On Hardy kernels as reproducing kernels

Jesús Oliva-Maza

Abstract. Hardy kernels are a useful tool to define integral operators on Hilbertian spaces like $L^2(\mathbb{R}^+)$ or $H^2(\mathbb{C}^+)$. These kernels entail an algebraic $L^1$-structure which is used in this work to study the range spaces of those operators as reproducing kernel Hilbert spaces. We obtain their reproducing kernels, which in the $L^2(\mathbb{R}^+)$ case turn out to be Hardy kernels as well. In the $H^2(\mathbb{C}^+)$ scenario, the reproducing kernels are given by holomorphic extensions of Hardy kernels. Other results presented here are theorems of Paley–Wiener type, and a connection with one-sided Hilbert transforms.

1 Introduction

Let $1 \leq p < \infty$, and let $H$ be a Hardy kernel of index $p$, that is, a mapping $H : (0, \infty) \times (0, \infty) \to \mathbb{C}$ which is homogenous of degree $-1$ and satisfies $\int_0^\infty |H(1, s)| s^{-1/p} \, ds < \infty$ (see Definition 2.1). As a straightforward consequence of the celebrated Hardy’s inequality [9, Theorem 319], one obtains that $H$ defines an operator $A_H$ given by

\begin{equation}
(A_H f)(r) := \int_0^\infty H(r, s) f(s) \, ds, \quad r > 0, \ f \in L^p(\mathbb{R}^+),
\end{equation}

which is bounded on $L^p(\mathbb{R}^+)$, where $\mathbb{R}^+ := (0, \infty)$. Hardy’s inequality also allows us to define a bounded operator $D_H$ on the Hardy spaces on the half plane $H^p(\mathbb{C}^+)$, where $\mathbb{C}^+ := \{ z \in \mathbb{C} \mid \Re z > 0 \}$, by

\begin{equation}
(D_H F)(z) := \int_0^\infty H(|z|, s) F^\theta(s) \, ds
\end{equation}

\begin{equation*}
= \int_0^\infty H(1, s) F(sz) \, ds, \quad z = |z| e^{i\theta} \in \mathbb{C}^+, \ F \in H^p(\mathbb{C}^+),
\end{equation*}

where $F^\theta(r) := F(re^{i\theta})$, for $r > 0$, $\theta \in (-\pi/2, \pi/2)$. Indeed, the last term in (1.2) shows that $D_H F$ is holomorphic (see, for example, [11]), and the boundedness of $D_H F$ follows by an application of Hardy’s inequality together with the realization of the norm of $H^p(\mathbb{C}^+)$ given in [19] by

\begin{equation}
\| F \|_{H^p} = \sup_{-\pi/2 < \theta < \pi/2} \left( \frac{1}{2\pi} \int_0^\infty |F^\theta(r)|^p \, dr \right)^{\frac{1}{p}}, \quad F \in H^p(\mathbb{C}^+).
\end{equation}
We will refer to these families of bounded operators on $L^p(\mathbb{R}^+)$ and $H^p(\mathbb{C}^+)$ as Hardy operators. These families have been actively studied, and are often labeled as Hausdorff operators due to its relation to the Hausdorff summability method through the function $\varphi(t) := H(t, 1)$ for $t > 0$ (see the survey articles [4, 15] for more details).

On the other hand, recall that a Hilbert space $X$ of complex-valued functions with domain $\Omega$ is said to be a reproducing kernel Hilbert space (RKHS) if and only if point evaluations $L_x f := f(x)$ are continuous functionals for all $x \in \Omega$. Then, by the Riesz representation theorem, for each $x \in \Omega$, there exists a unique $K_x \in X$ such that $f(x) = L_x f = (f | K_x)$ for all $f \in X$, where $(\cdot | \cdot)$ denotes the inner product in $X$. Then the reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}$ of $X$ is defined by

$$K(x, y) := K_y(x) = (K_y | K_x), \quad x, y \in \Omega.$$ 

The kernel $K$ determines the space $X$. More precisely, $X$ can be recovered from $K$ as the completion of span$\{K_x | x \in \Omega\}$ under the norm given by scalar product $(K_y | K_x) := K(x, y)$ (see the proof of the Moore–Aronszajn theorem [1]).

In this paper, we focus on the range spaces of Hardy operators in the Hilbertian case, that is, for $p = 2$. We show that these spaces are RKHSs and obtain their reproducing kernels. Our work is partly motivated by papers [7, 8], where the range spaces of generalized Cesàro operators $C_{\alpha}$ on $L^2(\mathbb{R}^+)$ and $H^2(\mathbb{C}^+)$ are analyzed as RKHSs. In this context, it is more appropriate to deal with Hardy operators using Hardy kernels $H$ rather than one-dimensional functions $\varphi$ associated with Hausdorff operators. Indeed, the set $\mathfrak{H}_p$ of Hardy kernels of index $p$ is naturally endowed with a structure of convolution (see [2, 6, 13]). More precisely, $\mathfrak{H}_p$ is a Banach algebra with multiplication $\bullet$ given by

$$\langle H \bullet G \rangle(r, s) = \int_{0}^{\infty} H(r, t) G(t, s) \, dt$$

(see Section 2).

In the setting of Hardy operators on $L^2(\mathbb{R}^+)$, our main result is that, for a Hardy kernel $H$ of index 2, the range space $\mathcal{A}(H) = A_H(L^2(\mathbb{R}^+))$ becomes an RKHS (continuously included in $L^2(\mathbb{R}^+)$) if and only if $H$ belongs to a certain ideal of $\mathfrak{H}_2$ (see Theorem 3.3). In this case, the reproducing kernel $K_H$ of $\mathcal{A}(H)$ is itself another Hardy kernel, given by

$$K_H = H \bullet H^*,$$

where $H^*$ is the adjoint kernel of $H$ (see Definition 2.2).

In the setting of Hardy spaces on the half plane, we prove in Theorem 4.3 that, for a given Hardy kernel $H$, the range space $\mathcal{D}(H)$ of a Hardy operator $D_H$ is an RKHS, continuously included in $H^2(\mathbb{C}^+)$, with reproducing kernel given by

$$\mathcal{K}_H = (H \bullet S \bullet H^*)^{hol}.$$ 

Here, $S$ is the Stieltjes kernel and $(\cdot)^{hol}$ denotes the extension to $\mathbb{C}^+ \times \mathbb{C}^+$, which is holomorphic in the first variable and anti-holomorphic in the second one, whenever such an extension exists (Theorem 4.3).

Next, we establish Paley–Wiener-type results in Section 5. We show that the Laplace transform $\mathcal{L}$ provides an isometric isomorphism between $\mathcal{A}(H)$ and $\mathcal{D}(H^T)$.
2 Banach algebras of Hardy kernels

In this section, we are concerned with arbitrary $p \in [1, \infty)$. 

**Definition 2.1** Let $1 \leq p < \infty$, and let $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ be a measurable map. $H$ is said to be a Hardy kernel of index $p$ if the following conditions hold.

(i) $H$ is homogeneous of degree $-1$; that is, for all $\lambda > 0$, $H(\lambda r, \lambda s) = \lambda^{-1}H(r, s)$ for all $r, s > 0$.

(ii) $\int_0^\infty |H(1, s)|s^{-1/p}ds < \infty$, which is equivalent to $\int_0^\infty |H(r, 1)|r^{-1/p'}dr < \infty$, where $p'$ is such that $1/p + 1/p' = 1$ (with $p' = \infty$ if $p = 1$ as usual).

