let $1 \le p < \infty$. We define $\mathcal{I}_p \subset \mathfrak{H}_p$ as $\mathcal{I}_p := \Phi_p^{-1}(L^1(\mathbb{R}) \cap L^{p'}(\mathbb{R}))$.

Clearly, $\mathcal{I}_p$ is a dense ideal of $\mathfrak{H}_p$ since so is $L^1(\mathbb{R}) \cap L^{p'}(\mathbb{R})$ in $L^1(\mathbb{R})$. We characterize its elements in the lemma below. For $H \in \mathfrak{H}_p$, define the family $(H_s)_{s\in\mathbb{R}^+}$ of complex-valued functions defined a.e. in $\mathbb{R}^+$, given by $\{H_s := H(\cdot,s) \mid s \in \mathbb{R}^+\}$. In particular, $H_r^\top = H(r,\cdot)$ for any $r>0$.

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***Lemma 2.3*** *Let $1 \le p < \infty$, and let $H \in \mathfrak{H}_p$. The following are equivalent.*

(i) $H \in \mathfrak{I}_p$.

(ii) $(H_r^\top)_{r>0} \subset L^{p'}(\mathbb{R}^+)$.

(iii) $H_1^\top \in L^{p'}(\mathbb{R}^+)$.

*In any of the above cases, one has that*

$$\|H_r^\top\|_{L^{p'}(\mathbb{R}^+)} = r^{-\frac{1}{p}} \|H_1^\top\|_{L^{p'}(\mathbb{R}^+)} = r^{-\frac{1}{p}} \|g_H\|_{L^{p'}(\mathbb{R})}, \qquad r > 0.$$

**Proof** All the statements of the equivalence are straightforward to obtain using the homogeneity of degree $-1$ of $H$ and the definition of the function $g_H$. Let us show the equivalence $(i) \iff (iii)$. For $1 < p < \infty$,

$$\|H_1^\top\|_{p'} = \left(\int_0^\infty |H(1,s)|^{p'}\, ds\right)^{1/p'} = \left(\int_{-\infty}^\infty |H(1,e^{-t})|^{p'} e^{-t}\, dt\right)^{1/p'} = \|g_H\|_{p'}.$$

For $p = 1$, it is straightforward that $\|H_1^\top\|_\infty = \|g_H\|_\infty$ since $g_H(t) = H(1, e^{-t})$ for a.e. $t > 0$. ∎

## 3 Hardy reproducing kernels on $\mathbb{R}^+ \times \mathbb{R}^+$

In this section, we analyze the range spaces of Hardy operators on $L^2(\mathbb{R}^+)$, although some minor results are also valid for general $p$. Our main motivation is to characterize the conditions for which these range spaces are RKHSs (Proposition 3.2).

***Definition 3.1*** Let $1 \le p < \infty$, and let $H \in \mathfrak{H}_p$. Let $\mathcal{A}(H)$ be the range space

$$\mathcal{A}(H) := \{A_H f \ : \ f \in L^p(\mathbb{R}^+)\}.$$

We endow $\mathcal{A}(H)$ with a Banach (Hilbert if $p = 2$) space structure through the canonical identification $\mathcal{A}(H) \cong L^p(\mathbb{R}^+)/\ker A_H$.

Let $C(\mathbb{R}^+)$ denote the space of continuous functions on $\mathbb{R}^+$.

***Lemma 3.1*** *Let $1 \le p < \infty$, and let $H \in \mathfrak{I}_p \subset \mathfrak{H}_p$. Then $\mathcal{A}(H) \subset C(\mathbb{R}^+)$.*

**Proof** Let $f \in L^p(\mathbb{R}^+)$. We have that

$$(A_H f)(r) = \int_0^\infty H(r,s) f(s)\, ds = \int_0^\infty H(1,t) f(rt)\, dt = \langle \tau_r f, H_1^\top \rangle, \quad \text{for all } r > 0,$$

where $(\tau_r f)(t) := f(rt)$ for $t > 0$, $\langle \cdot, \cdot \rangle$ denotes the dual product between $L^p(\mathbb{R}^+)$ and $L^{p'}(\mathbb{R}^+)$, and $H_1^\top$ is defined before Lemma 2.3.

Since the mapping $r \mapsto \tau_r f$ from $\mathbb{R}^+$ into $L^p(\mathbb{R}^+)$ is continuous for each $f \in L^p(\mathbb{R}^+)$, it follows that $\langle \tau_r f, H_1^\top \rangle = (A_H f)(r)$ is also continuous in $r$; that is, $A_H f \in C(\mathbb{R}^+)$, as we wanted to show. ∎

As a consequence of the lemma, point evaluations are well defined on $\mathcal{A}(H)$ whenever $H \in \mathfrak{I}_p$. Indeed, the proposition below adds a bit more information.

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***Proposition 3.2*** *Let $1 \le p < \infty$, and let $H \in \mathfrak{H}_p$. Then point evaluations are continuous functionals on $\mathcal{A}(H)$ if and only if $H \in \mathfrak{I}_p$. In this case, for all $f \in \mathcal{A}(H)$,*

$$|f(r)| \le r^{-1/p} \|H_1^\top\|_{p'} \|f\|_{\mathcal{A}(H)}, \qquad r > 0.$$

**Proof** Let us assume first that point evaluations are well defined and continuous on $\mathcal{A}(H)$, so for all $r > 0$, the mapping $\Omega_r : L^p(\mathbb{R}^+) \to \mathbb{C}$ given by $\Omega_r f := (A_H f)(r)$ is a well-defined continuous functional. Therefore, there exists $g_r \in L^{p'}(\mathbb{R}^+)$ such that $(A_H f)(r) = \int_0^\infty g_r(s) f(s) ds$ for all $f \in L^p(\mathbb{R}^+)$, which implies that $H(r,s) = g_r(s)$ for a.e. $s > 0$. By Lemma 2.3, one gets that $H \in \mathfrak{I}_p$.

