SECTIONS OF CONVEX BODIES IN JOHN'S AND MINIMAL SURFACE AREA POSITION

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Abstract. We prove several estimates for the volume, the mean width, and the value of the Wills functional of sections of convex bodies in John's position, as well as for their polar bodies. These estimates extend some well-known results for convex bodies in John's position to the case of lower-dimensional sections, which had mainly been studied for the cube and the regular simplex. Some estimates for centrally symmetric convex bodies in minimal surface area position are also obtained.

1. Introduction and notation

For any convex body $K \subseteq \mathbb{R}^n$ (i.e., a compact convex set with non-empty interior), it is said that any affine image of K is a position of K. Every position of the Euclidean ball, B_2^n, is called an ellipsoid. A well-known theorem by John (see [29]) states that every convex body $K \subseteq \mathbb{R}^n$ has a unique maximal volume ellipsoid, $\mathcal{E}(K)$, contained in it. The volume ratio of K is defined as

$$\text{v.rat}(K) := \left(\frac{|K|}{|\mathcal{E}(K)|}\right)^{1/n}.$$

Here and in what follows $|\cdot|$ denotes the volume of a convex body in the appropriate dimension. Notice that the volume ratio does not depend on the position of the convex body K.

A convex body $K \subseteq \mathbb{R}^n$ is said to be in John's position if the maximal volume ellipsoid contained in K is the Euclidean unit ball B_2^n. In other words, K is in John's position if B_2^n is contained in K and for every non-degenerate linear map $T \in GL(n)$ and every $a \in \mathbb{R}^n$ such that $a + T(B_2^n) \subseteq K$ we have that $|a + T(B_2^n)| = |T(B_2^n)| \leqslant |B_2^n|$. By the uniqueness of $\mathcal{E}(K)$, this position is uniquely determined up to orthogonal transformations. It is well known that, denoting by B_∞^n the n-dimensional cube and by S_n the centered regular simplex with inradius $r(S_n) = 1$ in $\mathbb{R}^n$, both B_∞^n and S_n are in John's position.

Ball proved in [6] that the simplex maximizes the volume ratio among all convex bodies in $\mathbb{R}^n$ and the cube maximizes the volume ratio among all the centrally symmetric convex bodies in $\mathbb{R}^n$. The proof consists of the following three steps: first, since the volume ratio of a convex body does not depend on its position, it can be assumed that the convex body is John's position; second, substitute the convex body by a polytope which contains the convex body and is also in John's

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position; finally, obtain an upper bound for the volume of such polytope by using Brascamp–Lieb inequality (see Theorem 2.1 below).

Dual to John's position is the so called Löwner's position. A convex body is said to be in Löwner's position if the minimal volume ellipsoid containing it is the Euclidean unit ball. It is well known (see, for instance, [4, Proposition 4.7]) that a convex body $K \subseteq \mathbb{R}^n$ is in John's position if and only if K° is in Löwner's position. Here K° denotes the polar body of K defined by

$$K^\circ = \{x \in \mathbb{R}^n \,:\, \langle x, y\rangle \leqslant 1\,,\, \forall\, y \in K\}.$$

Ball observed in [6] that a reverse form of the Brascamp–Lieb inequality would provide that, among all convex bodies in Löwner's position, the centered regular simplex $\tilde{S}_n$ with circumradius $R(\tilde{S}_n) = 1$ has the smallest volume. Moreover, among all centrally symmetric convex bodies in Löwner's position, the ℓ_1^n-ball, B_1^n, has the smallest volume. The needed reverse form of the Brascamp–Lieb inequality was obtained by Barthe in [9] (see Theorem 2.1 below).

Vaaler showed in [43] that if $F_0 \in G_{n,k}$ is a k-dimensional linear subspace of $\mathbb{R}^n$ in the Grassmannian manifold $G_{n,k}$, then $|B_\infty^n \cap F_0| \geqslant |B_\infty^k|$. Ball obtained in [5] a reverse inequality proving that $|B_\infty^n \cap F_0| \leqslant 2^{\frac{n-k}{2}} |B_\infty^k|$. He also obtained the bound

$$|B_\infty^n \cap F_0|^{1/k} \leqslant \sqrt{\frac{n}{k}}|B_\infty^k|^{1/k}, \tag{1.1}$$

which is optimal if and only if $k \mid n$ (see [26]).

It follows from results of Ball [7] that the k-dimensional sections of a regular simplex with largest volume are exactly its k-dimensional faces. Webb showed in [44] that for every hyperplane through the origin $F_0 \in G_{n,n-1}$,

$$|S_n \cap F_0|^{\frac{1}{n-1}} \leqslant \frac{1}{(\sqrt{2n(n+1)})^{\frac{1}{n-1}}}\sqrt{\frac{n(n+1)}{n-1}}|S_{n-1}|^{\frac{1}{n-1}}. \tag{1.2}$$

There is equality for the sections passing through the origin that contain $n-1$ of the vertices.

Dirksen proved in [15, Theorem 6.1] (see also [16]) the following estimate for the volume of k-dimensional sections of S_n through the origin:

$$|S_n \cap F_0|^{1/k} \leqslant \frac{1}{(k+1)^{\frac{n-k}{2k(n+1)}}}\sqrt{\frac{n(n+1)}{k(k+1)}}|S_k|^{1/k} \tag{1.3}$$

for every $F_0 \in G_{n,k}$. Besides, this estimate is asymptotically sharp.

The proof of these volume estimates for sections of the cube and the regular simplex follow the lines of Ball's upper bound of the volume ratio. However, only sections of B_∞^n and S_n, which are two particular convex bodies, are being considered. Passing to the general case, we observe that it is not possible to obtain an upper bound for the volume of sections of a general centered convex body K without any additional assumption since, considering different positions of K, we can obtain sections with volume as large as desired. However, B_∞^n and S_n are in John's position. We will consider convex bodies in John's position and generalize these (and other known results), to sections of such convex bodies. We will also obtain some estimates for sections of convex bodies in minimal surface area position.

1.1. **Volume of sections of convex bodies in John's position.** In a recent article [32], Markessinis claimed to have obtained an upper bound for the volume of k-dimensional central sections of convex bodies in John's position. However, although the estimate given for central sections of centrally symmetric convex bodies in John's position is correct, the proof in the not necessarily symmetric case is not correct. In the following theorem we give an upper bound for the volume of central (and non-central) k-dimensional sections of an arbitrary convex body in John's position.

Theorem 1.1. *Let $K\subseteq\mathbb{R}^n$ be a convex body in John's position and $F_0\in G_{n,k}$. Then*

$$|K\cap F_0|^{1/k}\leqslant\frac{1}{(k+1)^{\frac{n-k}{2k(n+1)}}}\sqrt{\frac{n(n+1)}{k(k+1)}}|S_k|^{1/k}.$$

Furthermore, if K is centrally symmetric

$$|K\cap F_0|^{1/k}\leqslant\sqrt{\frac{n}{k}}|B_\infty^k|^{1/k}.$$

Moreover, if F_h is a k-dimensional affine subspace at distance h from the origin and K is a convex body in John's position then

$$|K\cap F_h|^{1/k}\leqslant\sqrt{\frac{n(n+1)^{1+\frac{1}{k}}}{k(k+1)^{1+\frac{1}{k}}}}\left(\frac{n}{n+h^2}\right)^{\frac{1}{2k}}|S_k|^{1/k}.$$

Remark. The proof in the symmetric case is the same as the one given by Markessinis, which follows Ball's ideas in [5]. Nevertheless, we will reproduce it for the sake of completeness. We can also obtain it as a direct consequence of Theorem 1.5, as well as a consequence of Theorem 8.1 below (see Section 2.6). This estimate is a sharp generalization of Ball's estimate (1.1) for the cube. Moreover, the case $k=1$ gives one more proof of John's theorem in the symmetric case: if K is a centrally symmetric convex body in $\mathbb{R}^n$ whose maximal volume ellipsoid is B_2^n then $K\subseteq\sqrt{n}B_2^n$.

Remark. Many other generalizations and extensions of Ball's estimates for sections of B_∞^n have been obtained, for instance, in [28], [30], or [31].

Remark. The proof in the non-symmetric case follows Dirksen's ideas from [16]. However, the decomposition of the identity in a linear subspace of $\mathbb{R}^{n+1}$ in order to apply Theorem 2.1 is not obtained by projecting the vectors in an orthonormal basis of $\mathbb{R}^{n+1}$. It comes out by projecting the vectors providing a more general decomposition of the identity in $\mathbb{R}^{n+1}$. As a consequence, a different maximization problem from the one in Dirksen's proof has to be considered. Notice that we recover the estimate in (1.3), which is asymptotically sharp for the simplex. Besides, if we take non-central sections by k-dimensional subspaces a distance $h=\sqrt{\frac{n(n-k)}{(k+1)}}$ from the origin, which is the distance from the origin to any k-dimensional face of S_n we obtain that

$$|K\cap F_h|^{1/k}\leqslant\sqrt{\frac{n(n+1)}{k(k+1)}}|S_k|^{1/k},$$

which is exactly the volume of the k-dimensional faces of S_n. The estimate for general affine subspaces can also be obtained as a direct consequence of Theorem 8.1 below.

1.2. **Volume of projections of convex bodies in Löwner's position.** Let us recall that if $F_0 \in G_{n,k}$ is a linear subspace and $K \cap F_0$ contains the origin in its relative interior, the polar body of $K \cap F_0$ in F_0 is $P_{F_0} K^\circ$, the projection of the polar body of K onto F_0. Contrary to the case of the volume of sections of the cube, not much is known about the projections of the cross-polytope. For example, a dual statement to Vaaler's theorem, claiming that if $F_0 \in G_{n,k}$ then $|P_{F_0}(B_1^n)| \leqslant |B_1^k|$, has only been confirmed if $k = 2, 3$, and $n-1$ (see [8], [12], and [27]) and a dual statement of Ball's upper bound $|B_\infty^n \cap F_0| \leq 2^{\frac{n-k}{2}} |B_\infty^k|$ has only been proved when $k = 2$ or $k = n-1$ (see [12] and [27]). Nevertheless, concerning the volume of polar bodies of sections of convex bodies in John's position (i.e, projections of convex bodies in Löwner's position whenever they contain the origin in its interior), it was proved by Barthe in his PhD thesis (see also [1]) that in the case of the ℓ_p^n-balls, if $1 \leqslant p \leqslant 2$ and $F_0 \in G_{n,k}$ then

$$|P_{F_0}(B_p^n)|^{1/k} \geqslant \left(\frac{k}{n}\right)^{\frac{1}{p}-\frac{1}{2}} |B_p^k|^{1/k},$$

where P_{F_0} denotes the orthogonal projection onto F_0. In particular, we have the following estimate for the projections of B_1^n: for every $F_0 \in G_{n,k}$

$$|P_{F_0}(B_1^n)|^{1/k} \geqslant \sqrt{\frac{k}{n}} |B_1^k|^{1/k}.$$

We obtain a similar lower bound for the volume of k-dimensional projections of convex bodies in Löwner's position.

Theorem 1.2. *Let $K \subseteq \mathbb{R}^n$ be a convex body in Löwner's position and $F_0 \in G_{n,k}$. Then*

$$|P_{F_0}(K)|^{1/k} \geqslant \sqrt{\frac{k}{n}} |\tilde{S}_k|^{1/k}.$$

Furthermore, if K is centrally symmetric

$$|P_{F_0}(K)|^{1/k} \geqslant \sqrt{\frac{k}{n}} |B_1^k|^{1/k}.$$

Remark. In the symmetric case, the proof follows the idea of the proof of the aforementioned result for B_p^n. This relies on the use of the reverse Brascamp–Lieb inequality (see Theorem 2.1 below) together with the use of a decomposition of the identity in a linear subspace of $\mathbb{R}^n$. As in the proof of Theorem 1.1, unlike in the case in which $K = B_p^n$, the decomposition of the identity in the linear subspace does not arise by projecting the canonical basis, but by projecting the vectors in a general decomposition of the identity in $\mathbb{R}^n$.

In the non-symmetric case, the proof follows the idea of the proof of Ball's observation in [6] together with the reverse form of Brascamp–Lieb inequality. Again, in this case the decomposition of the identity in a linear subspace in $\mathbb{R}^{n+1}$ arises by projecting the vectors in a decomposition of the identity in $\mathbb{R}^{n+1}$ rather than an orthonormal basis in $\mathbb{R}^{n+1}$. Some estimates of the volume of projections of the regular simplex were obtained in [19]. However, the estimates do not rely on the use of the reverse Brascamp–Lieb inequality.

1.3. **The mean width.** The mean width of a convex body $K \subseteq \mathbb{R}^n$ is defined as

$$w(K) = \int_{S^{n-1}} h_K(\theta) d\sigma(\theta),$$

where, for every $x \in \mathbb{R}^n$, $h_K(x) := \sup\{\langle x, y\rangle : y \in K\}$ is the support function of K at x and $d\sigma$ denotes the uniform probability measure on S^{n-1}. In [40], the authors proved that among all centrally symmetric convex bodies in John's position in $\mathbb{R}^n$, $w(K)$ is maximized when $K = B_\infty^n$. The not necessarily symmetric case was treated in [10], where it was proved that among all convex bodies in John's position in $\mathbb{R}^n$, $w(K)$ is maximized when $K = S_n$, where S_n denotes the regular simplex in John's position. If we pass to the mean width of sections, then a direct consequence of [11, Theorem 10] is that for any k-dimensional linear subspace $F_0 \in G_{n,k}$,

$$w(B_\infty^n \cap F_0) \leqslant \sqrt{\frac{n}{k}} w(B_\infty^k),$$

and this estimate is sharp when $k \mid n$.