Hardy kernels are useful tools to construct bounded operators on the Lebesgue spaces $L^p(\mathbb{R}^+)$ through (1.1). This is a well-known result of Hardy, Littlewood, and Pólya (see [9, Theorem 319]), and it is part of folklore that such operators can be described as convolution operators by identifying a Hardy kernel $H$ with the function $g_H \in L^1(\mathbb{R})$ given by

$$g_H(t) := H(1, e^{-t})e^{-t/p'}, \ t \in \mathbb{R}$$

(see, for example, [2, 6]). If one wants $H \mapsto g_H$ to be a bijection, one must consider the following equivalence classes in the set of Hardy kernels of index $p$. We set that two Hardy kernels $H, G$ of index $p$ are equivalent, $H \sim G$, if and only if $H(r, 1) = G(r, 1)$ for a.e. $r > 0$. From now on, $\mathcal{H}_p$ will denote this set of equivalence classes of Hardy kernels of index $p$, and we will refer to $H \in \mathcal{H}_p$ as a Hardy kernel rather than an equivalence class of Hardy kernels, so we identify an equivalence class by any of its elements.

As a consequence, the mapping $\Phi_p : \mathcal{H}_p \to L^1(\mathbb{R})$ defined by $\Phi_p(H) := g_H$ is a bijection, with inverse given by $(\Phi_p^{-1}g)(r, s) = r^{-1/p}s^{-1/p'}(\log s)^{1/p}$, for a.e. $r, s > 0$, $g \in L^1(\mathbb{R})$.

Next, we endow the linear space $\mathcal{H}_p$ with the norm and product given, respectively, by

$$\|H\|_{\mathcal{H}_p} := \|\Phi_p(H)\|_{L^1(\mathbb{R})}, \quad H \cdot G := \Phi_p^{-1}((\Phi_p H) * (\Phi_p G)) \quad \text{for all} \quad H, G \in \mathcal{H}_p,$$

where $*$ stands for the usual convolution of two elements of $L^1(\mathbb{R})$. We will denote the Banach algebra of bounded linear operators on a Banach space $X$ by $B(X)$.

**Proposition 2.1** Let $1 \leq p < \infty$. The space $\mathcal{H}_p$ is a commutative Banach algebra if provided with the norm and product

$$\|H\|_{\mathcal{H}_p} = \int_0^\infty |H(1, s)|s^{-1/p}ds,$$

$$(H \cdot G)(r, s) = \int_0^\infty H(r, t)G(t, s)dt, \quad r, s > 0.$$
Moreover, the mappings \( A_p : \mathcal{S}_p \to \mathcal{B}(L^p(\mathbb{R}^+)) \), \( D_p : \mathcal{S}_p \to \mathcal{B}(H^p(\mathbb{C}^+)) \), given by \( A_p(H) := A_H \) and \( D_p(H) := D_H \), are bounded Banach algebra homomorphisms.

**Proof** It is readily seen that \( \|H\|_{\mathcal{S}_p} = \|g_H\|_{L^1(\mathbb{R})} = \int_0^\infty |H(1, s)|s^{-1/p} \, ds \). Let us prove the product identity. It follows that for \( H, G \in \mathcal{S}_p \),

\[
(H \cdot G)(r, s) = \Phi_p^{-1} \left( \Phi_p(H) \ast \Phi_p(G) \right) (r, s) = r^{-1/p}s^{-1/p'} \left( g_H \ast g_G \right) \left( \log \frac{r}{s} \right) = \frac{1}{r} \int_0^\infty G(1, u^{-1})H \left( 1, \frac{s}{r} u \right) \frac{du}{u} = \frac{1}{r} \int_0^\infty H \left( 1, \frac{v}{r} \right) G \left( 1, \frac{s}{v} \right) \frac{dv}{v} = \int_0^\infty H(r, v) G(v, s) \, dv, \quad r, s > 0.
\]

Next, it follows by Hardy’s inequality [9, Theorem 319] that \( \|A_H\|_{\mathcal{B}(L^p)} \leq \|H\|_{\mathcal{S}_p} \). Moreover, one has that

\[
(A_{H \cdot G}f)(r) = \int_0^\infty (H \cdot G)(r, s) f(s) \, ds = \int_0^\infty H(r, t) \int_0^\infty G(t, s) f(s) \, ds \, dt = (A_{HA}f)(r), \quad f \in L^p(\mathbb{R}^+), \text{ a.e. } r > 0.
\]

Note that \( (D_H F)^\theta = A_H F^\theta \). Thus, by (1.3) and what we have just proved,

\[
\|D_H F\|_{H^p} \leq \|A_H\|_{\mathcal{B}(L^p)} \sup_{-\pi/2 < \theta < \pi/2} \frac{1}{2\pi} \|F^\theta\|_{L^p} \leq \|H\|_{\mathcal{S}_p} \|F\|_{H^p}, \quad F \in H^p(\mathbb{C}^+).
\]

Similarly, \( (D_{H \cdot G} F)^\theta = A_{H \cdot G} F^\theta = A_{HA}F^\theta = (D_H D_G F)^\theta \) for any \( F \in H^p(\mathbb{C}^+) \) and \( \theta \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \), and thus \( D_{H \cdot G} = D_H D_G \).

Next, we give a few definitions and properties regarding Hardy kernels that will be needed later. Let us denote by \( \overline{z} \) the conjugate of \( z \in \mathbb{C} \).

**Definition 2.2** Let \( 1 < p < \infty \), and let \( H \in \mathcal{S}_p \). Set \( H^\top(r, s) := H(s, r) \) for all \( r, s > 0 \). Similarly, set \( H^*(r, s) := H(s, r) \) for all \( r, s > 0 \).

**Remark 2.2** Let \( 1 < p < \infty \), and let \( H, G \in \mathcal{S}_p \). One has that \( H^\top, H^* \in \mathcal{S}_{p'} \), that \( (H \cdot G)^\top = H^\top \cdot G^\top, (H \cdot G)^* = H^* \cdot G^* \), and that \( (H^\top)^* = (H^*)^\top \).

**Definition 2.3** Let \( 1 \leq p < \infty \). We define \( \mathcal{J}_p \subset \mathcal{S}_p \) as \( \mathcal{J}_p := \Phi_p^{-1}(L^1(\mathbb{R}) \cap L^p(\mathbb{R})) \).

Clearly, \( \mathcal{J}_p \) is a dense ideal of \( \mathcal{S}_p \) since so is \( L^1(\mathbb{R}) \cap L^p(\mathbb{R}) \) in \( L^1(\mathbb{R}) \). We characterize its elements in the lemma below. For \( H \in \mathcal{S}_p \), define the family \( \{H_s\}_{s \in \mathbb{R}^+} \) of complex-valued functions defined a.e. in \( \mathbb{R}^+ \), given by \( \{H_s := H(\cdot, s) \mid s \in \mathbb{R}^+\} \). In particular, \( H_r^\top = H(r, \cdot) \) for any \( r > 0 \).
In this section, we analyze the rangespaces of Hardy operators on $L^2(I^p)$. In any of the above cases, one has that

\[ \|H_1^\tau\|_p = r^{-\frac{p}{2}} \|H_1^\tau\|_{L^p(I^p)} = r^{-\frac{p}{2}} \|g_H\|_{L^p(I^p)}, \quad r > 0. \]

**Proof** All the statements of the equivalence are straightforward to obtain using the homogeneity of degree $-1$ of $H$ and the definition of the function $g_H$. Let us show the equivalence $(i) \iff (iii)$. For $1 < p < \infty$,

\[ \|H_1^\tau\|_p = \left( \int_0^\infty |H(1,s)|^p \, ds \right)^{1/p} = \left( \int_{-\infty}^\infty |H(1, e^{-t})|^p e^{-t} \, dt \right)^{1/p} = \|g_H\|_p. \]

For $p = 1$, it is straightforward that $\|H_1^\tau\|_1 = \|g_H\|_1$ since $g_H(t) = H(1, e^{-t})$ for a.e. $t > 0$. ■

### 3 Hardy reproducing kernels on $\mathbb{R}^+ \times \mathbb{R}^+$

In this section, we analyze the range spaces of Hardy operators on $L^2(\mathbb{R}^+)$, although some minor results are also valid for general $p$. Our main motivation is to characterize the conditions for which these range spaces are RKHSs (Proposition 3.2).