Now, let us assume that $H \in \mathfrak{I}_p$. By Lemma 3.1, it follows that point evaluations are well defined on $\mathcal{A}(H)$. By Lemma 2.3, one has that $(H_r^\top)_{r\in\mathbb{R}^+} \subset L^{p'}(\mathbb{R}^+)$. Let $f \in \mathcal{A}(H)$, $g \in L^p(\mathbb{R}^+)$ be such that $f = A_H g$. Let $[g + \ker A_H]$ be the quotient class of $L^p(\mathbb{R}^+)/\ker A_H$ containing $g$. It follows that, for all $\widetilde{g} \in [g + \ker A_H]$,

$$\begin{aligned} |f(r)| &= |(A_H \widetilde{g})(r)| = \left| \int_0^\infty H(r,s) \widetilde{g}(s)\, ds \right| = \langle H_r^\top, \widetilde{g} \rangle \\ &\le \inf_{\widetilde{g} \in [g + \ker A_H]} \|H_r^\top\|_{p'} \|\widetilde{g}\|_p = \|H_r^\top\|_{p'} \|[g + \ker A_H]\|_{L^p(\mathbb{R}^+)/\ker A_H} \\ &= r^{-1/p} \|H_1^\top\|_{p'} \|f\|_{\mathcal{A}(H)}, \qquad \forall r > 0, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual product between $L^{p'}(\mathbb{R}^+)$ and $L^p(\mathbb{R}^+)$. Therefore, point evaluations are continuous on $\mathcal{A}(H)$. ∎

The next theorem gives the reproducing kernel $K_H$ of $\mathcal{A}(H)$ for $H \in \mathfrak{I}_2$, which turns out to be a Hardy kernel as well.

***Theorem 3.3*** *Let $H \in \mathfrak{H}_2$. Then $\mathcal{A}(H)$ is an RKHS if and only if $H \in \mathfrak{I}_2$, and in this case, its reproducing kernel $K_H$ is continuous and given by*

$$K_H(r,s) = \int_0^\infty H(r,t) \overline{H(s,t)}\, dt, \quad \text{for } r, s > 0.$$

*Then $K_H \in \mathfrak{H}_2$, satisfying $K_H = H \bullet H^*$. As a consequence, $K_{H^*} = K_H$.*

**Proof** By Proposition 3.2, $\mathcal{A}(H)$ is an RKHS if and only if $H \in \mathfrak{I}_2$, and in this case, $\mathcal{A}(H)$ is isometrically isomorphic to $L^2(\mathbb{R}^+)/\ker A_H \cong (\ker A_H)^\perp$. Let us compute its reproducing kernel. Assume that $f \in \ker A_H$, so it follows that

$$(f \mid H_u^*)_{L^2} = \int_0^\infty H(u,v) f(v)\, dv = (A_H f)(u) = 0,$$

for all $u > 0$, where $H_u^*(v) = \overline{H(u,v)} = \overline{H_u^\top(v)}$ for a.e. $v > 0$. Therefore, one has that $H_u^* \in (\ker A_H)^\perp \subset L^2(\mathbb{R}^+)$ for all $u > 0$.

Now, let $h_u = A_H H_u^* \in \mathcal{A}(H)$, and let $f \in \mathcal{A}(H)$, so that $f = A_H g$ for a unique $g \in (\ker A_H)^\perp$. Since $\mathcal{A}(H) \cong (\ker A_H)^\perp$, it follows that, for all $u > 0$,

$$(f \mid h_u)_{\mathcal{A}(H)} = (g \mid H_u^*)_{(\ker A_H)^\perp} = \int_0^\infty H(u,v) g(v)\, dv = (A_H g)(u) = f(u).$$

Hence, $K_H(v,u) = h_u(v) = (A_H H_u^*)(v) = \int_0^\infty H(v,t)\overline{H(u,t)}\,dt$ for all $u, v > 0$, as we wanted to show.

Moreover, $K_H = H \bullet H^* = H^* \bullet H = K_{H^*}$ by Proposition 2.1 and Definition 2.2, and in particular $K_H$ turns out to be a Hardy kernel. The continuity of $K_H$ in each variable follows from the inclusion $\mathcal{A}(H) \subset C(\mathbb{R}^+)$ (Lemma 3.1) and the fact that $K_H(r,s) = \overline{K_H(s,r)}$, for $r, s > 0$ (see, for example, [17, Lemma I.1.2]). But then, $K_H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ is continuous jointly on both variables since $K_H(r,s) = s^{-1}K_H(r/s, 1)$. ■

Let $\mathcal{H}_+$ be the one-sided Hilbert transform on $L^p(\mathbb{R}^+)$ defined by

$$\mathcal{H}_+ f(x) := \text{p.v.}\frac{1}{\pi}\int_0^\infty \frac{f(r)}{x-r}\,dr, \qquad x > 0, \qquad f \in L^p(\mathbb{R}^+).$$

The boundedness of $\mathcal{H}_+$ on $L^p(\mathbb{R}^+)$ for $1 < p < \infty$ immediately follows from the boundedness of the Hilbert transform on $L^p(\mathbb{R})$ (see, for example, [5]). The following theorem has been inspired by [11, 14].

***Theorem 3.4*** *Let $1 < p < \infty$, and let $H \in \mathfrak{H}_p$. One has that $\mathcal{H}_+ A_H = A_H \mathcal{H}_+$. Therefore, $\mathcal{H}_+$ defines a bounded operator on $\mathcal{A}(H)$.*

**Proof** Let $f \in L^p(\mathbb{R}^+)$. Then, for all $x > 0$,

$$\begin{aligned}(\mathcal{H}_+(A_H f))(x) &= \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{(0,x-\varepsilon)\cup(x+\varepsilon,\infty)}\frac{1}{x-r}\int_0^\infty H(1,s)f(rs)\,ds\,dr \\ &= \lim_{\varepsilon\to 0^+}\int_0^\infty H(1,s)\frac{1}{\pi}\int_{(0,s(x-\varepsilon))\cup(s(x+\varepsilon),\infty)}\frac{f(v)}{xs-v}\,dv\,ds.\end{aligned} \tag{3.1}$$

Here, we have applied Fubini to commute the integrals since $\int_0^\infty |H(1,s)f(\cdot s)|\,ds \in L^p(\mathbb{R}^+)$ and $\frac{1}{x-(\cdot)}\chi_{(0,x-\varepsilon)\cap(x+\varepsilon,\infty)}(\cdot) \in L^{p'}(\mathbb{R}^+)$ for all $1 < p < \infty$.