Furthermore, it was proved in [40] that among all centrally symmetric convex bodies in Löwner's position in $\mathbb{R}^n$, $w(K)$ is minimized when $K = B_1^n$. Finally, in [41] it was proved that among all convex bodies in Löwner's position in $\mathbb{R}^n$, $w(K)$ is minimized if $K = \tilde{S}_n$. We will prove the following results on the mean width of sections of convex bodies in John's position:

Theorem 1.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body in John's position and $F_0 \in G_{n,k}$. Then*

$$w(K \cap F_0) \leqslant C \frac{n}{k} \sqrt{\frac{\log n}{\log k}} w(S_k),$$

where C is an absolute constant. Furthermore, if K is centrally symmetric then

$$w(K \cap F_0) \leqslant \sqrt{\frac{n}{k}} w(B_\infty^k).$$

We shall also prove the following result on the mean width of projections of convex bodies in Löwner's position:

Theorem 1.4. *Let $K \subseteq \mathbb{R}^n$ be a convex body in Löwner's position. Then, for any k-dimensional linear subspace $F_0 \in G_{n,k}$,*

$$w(P_{F_0}(K)) \geqslant \sqrt{\frac{k}{n}} w(\tilde{S}_k).$$

Furthermore, if K is centrally symmetric then

$$w(P_{F_0}(K)) \geqslant \sqrt{\frac{k}{n}} w(B_1^k).$$

Remark. The proofs of the latter two theorems follow the idea of the previously known results, by applying the Brascamp–Lieb inequality or its reverse form on a linear subspace of $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$. However, we were not able to handle the technical problems, arised from projecting a decomposition of the identity instead of an orthonormal basis, in the non-symmetric case in Theorem 1.3 and a different approach was considered.

1.4. **The Wills functional.** For any compact convex set $K \subseteq \mathbb{R}^n$, by Steiner's formula (see [42, Equation (4.1)]), the volume of $K + tB_2^n$ can be expressed as a polynomial in the variable t

$$|K + tB_2^n| = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i, \quad \forall t \geqslant 0,$$

where the numbers $W_i(K)$ are the so-called quermaßintegrals of K. We have that $W_0(K) = |K|$ is the volume of K, $nW_1(K) = |\partial K|$ is the surface area of K, and $W_{n-1} = |B_2^n| w(K)$ is a multiple of the mean width of K. If K is contained in a k-dimensional subspace $F_0 \in G_{n,k}$, we can compute its quermaßintegrals in $\mathbb{R}^n$, but also its quermaßintegrals with respect to the subspace F_0, which we identify with $\mathbb{R}^k$. If we denote these quermaßintegrals by $W_i^{(k)}(K)$, for $i = 0, \dots, k$, we have that (see e.g. [39, Property 3.1])

$$W_i^{(k)}(K) = \frac{\binom{n}{n-k+i}}{\binom{k}{i}} \frac{|B_2^i|}{|B_2^{n-k+i}|} W_{n-k+i}(K), \quad \forall 0 \leqslant i \leqslant k,$$

while $W_i(K) = 0$ for all $0 \leqslant i < n-k$. In order to avoid the issue that quermaßintegrals depend on the space where the convex body is embedded, McMullen [34] defined the intrinsic volumes of a compact convex set $K \subseteq \mathbb{R}^n$ as

$$V_i(K) = \frac{\binom{n}{i}}{|B_2^{n-i}|} W_{n-i}(K), \quad \forall 0 \leqslant i \leqslant n.$$

In [45], Wills introduced and studied the functional

$$\mathcal{W}(K) = \sum_{i=0}^{n} V_i(K) \tag{1.4}$$

because of its possible relation with the so-called lattice-point enumerator $G(K) = \#(K \cap \mathbb{Z}^n)$. It was proved in [2] that, among symmetric convex bodies in John's position, $\mathcal{W}(K)$ is maximized if $K = B_\infty^n$. Here, we prove the following:

Theorem 1.5. *Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body in John's position. Then, for any $F_0 \in G_{n,k}$ and every $\lambda \geqslant 0$,*

$$\mathcal{W}(\lambda(K \cap F_0)) \leqslant \mathcal{W}\left(\lambda \sqrt{\frac{n}{k}} B_\infty^k\right).$$

Remark. The proof of this Theorem follows the idea of the proof of the result in [2]. What is new here is the consideration of dilations of $K \cap F_0$ and $\sqrt{\frac{n}{k}} B_\infty^k$. This is indeed something different since the Wills functional is not homogeneous. Considering the dilations is important for the applications. Indeed, as direct consequences of obtaining Theorem 1.5 for such dilations, we can obtain the symmetric cases of Theorem 1.1 and Theorem 1.3, providing a different proof in those cases.

We also prove the following estimate for the Wills functional of projections of convex bodies in Löwner's position:

Theorem 1.6. *Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body in Löwner's position. Then, for any $F_0 \in G_{n,k}$,*

$$\mathcal{W}(P_{F_0}(K)) \geqslant \frac{1}{k^{k/2}}.$$

1.5. **Sections of convex bodies in minimal surface area position.** The main tool used to obtain most of the estimates above is the fact that a decomposition of the identity operator is associated to any convex body in John's position, and that this decomposition allows the use of the Brascamp–Lieb inequality (see Theorem 2.1 below). When K is a polytope in minimal surface area, then there is again a decomposition of the identity associated to K (see Section 2.7). A similar use of the Brascamp–Lieb inequality, together with an approximation by polytopes, will lead to similar estimates for sections of convex bodies in minimal surface area position. Namely, we can prove the following:

Theorem 1.7. *Let $K \subseteq \mathbb{R}^n$ be a convex body in minimal surface area position and let ΠK and $\Pi^* K$ denote its projection body and polar projection body, respectively. Then, for any k-dimensional linear subspace $F_0 \in G_{n,k}$ we have*

(a) $|\Pi^* K \cap F_0| \leqslant \frac{4^k n^k}{k!} \frac{1}{|\partial K|^k}$,

(b) $|P_{F_0}(\Pi K)| \geqslant \left(\frac{|\partial K|}{n}\right)^k$.

Furthermore, if K is centrally symmetric, then for any k-dimensional linear subspace $F_0 \in G_{n,k}$ we have

(i) $\mathcal{W}(K \cap F_0) \leqslant \mathcal{W}\left(\frac{n^2}{k}\frac{|K|}{|\partial K|} B_\infty^k\right)$,

(ii) $|K \cap F_0|^{1/k} \leqslant \frac{n^2}{k}\frac{|K|}{|\partial K|}|B_\infty^k|^{1/k}$,

(iii) $w(K \cap F_0) \leqslant \frac{n^2}{k}\frac{|K|}{|\partial K|} w(B_\infty^k)$,

(iv) $|(K \cap F_0)^\circ|^{1/k} \geqslant \frac{k}{n^2}\frac{|\partial K|}{|K|}|(B_\infty^k)^\circ|^{1/k}$,

(v) $w((K \cap F_0)^\circ) \geqslant \frac{k}{n^2}\frac{|\partial K|}{|K|} w((B_\infty^k)^\circ)$.

Remark. Notice that if $k = n$ then (a) recovers the right-hand side of (2.6), (b) recovers the left-hand side of (2.7), (ii) recovers the estimate given by Ball's reverse isoperimetric inequality in [6] and (iii) recovers the estimate given in [33, Theorem 7.1].

2. Preliminaries

2.1. **John's position.** As mentioned in the introduction, a convex body is said to be in John's position if the maximal volume ellipsoid contained in it is the Euclidean unit ball. A classical theorem of John [29] (see also [7]) states that K is in John's position if and only if $B_2^n \subseteq K$ and there exist $m = O(n^2)$ contact points $\{u_j\}_{j=1}^m \subseteq \partial K \cap S^{n-1}$ (the intersection of the boundary of K and the Euclidean unit sphere) and $\{c_j\}_{j=1}^m$ with $c_j > 0$ for every $1 \leqslant j \leqslant m$, such that

$$I_n = \sum_{j=1}^m c_j u_j \otimes u_j, \qquad \sum_{j=1}^m c_j u_j = 0 \quad \text{and} \quad \sum_{j=1}^m c_j = n. \tag{2.1}$$

Here I_n denotes the identity operator in $\mathbb{R}^n$, $u_j \otimes u_j(x) = \langle x, u_j\rangle u_j$ for every $x \in \mathbb{R}^n$, and the third inequality is obtained from the first one by taking traces.

Notice that, for any such decomposition of the identity, we have that for every $1 \leqslant k \leqslant m$

$$1 = |u_k|^2 = \sum_{j=1}^m c_j \langle u_k, u_j\rangle^2 \geqslant c_k \langle u_k, u_k\rangle^2 = c_k.$$

Thus, all the numbers $(c_j)_{j=1}^m$ are in the interval $(0, 1]$.

2.2. **Brascamp–Lieb inequality.** We will make use of the Brascamp–Lieb inequality (see [13]) and the reverse Brascamp–Lieb inequality due to Barthe (see [9]) in the following form, obtained by Ball (see [5]):

Theorem 2.1. *Let $m \geq n$, $\{u_j\}_{j=1}^m \subseteq S^{n-1}$, and $\{c_j\}_{j=1}^m \subseteq (0,1]$ be such that $I_n = \sum_{j=1}^m c_j u_j \otimes u_j$. Then, for any integrable functions $\{f_j\}_{j=1}^m : \mathbb{R} \to [0,\infty)$, we have that*

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{c_j}(\langle x, u_j\rangle)dx \leqslant \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j(t)dt\right)^{c_j} \qquad \textit{(Brascamp–Lieb inequality).}$$

Besides, for any integrable functions $\{h_j\}_{j=1}^m : \mathbb{R} \to [0,\infty)$ and $h : \mathbb{R}^n \to [0,\infty)$ satisfying

$$h\Big(\sum_{j=1}^m \theta_j c_j u_j\Big) \geqslant \prod_{j=1}^m h_j^{c_j}(\theta_j) \quad \textit{for every } \{\theta_j\}_{j=1}^m \subseteq \mathbb{R},$$

we have that

$$\int_{\mathbb{R}^n} h(x)dx \geqslant \prod_{j=1}^m \left(\int_{\mathbb{R}} h_j(t)dt\right)^{c_j} \qquad \textit{(Reverse Brascamp–Lieb inequality).}$$

2.3. **The regular simplex.** Let Δ_k denote the k-dimensional regular simplex

$$\Delta_k = \mathrm{conv}\{e_1, \dots, e_{k+1}\} \subseteq H_0,$$

where $H_0 = \left\{x \in \mathbb{R}^{k+1} : \sum_{i=1}^{k+1} x_i = 1\right\}$ is identified with $\mathbb{R}^k$ and $\left(\frac{1}{k+1}, \dots, \frac{1}{k+1}\right)$ is identified with the origin. It is well known that

- $|\Delta_k| = \frac{\sqrt{k+1}}{k!}$,
- $r(\Delta_k) = \frac{1}{\sqrt{k(k+1)}}$,
- $R(\Delta_k) = \sqrt{\frac{k}{k+1}}$,
- $\Delta_k^\circ = -(k+1)\Delta_k$.
- $w(\Delta_k) \simeq \sqrt{\frac{\log k}{k}}$,

where $a \simeq b$ denotes the fact that there exist two positive absolute constants c_1, c_2 such that $c_1 a \leqslant b \leqslant c_2 a$. Thus, $\frac{1}{r(\Delta_k)}\Delta_k$ is in John's position and $\frac{1}{R(\Delta_k)}\Delta_k$ is in Löwner's position. Then, if S_k denotes the k-dimensional simplex in John's position and $\tilde{S}_k$ denotes the k-dimensional simplex in Löwner's position, we have that

$$S_k = \sqrt{k(k+1)}\Delta_k \quad \text{and} \quad \tilde{S}_k = \sqrt{\frac{k+1}{k}}\Delta_k.$$

Therefore,

$$|S_k|^{1/k} = \frac{\sqrt{k(k+1)^{1+\frac{1}{k}}}}{(k!)^{1/k}} \quad \text{and} \quad |\tilde{S}_k|^{1/k} = \frac{1}{(k!)^{1/k}}\sqrt{\frac{(k+1)^{1+\frac{1}{k}}}{k}}.$$

Moreover,

$$w(S_k) \simeq \sqrt{k \log k} \quad \text{and} \quad w(\tilde{S}_k) \simeq \sqrt{\frac{\log k}{k}}.$$

2.4. **Mean width.** Let $K \subseteq \mathbb{R}^n$ be a convex body. The mean width of K is defined by

$$w(K) = \int_{S^{n-1}} h_K(\theta) d\sigma(\theta),$$

where, for every $\theta \in S^{n-1}$, $h_K(\theta)$ is the support function of K at θ and $d\sigma$ denotes the uniform probability measure on S^{n-1}. If we also assume that K contains the origin in its interior, then h_K is homogeneous of degree 1. There is a nice representation of the mean width in terms of the standard Gaussian random vector G in $\mathbb{R}^n$ (see, for instance, [3, Proof of Theorem 4.2.2]):

$$\mathbb{E} h_K(G) = c_n w(K), \tag{2.2}$$

where $c_n = \frac{n|B_2^n|\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2}\pi^{n/2}} = \frac{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$. Indeed, integrating in polar coordinates, one has

$$\begin{aligned} \mathbb{E} h_K(G) &= \int_{\mathbb{R}^n} h_K(x) \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{n/2}} dx = n|B_2^n| \int_0^\infty r^n \frac{e^{-\frac{r^2}{2}}}{(2\pi)^{n/2}} \int_{S^{n-1}} h_K(\theta) d\sigma(\theta) \\ &= c_n \int_{S^{n-1}} h_K(\theta) d\sigma(\theta) = c_n w(K). \end{aligned}$$

Likewise, since for any convex body containing the origin in its interior the support function of K° is $h_{K^\circ} = \|\cdot\|_K$, where $\|\cdot\|_K$ is the Minkowski gauge function of K, given by

$$\|x\|_K := \inf\{\lambda > 0 \,:\, x \in \lambda K\}$$

for all $x \in \mathbb{R}^n$, we have that if G is a standard Gaussian random vector in $\mathbb{R}^n$

$$\mathbb{E}\|G\|_K = c_n w(K^\circ). \tag{2.3}$$

We would like to refer the reader to [11], [17], [18], or [47] for more information on the use of the Gaussian measure of sections of convex bodies.

2.5. **Log-concave functions.** A function $f\colon \mathbb{R}^n \to [0,\infty)$ is called log-concave if $f(x) = e^{-v(x)}$ where $v : \mathbb{R}^n \to (-\infty, \infty]$ is a convex function. It is well-known that any integrable log-concave function $f\colon \mathbb{R}^n \to [0,\infty)$ is bounded and has moments of all orders. If $K \subseteq \mathbb{R}^n$ is a convex body then its indicator function χ_K is integrable and log-concave with integral $|K|$. If additionally K is a convex body containing the origin, then $e^{-\|\cdot\|_K}$ is integrable and log-concave with integral $n!|K|$.