**Definition 3.1** Let $1 \leq p < \infty$, and let $H \in \mathcal{S}_p$. Let $\mathcal{A}(H)$ be the range space

\[ \mathcal{A}(H) := \{ A_H f : f \in L^p(\mathbb{R}^+) \}. \]

We endow $\mathcal{A}(H)$ with a Banach (Hilbert if $p = 2$) space structure through the canonical identification $\mathcal{A}(H) \cong L^p(\mathbb{R}^+)/\ker A_H$.

Let $C(\mathbb{R}^+)$ denote the space of continuous functions on $\mathbb{R}^+$.

**Lemma 3.1** Let $1 \leq p < \infty$, and let $H \in \mathcal{S}_p \subset \mathcal{S}_p$. Then $\mathcal{A}(H) \subset C(\mathbb{R}^+)$. 

**Proof** Let $f \in L^p(\mathbb{R}^+)$. We have that

\[ (A_H f)(r) = \int_0^\infty H(r,s) f(s) \, ds = \int_0^\infty H(1,t) f(rt) \, dt = \langle \tau_r f, H_1^\tau \rangle, \quad \text{for all } r > 0, \]

where $(\tau_r f)(t) := f(rt)$ for $t > 0$, $\langle \cdot, \cdot \rangle$ denotes the dual product between $L^p(\mathbb{R}^+)$ and $L^{p'}(\mathbb{R}^+)$, and $H_1^\tau$ is defined before Lemma 2.3.

Since the mapping $r \mapsto \tau_r f$ from $\mathbb{R}^+$ into $L^p(\mathbb{R}^+)$ is continuous for each $f \in L^p(\mathbb{R}^+)$, it follows that $\langle \tau_r f, H_1^\tau \rangle = (A_H f)(r)$ is also continuous in $r$; that is, $A_H f \in C(\mathbb{R}^+)$, as we wanted to show. ■

As a consequence of the lemma, point evaluations are well defined on $\mathcal{A}(H)$ whenever $H \in \mathcal{S}_p$. Indeed, the proposition below adds a bit more information.
Proposition 3.2  Let $1 \leq p < \infty$, and let $H \in \mathcal{H}_p$. Then point evaluations are continuous functionals on $\mathcal{A}(H)$ if and only if $H \in \mathcal{I}_p$. In this case, for all $f \in \mathcal{A}(H)$,

$$|f(r)| \leq r^{-1/p} \|H^* \|_{\mathcal{P}^p} \|f\|_{\mathcal{A}(H)}, \quad r > 0.$$ 

Proof  Let us assume first that point evaluations are well defined and continuous on $\mathcal{A}(H)$, so for all $r > 0$, the mapping $\Omega_r : L^p(\mathbb{R}^+) \to \mathbb{C}$ given by $\Omega_r f := (A_H f)(r)$ is a well-defined continuous functional. Therefore, there exists $g_r \in L^p(\mathbb{R}^+)$ such that $(A_H f)(r) = \int_0^\infty g_r(s) f(s) ds$ for all $f \in L^p(\mathbb{R}^+)$, which implies that $H(r,s) = g_r(s)$ for a.e. $s > 0$. By Lemma 2.3, one gets that $H \in \mathcal{I}_p$.

Now, let us assume that $H \in \mathcal{I}_p$. By Lemma 3.1, it follows that point evaluations are well defined on $\mathcal{A}(H)$. By Lemma 2.3, one has that $(H^*_p)_r \in L^p(\mathbb{R}^+)$ be such that $f = A_H g$. Let $[g + \ker A_H]$ be the quotient class of $L^p(\mathbb{R}^+)/\ker A_H$ containing $g$. It follows that, for all $\tilde{g} \in [g + \ker A_H]$,

$$|f(r)| = |(A_H \tilde{g})(r)| = \left| \int_0^\infty H(r,s) \tilde{g}(s) ds \right| = \langle H^*_r, \tilde{g} \rangle$$

$$\leq \inf_{\tilde{g} \in [g + \ker A_H]} \|H^*_r \|_{\mathcal{P}^p} \|\tilde{g}\|_{\mathcal{P}^p} = \|H^*_r \|_{\mathcal{P}^p} \|g + \ker A_H\|_{L^p(\mathbb{R}^+)/\ker A_H}$$

$$= r^{-1/p} \|H^*_r \|_{\mathcal{P}^p} \|f\|_{\mathcal{A}(H)}, \quad \forall r > 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual product between $L^p(\mathbb{R}^+)$ and $L^p(\mathbb{R}^+)$. Therefore, point evaluations are continuous on $\mathcal{A}(H)$.

The next theorem gives the reproducing kernel $K_H$ of $\mathcal{A}(H)$ for $H \in \mathcal{I}_2$, which turns out to be a Hardy kernel as well.

Theorem 3.3  Let $H \in \mathcal{H}_2$. Then $\mathcal{A}(H)$ is an RKHS if and only if $H \in \mathcal{I}_2$, and in this case, its reproducing kernel $K_H$ is continuous and given by

$$K_H(r,s) = \int_0^\infty H(r,t) \overline{H(s,t)} dt, \quad \text{for } r, s > 0.$$ 

Then $K_H \in \mathcal{H}_2$, satisfying $K_H = H \bullet H^*$. As a consequence, $K_{H^*} = K_H$.

Proof  By Proposition 3.2, $\mathcal{A}(H)$ is an RKHS if and only if $H \in \mathcal{I}_2$, and in this case, $\mathcal{A}(H)$ is isometrically isomorphic to $L^2(\mathbb{R}^+)/\ker A_H \cong (\ker A_H)^\perp$. Let us compute its reproducing kernel. Assume that $f \in \ker A_H$, so it follows that

$$\langle f \mid H^*_u \rangle_{L^2} = \int_0^\infty H(u,v) f(v) dv = (A_H f)(u) = 0,$$

for all $u > 0$, where $H^*_u(v) = \overline{H(u,v)} = H^*_u(v)$ for a.e. $v > 0$. Therefore, one has that $H^*_u \in (\ker A_H)^\perp \subset L^2(\mathbb{R}^+)$ for all $u > 0$.

Now, let $h_u = A_H H^*_u \in \mathcal{A}(H)$, and let $f \in \mathcal{A}(H)$, so that $f = A_H g$ for a unique $g \in (\ker A_H)^\perp$. Since $\mathcal{A}(H) \cong (\ker A_H)^\perp$, it follows that, for all $u > 0$,

$$\langle f \mid h_u \rangle_{\mathcal{A}(H)} = \langle g \mid H^*_u \rangle_{(\ker A_H)^\perp} = \int_0^\infty H(u,v) g(v) dv = (A_H g)(u) = f(u).$$
Hence, \( K_H(v, u) = h_u(v) = \left(A_H H'_u\right)(v) = \int_0^\infty H(v, t)H(u, t)\,dt \) for all \( u, v > 0 \), as we wanted to show.