Recall that the maximal operator $\mathcal{M}\mathcal{H}_+$ defined as $(\mathcal{M}\mathcal{H}_+ f)(x) := \sup_{\varepsilon>0}|(\mathcal{H}_{+,\varepsilon}f)(x)|$, for all $x > 0$, belongs to $\mathcal{B}(L^p(\mathbb{R}^+))$, where

$$(\mathcal{H}_{+,\varepsilon}f)(x) := \frac{1}{\pi}\int_{(0,x-\varepsilon)\cup(x+\varepsilon,\infty)}\frac{1}{x-s}f(s)\,ds, \qquad \varepsilon > 0, \text{ for } x > 0$$

(see, for example, [5, Corollary 3.13]). As a consequence, the bound $|H(1,s)(\mathcal{H}_{+,s\varepsilon}f)(xs)| \le |H(1,s)|(\mathcal{M}\mathcal{H}_+ f)(xs)$ holds for every $\varepsilon > 0$, a.e. $s > 0$. By Hardy's inequality [9, Theorem 319], $|H(1,s)|(\mathcal{M}\mathcal{H}_+ f)(xs)$, as a function on $s > 0$, belongs to $L^1(\mathbb{R}^+)$ for a.e. $x > 0$. Thus, we apply the dominated convergence theorem to (3.1) to commute the limit and the integral $\int_0^\infty$. Hence, $\mathcal{A}(H)$ is $\mathcal{H}_+$ invariant, and then the continuity of $\mathcal{H}_+$ follows by the closed graph theorem. ■

## 4 Hardy reproducing kernels on $\mathbb{C}^+ \times \mathbb{C}^+$

Next, we proceed to analyze the range spaces of Hardy operators on the Hardy spaces of holomorphic functions on the right-hand half plane $H^2(\mathbb{C}^+)$.

For $1 \le p < \infty$, recall that the Hardy space $H^p(\mathbb{C}^+)$ is formed by all holomorphic functions $F$ on $\mathbb{C}^+$ such that

$$\|F\|_{H^p} := \sup_{x>0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(x+iy)|^p \, dy \right)^{1/p} < \infty.$$

It is known that $H^2(\mathbb{C}^+)$ is an RKHS, with reproducing kernel $\mathcal{K}$ given by

$$\mathcal{K}(z,w) = \frac{1}{z+\overline{w}}, \qquad z,\, w \in \mathbb{C}^+$$

(see [12, Chapter VI]). Notice that the restriction of $\mathcal{K}$ to $\mathbb{R}^+ \times \mathbb{R}^+$ defines the Stieltjes kernel $\mathcal{S}(r,s) = \frac{1}{r+s}$. One has that $\mathcal{S} \in \mathfrak{H}_p$ for all $1 \le p < \infty$. Recall that $D_H$ is the Hardy operator on $H^p(\mathbb{C}^+)$ associated with $H \in \mathfrak{H}_p$ through (1.2).

***Definition 4.1*** Let $1 \le p < \infty$, and let $H \in \mathfrak{H}_p$. Endow $\mathcal{D}(H) := D_H(H^p(\mathbb{C}^+))$ with the structure of Banach space induced by the canonical isomorphism $\mathcal{D}(H) \cong H^p(\mathbb{C}^+)/\ker D_H$.

***Remark 4.1*** Let $H \in \mathfrak{H}_2$. Using that $D_H F = \displaystyle\int_0^\infty H(1,s)F(s\,\cdot)\,ds$, it is simple to see that $(D_H F|G)_{H^2} = (F|D_{H^*}G)_{H^2}$ in the inner product in $H^2(\mathbb{C}^+)$. Since all Hardy operators commute between themselves (see Proposition 2.1), $D_H$ is a normal operator.

Next, we give the main theorem of this section, for which we will need the following lemma and definition. Set $\mathcal{K}_w := \mathcal{K}(\cdot, w)$ for all $w \in \mathbb{C}^+$, so $\mathcal{K}_w \in H^2(\mathbb{C}^+)$.

***Lemma 4.2*** *Let $H \in \mathfrak{H}_2$. For all $z \in \mathbb{C}^+$, one has that*

$$\int_0^\infty \|H(1,t)\mathcal{K}_{tz}\|_{H^2}\, dt < \infty.$$

**Proof** Note that the vector-valued function $t \mapsto H(1,t)\mathcal{K}_{tz}$ is strong measurable, since $t \mapsto H(1,t)$ is measurable, and $w \mapsto \mathcal{K}_w$ is continuous from $\mathbb{C}^+$ to $H^2(\mathbb{C}^2)$. Then, for all $z \in \mathbb{C}^+$,

$$\begin{aligned}\int_0^\infty |H(1,t)|\|\mathcal{K}_{tz}\|_{H^2}\, dt &= \int_0^\infty |H(1,t)|\sqrt{(\mathcal{K}_{tz}|\mathcal{K}_{tz})_{H^2}}\, dt = \int_0^\infty |H(1,t)|\sqrt{\mathcal{K}(tz,tz)}\, dt \\ &= \sqrt{\mathcal{K}(z,z)} \int_0^\infty |H(1,t)| t^{-1/2}\, dt = \sqrt{\mathcal{K}(z,z)}\,\|H\|_{\mathfrak{H}_2} < \infty.\end{aligned}$$

■

***Definition 4.2*** We define $\mathfrak{H}_p^{hol}$ to be the subset of $\mathfrak{H}_p$ consisting of those $H \in \mathfrak{H}_p$ with extension $H^{hol} : \mathbb{C}^+ \times \mathbb{C}^+ \to \mathbb{C}$ such that:

- $H(r,s) = H^{hol}(r,s)$ for $r,s > 0$,
- the map $z \mapsto H^{hol}(z,w)$ is holomorphic on $\mathbb{C}^+$ for all $w \in \mathbb{C}^+$, and

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- the map $w \mapsto H^{hol}(z,\overline{w})$ is holomorphic on $\mathbb{C}^+$ for all $z \in \mathbb{C}^+$.

Note that if $H \in \mathfrak{H}_p^{hol}$, the extension $H^{hol}$ is unique.