Given a log-concave function $f = e^{-v}$, where $v : \mathbb{R}^n \to (-\infty, \infty]$ is a convex function, its polar function is the function $f^\circ : \mathbb{R}^n \to [0,\infty)$ given by

$$f^\circ(x) = e^{-\mathcal{L}(v)(x)},$$

where $\mathcal{L}(v)$ denotes the Legendre transform

$$\mathcal{L}(v)(x) = \sup_{y\in\mathbb{R}^n} (\langle x, y\rangle - v(y)), \quad x \in \mathbb{R}^n.$$

For more information on log-concave functions we refer the reader to [14, Chapter 2].

2.6. **The Wills functional.** Let us recall that for any n-dimensional convex body K, its Wills functional is defined by

$$\mathcal{W}(K)=\sum_{i=0}^{n}V_i(K),$$

where $V_i(K)$ denotes the i-th intrinsic volume of K. Many properties of the Wills functional can be found in [46], [25], [35], or [2]. For our purposes, we emphasize the following two:

(1) (Hadwiger, see [25, (1.3)]) For any convex body $K\subseteq\mathbb{R}^n$,
$$\mathcal{W}(K)=\int_{\mathbb{R}^n}e^{-\pi d(x,K)^2}dx,$$
where $d(x,K)$ denotes the Euclidean distance from x to K.

(2) (Hadwiger, see [25, (2.3)]) If E is a linear subspace of $\mathbb{R}^n$, $K_1\subseteq E$ and $K_2\subseteq E^\perp$, then
$$\mathcal{W}(K_1\times K_2)=\mathcal{W}(K_1)\mathcal{W}(K_2).$$

In particular, if $K=[-a,a]\subseteq\mathbb{R}$, we have that

$$\mathcal{W}([-a,a])=2a+2\int_a^\infty e^{-\pi(x-a)^2}dx=2a+1$$

and if $K=aB_\infty^n\subseteq\mathbb{R}^n$ then $\mathcal{W}(aB_\infty^n)=(1+2a)^n$.

Let us point out that for any $\lambda>0$

$$\mathcal{W}(\lambda K)=\sum_{i=0}^{n}V_i(\lambda K)=1+\sum_{i=1}^{n}\lambda^iV_i(K).$$

Therefore, if two convex bodies $K,L\subseteq\mathbb{R}^n$ verify that $\mathcal{W}(\lambda K)\leqslant\mathcal{W}(\lambda L)$ for every $\lambda\geqslant 0$, then one immediately obtains that $V_n(K)\leqslant V_n(L)$ and $V_1(K)\leqslant V_1(L)$ or, equivalently, $|K|\leqslant|L|$ and $w(K)\leqslant w(L)$.

Notice that, for any convex body $K\subseteq\mathbb{R}^n$, the function given by $d(x,K)$ for every $x\in\mathbb{R}^n$ is convex on $\mathbb{R}^n$ (see [42, Lemma 1.5.9]) and, as the square function is convex on $\mathbb{R}$, $d(x,K)^2$ is convex on $\mathbb{R}^n$. Therefore, the first property above shows that, for any convex body $K\subseteq\mathbb{R}^n$, its Wills functional is the integral of the log-concave function $f_K:\mathbb{R}^n\to[0,\infty)$ given by

$$f_K(x)=e^{-\pi d(x,K)^2}.$$

Using a double polarity (both in the convex body and in the family of log-concave functions), for any convex body $K\subseteq\mathbb{R}^n$ containing the origin in its interior, we define the log-concave function $f^\circ_{K^\circ}$. It was proved in [2, Lemma 3.1] that for every $x\in\mathbb{R}^n$

$$f^\circ_{K^\circ}(x)=e^{-\frac{\|x\|_2^2}{4\pi}-\|x\|_K}. \tag{2.4}$$

The following lemma shows that if, for every $\lambda\geqslant 0$, the integral of $f^\circ_{(\lambda K)^\circ}(x)$ is bounded by the integral of $f^\circ_{(\lambda L)^\circ}(x)$, then $|K|\leqslant|L|$.

Lemma 2.1. *Let $K,L\subseteq\mathbb{R}^n$ be two convex bodies containing the origin in their interiors. Assume that there exist two numbers A and $\lambda_0>0$ such that, for any $\lambda\in(0,\lambda_0)$,*

$$\int_{\mathbb{R}^n}f^\circ_{(\lambda K)^\circ}(x)dx\leqslant A\int_{\mathbb{R}^n}f^\circ_{(\lambda L)^\circ}(x)dx.$$

Then $|K| \leqslant A|L|$.

Proof. Notice that for any convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior and any $\lambda > 0$,

$$\begin{aligned}\int_{\mathbb{R}^n} f^{\circ}_{(\lambda K)^\circ}(x)dx &= \int_{\mathbb{R}^n} e^{-\frac{\|x\|_2^2}{4\pi}} e^{-\|x\|_{\lambda K}} dx = \int_{\mathbb{R}^n} e^{-\frac{\|x\|_2^2}{4\pi}} e^{-\frac{\|x\|_K}{\lambda}} dx \\ &= \lambda^n \int_{\mathbb{R}^n} e^{-\frac{\lambda^2\|x\|_2^2}{4\pi}} e^{-\|x\|_K} dx.\end{aligned}$$

Therefore, we have that for every $\lambda \in (0, \lambda_0)$,

$$\int_{\mathbb{R}^n} e^{-\frac{\lambda^2\|x\|_2^2}{4\pi}} e^{-\|x\|_K} dx \leqslant A \int_{\mathbb{R}^n} e^{-\frac{\lambda^2\|x\|_2^2}{4\pi}} e^{-\|x\|_L} dx$$

and, taking the limit as λ tends to 0 we obtain that

$$n!|K| = \int_{\mathbb{R}^n} e^{-\|x\|_K} dx \leqslant A \int_{\mathbb{R}^n} e^{-\|x\|_L} dx = n!A|L|.$$

□

2.7. **Convex bodies in minimal surface area position.** A convex body $K \subseteq \mathbb{R}^n$ is said to be in minimal surface area position if it has minimal surface area among all of its volume preserving affine images. That is, if

$$|\partial K| = \min\left\{|\partial T(K)| \,:\, T \in SL(n)\right\},$$

where $SL(n)$ denotes the set of non-degenerate linear maps $T \in GL(n)$ with $|\det T| = 1$. The surface area measure of a convex body K is the measure on the sphere defined by

$$\sigma_K(A) := \nu\left(\{x \in \partial K \,:\, \nu_K(x) \in A\}\right) \quad \forall A \text{ Borel set in } S^{n-1},$$

where ν denotes the Hausdorff measure on ∂K and $\nu_K(x)$ is the outer normal vector to K at x, which is defined ν-almost everywhere.

The projection body ΠK and its polar, the polar projection body $\Pi^* K$, of a convex body K are the centrally symmetric convex bodies defined by

$$h_{\Pi K}(x) = \|x\|_{\Pi^* K} = |x||P_{x^\perp}(K)| = \frac{1}{2}\int_{S^{n-1}} |\langle x, \theta\rangle| d\sigma_K(\theta),$$

where, for any $x \neq 0$, $|P_{x^\perp}(K)|$ denotes the $(n-1)$-dimensional volume of the projection of K onto the hyperplane orthogonal to x and the last equality is the well-known Cauchy's formula (see, for instance, [42, Equation (5.80)]).

It was proved by Petty [36] (see also [21]) that K is in minimal surface area position if and only if σ_K is isotropic, i.e., if

$$I_n = \frac{n}{|\partial K|}\int_{S^{n-1}} u \otimes u d\sigma_K(u).$$

In [22] it was observed that the latter happens if and only if ΠK is in minimal mean width position, i.e.,

$$w(\Pi K) = \min\{w(T(\Pi K)) \,:\, T \in SL(n)\}.$$

Notice that if K is a polytope with facets $\{F_j\}_{j=1}^m$ with outer normal vectors $\{u_j\}_{j=1}^m$, then the surface area measure of K is

$$\sigma_K = \sum_{j=1}^m |F_j|\delta_{u_j},$$

where δ_j denotes the Dirac delta measure on u_j. Moreover, K is in minimal surface area position if and only if

$$I_n = \sum_{j=1}^{m} \frac{n|F_j|}{|\partial K|} u_j \otimes u_j.$$

In particular, if K is a polytope with facets $\{F_j\}_{j=1}^m$ and outer normal vectors $\{u_j\}_{j=1}^m$ then for every $x \in \mathbb{R}^n$

$$h_{\Pi K}(x) = \|x\|_{\Pi^* K} = \frac{1}{2} \sum_{j=1}^{m} |F_j||\langle x, u_j\rangle|. \tag{2.5}$$

It was proved in [21] that, as a consequence of a lemma obtained from the Brascamp–Lieb inequality (see [6]), if K is a convex body in minimal surface area position then

$$|B_2^n| \left(\frac{n|B_2^n|}{|B_2^{n-1}|}\right)^n \frac{1}{|\partial K|^n} \leqslant |\Pi^* K| \leqslant \frac{4^n n^n}{n!} \frac{1}{|\partial K|^n}. \tag{2.6}$$

Moreover, if K is a convex body in minimal surface area position then

$$\left(\frac{|\partial K|}{n}\right)^n \leqslant |\Pi K| \leqslant |B_2^n| \left(\frac{|B_2^{n-1}|}{n|B_2^n|}\right)^n |\partial K|^n. \tag{2.7}$$

This can be seen as a consequence of the Blaschke–Santaló inequality and its exact reverse for zonoids (see [23] and [38]), or as a direct consequence of the reverse form of Brascamp–Lieb inequality (see [22]).

3. General setting

In this section we introduce the notation for a setting that will be used in several of our proofs. We distinguish the cases in which we are dealing with non-symmetric convex bodies in John's position, symmetric convex bodies in John's position, or polytopes in minimal surface area position.

3.1. **Non-symmetric convex bodies in John's position.** Let $K \subseteq \mathbb{R}^n$ be a (not necessarily symmetric) convex body in John's position and let $\{u_j\}_{j=1}^m$ and $\{c_j\}_{j=1}^m$ be the contact points in $\partial K \cap S^{n-1}$ and positive weights satisfying John's condition (2.1). We will denote by $C \subseteq \mathbb{R}^n$ the convex body

$$C = \{x \in \mathbb{R}^n \,:\, \langle x, u_j\rangle \leqslant 1, \forall 1 \leqslant j \leqslant m\}. \tag{3.1}$$

It is easily verified that $K \subseteq C$. We will denote, for every $1 \leqslant j \leqslant m$,

- $v_j = \sqrt{\frac{n}{n+1}}(-u_j, \frac{1}{\sqrt{n}}) \in S^n$, and
- $\delta_j = \frac{n+1}{n} c_j$.

These vectors satisfy

$$I_{n+1} = \sum_{j=1}^{m} \delta_j v_j \otimes v_j, \qquad \sum_{j=1}^{m} \delta_j v_j = \left(0, \sqrt{n+1}\right) \quad \text{and} \quad \sum_{j=1}^{m} \delta_j = n+1. \tag{3.2}$$

Therefore, as seen in Section 2.1, $\delta_j \in (0,1]$ for every $1 \leqslant j \leqslant m$. Let the cone

$$L := \{y = (x, r) \in \mathbb{R}^{n+1} \,:\, \langle y, v_j\rangle \geq 0, \forall\, 1 \leqslant j \leqslant m\}. \tag{3.3}$$

The next lemma, which was proved in [6], relates L and C. We include its proof here for the sake of completeness.

Lemma 3.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body in John's position and let L be defined as in equation* (3.3)*. Then*

$$L = \left\{ (x,r) \in \mathbb{R}^{n+1} \,:\, r \geqslant 0, x \in \frac{r}{\sqrt{n}} C \right\}.$$

Proof. Let $y = (x,r) \in L$. By the definition of v_j we have that for each $1 \leqslant j \leqslant m$

$$\langle y, v_j \rangle = -\sqrt{\frac{n}{n+1}} \langle x, u_j \rangle + \frac{r}{\sqrt{n+1}}.$$

Assume that $r < 0$. Then, since $\langle y, v_j \rangle \geqslant 0$ for every $1 \leqslant j \leqslant m$, we have that

$$-\sqrt{\frac{n}{n+1}} \langle x, u_j \rangle + \frac{r}{\sqrt{n+1}} \geqslant 0 \quad \forall 1 \leqslant j \leqslant m.$$

Then, $\langle x, u_j \rangle < 0$ for every $1 \leqslant j \leqslant m$. As a consequence, since $\{c_j\}_{j=1}^m \subseteq (0, \infty)$,

$$\sum_{j=1}^m c_j \langle x, u_j \rangle < 0,$$

which contradicts the fact that $\sum_{j=1}^m c_j u_j = 0$. Therefore, if $y = (x,r) \in L$ then $r \geqslant 0$.

For any $r \geqslant 0$ and every $1 \leqslant j \leqslant m$ we have $\langle y, v_j \rangle \geqslant 0$ if and only if $\langle x, u_j \rangle \leqslant \frac{r}{\sqrt{n}}$. The latter condition is true for every $1 \leqslant j \leqslant m$ if and only if $x \in \frac{r}{\sqrt{n}} C$.