Moreover, \( K_H = H \bullet H^* = H^* \bullet H = K_{H^*} \) by Proposition 2.1 and Definition 2.2, and in particular \( K_H \) turns out to be a Hardy kernel. The continuity of \( K_H \) in each variable follows from the inclusion \( \mathcal{A}(H) \subset C(\mathbb{R}^+) \) (Lemma 3.1) and the fact that \( K_H(r, s) = K_{H^*}(s, r) \), for \( r, s > 0 \) (see, for example, [17, Lemma I.1.2]). But then, \( K_H : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{C} \) is continuous jointly on both variables since \( K_H(r, s) = s^{-1}K_H(r/s, 1) \).

Let \( \mathcal{H}_+ \) be the one-sided Hilbert transform on \( L^p(\mathbb{R}^+) \) defined by

\[
\mathcal{H}_+ f(x) := \text{p.v.} \frac{1}{\pi} \int_0^\infty \frac{f(r)}{x-r} \,dr, \quad x > 0, \quad f \in L^p(\mathbb{R}^+).
\]

The boundedness of \( \mathcal{H}_+ \) on \( L^p(\mathbb{R}^+) \) for \( 1 < p < \infty \) immediately follows from the boundedness of the Hilbert transform on \( L^p(\mathbb{R}) \) (see, for example, [5]). The following theorem has been inspired by [11, 14].

**Theorem 3.4** Let \( 1 < p < \infty \), and let \( H \in \mathcal{S}_p \). One has that \( \mathcal{H}_+ A_H = A_H \mathcal{H}_+ \). Therefore, \( \mathcal{H}_+ \) defines a bounded operator on \( \mathcal{A}(H) \).

**Proof** Let \( f \in L^p(\mathbb{R}^+) \). Then, for all \( x > 0 \),

\[
\left(\mathcal{H}_+ (A_H f)\right)(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_0^\infty H(1, s) f(rs) ds dr
\]

\[
= \lim_{\varepsilon \to 0^+} \int_0^\infty H(1, s) \frac{1}{\pi} \int_0^\infty \frac{f(r)}{x-r} \,dr \,ds dr
\]

\[
= \lim_{\varepsilon \to 0^+} \int_0^\infty H(1, s) \frac{1}{\pi} \int_0^\infty \frac{f(r)}{x-r} \,dr \,ds dr
\]

Here, we have applied Fubini to commute the integrals since \( \int_0^\infty |H(1, s)f(s)| \,ds \in L^p(\mathbb{R}^+) \) and \( \frac{1}{\pi} \chi(0, x-\varepsilon) \mathcal{H}_+(f) \chi(x+\varepsilon, \infty)(\cdot) \in L^p(\mathbb{R}^+) \) for all \( 1 < p < \infty \).

Recall that the maximal operator \( M\mathcal{H}_+ \) defined as \( (M\mathcal{H}_+ f)(x) := \sup_{\varepsilon > 0} |(\mathcal{H}_+ f)(x)| \), for all \( x > 0 \), belongs to \( B(L^p(\mathbb{R}^+)) \), where

\[
(M\mathcal{H}_+ f)(x) := \frac{1}{\pi} \int_{(0, x-\varepsilon) \cup (x+\varepsilon, \infty)} \frac{f(s)}{x-s} \,ds, \quad \varepsilon > 0, \text{ for } x > 0
\]

(see, for example, [5, Corollary 3.13]). As a consequence, the bound \( |H(1, s)(\mathcal{H}_+ f)(xs)| \leq |H(1, s)|(M\mathcal{H}_+ f)(xs) \) holds for every \( \varepsilon > 0 \), a.e. \( s > 0 \). By Hardy’s inequality [9, Theorem 3.19], \( |H(1, s)|(M\mathcal{H}_+ f)(xs) \), as a function on \( s > 0 \), belongs to \( L^1(\mathbb{R}^+) \) for a.e. \( x > 0 \). Thus, we apply the dominated convergence theorem to (3.1) to commute the limit and the integral \( \int_0^\infty \). Hence, \( \mathcal{A}(H) \) is \( \mathcal{H}_+ \) invariant, and then the continuity of \( \mathcal{H}_+ \) follows by the closed graph theorem.

4 **Hardy reproducing kernels on \( \mathbb{C}^+ \times \mathbb{C}^+ \)**

Next, we proceed to analyze the range spaces of Hardy operators on the Hardy spaces of holomorphic functions on the right-hand half plane \( H^2(\mathbb{C}^+) \).
Remark 4.1 Let $H$ with extension (see [12, Chapter VI]). Notice that the restriction of $\mathcal{F}$ on $\mathcal{F}$. One has that $\mathcal{F} = \mathcal{F}_p$ for all $1 \leq p < \infty$. Recall that $D_H$ is the Hardy operator on $H^p(\mathbb{C}^+)$ associated with $H \in \mathcal{F}_p$ through (1.2).

Definition 4.2 Let $1 \leq p < \infty$, and let $H \in \mathcal{F}_p$. Endow $\mathcal{D}(H) := D_H(H^p(\mathbb{C}^+))$ with the structure of Banach space induced by the canonical isomorphism $\mathcal{D}(H) \cong H^p(\mathbb{C}^+)/\ker D_H$.

Remark 4.1 Let $H \in \mathcal{F}_p$. Using that $D_H F = \int_0^\infty H(1, s) F(s \cdot) ds$, it is simple to see that $(D_H F| G)_{H^2} = (F| D_H G)_{H^2}$ in the inner product in $H^2(\mathbb{C}^+)$. Since all Hardy operators commute between themselves (see Proposition 2.1), $D_H$ is a normal operator.

Next, we give the main theorem of this section, for which we will need the following lemma and definition. Set $\mathcal{K}_w := \mathcal{K}(\cdot, w)$ for all $w \in \mathbb{C}^+$, so $\mathcal{K}_w \in H^2(\mathbb{C}^+)$.

Lemma 4.2 Let $H \in \mathcal{F}_p$. For all $z \in \mathbb{C}^+$, one has that

$$\int_0^\infty \|H(1, t)\mathcal{K}_{tz}\|_{H^2} dt < \infty.$$  

Proof Note that the vector-valued function $t \mapsto H(1, t)\mathcal{K}_{tz}$ is strong measurable, since $t \mapsto H(1, t)$ is measurable, and $w \mapsto \mathcal{K}_w$ is continuous from $\mathbb{C}^+$ to $H^2(\mathbb{C}^2)$. Then, for all $z \in \mathbb{C}^+$,

$$\int_0^\infty |H(1, t)|\mathcal{K}_{tz}\|_{H^2} dt = \int_0^\infty |H(1, t)|\sqrt{(\mathcal{K}_{tz}\|_{H^2})} dt = \int_0^\infty |H(1, t)|\sqrt{\mathcal{K}(tz, tz)} dt$$

$$= \frac{\sqrt{\mathcal{K}(z, z)}}{\mathcal{K}(z, z)} \int_0^\infty |H(1, t)| t^{-1/2} dt = \sqrt{\mathcal{K}(z, z)} \|H\|_{\mathcal{F}_p} < \infty.$$

Definition 4.2 We define $\mathcal{F}_{p, hol}$ to be the subset of $\mathcal{F}_p$ consisting of those $H \in \mathcal{F}_p$ with extension $H^{hol} : \mathbb{C}^+ \times \mathbb{C}^+ \to \mathbb{C}$ such that:

- $H(r, s) = H^{hol}(r, s)$ for $r, s > 0$,
- the map $z \mapsto H^{hol}(z, w)$ is holomorphic on $\mathbb{C}^+$ for all $w \in \mathbb{C}^+$, and
the map \( w \mapsto H^{hol}(z, w) \) is holomorphic on \( \mathbb{C}^+ \) for all \( z \in \mathbb{C}^+ \).