Notice that the Stieltjes kernel $\mathcal{S}$ satisfies that $\mathcal{S} \in \mathfrak{H}_p^{hol}$ with $\mathcal{S}^{hol} = \mathcal{K}$.

***Theorem 4.3*** *Let $H \in \mathfrak{H}_2$. One has that $H \bullet \mathcal{S} \bullet H^* \in \mathfrak{H}_2^{hol}$, and that $\mathcal{D}(H)$ is an RKHS with reproducing kernel $\mathcal{K}_H$ given by*

$$\mathcal{K}_H = (H \bullet \mathcal{S} \bullet H^*)^{hol}.$$

**Proof** Let $G$ be in $(\ker D_H)^\perp$ such that $F = D_H(G) \in \mathcal{D}(H)$.

$$\|F\|_{H^2} = \|D_H G\|_{H^2} \le \|D_H\|_{\mathcal{B}(H^2)}\|G\|_{H^2} = \|D_H\|_{\mathcal{B}(H^2)}\|F\|_{\mathcal{D}(H)}.$$

Since $H^2(\mathbb{C}^+)$ is an RKHS, it follows from above that $\mathcal{D}(H)$ is an RKHS too. Let us compute its reproducing kernel $\mathcal{K}_H$. As before, set $\mathcal{K}_w(z) = \mathcal{K}(z,w)$. For $F = D_H G \in \mathcal{D}(H)$, $z = |z|e^{i\theta} \in \mathbb{C}^+$, and $G \in (\ker D_H)^\perp$, we have

(4.1)
$$\begin{aligned} F(z) &= \int_0^\infty H(|z|,s)G^\theta(s)\,ds = \int_0^\infty H(|z|,s)(G|\mathcal{K}_{se^{i\theta}})_{H^2}\,ds = \int_0^\infty H(1,t)(G|\mathcal{K}_{tz})_{H^2}\,dt \\ &= \int_0^\infty (G|\overline{H(1,t)}\mathcal{K}_{tz})_{H^2}\,dt = \left(G\,\middle|\,\int_0^\infty \overline{H(1,t)}\mathcal{K}_{tz}\,dt\right)_{H^2}, \end{aligned}$$

where one can intertwine the integral sign with the inner product by Lemma 4.2.

Let $J \in \ker D_H$. By substituting $F$ by $D_H J = 0$ and $G$ by $J$ in (4.1), one concludes that $\int_0^\infty \overline{H(1,t)}\mathcal{K}_{tz}dt \in (\ker D_H)^\perp$. Then, we have that

$$F(z) = \left(G\,\middle|\,\int_0^\infty \overline{H(1,t)}\mathcal{K}_{tz}\,dt\right)_{H^2} = \left(F\,\middle|\,D_H\left(\int_0^\infty \overline{H(1,t)}\mathcal{K}_{tz}\,dt\right)\right)_{\mathcal{D}(H)}.$$

Therefore, after rearranging some variables, one gets that the reproducing kernel $\mathcal{K}_H$ of $\mathcal{D}(H)$ is given by

$$\mathcal{K}_H(z,w) = \left[D_H\left(\int_0^\infty \overline{H(1,t)}\mathcal{K}_{tw}\,dt\right)\right](z), \qquad z,w \in \mathbb{C}^+.$$

Now, let us see that the expression above coincides with the one given in the statement for all $z,w \in \mathbb{R}^+$:

$$\begin{aligned} &\left[D_H\left(\int_0^\infty \overline{H(1,t)}\mathcal{K}_{tw}dt\right)\right](z) = \int_0^\infty H(z,s)\left(\int_0^\infty \overline{H(1,t)}\mathcal{K}_{tw}dt\right)(s)\,ds \\ &= \int_0^\infty H(z,s)\int_0^\infty \overline{H(1,t)}\mathcal{K}(s,tw)\,dt\,ds = \int_0^\infty H(z,s)\int_0^\infty \mathcal{S}(s,u)\overline{H(w,u)}\,du\,ds \\ &= \int_0^\infty H(z,s)\int_0^\infty \mathcal{S}(s,u)H^*(u,w)\,du\,ds = \int_0^\infty H(z,s)(\mathcal{S}\bullet H^*)(s,w)\,ds \\ &= (H\bullet\mathcal{S}\bullet H^*)(z,w). \end{aligned}$$

Therefore, $\mathcal{K}_H(z,w) = (H\bullet\mathcal{S}\bullet H^*)(z,w)$ for all $z,w \in \mathbb{R}^+$.

Since all the elements in $\mathcal{D}(H) \subset H^2(\mathbb{C}^+)$ are holomorphic, we have that $\mathcal{K}_H(z,w)$ is holomorphic in $z$, so it is determined for all $(z,w) \in \mathbb{C}^+ \times \mathbb{R}^+$ by its restriction at $\mathbb{R}^+ \times \mathbb{R}^+$. Moreover, since $\mathcal{K}_H$ is a reproducing kernel, we have that $\mathcal{K}_H(z,w) = \overline{\mathcal{K}_H(w,z)}$ (see [17, Lemma I.1.2]), and as a consequence, $\mathcal{K}_H(z,w)$ is anti-holomorphic in $w$, and by the same reasoning as before, $\mathcal{K}_H(z,w)$ is determined for all $z, w \in \mathbb{C}^+$ by its restriction to $\mathbb{C}^+ \times \mathbb{R}^+$. All these statements imply that $H \bullet \mathcal{S} \bullet H^* \in \mathfrak{H}_2^{hol}$ and that its holomorphic extension is precisely $\mathcal{K}_H$. ∎

## 5 Paley–Wiener theorems for range spaces

We wish to start this section with the following remark. Paley–Wiener's theorem states that $\mathcal{L} : L^2(\mathbb{R}^+) \to H^2(\mathbb{C}^+)$ is an isometric isomorphism, where $\mathcal{L}$ is the Laplace transform given by

$$(\mathcal{L}f)(z) := \int_0^\infty e^{-rz} f(r)dr, \qquad f \in L^2(\mathbb{R}^+), z \in \mathbb{C}^+ \tag{5.1}$$

(see [18, Theorem V]).