Conversely, assume that $y = (x,r)$ verifies that $r \geqslant 0$ and $x \in \frac{r}{\sqrt{n}} C$, which happens if and only if $\langle x, u_j \rangle \leqslant \frac{r}{\sqrt{n}}$ for every $1 \leqslant j \leqslant m$. Then, for every $1 \leqslant j \leqslant m$,

$$\langle y, v_j \rangle = -\sqrt{\frac{n}{n+1}} \langle x, u_j \rangle + \frac{r}{\sqrt{n+1}} \geqslant 0.$$

Thus, $y \in L$. □

Given any k-dimensional affine subspace F_h in $\mathbb{R}^n$ at distance h from the origin, we will consider the linear $(k+1)$-dimensional subspace in $\mathbb{R}^{n+1}$

$$H = \operatorname{span}\{ \left(x, \sqrt{n}\right) \,:\, x \in F_h \} \in G_{n+1,k+1}. \tag{3.4}$$

Notice that if $F_0 \in G_{n,k}$ is a linear subspace then H equals the cartesian product $H = F_0 \times \mathbb{R} = \{(x,r) \in \mathbb{R}^{n+1} \,:\, x \in F_0,\, r \in \mathbb{R}\}$. Furthermore, assume that F_h is at distance h from the origin and $f : L \cap H \to [0, \infty)$ is an integrable function. By Lemma 3.1, and taking into account that $\mathbb{R}^n \times \{0\}$ and $P_H(\{0\} \times \mathbb{R})$ provide an orthogonal decomposition of H, we have that

$$\int_{L \cap H} f(x,r) dr dx = \int_0^\infty \int_{\frac{r}{\sqrt{n}}(C \cap F_h) \times \{r\}} f(x,r) dx \sqrt{\frac{n+h^2}{n}} dr. \tag{3.5}$$

Set $J = \{1 \leqslant j \leqslant m \,:\, P_H v_j \neq 0\}$ and, for every $j \in J$, we define

- $w_j = \frac{P_H v_j}{\|P_H v_j\|_2}$,
- $\kappa_j = \frac{n+1}{n} c_j \|P_H v_j\|_2^2 = \delta_j \|P_H v_j\|_2^2$.

Then, we have that

$$I_H = \sum_{j\in J} \kappa_j w_j \otimes w_j \quad \text{and} \quad \sum_{j\in J} \kappa_j = k+1, \tag{3.6}$$

where I_H denotes the identity in the linear subspace H. Furthermore, denoting by $s_j = \frac{1}{\|P_H v_j\|_2}$ for every $j \in J$, one has that for every $y = (x, r) \in H \subseteq \mathbb{R}^{n+1}$

$$\begin{aligned} \sum_{j\in J} \kappa_j s_j \langle y, w_j\rangle &= \sum_{j\in J} \delta_j \langle y, P_H v_j\rangle = \sum_{j=1}^m \delta_j \langle y, P_H v_j\rangle = \sum_{j=1}^m \delta_j \langle y, v_j\rangle \\ &= r\sqrt{n+1}. \end{aligned} \tag{3.7}$$

The following lemma shows that, whenever F_0 is a linear subspace, we have a strictly positive lower bound for the Euclidean norm of $P_H v_j$ for every $1 \leqslant j \leqslant m$. Consequently, if $F_0 \in G_{n,k}$, the set J defined above equals $J = \{1, \dots, m\}$.

Lemma 3.2. *Let $\{u_j\}_{j=1}^m \subseteq S^{n-1}$, $\{c_j\}_{j=1}^m$ be such that (2.1) holds, $F_0 \in G_{n,k}$, $H = F_0 \times \mathbb{R} \in G_{n+1,k+1}$, and $\{v_j\}_{j=1}^m \subseteq S^n$ be defined as in (3.2). Then, for every $1 \leqslant j \leqslant m$, we have*

$$\frac{1}{n+1} \leqslant \|P_H v_j\|_2^2 \leqslant 1.$$

Proof. Let $c = \left(0, \frac{1}{\sqrt{n+1}}\right) \in H$ and notice that for every $1 \leqslant j \leqslant m$

$$\langle P_H(v_j - c), c\rangle = \langle v_j - c, c\rangle = \frac{1}{n+1} - \frac{1}{n+1} = 0.$$

Since $c \in H$, we have that $P_H c = c$ and then $P_H v_j = c + P_H(v_j - c)$. Thus,

$$\|P_H v_j\|_2^2 = \|c + (P_H(v_j - c))\|_2^2 = \|c\|_2^2 + \|P_H(v_j - c)\|_2^2 \geq \|c\|_2^2 = \frac{1}{n+1}.$$

Thus, for every $1 \leqslant j \leqslant m$, we have that

$$\frac{1}{n+1} \leqslant \|P_H v_j\|_2^2 \leqslant 1.$$

□

3.2. **Symmetric convex bodies in John's position.** Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body in John's position and, like in the not necessarily symmetric case, let $\{u_j\}_{j=1}^m$ and $\{c_j\}_{j=1}^m$ be the contact points of ∂K and S^{n-1} and positive weights satisfying (2.1). We will also denote by C_0 the symmetric convex body

$$C_0 = \{x \in \mathbb{R}^n : |\langle x, u_j\rangle| \leqslant 1, \forall\, 1 \leqslant j \leqslant m\}. \tag{3.8}$$

Clearly, K is a subset of C_0. If $F_0 \in G_{n,k}$ is a linear subspace, we set $J_0 = \{1 \leqslant j \leqslant m : P_{F_0} u_j \neq 0\}$ and for every $j \in J_0$, we define

- $v_j^0 = \frac{P_{F_0} u_j}{\|P_{F_0} u_j\|_2} \in S^{n-1} \cap F_0$,
- $\delta_j^0 = c_j \|P_{F_0} u_j\|_2^2$.

Then, we have that

$$I_{F_0} = \sum_{j\in J_0} \delta_j^0 v_j^0 \otimes v_j^0 \qquad \text{and} \qquad \sum_{j\in J_0} \delta_j^0 = k, \tag{3.9}$$

where I_{F_0} denotes the identity operator in F_0, and also

$$K \cap F_0 \subseteq C_0 \cap F_0 \quad = \quad \{x \in F_0 : |\langle x, u_j\rangle| \leqslant 1, \forall 1 \leqslant j \leqslant m\}$$

$$\begin{aligned} &= \{x \in F_0 \,:\, |\langle x, P_{F_0} u_j\rangle| \leqslant 1, \forall 1 \leqslant j \leqslant m\} \\ &= \{x \in F_0 \,:\, |\langle x, P_{F_0} u_j\rangle| \leqslant 1, \forall j \in J_0\} \\ &= \left\{x \in F_0 \,:\, |\langle x, v_j^0\rangle| \leqslant t_j, \forall j \in J_0\right\}, \end{aligned}$$

where $t_j = \frac{1}{\|P_F u_j\|_2} = \left(\frac{c_j}{\delta_j^0}\right)^{1/2}$ for every $j \in J_0$. Furthermore,

$$\begin{aligned} (K \cap F_0)^\circ &\supseteq (C_0 \cap F_0)^\circ = P_{F_0}(C_0^\circ) = P_{F_0}\left(\operatorname{conv}\{\pm u_j \,:\, 1 \leqslant j \leqslant m\}\right) \\ &= \operatorname{conv}\{\pm P_{F_0} u_j \,:\, 1 \leqslant j \leqslant m\} \\ &= \operatorname{conv}\{\pm P_{F_0} u_j \,:\, j \in J_0\}. \end{aligned}$$

Thus,

$$K \cap F_0 \subseteq C_0 \cap F_0 = \left\{x \in F_0 \,:\, |\langle x, v_j^0\rangle| \leqslant t_j, \forall j \in J_0\right\} \tag{3.10}$$

and

$$(K \cap F_0)^\circ \supseteq (C_0 \cap F_0)^\circ = \operatorname{conv}\{\pm P_{F_0} u_j \,:\, j \in J_0\}. \tag{3.11}$$

3.3. **Polytopes in minimal surface area position.** Let K be a (not necessarily centrally symmetric) polytope in minimal surface area position with facets $\{F_j\}_{j=1}^m$ and outer normal vectors $\{u_j\}_{j=1}^m$, and let $F_0 \in G_{n,k}$ be a k-dimensional linear subspace. Then,

$$K = \{x \in \mathbb{R}^n \,:\, \langle x, u_j\rangle \leqslant h_K(u_j), \forall\, 1 \leqslant j \leqslant m\}$$

and

$$I_n = \sum_{j=1}^m \frac{n|F_j|}{|\partial K|} u_j \otimes u_j = \sum_{j=1}^m c_j u_j \otimes u_j, \tag{3.12}$$

where $c_j = \frac{n|F_j|}{|\partial K|}$ for every $1 \leqslant j \leqslant m$. Besides (see, for instance, [24, Theorem 18.2])

$$\sum_{j=1}^m c_j u_j = \frac{n}{|\partial K|} \sum_{j=1}^m |F_j| u_j = 0.$$

and

$$\sum_{j=1} c_j h_K(u_j) = \sum_{j=1}^m \frac{n|F_j|}{|\partial K|} h_K(u_j) = \frac{n^2|K|}{|\partial K|}. \tag{3.13}$$

Note also that if K is a centrally symmetric polytope in minimal surface area position, with facets $\{F_j\}_{j=1}^m$ and outer normal vectors $\{u_j\}_{j=1}^m$, and if $F_0 \in G_{n,k}$ is a k-dimensional linear subspace, then

$$K = \{x \in \mathbb{R}^n \,:\, |\langle x, u_j\rangle| \leqslant h_K(u_j), \forall\, 1 \leqslant j \leqslant m\}.$$

As in the case where the decomposition of the identity comes from a centrally symmetric convex body in John's position, we set $J_0 = \{1 \leqslant j \leqslant m \,:\, P_{F_0} u_j \neq 0\}$ and, for every $j \in J_0$, we define

- $v_j^0 = \frac{P_{F_0} u_j}{\|P_{F_0} u_j\|_2}$,
- $\delta_j^0 = c_j \|P_{F_0} u_j\|_2^2 = \frac{n|F_j| \|P_{F_0} u_j\|_2^2}{|\partial K|}$.

We have that

$$I_{F_0} = \sum_{j=1}^{m} c_j P_{F_0} u_j \otimes P_{F_0} u_j = \sum_{j \in J_0} \delta_j^0 v_j^0 \otimes v_j^0. \tag{3.14}$$

Besides, if we denote $t_j = \frac{1}{\|P_{F_0} u_j\|_2} = \left(\frac{c_j}{\delta_j^0}\right)^{1/2}$ for every $j \in J_0$, then

$$K \cap F_0 = \{x \in F_0 \,:\, |\langle x, v_j^0 \rangle| \leqslant t_j h_K(u_j), \forall\, j \in J_0\} \tag{3.15}$$

and

$$\begin{aligned} (K \cap F_0)^\circ &= \operatorname{conv}\left\{ \pm \frac{P_{F_0}(u_j)}{h_K(u_j)} \,:\, 1 \leqslant j \leqslant m \right\} \\ &= \operatorname{conv}\left\{ \pm \frac{P_{F_0}(u_j)}{h_K(u_j)} \,:\, j \in J_0 \right\}. \end{aligned} \tag{3.16}$$

4. Volume of sections of convex bodies in John's position

In this section we will give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us start with the symmetric case. Assume that K is a centrally symmetric convex body in John's position and $F_0 \in G_{n,k}$ is a linear k-dimensional subspace. We follow the notation in Section 3.2. By (3.10), we have that

$$K \cap F_0 \subseteq C_0 \cap F_0 = \left\{ x \in F_0 \,:\, |\langle x, v_j^0 \rangle| \leqslant t_j, \, \forall j \in J_0 \right\},$$

where $t_j = \frac{1}{\|P_{F_0} u_j\|_2} = \left(\frac{c_j}{\delta_j^0}\right)^{1/2}$ for all $j \in J_0$. Therefore, by (3.9) and the Brascamp–Lieb inequality (Theorem 2.1),

$$\begin{aligned} |K \cap F_0| &\leqslant |C_0 \cap F_0| = \int_{F_0} \prod_{j \in J_0} \chi_{[-t_j, t_j]}(\langle x, v_j^0 \rangle) dx = \int_{F_0} \prod_{j \in J_0} \chi_{[-t_j, t_j]}^{\delta_j^0}(\langle x, v_j^0 \rangle) dx \\ &\leqslant \prod_{j \in J_0} \left(\int_{\mathbb{R}} \chi_{[-t_j, t_j]}(t) \right)^{\delta_j^0} = \prod_{j \in J_0} (2t_j)^{\delta_j^0} = 2^k \prod_{j \in J_0} \left(\frac{c_j}{\delta_j^0} \right)^{\frac{\delta_j^0}{2}}. \end{aligned}$$

By the arithmetic-geometric mean inequality and (2.1), we get

$$\prod_{j \in J_0} \left(\frac{c_j}{\delta_j^0} \right)^{\frac{\delta_j^0}{k}} \leqslant \sum_{j \in J_0} \frac{\delta_j^0}{k} \frac{c_j}{\delta_j^0} = \frac{1}{k} \sum_{j \in J_0} c_j \leqslant \frac{1}{k} \sum_{j=1}^{m} c_j = \frac{n}{k}.$$

Hence,

$$|K \cap F_0|^{1/k} \leqslant 2\sqrt{\frac{n}{k}} = \sqrt{\frac{n}{k}} |B_\infty^k|^{1/k}.$$

Assume now that $K \subseteq \mathbb{R}^n$ is a (not necessarily symmetric) convex body in John's position and $F_0 \in G_{n,k}$ a k-dimensional linear subspace. We follow the notation introduced in Section 3.1. Applying Lemma 3.1, one gets

$$L \cap H = \left\{ (x, r) \in F_0 \times \mathbb{R} \,:\, r \geqslant 0, x \in \frac{r}{\sqrt{n}} (C \cap F_0) \right\}.$$

Denote $s_j = \frac{1}{\|P_H v_j\|_2}$ as in (3.7). Using the Brascamp–Lieb inequality (Theorem 2.1) we get

$$\int_{L\cap H} e^{-\sum_{j=1}^m \kappa_j s_j \langle y,w_j\rangle} dy = \int_H \prod_{j=1}^m \left(\chi_{[0,\infty)}(\langle y,v_j\rangle) e^{-s_j\langle y,w_j\rangle}\right)^{\kappa_j} dy$$
$$= \int_H \prod_{j=1}^m \left(\chi_{[0,\infty)}(\langle y,P_H v_j\rangle) e^{-s_j\langle y,w_j\rangle}\right)^{\kappa_j} dy$$
$$= \int_H \prod_{j=1}^m \left(\chi_{[0,\infty)}(\langle y,w_j\rangle) e^{-s_j\langle y,w_j\rangle}\right)^{\kappa_j} dy \leqslant \prod_{j=1}^m \left(\int_0^\infty e^{-s_j t} dt\right)^{\kappa_j}$$
$$= \prod_{j=1}^m \|P_H v_j\|_2^{\delta_j \|P_H v_j\|_2^2}.$$

On the other hand, taking into account (3.7), we see that

$$\int_{L\cap H} e^{-\sum_{j=1}^m \kappa_j s_j \langle y,w_j\rangle} dy = \int_0^\infty \int_{\frac{r}{\sqrt{n}}(C\cap F_0)} e^{-r\sqrt{n+1}} dx dr$$
$$= \int_0^\infty \frac{r^k}{n^{\frac{k}{2}}} |C\cap F_0| e^{-r\sqrt{n+1}} dr = \frac{k!}{n^{\frac{k}{2}}(n+1)^{\frac{k+1}{2}}} |C\cap F_0|$$
$$= \frac{k^{\frac{k}{2}}(k+1)^{\frac{k+1}{2}}}{n^{\frac{k}{2}}(n+1)^{\frac{k+1}{2}}} \frac{|C\cap F_0|}{|S_k|}.$$

Let us maximize $\prod_{j=1}^m \|P_H v_j\|_2^{\delta_j\|P_H v_j\|_2^2}$ under the constraints

- $\frac{1}{n+1} \leqslant \|P_H v_j\|_2^2 \leqslant 1\ \forall 1 \leqslant j \leqslant m$,
- $\sum_{j=1}^m \delta_j \|P_H v_j\|_2^2 = k+1$,
- $\sum_{j=1}^m \delta_j = n+1$,
- $0 \leqslant \delta_j \leqslant 1$.