Note that if \( H \in \mathcal{S}_p^{hol} \), the extension \( H^{hol} \) is unique.

Notice that the Stieltjes kernel \( S \) satisfies that \( S \in \mathcal{S}_p^{hol} \) with \( S^{hol} = \mathcal{K} \).

**Theorem 4.3** Let \( H \in \mathcal{S}_2 \). One has that \( H \cdot S \cdot H^* \in \mathcal{S}_2^{hol} \), and that \( \mathcal{D}(H) \) is an RKHS with reproducing kernel \( \mathcal{K}_H \) given by

\[
\mathcal{K}_H = (H \cdot S \cdot H^*)^{hol}.
\]

**Proof** Let \( G \) be in \((\ker D_H)^\perp \) such that \( F = D_H(G) \in \mathcal{D}(H) \).

\[
\| F \|_{\mathcal{H}} = \| D_H G \|_{\mathcal{H}} \leq \| D_H \|_{\mathcal{B}(\mathcal{H})} \| G \|_{\mathcal{H}} = \| D_H \|_{\mathcal{B}(\mathcal{H})} \| F \|_{\mathcal{D}(H)}.
\]

Since \( \mathcal{H}^2(\mathbb{C}^+) \) is an RKHS, it follows from above that \( \mathcal{D}(H) \) is an RKHS. Let us compute its reproducing kernel \( \mathcal{K}_H \). As before, set \( \mathcal{K}_w(z) = \mathcal{K}(z, w) \). For \( F = D_H G \in \mathcal{D}(H), z = |z|e^{i\theta} \in \mathbb{C}^+ \), and \( G \in (\ker D_H)^\perp \), we have

\[
(4.1) \quad F(z) = \int_0^\infty H(|z|, s) G^d(s) \, ds = \int_0^\infty H(|z|, s) (G|\mathcal{K}_{e^{i\theta}})_{\mathcal{H}^2} \, ds = \int_0^\infty H(1, t) (G|\mathcal{K}_{iz})_{\mathcal{H}^2} \, dt
= \int_0^\infty (G|H(1, t) \mathcal{K}_{iz})_{\mathcal{H}^2} \, dt = \left( G \bigg| \int_0^\infty H(1, t) \mathcal{K}_{iz} \, dt \right)_{\mathcal{H}^2},
\]

where one can intertwine the integral sign with the inner product by Lemma 4.2.

Let \( J \in \ker D_H \). By substituting \( F \) by \( D_H J = 0 \) and \( G \) by \( J \) in (4.1), one concludes that \( \int_0^\infty H(1, t) \mathcal{K}_{iz} \, dt \in (\ker D_H)^\perp \). Then, we have that

\[
F(z) = \left( G \bigg| \int_0^\infty H(1, t) \mathcal{K}_{iz} \, dt \right)_{\mathcal{H}^2} = \left( F \bigg| D_H \left( \int_0^\infty H(1, t) \mathcal{K}_{iz} \, dt \right) \right)_{\mathcal{D}(H)}.
\]

Therefore, after rearranging some variables, one gets that the reproducing kernel \( \mathcal{K}_H \) of \( \mathcal{D}(H) \) is given by

\[
\mathcal{K}_H(z, w) = \left[ D_H \left( \int_0^\infty H(1, t) \mathcal{K}_{tw} \, dt \right) \right](z), \quad z, w \in \mathbb{C}^+.
\]

Now, let us see that the expression above coincides with the one given in the statement for all \( z, w \in \mathbb{R}^+ \):

\[
\left[ D_H \left( \int_0^\infty H(1, t) \mathcal{K}_{tw} \, dt \right) \right](z) = \int_0^\infty H(z, s) \left( \int_0^\infty H(1, t) \mathcal{K}_{tw} \, dt \right)(s) \, ds
= \int_0^\infty H(z, s) \int_0^\infty H(1, t) \mathcal{K}(s, tw) \, dt \, ds = \int_0^\infty H(z, s) \int_0^\infty S(s, u) H(w, u) \, du \, ds
= \int_0^\infty H(z, s) \int_0^\infty S(s, u) H^*(u, w) \, du \, ds = \int_0^\infty H(z, s) (S \cdot H^*)(s, w) \, ds
= (H \cdot S \cdot H^*)(z, w).
\]

Therefore, \( \mathcal{K}_H(z, w) = (H \cdot S \cdot H^*)(z, w) \) for all \( z, w \in \mathbb{R}^+ \).

Since all the elements in \( \mathcal{D}(H) \subset H^2(\mathbb{C}^+) \) are holomorphic, we have that \( \mathcal{K}_H(z, w) \) is holomorphic in \( z \), so it is determined for all \( (z, w) \in \mathbb{C}^+ \times \mathbb{R}^+ \) by its

https://doi.org/10.4153/S0008439522000406 Published online by Cambridge University Press
restriction at $\mathbb{R}^+ \times \mathbb{R}^+$. Moreover, since $\mathcal{K}_H$ is a reproducing kernel, we have that $\mathcal{K}_H(z, w) = \mathcal{K}_H(w, z)$ (see [17, Lemma I.1.2]), and as a consequence, $\mathcal{K}_H(z, w)$ is anti-holomorphic in $w$, and by the same reasoning as before, $\mathcal{K}_H(z, w)$ is determined for all $z, w \in \mathbb{C}^+$ by its restriction to $\mathbb{C}^+ \times \mathbb{R}^+$. All these statements imply that $H \cdot S \cdot H^* \in S_2^{hol}$ and that its holomorphic extension is precisely $\mathcal{K}_H$. ■

5 Paley–Wiener theorems for range spaces

We wish to start this section with the following remark. Paley–Wiener’s theorem states that $\mathcal{L} : L^2(\mathbb{R}^+) \rightarrow H^2(\mathbb{C}^+)$ is an isometric isomorphism, where $\mathcal{L}$ is the Laplace transform given by

(5.1) \[ (\mathcal{L} f)(z) := \int_0^\infty e^{-rz} f(r) dr, \quad f \in L^2(\mathbb{R}^+), z \in \mathbb{C}^+ \]

(see [18, Theorem V]).

This classical $L^2-H^2$ Paley–Wiener theorem can be used to prove that $H^2(\mathbb{C}^+)$ is an RKHS with kernel $\mathcal{K}(z, w) = \frac{1}{z + \bar{w}}$ [10, Proposition 1.8]. Conversely, one can reverse the implications of such a proof to obtain the $L^2-H^2$ Paley–Wiener theorem using RKHS theory (note that the kernel $\mathcal{K}$ of the space $H^2(\mathbb{C}^+)$ can be obtained independently of Paley–Wiener’s theorem; see, for instance, [12, Chapter VI]), as we show next.