This classical $L^2 - H^2$ Paley–Wiener theorem can be used to prove that $H^2(\mathbb{C}^+)$ is an RKHS with kernel $\mathcal{K}(z,w) = \frac{1}{z+\overline{w}}$ [10, Proposition 1.8]. Conversely, one can reverse the implications of such a proof to obtain the $L^2 - H^2$ Paley–Wiener theorem using RKHS theory (note that the kernel $\mathcal{K}$ of the space $H^2(\mathbb{C}^+)$ can be obtained independently of Paley–Wiener's theorem; see, for instance, [12, Chapter VI]), as we show next.

Since the Laplace transform $\mathcal{L}$ acting on $L^2(\mathbb{R}^+)$ is injective, one can endow the range space $\mathcal{L}(L^2(\mathbb{R}^+))$ with the structure of Hilbert space induced by the bijection $\mathcal{L} : L^2(\mathbb{R}^+) \to \mathcal{L}(L^2(\mathbb{R}^+))$. For $F = \mathcal{L}f \in \mathcal{L}(L^2(\mathbb{R}^+))$, one has

$$F(z) = \int_0^\infty e^{-rz} f(r)\, dr = (f|e^{-r\overline{z}})_{L^2} = (F|\mathcal{L}(e^{-r\overline{z}}))_{\mathcal{L}(L^2)}, \quad z \in \mathbb{C}^+.$$

As a consequence, $\mathcal{L}(L^2(\mathbb{R}^+))$ is an RKHS with kernel $K_{\mathcal{L}}$ given by

$$K_{\mathcal{L}}(z,w) = \mathcal{L}(e^{-r\overline{w}})(z) = \int_0^\infty e^{-rz}e^{-r\overline{w}}\, dr = \frac{1}{z+\overline{w}} = \mathcal{K}(z,w), \quad z,w \in \mathbb{C}^+.$$

That is, both $\mathcal{L}(L^2(\mathbb{R}^+))$ and $H^2(\mathbb{C}^+)$ are RKHSs with the same kernel $K_{\mathcal{L}} = \mathcal{K}$, so $\mathcal{L}(L^2(\mathbb{R}^+)) = H^2(\mathbb{C}^+)$ as Hilbert spaces (see, for instance, [17, Lemma I.1.5]), and the claim follows.

Now, we establish results of Paley–Wiener type for range spaces. We first show that $\mathcal{L}$ is an intertwining operator.

***Proposition 5.1*** *$\mathcal{L}A_H = D_{H^\intercal}\mathcal{L}$ on $L^2(\mathbb{R}^+)$ for all $H \in \mathfrak{H}_2$.*

**Proof** Let $z \in \mathbb{C}^+$ and $f \in L^2(\mathbb{R}^+)$. One has

$$\begin{aligned}(\mathcal{L}A_H f)(z) &= \int_0^\infty e^{-rz}\int_0^\infty H(r,t)f(t)dtdr = \int_0^\infty e^{-rz}\int_0^\infty H(1,s)f(rs)dsdr \\ &= \int_0^\infty H(1,s)\int_0^\infty e^{-rz}f(rs)drds = \int_0^\infty H(1,s)(\mathcal{L}f)\left(\frac{z}{s}\right)\frac{ds}{s} \\ &= \int_0^\infty H(u,1)(\mathcal{L}f)(uz)\,du = (D_{H^\intercal}\mathcal{L}f)(z),\end{aligned}$$

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where we have applied Fubini's theorem since both $r \mapsto \int_0^\infty |H(r,t)f(t)|\,dt$ and $r \mapsto e^{-rz}$ are in $L^2(\mathbb{R}^+)$. ∎

***Theorem 5.2*** *Let $H \in \mathfrak{H}_2$. The Laplace transform $\mathcal{L}$ restricted to $\mathcal{A}(H)$ is an isometric isomorphism onto $\mathcal{D}(H^\top)$, $\mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H^\top)$.*

**Proof** By the definition of $\mathcal{A}(H)$ and $\mathcal{D}(H^\top)$, the restrictions $\widetilde{A_H} : (\ker A_H)^\perp \to \mathcal{A}(H)$ and $\widetilde{D_{H^\top}} : (\ker D_{H^\top})^\perp \to \mathcal{D}(H^\top)$ are isometric isomorphisms. By the $L^2 - H^2$ Paley–Wiener theorem and Proposition 5.1, it follows that $(\ker D_{H^\top})^\perp = \mathcal{L}((\ker A_H)^\perp)$. Indeed, by Proposition 5.1, it easily follows that $\mathcal{L}(\ker A_H) = \ker D_{H^\top}$, and thus $(f\,|\,g)_{L^2} = 0$ for all $g \in \ker A_H$ if and only if $(\mathcal{L}f\,|\,G)_{H^2} = 0$ for all $G \in \mathcal{L}(\ker A_H) = \ker D_{H^\top}$.

Therefore, by Proposition 5.1 again, we obtain $\mathcal{L}f = \widetilde{D_{H^\top}}\mathcal{L}(\widetilde{A_H})^{-1}f$ for all $f \in \mathcal{A}(H)$, where all the mappings $\widetilde{D_{H^\top}}$, $\mathcal{L}$ (seen as an operator from the subspace $(\ker A_H)^\perp \subset L^2(\mathbb{R}^+)$ to the subspace $(\ker D_{H^\top})^\perp \subset H^2(\mathbb{C}^+)$), and $(\widetilde{A_H})^{-1}$ are in fact unitary operators. As a consequence, $\mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H^\top)$ defines an isometric isomorphism. ∎

***Corollary 5.3*** *Let $H \in \mathfrak{H}_2$. The Laplace transform defines an isometric isomorphism $\mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H)$ if and only if $H \bullet H^*$ is a real-valued kernel.*

**Proof** By the theorem above, we have that $\mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H)$ is an isometric isomorphism if and only if $\mathcal{D}(H) = \mathcal{D}(H^\top)$ as Hilbert spaces, and this happens if and only if their reproducing kernels are the same, $\mathcal{K}_H = \mathcal{K}_{H^\top}$. By Theorem 4.3, this is equivalent to $\mathcal{S} \bullet H \bullet H^* = \mathcal{S} \bullet H^\top \bullet (H^\top)^*$. The injectivity of the Stieltjes transform $A_\mathcal{S}$ (which can be proved via the Mellin transform; see, for example, [6]) implies that this holds if and only if $H \bullet H^* = H^\top \bullet (H^\top)^* = (H \bullet H^*)^\top$. Then, the claim follows from the fact that $(H \bullet H^*)^\top = \overline{H \bullet H^*}$ for all $H \in \mathfrak{H}_2$. ∎