Equivalently, let us maximize $F(x,\delta) = \frac{1}{2}\sum_{j=1}^m \delta_j x_j \log x_j$ under the constraints

- $\frac{1}{n+1} \leqslant x_j \leqslant 1\ \forall 1 \leqslant j \leqslant m$,
- $\sum_{j=1}^m \delta_j x_j = k+1$,
- $\sum_{j=1}^m \delta_j = n+1$,
- $0 \leqslant \delta_j \leqslant 1$.

First notice that the function $F(x,\delta)$ is continuous on a compact domain M in $\mathbb{R}^{2m}$, which is given by the constraints. Therefore, it attains its maximum. For every $x = (x_1,\dots,x_m)$ with $\frac{1}{n+1} \leqslant x_j \leqslant 1$ for all $1 \leqslant j \leqslant m$, let $F_x(\delta)$ be the function

$$F_x(\delta) = \frac{1}{2}\sum_{j=1}^m \delta_j x_j \log x_j.$$

Notice that F_x is a convex function. Since the set

$$A = \left\{\delta \in \mathbb{R}^m : \sum_{j=1}^m \delta_j x_j = k+1,\ \sum_{j=1}^m \delta_j = n+1,\ 0 \leqslant \delta_j \leqslant 1\, \forall 1 \leqslant j \leqslant m\right\}$$

is a compact convex set, F_x attains its maximum on some extreme point of A. These are the points of intersection of the 2-dimensional faces of the cube

$$\{\delta \in \mathbb{R}^m \,:\, 0 \leqslant \delta_j \leqslant 1 \ \forall 1 \leqslant j \leqslant m\}$$

with the $(m-2)$-dimensional affine subspace

$$\Big\{\delta \in \mathbb{R}^m \,:\, \sum_{j=1}^m \delta_j x_j = k+1,\, \sum_{j=1}^m \delta_j = n+1\Big\}.$$

Therefore, a maximizer of the function F_x has to be a point of the form

$$\delta_\lambda = (\underbrace{1,1,\ldots,1}_{n}, \lambda, 1-\lambda, \underbrace{0,\ldots,0}_{m-n-2})$$

for some $\frac{1}{2} \leqslant \lambda \leqslant 1$ (or a permutation of it), such that $\sum_{j=1}^m \delta_j x_j = k+1$ is satisfied. For every δ_λ with $\frac{1}{2} \leqslant \lambda \leqslant 1$ we will find the maximizer of the function

$$F_{\delta_\lambda}(x) = \frac{1}{2}\sum_{j=1}^m \delta_j x_j \log x_j$$

on the compact convex set

$$B_\lambda = \Big\{x \in \mathbb{R}^m \,:\, \sum_{j=1}^m \delta_{\lambda,j} x_j = k+1,\, \frac{1}{n+1} \leqslant x_j \leqslant 1 \ \forall 1 \leqslant j \leqslant m\Big\}.$$

If δ_λ^* is the decreasing rearrangement of δ_λ, we can assume without loss of generality that $\delta_\lambda = \delta_\lambda^*$. Let

$$D = \frac{k+1}{n+1}$$

and

$$\tilde{x} = \left(\underbrace{1,1,\ldots,1}_{k}, D, \underbrace{\frac{1}{n+1},\ldots,\frac{1}{n+1}}_{m-k-1}\right).$$

We check that $\frac{1}{n+1} \leqslant D \leqslant 1$ and

$$\sum_{j=1}^m \delta_{\lambda,j}\tilde{x}_j = k + D + \frac{n-k}{n+1} = k+1.$$

For every $x = (x_1,\ldots,x_m) \in B_\lambda$, we have $\tilde{x} \succ (x_1,\ldots,x_m)$, since the first $k+1$ coordinates of $\tilde{x}$ are as large as they can. Here, the notation $\tilde{x} \succ (x_1,\ldots,x_m)$ means that

- $\displaystyle\sum_{j=1}^m \delta_{\lambda,j}\tilde{x}_j = \sum_{j=1}^m \delta_{\lambda,j} x_j = k+1,$
- $\displaystyle\sum_{j=1}^l \delta_{\lambda,j}\tilde{x}_j \geqslant \sum_{j=1}^l \delta_{\lambda,j} x_j \ \ \forall 1 \leqslant l \leqslant m.$

Therefore, by the weighted Karamata's inequality (see [20]), we have that for every $x \in B_\lambda$

$$F_{\delta_\lambda}(x) \leqslant F_{\delta_\lambda}(\tilde{x}) \leqslant \max_{(\delta,x)\in M} F(\delta,x).$$

Since

$$\max_{(\delta,x)\in M} F(\delta,x) \leqslant \max_{\lambda\in[\frac{1}{2},1],x\in B_\lambda} F_{\delta_\lambda}(x),$$

we see that

$$\begin{aligned}\max_{(\delta,x)\in M} F(\delta,x) &= \max_{\lambda\in[\frac{1}{2},1]} F_{\delta_\lambda}(\tilde{x}) = \max_{\lambda\in[\frac{1}{2},1]}\left\{\frac{1}{2}D\log D + \frac{n-k}{2(n+1)}\log\left(\frac{1}{n+1}\right)\right\}\\ &= \frac{1}{2}D\log D - \frac{n-k}{n+1}\log(n+1).\end{aligned}$$

Thus,

$$\prod_{j=1}^{m} \|P_H v_j\|_2^{\delta_j\|P_H v_j\|_2^2} \leqslant e^{\frac{1}{2}D\log D - \frac{n-k}{2(n+1)}\log(n+1)} = \frac{(k+1)^{\frac{k+1}{2(n+1)}}}{(n+1)^{\frac{n+1}{2(n+1)}}}.$$

Since $|K\cap F_0| \le |C\cap F_0|$, it follows that

$$|K\cap F|^{1/k} \leqslant \frac{1}{(k+1)^{\frac{n-k}{2k(n+1)}}}\sqrt{\frac{n(n+1)}{k(k+1)}}|S_k|^{1/k}.$$

Finally, assume now that $K\subseteq\mathbb{R}^n$ is a (not necessarily symmetric) convex body in John's position and F_h is a k-dimensional affine subspace at distance h from 0. We continue to follow the notation introduced in Section 3.1. Given the k-dimensional affine subspace F_h, we take the linear subspace $H = \mathrm{span}\{(x,\sqrt{n}) : x\in F_h\}\in G_{n+1,k+1}$ as in (3.4). By Lemma 3.1, we see that

$$L\cap H = \left\{(x,r)\in\mathbb{R}^{n+1} : r\geqslant 0, x\in\frac{r}{\sqrt{n}}(C\cap F_h)\right\}.$$

Recall that $J=\{1\leqslant j\leqslant m : P_H v_j\neq 0\}$ and $s_j = \frac{1}{\|P_H v_j\|_2}$ for all $j\in J$. Using (3.6) and the Brascamp–Lieb inequality (Theorem 2.1), we have that

$$\begin{aligned}&\int_{L\cap H} e^{-\sum_{j\in J}\kappa_j s_j\langle y,w_j\rangle}dy = \int_H \prod_{j\in J}\left(\chi_{[0,\infty)}(\langle y,v_j\rangle)e^{-s_j\langle y,w_j\rangle}\right)^{\kappa_j}dy\\ &= \int_H \prod_{j\in J}\left(\chi_{[0,\infty)}(\langle y,P_H v_j\rangle)e^{-s_j\langle y,w_j\rangle}\right)^{\kappa_j}dy\\ &= \int_H \prod_{j\in J}\left(\chi_{[0,\infty)}(\langle y,w_j\rangle)e^{-s_j\langle y,w_j\rangle}\right)^{\kappa_j}dy\\ &\leqslant \prod_{j\in J}\left(\int_0^\infty e^{-s_j t}dt\right)^{\kappa_j} = \prod_{j\in J}\|P_H v_j\|_2^{\delta_j\|P_H v_j\|_2^2} \leqslant 1.\end{aligned}$$

Taking into account (3.5) and (3.7), we obtain

$$\begin{aligned}\int_{L\cap H} e^{-\sum_{j\in J}\kappa_j s_j\langle y,w_j\rangle}dy &= \int_0^\infty \int_{\frac{r}{\sqrt{n}}(C\cap F_h)} e^{-r\sqrt{n+1}}dx\sqrt{\frac{n+h^2}{n}}dr\\ &= \int_0^\infty \frac{r^k(n+h^2)^{\frac{1}{2}}}{n^{\frac{k+1}{2}}}|C\cap F_h|e^{-r\sqrt{n+1}}dr\\ &= \frac{(n+h^2)^{\frac{1}{2}}k!}{n^{\frac{k+1}{2}}(n+1)^{\frac{k+1}{2}}}|C\cap F_h|.\end{aligned}$$

Since $|K\cap F_h|\leq |C\cap F_h|$, we get

$$|K\cap F_h|\leqslant \frac{n^{\frac{k}{2}}(n+1)^{\frac{k+1}{2}}}{k!}\sqrt{\frac{n}{n+h^2}}$$

or, equivalently,

$$|K\cap F_h|^{1/k}\leqslant \sqrt{\frac{n(n+1)^{1+\frac{1}{k}}}{k(k+1)^{1+\frac{1}{k}}}}\left(\frac{n}{n+h^2}\right)^{\frac{1}{2k}}|S_k|^{1/k}.$$

□

5. Volume of projections of convex bodies in Löwner's position

In this section we will give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let us start with the symmetric case. Assume that K is a centrally symmetric convex body in John's position and $F_0\in G_{n,k}$ a k-dimensional linear subspace. We follow the notation in Section 3.2. By (3.11), we get that $K\cap F_0\subseteq C_0\cap F_0$. This implies that

$$(K\cap F_0)^\circ \supseteq (C_0\cap F_0)^\circ = \operatorname{conv}\{\pm P_{F_0}u_j\ :\ j\in J_0\}.$$

It follows that for every $x\in F_0$

$$h_{K\cap F_0}(x)\leqslant h_{C_0\cap F_0}(x)=\|x\|_{(C_0\cap F_0)^\circ}=\inf\left\{\sum_{j\in J_0}|\alpha_j|\ :\ x=\sum_{j\in J_0}\alpha_j P_{F_0}u_j\right\}$$

$$=\inf\left\{\sum_{j\in J_0}|\alpha_j|\ :\ x=\sum_{j\in J_0}\alpha_j\|P_{F_0}u_j\|_2 v_j^0\right\}=\inf\left\{\sum_{j\in J_0}\frac{|\beta_j|}{\|P_{F_0}u_j\|_2}\ :\ x=\sum_{j\in J_0}\beta_j v_j^0\right\}$$

$$=\inf\left\{\sum_{j\in J_0}\delta_j^0|\theta_j|t_j\ :\ x=\sum_{j\in J_0}\delta_j^0\theta_j v_j^0\right\},$$

where $t_j=\frac{1}{\|P_{F_0}u_j\|_2}=\left(\frac{c_j}{\delta_j^0}\right)^{\frac{1}{2}}$ for all $j\in J_0$. For every $j\in J_0$, we set

$$f_j(t):=e^{-|t|t_j},\quad t\in\mathbb{R}.$$

Then, if $x=\sum_{j\in J_0}\delta_j^0\theta_j v_j^0$ for some $\{\theta_j\}_{j\in J_0}\subseteq\mathbb{R}$, we have

$$\prod_{j\in J_0}f_j^{\delta_j^0}(\theta_j)=e^{-\sum_{j\in J_0}\delta_j|\theta_j|t_j}\leqslant e^{-h_{K\cap F_0}(x)}.$$

Using the reverse Brascamp–Lieb inequality (Theorem 2.1), one obtains

$$\begin{aligned} k!|(K\cap F_0)^\circ| &= \int_F e^{-h_{K\cap F_0}(x)}dx\geqslant \prod_{j\in J}\left(\int_{\mathbb{R}}e^{-|t|t_j}dt\right)^{\delta_j}\\ &= \frac{2^k}{\prod_{j\in J_0}t_j^{\delta_j}}=\frac{2^k}{\prod_{j\in J_0}\left(\frac{c_j}{\delta_j}\right)^{\frac{\delta_j}{2}}}. \end{aligned}$$

As we have seen in the proof of Theorem 1.1

$$\prod_{j\in J_0}\left(\frac{c_j}{\delta_j^0}\right)^{\frac{\delta_j}{k}} \leqslant \sum_{j\in J_0}\frac{\delta_j^0}{k}\frac{c_j}{\delta_j^0}=\frac{1}{k}\sum_{j\in J_0}c_j\leqslant\frac{1}{k}\sum_{j=1}^m c_j=\frac{n}{k}.$$

Taking into account that $|(B_\infty^k)^\circ|=|B_1^k|=\frac{2^k}{k!}$, we obtain

$$|(K\cap F_0)^\circ|^{1/k}\geqslant\sqrt{\frac{k}{n}}|(B_\infty^k)^\circ|^{1/k}.$$