Since the Laplace transform $\mathcal{L}$ acting on $L^2(\mathbb{R}^+)$ is injective, one can endow the range space $\mathcal{L}(L^2(\mathbb{R}^+))$ with the structure of Hilbert space induced by the bijection $\mathcal{L} : L^2(\mathbb{R}^+) \rightarrow \mathcal{L}(L^2(\mathbb{R}^+))$. For $F = \mathcal{L} f \in \mathcal{L}(L^2(\mathbb{R}^+))$, one has

\[ F(z) = \int_0^\infty e^{-rz} f(r) dr = (f[e^{-rz}])_{L^2} = (F[\mathcal{L}(e^{-rz})])_{\mathcal{L}(L^2)}, \quad z \in \mathbb{C}^+. \]

As a consequence, $\mathcal{L}(L^2(\mathbb{R}^+))$ is an RKHS with kernel $K_\mathcal{L}$ given by

\[ K_\mathcal{L}(z, w) = \mathcal{L}(e^{-r\bar{w}})(z) = \int_0^\infty e^{-rz} e^{-r\bar{w}} dr = \frac{1}{z + \bar{w}} = \mathcal{K}(z, w), \quad z, w \in \mathbb{C}^+. \]

That is, both $\mathcal{L}(L^2(\mathbb{R}^+))$ and $H^2(\mathbb{C}^+)$ are RKHSs with the same kernel $K_\mathcal{L} = \mathcal{K}$, so $\mathcal{L}(L^2(\mathbb{R}^+)) = H^2(\mathbb{C}^+)$ as Hilbert spaces (see, for instance, [17, Lemma I.1.5]), and the claim follows.

Now, we establish results of Paley–Wiener type for range spaces. We first show that $\mathcal{L}$ is an intertwining operator.

**Proposition 5.1** $\mathcal{L}A_H = D_{H^*} \mathcal{L}$ on $L^2(\mathbb{R}^+)$ for all $H \in S_2$.

**Proof** Let $z \in \mathbb{C}^+$ and $f \in L^2(\mathbb{R}^+)$. One has

\[
(\mathcal{L}A_H f)(z) = \int_0^\infty e^{-rz} \int_0^\infty H(r, t) f(t) dt dr = \int_0^\infty e^{-rz} \int_0^s H(1, s) f(rs) ds dr
\]

\[
= \int_0^\infty H(1, s) \int_0^\infty e^{-rz} f(rs) dr ds = \int_0^\infty H(1, s)(\mathcal{L} f)(\frac{z}{s}) \frac{ds}{s}
\]

\[
= \int_0^\infty H(u, 1)(\mathcal{L} f)(uz) du = (D_{H^*} \mathcal{L} f)(z),
\]
where we have applied Fubini’s theorem since both \( r \mapsto \int_0^\infty |H(r, t) f(t)| \, dt \) and \( r \mapsto e^{-rt} \) are in \( L^2(\mathbb{R}^+) \).

**Theorem 5.2** Let \( H \in \mathcal{S}_2 \). The Laplace transform \( \mathcal{L} \) restricted to \( \mathcal{A}(H) \) is an isometric isomorphism onto \( \mathcal{D}(H^\top) \), \( \mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H^\top) \).

**Proof** By the definition of \( \mathcal{A}(H) \) and \( \mathcal{D}(H^\top) \), the restrictions \( \mathcal{L}_H : (\ker A_H)^\perp \to \mathcal{A}(H) \) and \( D_{H^\top} : (\ker D_{H^\top})^\perp \to \mathcal{D}(H^\top) \) are isometric isomorphisms. By the \( L^2 - H^2 \) Paley–Wiener theorem and Proposition 5.1, it follows that \( (\ker D_{H^\top})^\perp = \mathcal{L}( (\ker A_H)^\perp ) \). Indeed, by Proposition 5.1, it easily follows that \( \mathcal{L}(\ker A_H) = \ker D_{H^\top} \), and thus \( (f, g)_{L^2} = 0 \) for all \( g \in \ker A_H \) if and only if \( (\mathcal{L} f, g)_{H^\top} = 0 \) for all \( g \in \mathcal{L}(\ker A_H) = \ker D_{H^\top} \).

Therefore, by Proposition 5.1 again, we obtain \( \mathcal{L} f = D_{H^\top} (\mathcal{L}_H)^{-1} f \) for all \( f \in \mathcal{A}(H) \), where all the mappings \( D_{H^\top} \), \( \mathcal{L} \) (seen as an operator from the subspace \( (\ker A_H)^\perp \subset L^2(\mathbb{R}^+) \) to the subspace \( (\ker D_{H^\top})^\perp \subset H^2(\mathbb{C}^+) \)) are in fact unitary operators. As a consequence, \( \mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H^\top) \) defines an isometric isomorphism.

**Corollary 5.3** Let \( H \in \mathcal{S}_2 \). The Laplace transform defines an isometric isomorphism \( \mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H) \) if and only if \( H \cdot H^* \) is a real-valued kernel.

**Proof** By the theorem above, we have that \( \mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H) \) is an isometric isomorphism if and only if \( \mathcal{D}(H) = \mathcal{D}(H^\top) \) as Hilbert spaces, and this happens if and only if their reproducing kernels are the same, \( \mathcal{K}_H = \mathcal{K}_{H^\top} \). By Theorem 4.3, this is equivalent to \( \mathcal{S} \cdot H \cdot H^* = \mathcal{S} \cdot H^\top \cdot (H^\top)^* \). The injectivity of the Stieltjes transform \( A_S \) (which can be proved via the Mellin transform; see, for example, [6]) implies that this holds if and only if \( H \cdot H^* = H^\top \cdot (H^\top)^* = (H \cdot H^*)^\top \). Then, the claim follows from the fact that \( (H \cdot H^*)^\top = H \cdot H^* \) for all \( H \in \mathcal{S}_2 \).

**Corollary 5.4** Let \( H \in \mathcal{S}_2 \). Either if \( H \) is symmetric, that is, \( H = H^\top \), or if \( H \) is real-valued, the Laplace transform \( \mathcal{L} \) restricts to an isometric isomorphism from \( \mathcal{A}(H) \) onto \( \mathcal{D}(H) \), \( \mathcal{L} : \mathcal{A}(H) \to \mathcal{D}(H) \).

We will see in Theorem 6.4 that, for any \( H \in \mathcal{S}_2 \), there exist isometric isomorphisms \( \mathcal{P}, \mathcal{Q} : \mathcal{A}(H) \to \mathcal{D}(H) \) related to the Poisson kernel.

### 6 Examples and applications

Here, we illustrate the theory given above with some examples and applications.

1. **Generalized Poisson operators.** For \( \alpha, \beta, \mu \) real numbers, let \( P_{\alpha, \beta, \mu}(r, s) = r^{\alpha \mu - \beta} s^{\beta - 1} (r^\alpha + s^\alpha)^{-\mu} \) for all \( r, s > 0 \). The spectral properties of its associated Hardy operator have been studied in [16]. Regarding the properties considered in the present paper, we have that, for \( p \in [1, \infty) \) and \( \alpha > 0 \), \( P_{\alpha, \beta, \mu} \in \mathcal{S}_p \) if and only if \( 0 < \beta - 1/p < \alpha \mu \), and in this case, \( P_{\alpha, \beta, \mu} \in \mathcal{F}_p \). For \( p = 2 \), one has

\[
K_{P_{\alpha, \beta, \mu}}(r, s) = \frac{s^{\beta - 1}}{\alpha r^\beta} B \left( \frac{2\beta - 1}{\alpha}, 2\mu - \frac{2\beta - 1}{\alpha} \right) _2 F_1 \left( \mu, \frac{2\beta - 1}{\alpha}; 2\mu; 1 - \left( \frac{s}{r} \right)^\alpha \right), \quad r, s > 0,
\]
where \( B \) is the Euler Beta function and \( _2F_1 \) is the hypergeometric Gaussian function.

As particular cases, one has the following.

**Stieltjes kernel.** For \( \alpha = \beta = \mu = 1 \), we obtain \( P_{1,1,1}(r, s) = S(r, s) = \frac{1}{r + s} \) for \( r, s > 0 \).