***Corollary 5.4*** *Let $H \in \mathfrak{H}_2$. Either if $H$ is symmetric, that is, $H = H^\top$, or if $H$ is real-valued, the Laplace transform $\mathcal{L}$ restricts to an isometric isomorphism from $\mathcal{A}(H)$ onto $\mathcal{D}(H)$, $\mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H)$.*

We will see in Theorem 6.4 that, for any $H \in \mathfrak{H}_2$, there exist isometric isomorphisms $\mathscr{P}, \mathscr{Q} : \mathcal{A}(H) \to \mathcal{D}(H)$ related to the Poisson kernel.

## 6 Examples and applications

Here, we illustrate the theory given above with some examples and applications.

(1) *Generalized Poisson operators.* For $\alpha, \beta, \mu$ real numbers, let $P_{\alpha,\beta,\mu}(r,s) = r^{\alpha\mu-\beta}s^{\beta-1}(r^\alpha + s^\alpha)^{-\mu}$ for all $r, s > 0$. The spectral properties of its associated Hardy operator have been studied in [16]. Regarding the properties considered in the present paper, we have that, for $p \in [1,\infty)$ and $\alpha > 0$, $P_{\alpha,\beta,\mu} \in \mathfrak{H}_p$ if and only if $0 < \beta - 1/p < \alpha\mu$, and in this case, $P_{\alpha,\beta,\mu} \in \mathfrak{I}_p$. For $p = 2$, one has

$$K_{P_{\alpha,\beta,\mu}}(r,s) = \frac{s^{\beta-1}}{\alpha r^\beta} B\left(\frac{2\beta-1}{\alpha}, 2\mu - \frac{2\beta-1}{\alpha}\right) {}_2F_1\left(\mu, \frac{2\beta-1}{\alpha}; 2\mu; 1 - \left(\frac{s}{r}\right)^\alpha\right), \qquad r, s > 0,$$

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where $B$ is the Euler Beta function and ${}_2F_1$ is the hypergeometric Gaussian function. As particular cases, one has the following.

*Stieltjes kernel.* For $\alpha = \beta = \mu = 1$, we obtain $P_{1,1,1}(r,s) = \mathcal{S}(r,s) = \dfrac{1}{r+s}$ for $r,s > 0$. By Theorem 3.3, $\mathcal{A}(\mathcal{S})$ is an RKHS with kernel

$$K_{\mathcal{S}}(r,s) = \int_0^\infty \frac{1}{r+t}\frac{1}{t+s}\,dt = \begin{cases} \frac{1}{r-s}\log\frac{r}{s}, & \text{if } r \neq s, \\ \frac{1}{r}, & \text{if } r = s, \end{cases} \quad \text{for } r,s > 0. \tag{6.1}$$

*Poisson kernel and conjugate Poisson kernel.* Recall that, for $x > 0$, the Poisson kernel $P^x$ and conjugate Poisson kernel $Q^x$ on the half-right plane $\mathbb{C}^+$ are given by

$$P^x(y) = \frac{1}{\pi}\frac{x}{x^2+y^2}, \qquad Q^x(y) = \frac{1}{\pi}\frac{y}{x^2+y^2}, \qquad s > 0.$$

These kernels give rise to Hardy kernels $P, Q$ as follows:

$$P(r,s) := P^r(s) = P_{2,1,1}(r,s), \qquad Q(r,s) := Q^r(s) = P^*_{2,1,1}(r,s), \qquad r,s > 0.$$

These kernels are related to the operators $\mathscr{P}, \mathscr{Q} : L^2(\mathbb{R}^+) \to H^2(\mathbb{C}^+)$ given by

$$(\mathscr{P}f)(z) := \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{z}{z^2+s^2}f(s)\,ds, \qquad (\mathscr{Q}f)(z) := \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{s}{z^2+s^2}f(s)\,ds,$$

for any $z \in \mathbb{C}^+$, $f \in L^2(\mathbb{R}^+)$. Indeed, $(\mathscr{P}f)(r) = \sqrt{2\pi}(A_P f)(r)$ and $(\mathscr{Q}f)(r) = \sqrt{2\pi}(A_Q f)(r)$ for $r > 0$, $f \in L^2(\mathbb{R}^+)$. It is a matter of fact that $\mathscr{P}, \mathscr{Q}$ are isometric isomorphisms (see Remark 6.2). Here, we provide a proof of it based on results of this paper. Set

$$L^2_{hol}(\mathbb{R}^+) := \{f : \mathbb{R}^+ \to \mathbb{C}^+ \mid f(r) = F(r),\ r > 0,\ \text{for some } F \in H^2(\mathbb{C}^+)\}.$$

Since any holomorphic function in $\mathbb{C}^+$ is determined by its restriction to $\mathbb{R}^+$, the space $L^2_{hol}(\mathbb{R}^+)$, regarded as a range space of $H^2(\mathbb{C}^+)$, is an RKHS isometrically isomorphic to $H^2(\mathbb{C}^+)$ with kernel $\mathcal{S}(r,s) = \mathcal{K}(r,s) = \frac{1}{r+s}$, $r,s > 0$. To see this, take $F \in H^2(\mathbb{C}^+)$, $f = F|_{\mathbb{R}^+}$, and $s > 0$. Then,

$$f(s) = F(s) = (F \mid \mathcal{K}_s)_{H^2} = (f \mid \mathcal{K}_s|_{\mathbb{R}^+})_{L^2_{hol}} = (f \mid \mathcal{S}_s)_{L^2_{hol}},$$

as claimed.