Assume now that $K\subseteq\mathbb{R}^n$ is a (not necessarily symmetric) convex body in John's position and $F_0\in G_{n,k}$ is a k-dimensional linear subspace. We follow the notation introduced in Section 3.1. For $H=F_0\times\mathbb{R}\in G_{n+1,k+1}$ as in (3.4), we have that

$$\begin{aligned}(C\cap F_0)^\circ=P_{F_0}(C^\circ)&=P_{F_0}\left(\mathrm{conv}\{u_j\,:\,1\leqslant j\leqslant m\}\right)\\&=\mathrm{conv}\{P_{F_0}u_j\,:\,1\leqslant j\leqslant m\}.\end{aligned}$$

For any $y=(x,r)\in H=F_0\times\mathbb{R}$, we write

$$N(y)=\inf\left\{\sum_{j=1}^m\frac{\kappa_j\theta_j}{\sqrt{n\|P_{F_0}u_j\|_2^2+1}}\,:\,\theta_j\geqslant 0,\,y=\sum_{j=1}^m\kappa_j\theta_jw_j\right\},$$

where the latter infimum is understood as ∞ if there do not exist $\{\theta_j\}_{j=1}^m$ with $\theta_j\geqslant 0$ such that $y=\sum_{j=1}^m\kappa_j\theta_jw_j$. Notice that for any $\{\theta_j\}_{j=1}^m\subseteq\mathbb{R}$,

$$\begin{aligned}y=\sum_{j\in J}\kappa_j\theta_jw_j\quad&\Leftrightarrow\quad(x,r)=\left(-\sum_{j=1}^m\frac{\kappa_j\theta_jP_{F_0}(u_j)}{\sqrt{\|P_{F_0}u_j\|_2^2+\frac{1}{n}}},\sum_{j=1}^m\frac{\kappa_j\theta_j}{\sqrt{n\|P_{F_0}u_j\|_2^2+1}}\right)\\&\Leftrightarrow\quad(x,r)=\left(-r\sqrt{n}\sum_{j=1}^m\frac{\kappa_j\theta_jP_{F_0}(u_j)}{r\sqrt{n\|P_{F_0}u_j\|_2^2+1}},\sum_{j=1}^m\frac{\kappa_j\theta_j}{\sqrt{n\|P_{F_0}u_j\|_2^2+1}}\right).\end{aligned}$$

Then there exist $\{\theta_j\}_{j=1}^m\subseteq\mathbb{R}$ with $\theta_j\geq 0$ for every $1\leqslant j\leqslant m$ such that the latter equality holds if and only if

$$(x,r)\in L_1:=\left\{(x,r)\in F_0\times\mathbb{R}\,:\,r\geq 0\,:\,x\in -r\sqrt{n}(C\cap F_0)^\circ\right\},$$

and for all such $y=(x,r)\in L_1$, we have that $N(y)=r$. Therefore, for every $y\in H$

$$\sup_{y=\sum_{j=1}^m\kappa_j\theta_jw_j}\prod_{j=1}^m\left(\chi_{[0,\infty)}(\theta_j)e^{-\frac{\theta_j}{\sqrt{n\|P_{F_0}u_j\|_2^2+1}}}\right)^{\kappa_j}=e^{-N(y)}.$$

Thus, by (3.6) and the reverse Brascamp–Lieb inequality (Theorem 2.1),

$$\int_H e^{-N(y)}dy\geqslant\prod_{j=1}^m\left(\int_0^\infty e^{-\frac{t}{\sqrt{n\|P_{F_0}u_j\|_2^2+1}}}dt\right)^{\kappa_j}.$$

On the one hand,

$$\int_H e^{-N(y)}dy=\int_0^\infty e^{-r}\left|-r\sqrt{n}(C\cap F_0)^\circ\right|dr=k!n^{k/2}|(C\cap F_0)^\circ|.$$

On the other hand, for every $1\leqslant j\leqslant m$

$$\int_0^\infty e^{-\frac{t}{\sqrt{n\|P_{F_0}u_j\|_2^2+1}}}dt=\sqrt{n\|P_{F_0}u_j\|_2^2+1}=\sqrt{n+1}\|P_Hv_j\|_2.$$

Since $(K\cap F_0)^\circ \supseteq (C\cap F_0)^\circ$, we obtain

$$\begin{aligned} |(K\cap F_0)^\circ| &\geqslant \frac{(n+1)^{\frac{k+1}{2}}\prod_{j=1}^m \|P_H v_j\|_2^{\delta_j\|P_H v_j\|_2^2}}{k! n^{k/2}} \\ &= \left(\frac{n+1}{k+1}\right)^{\frac{k+1}{2}}\left(\frac{k}{n}\right)^{k/2}\prod_{j=1}^m \|P_H v_j\|_2^{\delta_j\|P_H v_j\|_2^2}|S_k^\circ|. \end{aligned}$$

For the convex function $f(x)=x\log x$ we apply Jensen's inequality to get

$$\sum_{j=1}^m \frac{\delta_j}{n+1}\|P_H v_j\|^2\log\|P_H v_j\|_2^2 \geq f\left(\sum_{j=1}^m \frac{\delta_j\|P_H v_j\|_2^2}{n+1}\right) = f\left(\frac{k+1}{n+1}\right).$$

Thus,

$$\prod_{j=1}^m \|P_H v_j\|_2^{\delta_j\|P_H v_j\|_2^2} = e^{\frac{n+1}{2}\sum_{j=1}^m \frac{\delta_j}{n+1}\|P_H v_j\|^2\log\|P_H v_j\|_2^2} \geq \left(\frac{k+1}{n+1}\right)^{\frac{k+1}{2}}.$$

Therefore,

$$|(K\cap F_0)^\circ|^{\frac{1}{k}} \geqslant \sqrt{\frac{k}{n}}|S_k^\circ|^{\frac{1}{k}}.$$

□

6. Mean width of sections of convex bodies in John's position

In this section we will prove Theorem 1.3.

Proof of Theorem 1.3. Let us start with the symmetric case. Assume that K is a centrally symmetric convex body in John's position and $F_0\in G_{n,k}$ is a k-dimensional linear subspace. We follow the notation in Section 3.2. By (3.11), we have that $K\cap F_0\subseteq C_0\cap F_0$ and

$$(K\cap F_0)^\circ \supseteq (C_0\cap F_0)^\circ = \mathrm{conv}\{\pm P_{F_0}u_j \,:\, j\in J_0\}.$$

It follows that for every $x\in F_0$

$$h_{K\cap F_0}(x) \leqslant h_{C_0\cap F_0}(x) = \inf\left\{\sum_{j\in J_0}|\alpha_j| \,:\, x=\sum_{j\in J_0}\alpha_j P_{F_0}u_j\right\}$$

$$= \inf\left\{\sum_{j\in J_0}|\alpha_j| \,:\, x=\sum_{j\in J_0}\alpha_j\|P_{F_0}u_j\|_2 v_j^0\right\} = \inf\left\{\sum_{j\in J_0}|\beta_j|t_j \,:\, x=\sum_{j\in J_0}\beta_j v_j^0\right\},$$

where $t_j=\frac{1}{\|P_{F_0}u_j\|_2}$ for all $j\in J_0$. For every $x\in F_0$, we write $x=\sum_{j\in J_0}\delta_j^0\langle x,v_j^0\rangle v_j^0$, therefore

$$h_{K\cap F_0}(x) \leqslant \sum_{j\in J_0}\delta_j^0 t_j|\langle x,v_j^0\rangle|. \tag{6.1}$$

If G_1 is a standard Gaussian random vector in F_0 and G_2 is a standard Gaussian random vector on $\mathbb{R}^k$, using (6.1), we get

$$\mathbb{E}h_{K\cap F_0}(G_1) \leqslant \sum_{j\in J_0}\delta_j^0 t_j\mathbb{E}|\langle G_1,v_j\rangle| = \mathbb{E}|\langle G_2,e_1\rangle|\sum_{j\in J_0}\delta_j^0 t_j$$

$$= \frac{1}{k}\mathbb{E}\|G_2\|_1 \sum_{j\in J_0} \delta_j^0 t_j = \frac{1}{k}\sum_{j\in J_0}\delta_j^0 t_j \mathbb{E} h_{B_\infty^k}(G_2).$$

By Hölder's inequality and (3.9),

$$\begin{aligned} \frac{1}{k}\sum_{j\in J_0}\delta_j^0 t_j &= \frac{1}{k}\sum_{j\in J_0} c_j\|P_{F_0}u_j\|_2 \leqslant \frac{1}{k}\left(\sum_{j\in J_0} c_j\right)^{\frac{1}{2}}\left(\sum_{j\in J_0} c_j\|P_{F_0}u_j\|_2^2\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{k}\left(\sum_{j=1}^m c_j\right)^{\frac{1}{2}}\left(\sum_{j=1}^m c_j\|P_{F_0}u_j\|_2^2\right)^{\frac{1}{2}} = \frac{\sqrt{nk}}{k} = \sqrt{\frac{n}{k}}. \end{aligned}$$

Hence,

$$\mathbb{E} h_{K\cap F_0}(G_1) \leqslant \sqrt{\frac{n}{k}}\mathbb{E} h_{B_\infty^k}(G_2).$$

Equivalently, by (2.2),

$$w(K\cap F_0) \leqslant \sqrt{\frac{n}{k}} w(B_\infty^k).$$

Let us now assume that K is a not necessarily symmetric convex body in John's position and let $F_0 \in G_{n,k}$. We follow the notation introduced in Section 3.1. For every $x \in F_0$, we have

$$h_{K\cap F_0}(x) \leqslant h_{C\cap F_0}(x) = \inf\left\{\sum_{j=1}^m a_j : x = \sum_{j=1}^m a_j P_{F_0}u_j, a_j \geqslant 0\right\}.$$

Let $\theta \in S^{n-1}\cap F_0$. By (2.1), we have $\sum_{j=1}^m c_j P_{F_0}u_j = 0$, so we may write

$$\theta = \sum_{j=1}^m c_j\left(\langle\theta, P_{F_0}u_j\rangle - \min_{1\leqslant k\leqslant m}\langle\theta, P_{F_0}u_k\rangle\right)P_{F_0}u_j.$$

Setting (like in the symmetric case before) $J_0 = \{1 \leqslant j \leqslant m : P_{F_0}u_j \neq 0\}$ and $v_j^0 = \frac{P_{F_0}u_j}{\|P_{F_0}u_j\|_2}$ for $j \in J_0$, we get

$$\begin{aligned} &w(K\cap F_0) \leqslant w(C\cap F_0) = \int_{S^{n-1}\cap F_0} h_{C\cap F_0}(\theta)d\sigma(\theta) \\ &\leqslant \int_{S^{n-1}\cap F_0}\sum_{j=1}^m c_j\left(\langle\theta, P_{F_0}u_j\rangle - \min_{1\leqslant k\leqslant m}\langle\theta, P_{F_0}u_k\rangle\right)d\sigma(\theta) \\ &= n\int_{S^{n-1}\cap F_0}\max_{1\leqslant k\leqslant m}\langle\theta, -P_{F_0}u_k\rangle d\sigma(\theta) \leqslant n\int_{S^{n-1}\cap F_0}\max_{1\leqslant k\leqslant m}|\langle\theta, P_{F_0}u_k\rangle|\, d\sigma(\theta) \\ &\leqslant n\int_{S^{n-1}\cap F_0}\max_{1\leqslant k\leqslant m}\|P_{F_0}u_k\|_2\max_{k\in J_0}\left|\langle\theta, v_k^0\rangle\right| d\sigma(\theta) \leqslant n\int_{S^{n-1}\cap F_0}\max_{k\in J_0}\left|\langle\theta, v_k^0\rangle\right| d\sigma(\theta). \end{aligned}$$

It is a well-known fact (see, for instance, [3, Proposition 9.1.5 and Lemma 5.2.11]) that for any $\{\theta_i\}_{i=1}^N \subseteq S^{n-1}$, one has that

$$\int_{S^{n-1}}\max_{1\leqslant k\leqslant N}|\langle\theta, \theta_k\rangle|\, d\sigma(\theta) \simeq \sqrt{\frac{\log N}{n}}.$$

Therefore, there exists an absolute constant C_1 such that

$$\int_{S^{n-1}\cap F}\max_{k\in J_0}\left|\langle\theta,v_k^0\rangle\right|d\sigma(\theta)\leqslant C_1\sqrt{\frac{\log m}{k}}.$$

Taking into account that $m=O(n^2)$ and that $w(S_k)\simeq\sqrt{k\log k}$, we obtain that there exists an absolute constant $C_2>0$ such that

$$w(K\cap F_0)\leqslant C_2\frac{n}{k}\sqrt{\frac{\log n}{\log k}}w(S_k).$$

□

7. Mean width of projections of convex bodies in Löwner's position

In this section we will prove Theorem 1.4. We will make use of the following lemma.

Lemma 7.1. *Let $K\subseteq\mathbb{R}^n$ be a (not necessarily symmetric) convex body and F_h be a k-dimensional affine subspace at distance h from the origin. Take some $\alpha\in\mathbb{R}$ and $\beta\leqslant 0$. We identify F_h with $\mathbb{R}^k$ with the origin at the closest point in F_h to 0 and let γ_k be the k-dimensional Gaussian measure on F_h. Then,*

$$\sqrt{\frac{n+h^2}{n}}\int_0^\infty\frac{e^{-\frac{(r-\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}}e^{\beta r\sqrt{n+1}}\gamma_k\left(\frac{r}{\sqrt{n}}(C\cap F_h)\right)dr\leqslant$$
$$\leqslant\int_0^\infty\frac{e^{-\frac{(r-\alpha d_1)^2}{2}}}{\sqrt{2\pi}}e^{\beta r\sqrt{k+1}}\gamma_k\left(r\sqrt{k+1}\Delta_k\right)dr,$$

where C is defined as in (3.1), Δ_k denotes the regular k-dimensional simplex as introduced in Section 2.3, $d_1=\frac{1}{\sqrt{k+1}}\sum_{j\in J}\delta_j\|P_Hv_j\|_2$, and H is defined as in (3.4), δ_j and v_j as in (3.2), and J as in (3.6).