By Theorem 3.3, \( A(S) \) is an RKHS with kernel

\[
K_S(r, s) = \int_0^\infty \frac{1}{r + t + s} dt = \begin{cases} \frac{1}{r-s} \log \frac{r-s}{r} & \text{if } r \neq s, \\ \frac{1}{r} & \text{if } r = s, \end{cases} \quad \text{for } r, s > 0.
\]

**Poisson kernel and conjugate Poisson kernel.** Recall that, for \( x > 0 \), the Poisson kernel \( P^x \) and conjugate Poisson kernel \( Q^x \) on the half-right plane \( \mathbb{C}^+ \) are given by

\[
P^x(y) = \frac{x}{\pi x^2 + y^2}, \quad Q^x(y) = \frac{y}{\pi x^2 + y^2}, \quad s > 0.
\]

These kernels give rise to Hardy kernels \( P, Q \) as follows:

\[
P(r, s) := P^x(s) = P_{2,1,1}(r, s), \quad Q(r, s) := Q^x(s) = P_{2,1,1}^*(r, s), \quad r, s > 0.
\]

These kernels are related to the operators \( \mathcal{P}, \mathcal{Q} : L^2(\mathbb{R}^+) \rightarrow H^2(\mathbb{C}^+) \) given by

\[
(\mathcal{P}f)(z) := \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{z}{z^2 + s^2} f(s) ds, \quad (\mathcal{Q}f)(z) := \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{z^2 + s^2} f(s) ds,
\]

for any \( z \in \mathbb{C}^+, f \in L^2(\mathbb{R}^+) \). Indeed, \( (\mathcal{P}f)(r) = \sqrt{2\pi}(A_{P}f)(r) \) and \( (\mathcal{Q}f)(r) = \sqrt{2\pi}(A_{Q}f)(r) \) for \( r > 0, f \in L^2(\mathbb{R}^+) \). It is a matter of fact that \( \mathcal{P}, \mathcal{Q} \) are isometric isomorphisms (see Remark 6.2). Here, we provide a proof of it based on results of this paper.

Set

\[
L^2_{hol}(\mathbb{R}^+) := \{ f : \mathbb{R}^+ \rightarrow \mathbb{C}^+ \mid f(r) = F(r), r > 0, \text{ for some } F \in H^2(\mathbb{C}^+) \}.
\]

Since any holomorphic function in \( \mathbb{C}^+ \) is determined by its restriction to \( \mathbb{R}^+ \), the space \( L^2_{hol}(\mathbb{R}^+) \), regarded as a range space of \( H^2(\mathbb{C}^+) \), is an RKHS isometrically isomorphic to \( H^2(\mathbb{C}^+) \) with kernel \( S(r, s) = K(r, s) = \frac{1}{r+s}, r, s > 0 \). To see this, take \( F \in H^2(\mathbb{C}^+), f = F|_{\mathbb{R}^+}, \text{ and } s > 0 \). Then,

\[
f(s) = F(s) = (f|\mathcal{K}_s)_{H^2} = (f|\mathcal{K}_s|_{\mathbb{R}^+})_{L^2_{hol}} = (f|S_s)_{L^2_{hol}},
\]

as claimed.

**Proposition 6.1** Both \( \mathcal{P}, \mathcal{Q} : L^2(\mathbb{R}^+) \rightarrow H^2(\mathbb{C}^+) \) are isometric isomorphisms.

**Proof** Since \( P = Q^*, \) Theorem 3.3 implies that \( A(\sqrt{2\pi}P) = A(\sqrt{2\pi}Q) \) is an RKHS on \( \mathbb{R}^+ \) with kernel \( K_{\sqrt{2\pi}P} = K_{\sqrt{2\pi}Q} \) given by

\[
K_{\sqrt{2\pi}P}(r, s) = \frac{2}{\pi} \int_0^\infty \frac{r}{r^2 + t^2} \frac{s}{t^2 + s^2} dt = \frac{1}{r+s} S(r, s), \quad r, s > 0.
\]

Therefore, \( A(\sqrt{2\pi}P) = A(\sqrt{2\pi}Q) = L^2_{hol}(\mathbb{R}^+) \) as Hilbert spaces. Thus, all is left to prove is that \( \mathcal{P}f, \mathcal{Q}f \) are holomorphic on \( \mathbb{C}^+ \) and that both \( \mathcal{P}, \mathcal{Q} \) are injective operators. First, claim which follows by an application of Morera’s theorem. For the
second one, note that the Stieltjes transform $A_S$ is an injective operator and that
$A_S = A_P \circ Q = A_P A_Q = A_Q A_P$. Thus, both $A_P, A_Q$ are injective, and so are $\mathcal{P}, \mathcal{Q}$.

**Remark 6.2** The proposition above is equivalent to Paley–Wiener’s theorem. To see this, let $L^2_{\text{even}}(\mathbb{R})$ as the subset of even functions of $L^2(\mathbb{R})$, and note that the Fourier transform $\mathcal{F}$ restricts to an isometric mapping from $L^2_{\text{even}}(\mathbb{R})$ onto itself. Set $t : L^2_{\text{even}}(\mathbb{R}) \rightarrow L^2(\mathbb{R}^+)$ by $(t(f))(r) := f(r)$ for a.e. $r > 0$. Then $\mathcal{F}t^{-1}$ is a unitary operator on $L^2(\mathbb{R}^+)$, and one easily obtains that $\mathcal{P} = \mathcal{L}t\mathcal{F}t^{-1}$. Hence, $\mathcal{P}$ is an isometric isomorphism if and only if $\mathcal{L}$ is an isometric isomorphism.

By considering the subset of odd functions of $L^2(\mathbb{R})$, one obtains an analogous statement for $\mathcal{Q}$.

Some other consequences of results of this paper are the following.

**Corollary 6.3** As a range space, $\mathcal{L}(H^2(\mathbb{C}^+))$ is an RKHS with kernel $K$ given by

$$K(z, w) = \begin{cases} 
\frac{1}{\pi} \log \frac{z}{w}, & \text{if } z \neq w, \\
1, & \text{if } z = w,
\end{cases} \text{ for } z, w \in \mathbb{C}^+.$$

Here, we consider $\mathcal{L} : H^2(\mathbb{C}^+) \rightarrow H^2(\mathbb{C}^+)$ given by $(\mathcal{L}F)(z) := \int_0^\infty e^{-rz} F(r) \, dr$.

**Proof** By Proposition 5.1, one has $\mathcal{L} = \sqrt{2\pi} D_P \mathcal{L}$. Thus, $\mathcal{L}(H^2(\mathbb{C}^+)) = \mathcal{L}(L^2(\mathbb{R}^+)) = \mathcal{D}(\sqrt{2\pi}P)$ regarded as Hilbert spaces, since all the operators considered in the equalities are isometric isomorphisms. Hence, by Theorems 3.3 and 6.2,

$$K = K_{\sqrt{2\pi}P} = (K_{\sqrt{2\pi}P} \bullet S)^{hol} = (S \bullet S)^{hol} = (KS)^{hol},$$

and the claim follows by (6.1).

Next, we show that $\mathcal{A}(H)$ and $\mathcal{D}(H)$ are isometrically isomorphic for any $H \in \mathcal{S}_{\mathcal{H}}$.

**Corollary 6.4** Let $H \in \mathcal{S}_{\mathcal{H}}$. Then $\mathcal{A}_H = D_H \mathcal{P}$ and $\mathcal{D}_H = D_H \mathcal{Q}$. Hence, both $\mathcal{P}, \mathcal{Q} : \mathcal{A}(H) \rightarrow \mathcal{D}(H)$ are isometric isomorphisms.