***Proposition 6.1*** *Both $\mathscr{P}, \mathscr{Q} : L^2(\mathbb{R}^+) \to H^2(\mathbb{C}^+)$ are isometric isomorphisms.*

**Proof** Since $P = Q^*$, Theorem 3.3 implies that $\mathcal{A}(\sqrt{2\pi}P) = \mathcal{A}(\sqrt{2\pi}Q)$ is an RKHS on $\mathbb{R}^+$ with kernel $K_{\sqrt{2\pi}P} = K_{\sqrt{2\pi}Q}$ given by

$$K_{\sqrt{2\pi}P}(r,s) = \frac{2}{\pi}\int_0^\infty \frac{r}{r^2+t^2}\frac{s}{t^2+s^2}\,dt = \frac{1}{r+s} = \mathcal{S}(r,s), \qquad r,s > 0. \tag{6.2}$$

Therefore, $\mathcal{A}(\sqrt{2\pi}P) = \mathcal{A}(\sqrt{2\pi}Q) = L^2_{hol}(\mathbb{R}^+)$ as Hilbert spaces. Thus, all is left to prove is that $\mathscr{P}f, \mathscr{Q}f$ are holomorphic on $\mathbb{C}^+$ and that both $\mathscr{P}, \mathscr{Q}$ are injective operators. First, claim which follows by an application of Morera's theorem. For the second one, note that the Stieltjes transform $A_{\mathcal{S}}$ is an injective operator and that $A_{\mathcal{S}} = A_{P\bullet Q} = A_P A_Q = A_Q A_P$. Thus, both $A_P, A_Q$ are injective, and so are $\mathscr{P}, \mathscr{Q}$. ■

***Remark 6.2*** The proposition above is equivalent to Paley–Wiener's theorem. To see this, set $L^2_{even}(\mathbb{R})$ as the subset of even functions of $L^2(\mathbb{R})$, and note that the Fourier transform $\mathcal{F}$ restricts to an isometric mapping from $L^2_{even}(\mathbb{R})$ onto itself. Set $\iota : L^2_{even}(\mathbb{R}) \to L^2(\mathbb{R}^+)$ by $(\iota f)(r) := f(r)$ for a.e. $r > 0$. Then $\iota\mathcal{F}\iota^{-1}$ is a unitary operator on $L^2(\mathbb{R}^+)$, and one easily obtains that $\mathscr{P} = \mathcal{L}\iota\mathcal{F}\iota^{-1}$. Hence, $\mathscr{P}$ is an isometric isomorphism if and only if $\mathcal{L}$ is an isometric isomorphism.

By considering the subset of odd functions of $L^2(\mathbb{R})$, one obtains an analogous statement for $\mathscr{Q}$.

Some other consequences of results of this paper are the following.

***Corollary 6.3*** *As a range space, $\mathcal{L}(H^2(\mathbb{C}^+))$ is an RKHS with kernel $K$ given by*

$$K(z,w) = \begin{cases} \frac{1}{z-\overline{w}} \log \frac{z}{\overline{w}}, & \text{if } z \neq \overline{w}, \\ \frac{1}{z}, & \text{if } z = \overline{w}, \end{cases} \qquad \text{for } z, w \in \mathbb{C}^+.$$

Here, we consider $\mathcal{L} : H^2(\mathbb{C}^+) \to H^2(\mathbb{C}^+)$ given by $(\mathcal{L}F)(z) := \int_0^\infty e^{-zr}F(r)\,dr$.

**Proof** By Proposition 5.1, one has $\mathcal{L}\mathscr{Q} = \sqrt{2\pi}D_P\mathcal{L}$. Thus, $\mathcal{L}(H^2(\mathbb{C}^+)) = \mathcal{L}\mathscr{Q}(L^2(\mathbb{R}^+)) = \mathcal{D}(\sqrt{2\pi}P)$ regarded as Hilbert spaces, since all the operators considered in the equalities are isometric isomorphisms. Hence, by Theorems 3.3 4.3 and (6.2),

$$K = \mathcal{K}_{\sqrt{2\pi}P} = (K_{\sqrt{2\pi}P} \bullet \mathcal{S})^{hol} = (\mathcal{S} \bullet \mathcal{S})^{hol} = (K_{\mathcal{S}})^{hol},$$

and the claim follows by (6.1). ■

Next, we show that $\mathcal{A}(H)$ and $\mathcal{D}(H)$ are isometrically isomorphic for any $H \in \mathfrak{H}_2$.

***Corollary 6.4*** *Let $H \in \mathfrak{H}_2$. Then $\mathscr{P}A_H = D_H\mathscr{P}$ and $\mathscr{Q}A_H = D_H\mathscr{Q}$. Hence, both $\mathscr{P}, \mathscr{Q} : \mathcal{A}(H) \to \mathcal{D}(H)$ are isometric isomorphisms.*

**Proof** Let us show the claim for $\mathscr{P}$, since the proof for $\mathscr{Q}$ is completely analogous. Let $r > 0$ and $f \in L^2(\mathbb{R}^+)$. Then,

$$(\mathscr{P}A_H f)(r) = \sqrt{\frac{2}{\pi}}(A_P A_H f)(r) = \sqrt{\frac{2}{\pi}}(A_H A_P f)(r) = (D_H \mathscr{P} f)(r),$$

where we have used that $A_P A_H = A_H A_P$. It follows by analytic continuation that $\mathscr{P}A_H = D_H\mathscr{P}$ (Proposition 2.1). Then, reasoning as in the proof of Theorem 5.2, we obtain $\mathscr{P} : \mathcal{A}(H) \to \mathcal{D}(H)$ is a well-defined isometric isomorphism. ■

We define the one-sided Hilbert-like operator $\mathcal{H}_+^{\mathbb{C}^+} : H^2(\mathbb{C}^+) \to H^2(\mathbb{C}^+)$ by

$$(\mathcal{H}_+^{\mathbb{C}^+} F)(z) = \frac{1}{\pi}\text{p.v.}\int_{\gamma_z} \frac{F(w)}{z-w}\,dw = \frac{1}{\pi}\text{p.v.}\int_0^\infty \frac{F(sz)}{1-s}\,ds, \quad z \in \mathbb{C}^+,\ F \in H^2(\mathbb{C}^+),$$

where $\gamma_z : (0,\infty) \to \mathbb{C}^+$, $\gamma_z(s) = sz$.