Proof. Following the notation introduced in Section 3.1, let L be the cone defined in (3.3). By Lemma 3.1 we have that

$$L\cap H=\Big\{(x,r)\,:\,r\geqslant 0,\,x\in\frac{r}{\sqrt{n}}(C\cap F_h)\Big\}.$$

For any $\alpha,\beta\in\mathbb{R}$, let $\mu_{\alpha,\beta}$ be the measure on H whose density with respect to the Lebesgue measure at a point $y=(x,r)$ is

$$d\mu_{\alpha,\beta}(y)=\frac{e^{-\frac{\|y\|_2^2}{2}}}{(2\pi)^{\frac{k+1}{2}}}e^{(\alpha+\beta)r\sqrt{n+1}}dy.$$

For any $\alpha,\beta\in\mathbb{R}$, taking into account (3.5), we have that

$$\begin{aligned}\mu_{\alpha,\beta}(L\cap H)&=\int_0^\infty\frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}}e^{\alpha r\sqrt{n+1}}e^{\beta r\sqrt{n+1}}\gamma_k\left(\frac{r}{\sqrt{n}}(C\cap F_h)\right)\sqrt{\frac{n+h^2}{n}}dr\\&=e^{\frac{\alpha^2(n+1)}{2}}\int_0^\infty\frac{e^{-\frac{(r-\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}}e^{\beta r\sqrt{n+1}}\gamma_k\left(\frac{r}{\sqrt{n}}(C\cap F_h)\right)\sqrt{\frac{n+h^2}{n}}dr.\end{aligned}$$

Recall that $J = \{1 \leqslant j \leqslant m : P_H v_j \neq 0\}$, $s_j = \frac{1}{\|P_H v_j\|_2}$ for every $j \in J$ and $y = (x, r) \in H$. Using the definition of L, the identity (3.7), and the Brascamp–Lieb inequality (Theorem 2.1), we have

$$\begin{aligned}
&\mu_{\alpha,\beta}(L\cap H) = \int_H \frac{e^{-\frac{\|y\|_2^2}{2}}}{(2\pi)^{(k+1)/2}} e^{\alpha r\sqrt{n+1}} e^{\beta r\sqrt{n+1}} \prod_{j\in J} \chi_{[0,\infty)}(\langle y, v_j\rangle)dy \\
= &\int_H \frac{e^{-\frac{\|y\|_2^2}{2}}}{(2\pi)^{(k+1)/2}} e^{\alpha r\sqrt{n+1}} e^{\beta r\sqrt{n+1}} \prod_{j\in J} \chi_{[0,\infty)}(\langle y, w_j\rangle)dy \\
= &\int_H \frac{e^{-\frac{\sum_{j\in J}\kappa_j\langle y,w_j\rangle^2}{2}}}{(2\pi)^{(k+1)/2}} e^{\sum_{j\in J}\kappa_j(\alpha+\beta)s_j\langle y,w_j\rangle} \prod_{j\in J} \chi_{[0,\infty)}(\langle y, w_j\rangle)dy \\
= &\int_H \prod_{j\in J}\left(\frac{e^{-\frac{\langle y,w_j\rangle^2}{2}}}{\sqrt{2\pi}} e^{(\alpha+\beta)s_j\langle y,w_j\rangle}\chi_{[0,\infty)}(\langle y, w_j\rangle)\right)^{\kappa_j} dy \\
\leqslant &\prod_{j\in J}\left(\int_0^\infty \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} e^{\alpha s_j t} e^{\beta s_j t}dt\right)^{\kappa_j} = e^{\frac{\alpha^2\sum_{j\in J}\kappa_j s_j^2}{2}} \prod_{j=1}^m \left(\int_0^\infty \frac{e^{-\frac{(t-\alpha s_j)^2}{2}}}{\sqrt{2\pi}} e^{\beta s_j t}dt\right)^{\kappa_j} \\
= &e^{\frac{\alpha^2(n+1)}{2}} \prod_{j\in J}\left(\int_0^\infty \frac{e^{-\frac{(t-\alpha s_j)^2}{2}}}{\sqrt{2\pi}} e^{\beta s_j t}dt\right)^{\kappa_j}
\end{aligned}$$

Therefore, for any $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
\sqrt{\frac{n+h^2}{n}} \int_0^\infty \frac{e^{-\frac{(r-\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}} e^{\beta r\sqrt{n+1}} \gamma_k\left(\frac{r}{\sqrt{n}}(C\cap F_h)\right)dr \\
\leqslant \prod_{j\in J}\left(\int_0^\infty \frac{e^{-\frac{(t-\alpha s_j)^2}{2}}}{\sqrt{2\pi}} e^{\beta s_j t}dt\right)^{\kappa_j}.
\end{aligned}$$

Notice that $s_j = \frac{1}{\|P_H v_j\|_2} \geqslant 1$ for every $j \in J$, which implies that, for any $\beta \leqslant 0$, one has $\beta s_j \leqslant \beta$ for every $j \in J$. Using this inequality in the second following inequality and the Prékopa–Leindler inequality (see [37, Lemma 1.2]) in the third following inequality, we have that for any $\beta \leqslant 0$

$$\begin{aligned}
&\sqrt{\frac{n+h^2}{n}} \int_0^\infty \frac{e^{-\frac{(r-\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}} e^{\beta r\sqrt{n+1}} \gamma_k\left(\frac{r}{\sqrt{n}}(C\cap F_h)\right)dr \\
\leqslant &\prod_{j\in J}\left(\int_0^\infty \frac{e^{-\frac{(t-\alpha s_j)^2}{2}}}{\sqrt{2\pi}} e^{\beta s_j t}dt\right)^{\kappa_j} \\
\leqslant &\prod_{j\in J}\left(\int_0^\infty \frac{e^{-\frac{(t-\alpha s_j)^2}{2}}}{\sqrt{2\pi}} e^{\beta t}dt\right)^{\kappa_j} \leqslant \left(\int_0^\infty \frac{e^{-\frac{\left(t-\frac{\alpha}{k+1}\sum_{j\in J}\kappa_j s_j\right)^2}{2}}}{\sqrt{2\pi}} e^{\beta t}dt\right)^{k+1} \\
= &\int_{[0,\infty)^{k+1}} \prod_{i=1}^{k+1} \frac{e^{-\frac{\left(t_i-\frac{\alpha d_1}{\sqrt{k+1}}\right)^2}{2}}}{\sqrt{2\pi}} e^{\beta t_i}dt
\end{aligned}$$

$$= \int_{\left[-\frac{\alpha d_1}{\sqrt{k+1}},\infty\right)^{k+1}} \prod_{i=1}^{k+1} \frac{e^{-\frac{\|t\|_2^2}{2}}}{\sqrt{2\pi}} e^{\beta\sqrt{k+1}\langle t,v_0\rangle} e^{\beta\alpha d_1\sqrt{k+1}} dt,$$

where $v_0 = \left(\frac{1}{\sqrt{k+1}}, \dots, \frac{1}{\sqrt{k+1}}\right)$. Therefore, for any $\alpha \in \mathbb{R}$ and any $\beta \leqslant 0$,

$$\begin{aligned}
&\sqrt{\frac{n+h^2}{n}} \int_0^\infty \frac{e^{-\frac{(r-\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}} e^{\beta r\sqrt{n+1}} \gamma_k\left(\frac{r}{\sqrt{n}}(C \cap F_h)\right) dr \\
\leqslant\ & \int_{\left[-\frac{\alpha d_1}{\sqrt{k+1}},\infty\right)^{k+1}} \prod_{i=1}^{k+1} \frac{e^{-\frac{\|t\|_2^2}{2}}}{\sqrt{2\pi}} e^{\beta\sqrt{k+1}\langle t,v_0\rangle} e^{\beta\alpha d_1\sqrt{k+1}} dt \\
=\ & \int_{-\alpha d_1}^\infty \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} e^{\beta\sqrt{k+1}t} e^{\beta\alpha d_1\sqrt{k+1}} \gamma_k\left((t+\alpha d_1)\sqrt{k+1}\Delta_k\right) dt \\
=\ & \int_0^\infty \frac{e^{-\frac{(r-\alpha d_1)^2}{2}}}{\sqrt{2\pi}} e^{\beta\sqrt{k+1}r} \gamma_k\left(r\sqrt{k+1}\Delta_k\right) dr,
\end{aligned}$$

where Δ_k denotes the regular k-dimensional simplex as introduced in Section 2.3. □

Proof of Theorem 1.4. Let us start with the symmetric case. Assume that K is a centrally symmetric convex body in John's position and $F_0 \in G_{n,k}$ is a k-dimensional linear subspace. We want to prove that

$$w((K \cap F_0)^\circ) \geqslant w\left(\left(\sqrt{\frac{n}{k}} B_\infty^k\right)^\circ\right).$$

Equivalently, by (2.3), we want to prove that

$$\mathbb{E}\|G_1\|_{K\cap F_0} \geqslant \mathbb{E}\|G_2\|_{\sqrt{\frac{n}{k}}B_\infty^k},$$

where G_1 is a standard Gaussian random vector on F_0 and G_2 is a standard Gaussian random vector on $\mathbb{R}^k$. If $L \subseteq \mathbb{R}^k$ is a convex body containing the origin in its interior and G is a standard Gaussian random vector then

$$\mathbb{E}\|G\|_L = \int_0^\infty \mathbb{P}(\|G\|_L \geqslant t)dt = \int_0^\infty \gamma_k(\mathbb{R}^n \setminus tL)dt, \tag{7.1}$$

where $\gamma_k(A)$ denotes the Gaussian measure of the k-dimensional set A. Therefore, the statement we want to prove is equivalent to

$$\int_0^\infty \gamma_k(F_0 \setminus t(K \cap F_0))dt \geqslant \int_0^\infty \gamma_k\left(\mathbb{R}^n \setminus t\sqrt{\frac{n}{k}} B_\infty^k\right) dt$$

or, equivalently,

$$\int_0^\infty (1 - \gamma_k(t(K \cap F_0))dt \geqslant \int_0^\infty \left(1 - \gamma_k\left(t\sqrt{\frac{n}{k}} B_\infty^k\right)\right) dt.$$

We are going to prove that for any $t \geqslant 0$

$$\gamma_k(t(K \cap F_0)) \leqslant \gamma_k\left(t\sqrt{\frac{n}{k}} B_\infty^k\right),$$

which implies the latter inequality.

We follow the notation in Section 3.2. By (3.10) we have that

$$K \cap F_0 \subseteq C_0 \cap F_0 = \left\{x \in F_0 \,:\, |\langle x, v_j^0\rangle| \leqslant t_j,\, \forall j \in J_0\right\},$$

where $t_j = \frac{1}{\|P_{F_0}u_j\|_2} = \left(\frac{c_j}{\delta_j^0}\right)^{1/2}$ for all $j \in J_0$. Therefore for every $t \geqslant 0$,

$$t(K \cap F_0) \subseteq t(C_0 \cap F_0) = \left\{x \in F_0 \,:\, |\langle x, v_j^0\rangle| \leqslant tt_j,\, \forall j \in J_0\right\}.$$

By (3.9) and the Brascamp–Lieb inequality (Theorem 2.1),

$$\begin{aligned}
\gamma_k(t(K\cap F_0)) &\leqslant \gamma_k(t(C_0\cap F_0)) = \int_{F_0}\left(\prod_{j\in J}\chi_{[-tt_j,tt_j]}(\langle x, v_j^0\rangle)\right)\frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{k/2}}dx\\
&= \int_{F_0}\left(\prod_{j\in J_0}\chi_{[-tt_j,tt_j]}(\langle x, v_j^0\rangle)\right)\frac{e^{-\frac{\sum_{j\in J_0}\delta_j^0\langle x,v_j^0\rangle^2}{2}}}{(2\pi)^{k/2}}dx\\
&= \int_{F_0}\prod_{j\in J_0}\left(\chi_{[-tt_j,tt_j]}(\langle x, v_j^0\rangle)\frac{e^{-\frac{\langle x,v_j^0\rangle^2}{2}}}{\sqrt{2\pi}}\right)^{\delta_j^0}dx\\
&\leqslant \prod_{j\in J_0}\left(\int_{-tt_j}^{tt_j}\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}dt\right)^{\delta_j^0} = \left(\prod_{j\in J_0}\gamma_1(tt_j[-e_1,e_1])^{\frac{\delta_j^0}{k}}\right)^k.
\end{aligned}$$

Since γ_1 is log-concave, we obtain

$$\gamma_k(t(K\cap F_0)) \leqslant \gamma_1\left(\left(t\sum_{j\in J_0}\frac{t_j\delta_j^0}{k}\right)[-e_1,e_1]\right)^k = \gamma_k\left(\left(t\sum_{j\in J_0}\frac{t_j\delta_j^0}{k}\right)B_\infty^k\right).$$

By Hölder's inequality and (3.9), we have that

$$\begin{aligned}
\sum_{j\in J_0}\frac{t_j\delta_j^0}{k} &= \sum_{j\in J_0}\frac{\sqrt{c_j\delta_j^0}}{k} \leqslant \frac{1}{k}\left(\sum_{j\in J_0}c_j\right)^{1/2}\left(\sum_{j\in J_0}\delta_j^0\right)^{1/2}\\
&\leqslant \frac{1}{k}\left(\sum_{j=1}^m c_j\right)^{1/2}\left(\sum_{j\in J_0}\delta_j^0\right)^{1/2} = \sqrt{\frac{n}{k}}.
\end{aligned}$$

Thus, for every $t \geqslant 0$,

$$\gamma_k(t(K\cap F_0)) \leqslant \gamma_k\left(t\sqrt{\frac{n}{k}}B_\infty^k\right).$$

Assume now that $K \subseteq \mathbb{R}^n$ is a (not necessarily symmetric) convex body in John's position and $F_0 \in G_{n,k}$ is a linear subspace. Following the notation introduced in Section 3.1, we denote by $H \in G_{n+1,k+1}$ the $(k+1)$-dimensional linear subspace $H = \mathrm{span}\{(x, \sqrt{n}) \,:\, x \in F\} = F_0 \times \mathbb{R}$, as in (3.4). By Lemma 3.1, we have that

$$L \cap H = \left\{(x,r) \in F_0 \times \mathbb{R} \,:\, r \geqslant 0,\, x \in \frac{r}{\sqrt{n}}(C\cap F_0)\right\}.$$