**Proof** Let us show the claim for $\mathcal{P}$, since the proof for $\mathcal{Q}$ is completely analogous. Let $r > 0$ and $f \in L^2(\mathbb{R}^+)$. Then,

$$(\mathcal{P}A_H f)(r) = \sqrt{\frac{2}{\pi}} (A_P A_H f)(r) = \sqrt{\frac{2}{\pi}} (A_H A_P f)(r) = (D_H \mathcal{P} f)(r),$$

where we have used that $A_P A_H = A_H A_P$. It follows by analytic continuation that $\mathcal{P}A_H = D_H \mathcal{P}$ (Proposition 2.1). Then, reasoning as in the proof of Theorem 5.2, we obtain $\mathcal{P} : \mathcal{A}(H) \rightarrow \mathcal{D}(H)$ is a well-defined isometric isomorphism.

We define the one-sided Hilbert-like operator $\mathcal{H}_{+}^{\mathbb{C}^+} : H^2(\mathbb{C}^+) \rightarrow H^2(\mathbb{C}^+)$ by

$$(\mathcal{H}_{+}^{\mathbb{C}^+} F)(z) = \frac{1}{\pi} \text{p.v.} \int_{y_z} \frac{F(w)}{w} \, dw = \frac{1}{\pi} \text{p.v.} \int_{0}^{\infty} \frac{F(sz)}{1-s} \, ds, \quad z \in \mathbb{C}^+, F \in H^2(\mathbb{C}^+),$$

where $y_z : (0, \infty) \rightarrow \mathbb{C}^+$, $y_z(s) = sz$. 

https://doi.org/10.4153/S0008439522000406 Published online by Cambridge University Press
Corollary 6.5 \( \mathcal{H}_+^{c^*} \) is a well-defined bounded operator on \( H^2(\mathbb{C}^+) \) and on \( \mathcal{D}(H) \) for any \( H \in \mathcal{S}_2 \).

Proof By Proposition 6.1, Corollary 6.4, and Theorem 3.4, the claim will follow once we prove that \( \mathcal{H}_+^{c^*} = \mathcal{P_+} \mathcal{H}_+ \mathcal{P}^{-1} \). For \( \theta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \), set \( P_\theta(r,s) := \frac{r e^{i \theta}}{r^2 + s^2} \) for \( r, s > 0 \). Then, \( P_\theta \in \mathcal{S}_2 \), so \( A_{P_\theta} \mathcal{H}_+ = \mathcal{H}_+ A_{P_\theta} \) on \( L^2(\mathbb{R}^+) \) by Theorem 3.4, and it is readily seen that \( \mathcal{P} f = \sqrt{2 \pi} A_{P_\theta} f \) for any \( f \in L^2(\mathbb{R}^+) \). Furthermore, notice that \( F^\theta \in L^2(\mathbb{R}^+) \) for any \( F \in H^2(\mathbb{C}^+) \) by (1.3). Then,

\[
(\mathcal{P} \mathcal{H}_+ \mathcal{P}^{-1} F)^\theta = \sqrt{2 \pi} A_{P_\theta} \mathcal{H}_+ \mathcal{P}^{-1} F = \sqrt{2 \pi} \mathcal{H}_+ A_{P_\theta} \mathcal{P}^{-1} F = \mathcal{H}_+(\mathcal{P} \mathcal{P}^{-1} F)^\theta,
\]

\( \theta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \), and the claim follows. Analogously, one can prove that \( \mathcal{H}_+^{c^*} = \mathcal{P_+} \mathcal{H}_+ \mathcal{P}^{-1} \).

(2) Fractional kernels. Let \( \alpha > 0 \), and let \( (x)_+ = x \), if \( x \geq 0 \), and \( (x)_+ = 0 \) otherwise. Set \( C_\alpha(r,s) = a(r - s)^{\alpha - 1}, r, s > 0 \). These kernels are related to the Riemann–Liouville and Weyl fractional integrals of order \( \alpha \). Their range spaces have been studied in [8], where they are realized as spaces of Sobolev type of absolutely continuous functions of fractional order on \( \mathbb{R}^+ \). Using the theory developed, we recover, with simpler proofs, some results given in [8].

Theorem 6.6 The range space \( \mathcal{A}(C_\alpha) = \mathcal{A}(C_\alpha^*) \) is an RKHS if and only if \( \alpha > 1/2 \). In this case, its kernel \( K_{C_\alpha} \) is given by

\[
K_{C_\alpha}(r,s) = \frac{\alpha}{\max(r,s)} \mathcal{F}_1 \left( 1 - \alpha, 1; \frac{\min(r,s)}{\max(r,s)} \right), \quad \alpha > \frac{1}{2}, r \neq s > 0,
\]

and \( K_{C_\alpha}(r,r) = \frac{a^2}{\alpha - 1} \frac{1}{r^2} \), \( r > 0 \). For \( \alpha > 0 \), the range space \( \mathcal{D}(C_\alpha) = \mathcal{D}(C_\alpha^*) \) is an RKHS with kernel \( K_{C_\alpha} \) given by

\[
K_{C_\alpha}(z,w) = \alpha^2 \int_0^1 \int_0^1 \frac{(1 - x)^{\alpha - 1}(1 - y)^{\alpha - 1}}{xz + yw} dx dy, \quad z, w \in \mathbb{C}^+.
\]

In addition, the Laplace transform \( L \) defines an isometric isomorphism \( L : \mathcal{A}(C_\alpha) \rightarrow \mathcal{D}(C_\alpha) \) for any \( \alpha > 0 \).

Proof It is readily seen that \( C_\alpha \in \mathcal{J}_p \) if and only if \( \alpha > 1/p \). Hence, the claim is an immediate consequence of Theorems 3.3, 4.3 and Corollary 5.4.

Another kernel related to fractional theory, in particular with the Hadamard fractional integral (see [3]), is \( D_{\alpha,c} := \frac{1}{\Gamma(\alpha)} \left( \frac{t}{r} \right)^c (\log t)^{\alpha - 1} \chi_{(0,r)}(s) \) \( (r, s > 0) \), for \( \alpha > 0 \) and \( c \in \mathbb{R} \). It is readily seen that \( D_{\alpha,c} \in \mathcal{J}_p \) if and only if \( c > 1/p \), and in this case, \( D_{\alpha,c} \in \mathcal{J}_p \) if and only if \( \alpha > 1/p \). In particular, if \( \alpha, c > 1/2 \), then

\[
K_{D_{\alpha,c}}(r,s) = \frac{1}{\Gamma(\alpha)^2} \int_0^{\min(r,s)} \left( \frac{t^2}{rs} \right)^{\mu} (\log r - \log s)^{\alpha - 1} dt, \quad r, s > 0,
\]
On Hardy kernels as reproducing kernels

and

$$K_{D_{\alpha}}(z, w) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^{\min\{1, x\}} \left( \frac{y^2}{x} \right)^{\mu} \left( \log \frac{1}{y} \log \frac{x}{y} \right)^{\alpha - 1} \frac{1}{z + xw} \, dy \, dx,$$

for $z, w \in \mathbb{C}^+$. 

Acknowledgment  The author thanks J. E. Galé for several ideas, comments, and additional information, in particular for his helpful reviews. The author also thanks the referee for a meticulous review. Both of them have led to considerably improve the final version of the paper.

References


Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, Pedro Cerbuna 12, Zaragoza 50009, Spain

e-mail: joliva@unizar.es