***Corollary 6.5*** *$\mathcal{H}_+^{\mathbb{C}^+}$ is a well-defined bounded operator on $H^2(\mathbb{C}^+)$ and on $\mathcal{D}(H)$ for any $H \in \mathfrak{H}_2$.*

**Proof** By Proposition 6.1, Corollary 6.4, and Theorem 3.4, the claim will follow once we prove that $\mathcal{H}_+^{\mathbb{C}^+} = \mathscr{P}\mathcal{H}_+\mathscr{P}^{-1}$. For $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, set $P_\theta(r,s) := \frac{re^{i\theta}}{r^2e^{2i\theta}+s^2}$ for $r, s > 0$. Then, $P_\theta \in \mathfrak{H}_2$, so $A_{P_\theta}\mathcal{H}_+ = \mathcal{H}_+A_{P_\theta}$ on $L^2(\mathbb{R}^+)$ by Theorem 3.4, and it is readily seen that $(\mathscr{P}f)^\theta = \sqrt{\frac{2}{\pi}}A_{P_\theta}f$ for any $f \in L^2(\mathbb{R}^+)$. Furthermore, notice that $F^\theta \in L^2(\mathbb{R}^+)$ for any $F \in H^2(\mathbb{C}^+)$ by (1.3). Then,

$$
\begin{aligned}
(\mathscr{P}\mathcal{H}_+\mathscr{P}^{-1}F)^\theta &= \sqrt{\frac{2}{\pi}}A_{P_\theta}\mathcal{H}_+\mathscr{P}^{-1}F = \sqrt{\frac{2}{\pi}}\mathcal{H}_+A_{P_\theta}\mathscr{P}^{-1}F = \mathcal{H}_+(\mathscr{P}\mathscr{P}^{-1}F)^\theta \\
&= \mathcal{H}_+F^\theta = (\mathcal{H}_+^{\mathbb{C}^+}F)^\theta, \qquad \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), F \in H^2(\mathbb{C}^+),
\end{aligned}
$$

and the claim follows. Analogously, one can prove that $\mathcal{H}_+^{\mathbb{C}^+} = \mathscr{Q}\mathcal{H}_+\mathscr{Q}^{-1}$. ■

(2) *Fractional kernels.* Let $\alpha > 0$, and let $(x)_+ = x$, if $x \ge 0$, and $(x)_+ = 0$ otherwise. Set $\mathcal{C}_\alpha(r,s) = \alpha(r-s)_+^{\alpha-1}r^{-\alpha}$, $r, s > 0$. These kernels are related to the Riemann–Liouville and Weyl fractional integrals of order $\alpha$. Their range spaces have been studied in [8], where they are realized as spaces of Sobolev type of absolutely continuous functions of fractional order on $\mathbb{R}^+$. Using the theory developed, we recover, with simpler proofs, some results given in [8].

***Theorem 6.6*** *The range space $\mathcal{A}(\mathcal{C}_\alpha) = \mathcal{A}(\mathcal{C}_\alpha^*)$ is an RKHS if and only if $\alpha > 1/2$. In this case, its kernel $K_{\mathcal{C}_\alpha}$ is given by*

$$K_{\mathcal{C}_\alpha}(r,s) = \frac{\alpha}{\max(r,s)}\, {}_2F_1\left(1-\alpha, 1; \alpha+1; \frac{\min(r,s)}{\max(r,s)}\right), \qquad \alpha > \frac{1}{2}, \ r \ne s > 0,$$

*and $K_{\mathcal{C}_\alpha}(r,r) = \frac{\alpha^2}{2\alpha-1}\frac{1}{r}$, $r > 0$. For $\alpha > 0$, the range space $\mathcal{D}(\mathcal{C}_\alpha) = \mathcal{D}(\mathcal{C}_\alpha^*)$ is an RKHS with kernel $\mathcal{K}_{\mathcal{C}_\alpha}$ given by*

$$\mathcal{K}_{\mathcal{C}_\alpha}(z,w) = \alpha^2 \int_0^1 \int_0^1 \frac{(1-x)^{\alpha-1}(1-y)^{\alpha-1}}{xz + y\overline{w}}\,dx\,dy, \qquad z, w \in \mathbb{C}^+.$$

*In addition, the Laplace transform $\mathcal{L}$ defines an isometric isomorphism $\mathcal{L} : \mathcal{A}(\mathcal{C}_\alpha) \to \mathcal{D}(\mathcal{C}_\alpha)$ for any $\alpha > 0$.*

**Proof** It is readily seen that $\mathcal{C}_\alpha \in \mathcal{I}_p$ if and only if $\alpha > 1/p$. Hence, the claim is an immediate consequence of Theorems 3.3 4.3 and Corollary 5.4. ■

Another kernel related to fractional theory, in particular with the Hadamard fractional integral (see [3]), is $D_{\alpha,c} := \frac{1}{\Gamma(\alpha)}\left(\frac{s}{r}\right)^c\left(\log\frac{r}{s}\right)^{\alpha-1}\frac{1}{s}\chi_{(0,r)}(s)$ $(r, s > 0)$, for $\alpha > 0$ and $c \in \mathbb{R}$. It is readily seen that $D_{\alpha,c} \in \mathfrak{H}_p$ if and only if $c > 1/p$, and in this case, $D_{\alpha,c} \in \mathcal{I}_p$ if and only if $\alpha > 1/p$. In particular, if $\alpha, c > 1/2$, then

$$K_{D_{\alpha,c}}(r,s) = \frac{1}{\Gamma(\alpha)^2}\int_0^{\min\{r,s\}}\left(\frac{t^2}{rs}\right)^\mu\left(\log\frac{r}{t}\log\frac{s}{t}\right)^{\alpha-1}\frac{dt}{t^2}, \qquad r, s > 0,$$

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and

$$\mathcal{K}_{D_{\alpha,c}}(z,w) = \frac{1}{\Gamma(\alpha)^2}\int_0^\infty \int_0^{\min\{1,x\}} \left(\frac{y^2}{x}\right)^\mu \left(\log\frac{1}{y}\log\frac{x}{y}\right)^{\alpha-1} \frac{1}{z+x\overline{w}}\,dydx,$$

for $z, w \in \mathbb{C}^+$.

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