Using Lemma 7.1 for $\beta = 0$, the linear subspace $F_0 \in G_{n,k}$ and an arbitrary $\alpha \in \mathbb{R}$, we get

$$\int_0^\infty \frac{e^{-\frac{(r-\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}}\gamma_k\left(\frac{r}{\sqrt{n}}(C\cap F_0)\right)dr \leqslant \int_0^\infty \frac{e^{-\frac{(r-\alpha d_1)^2}{2}}}{\sqrt{2\pi}}\gamma_k\left(r\sqrt{k+1}\Delta_k\right)dr,$$

where $d_1 = \frac{1}{\sqrt{k+1}}\sum_{j\in J}\delta_j\|P_H v_j\|_2$, δ_j and v_j are defined as in (3.2), and J as in (3.6). Applying the latter inequality to $-\alpha$, we also get

$$\int_0^\infty \frac{e^{-\frac{(r+\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}}\gamma_k\left(\frac{r}{\sqrt{n}}(C\cap F_0)\right)dr \leqslant \int_0^\infty \frac{e^{-\frac{(r+\alpha d_1)^2}{2}}}{\sqrt{2\pi}}\gamma_k\left(r\sqrt{k+1}\Delta_k\right)dr$$

or, equivalently,

$$\int_{-\infty}^0 \frac{e^{-\frac{(r-\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}}\gamma_k\left(\frac{|r|}{\sqrt{n}}(C\cap F_0)\right)dr \leqslant \int_{-\infty}^0 \frac{e^{-\frac{(r-\alpha d_1)^2}{2}}}{\sqrt{2\pi}}\gamma_k\left(|r|\sqrt{k+1}\Delta_k\right)dr.$$

Therefore, for any $\alpha\in\mathbb{R}$,

$$\int_{-\infty}^\infty \frac{e^{-\frac{(r-\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}}\gamma_k\left(\frac{|r|}{\sqrt{n}}(C\cap F_0)\right)dr \leqslant \int_{-\infty}^\infty \frac{e^{-\frac{(r-\alpha d_1)^2}{2}}}{\sqrt{2\pi}}\gamma_k\left(|r|\sqrt{k+1}\Delta_k\right)dr.$$

Hence,

$$\int_{-\infty}^\infty \frac{e^{-\frac{(r-\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}}\gamma_k(F_0\backslash\left(\left(\frac{|r|}{\sqrt{n}}(C\cap F_0)\right)\right)dr \geqslant \int_{-\infty}^\infty \frac{e^{-\frac{(r-\alpha d_1)^2}{2}}}{\sqrt{2\pi}}\gamma_k\left(\mathbb{R}^k\setminus(|r|\sqrt{k+1}\Delta_k)\right)dr.$$

Integrating in $\alpha\in\mathbb{R}$, we obtain

$$\frac{1}{\sqrt{n+1}}\int_{-\infty}^\infty \gamma_k\left(F_0\setminus\left(\frac{|r|}{\sqrt{n}}(C\cap F_0)\right)\right)dr \geqslant \frac{1}{d_1}\int_{-\infty}^\infty \gamma_k\left(\mathbb{R}^k\setminus(|r|\sqrt{k+1}\Delta_k)\right)dr.$$

Equivalently,

$$\frac{1}{\sqrt{n+1}}\int_0^\infty \gamma_k\left(F_0\setminus\left(\frac{r}{\sqrt{n}}(C\cap F_0)\right)\right)dr \geqslant \frac{1}{d_1}\int_0^\infty \gamma_k\left(\mathbb{R}^k\setminus(r\sqrt{k+1}\Delta_k)\right)dr,$$

or

$$\sqrt{\frac{n}{n+1}}\int_0^\infty \gamma_k\left(F_0\setminus(r(C\cap F_0))\right)dr \geqslant \frac{1}{d_1\sqrt{k+1}}\int_0^\infty \gamma_k\left(\mathbb{R}^k\setminus(r\Delta_k)\right)dr.$$

Using (7.1), (2.3) and the fact that $K\subseteq C$, we obtain

$$w((K\cap F_0)^\circ) \geqslant \frac{1}{d_1}\sqrt{\frac{n+1}{n(k+1)}}w((\Delta_k)^\circ).$$

If S_k denotes the k-dimensional regular simplex in John's position, then

$$\sqrt{k(k+1)}\Delta_k = S_k.$$

Therefore, for any k-dimensional linear subspace F_0, we have

$$w((K\cap F_0)^\circ) \geqslant \sqrt{\frac{k(n+1)}{n}}\frac{1}{d_1}w((S_k)^\circ).$$

By Hölder's inequality and (3.6), we have that

$$d_1\sqrt{k+1} = \sum_{j\in J}\delta_j\|P_H v_j\|_2 \leqslant \left(\sum_{j=1}^m \delta_j\right)^{\frac{1}{2}}\left(\sum_{j=1}^m \delta_j\|P_H v_j\|_2^2\right)^{\frac{1}{2}} = \sqrt{(n+1)(k+1)}.$$

Thus,

$$w((K\cap F_0)^\circ) \geqslant \sqrt{\frac{k}{n}}w(S_k^\circ).$$

□

8. The Wills functional of sections of convex bodies in John's position

In this section we will give the proof of Theorem 1.5.

Proof of Theorem 1.5. Let K be a centrally symmetric convex body in John's position and $F_0 \in G_{n,k}$ a k-dimensional linear subspace. We follow the notation in Section 3.2. By (3.10), we have that for every $\lambda \geqslant 0$

$$\lambda(K\cap F_0)\subseteq \lambda(C_0\cap F_0)=\left\{x\in F_0\,:\,|\langle x,v_j^0\rangle|\leqslant \lambda t_j,\,\forall j\in J_0\right\},$$

where $t_j=\frac{1}{\|P_{F_0}u_j\|_2}=\left(\frac{c_j}{\delta_j^0}\right)^{1/2}$ for all $j\in J_0$. For every $j\in J_0$, we define $f_j:\mathbb{R}\to[0,\infty)$ to be the function

$$f_j(t)=e^{-\pi d(tv_j^0,P_{\langle v_j^0\rangle}(\lambda(C_0\cap F_0)))^2}\qquad \forall t\in\mathbb{R},$$

where $\langle v_j^0\rangle$ denotes the 1-dimensional subspace spanned by v_j^0. Then,

$$\int_{\mathbb{R}} f_j(t)dt=\int_{\mathbb{R}} e^{-\pi d(tv_j^0,P_{\langle v_j^0\rangle}(\lambda(C_0\cap F_0)))^2}dt=\mathcal{W}(P_{\langle v_j^0\rangle}(\lambda(C_0\cap F_0)))$$

and

$$P_{\langle v_j^0\rangle}(\lambda(C_0\cap F_0))\subseteq[-\lambda t_j,\lambda t_j]\,v_j.$$

It follows that for every $j\in J_0$,

$$\int_{\mathbb{R}} f_j(t)dt\leqslant \mathcal{W}\left([-\lambda t_j,\lambda t_j]\,v_j\right)=(1+2\lambda t_j)\,.$$

Therefore, by (3.9) and the Brascamp–Lieb inequality (Theorem 2.1),

$$\int_{F_0} e^{-\pi\sum_{j\in J_0}\delta_j^0 d\left(\langle x,v_j^0\rangle v_j^0,P_{\langle v_j^0\rangle}(\lambda(C_0\cap F_0))\right)^2}=\int_{F_0}\prod_{j\in J_0} f_j^{\delta_j^0}(\langle x,v_j^0\rangle)dx$$

$$\leqslant\prod_{j\in J_0}\left(\int_{\mathbb{R}} f_j(t)dt\right)^{\delta_j^0}\leqslant\prod_{j\in J_0}(1+2\lambda t_j)^{\delta_j^0}\,.$$

By the arithmetic-geometric mean inequality, (2.1), and (3.9), we have

$$\begin{aligned}
\prod_{j\in J_0}(1+2\lambda t_j)^{\frac{\delta_j^0}{k}} &\leqslant \sum_{j\in J_0}\frac{\delta_j^0}{k}(1+2\lambda t_j)\leqslant 1+\frac{2\lambda}{k}\sum_{j\in J_0}c_j\|P_{F_0}u_j\|_2\\
&\leqslant 1+\frac{2\lambda}{k}\left(\sum_{j\in J_0}c_j\right)^{\frac12}\left(\sum_{j\in J_0}c_j\|P_{F_0}u_j\|_2^2\right)^{\frac12}\\
&\leqslant 1+\frac{2\lambda}{k}\left(\sum_{j=1}^{m}c_j\right)^{\frac12}\left(\sum_{j\in J_0}c_j\|P_{F_0}u_j\|_2^2\right)^{\frac12}\\
&= 1+2\lambda\sqrt{\frac{n}{k}}.
\end{aligned}$$

It follows that

$$\prod_{j\in J_0}(1+2\lambda t_j)^{\delta_j^0}\leqslant\left(1+2\lambda\sqrt{\frac{n}{k}}\right)^k=\mathcal{W}\left(\lambda\sqrt{\frac{n}{k}}B_\infty^k\right).$$

On the other hand, let $x_0\in\lambda(C_0\cap F_0)$. Then, for every $x\in F_0$ and every $j\in J_0$

$$d\left(\langle x,v_j^0\rangle v_j^0,P_{\langle v_j^0\rangle}(\lambda(C_0\cap F_0))\right)^2\leqslant d\left(\langle x,v_j^0\rangle v_j^0,\langle x_0,v_j^0\rangle v_j^0\right)^2=\langle x-x_0,v_j\rangle^2.$$

Thus, for every $x_0\in\lambda(C_0\cap F_0)$ and every $x\in F_0$,

$$\sum_{j\in J_0}\delta_j^0 d\left(\langle x,v_j^0\rangle v_j^0,P_{\langle v_j^0\rangle}(\lambda(C_0\cap F_0))\right)^2\leqslant\sum_{j\in J_0}\delta_j^0\langle x-x_0,v_j^0\rangle^2=|x-x_0|^2.$$

Hence, for every $x\in F_0$,

$$\sum_{j\in J_0}\delta_j^0 d\left(\langle x,v_j^0\rangle v_j^0,P_{\langle v_j^0\rangle}(\lambda(C_0\cap F_0))\right)^2\leqslant d(x,\lambda(C_0\cap F_0))^2.$$

Consequently,

$$\begin{aligned}\mathcal{W}(\lambda(C_0\cap F_0))&=\int_F e^{-\pi d(x,\lambda(C_0\cap F_0))^2}dx\\&\leqslant\int_F e^{-\pi\sum_{j\in J_0}\delta_j^0 d\left(\langle x,v_j^0\rangle v_j^0,P_{\langle v_j^0\rangle}(\lambda(C_0\cap F_0))\right)^2}dx\leqslant\mathcal{W}\left(\lambda\sqrt{\frac{n}{k}}B_\infty^k\right).\end{aligned}$$

Since $K\cap F_0\subseteq C_0\cap F_0$ and by the monotonicity of the Wills functional, we get

$$\mathcal{W}(\lambda(K\cap F_0))\leqslant\mathcal{W}(\lambda(C_0\cap F_0))\leqslant\mathcal{W}\left(\lambda\sqrt{\frac{n}{k}}B_\infty^k\right).$$

□

The following result gives a similar upper bound for a quantity defined via a double polarity, both on the convex body and on the log-concave function. For any k dimensional affine subspace F_h and any convex body $K\subseteq F_h$, we will consider

$$f_K(x)=e^{-\pi d^2(x,K)}\quad\forall x\in F_h,$$

as defined in Section 2.6.

Theorem 8.1. *Let $K\subseteq\mathbb{R}^n$ be a convex body in John's position and let F_h be a k-dimensional affine subspace at distance h from 0. Assume that the closest point to the origin in F_h belongs to the relative interior to $K\cap F_h$. Then, for every $\lambda>0$,*

$$\int_{F_h}f^\circ_{(\lambda(K\cap F_h))^\circ}(x)dx\leqslant\frac{n+1}{k+1}\sqrt{\frac{n}{n+h^2}}\int_{\mathbb{R}^k}f^\circ_{\left(\lambda\sqrt{\frac{n(n+1)}{k(k+1)}}S_k\right)^\circ}(x)dx,$$

where the polarity is taken with respect to the closest point to the origin in F_h.

Furthermore, if K is centrally symmetric and $F_0\in G_{n,k}$ is a k-dimensional linear subspace then, for every $\lambda>0$,

$$\int_{F_0}f^\circ_{(\lambda(K\cap F_0))^\circ}(x)dx\leqslant\int_{\mathbb{R}^k}f^\circ_{(\lambda\sqrt{\frac{n}{k}}B_\infty^k)^\circ}(x)dx.$$

Proof. Let $K\subseteq\mathbb{R}^n$ be a centrally symmetric convex body in John's position and let $F_0\in G_{n,k}$ be a k-dimensional linear subspace. From the definition of $f^\circ_{(\lambda(K\cap F_0))^\circ}$ and (2.4), we have that, for every $\lambda>0$,

$$\begin{aligned}\int_{F_0}f^\circ_{(\lambda(K\cap F_0))^\circ}(x)dx&=\int_{F_0}e^{-\frac{\|x\|_2^2}{4\pi}}e^{-\|x\|_{\lambda(K\cap F_0)}}dx=\int_{F_0}e^{-\frac{\|x\|_2^2}{4\pi}}\int_{\|x\|_{\lambda(K\cap F_0)}}^\infty e^{-t}dtdx\\&=\int_0^\infty e^{-t}\int_{t\lambda(K\cap F_0)}e^{-\frac{\|x\|_2^2}{4\pi}}dtdx\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^k \int_0^\infty e^{-t} \int_{\frac{t\lambda}{\sqrt{2\pi}}(K\cap F_0)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(\sqrt{2\pi})^k} dx dt \\
&= (2\pi)^k \int_0^\infty e^{-t} \gamma_k\left(\frac{t\lambda}{\sqrt{2\pi}}(K\cap F_0)\right) dt.
\end{aligned}
$$

Similarly, for every $\lambda > 0$,

$$
\int_{\mathbb{R}^k} f^\circ_{(\lambda\sqrt{\frac{n}{k}}B_\infty^k)^\circ}(x)dx = (2\pi)^k \int_0^\infty e^{-t}\gamma_k\left(\frac{t\lambda}{\sqrt{2\pi}}\sqrt{\frac{n}{k}}B_\infty^k\right) dt.
$$

As we have seen in the proof of Theorem 1.4, for every $t \geqslant 0$ and every $\lambda > 0$,

$$
\gamma_k\left(\frac{t\lambda}{\sqrt{2\pi}}(K\cap F_0)\right) \leqslant \gamma_k\left(\frac{t\lambda}{\sqrt{2\pi}}\sqrt{\frac{n}{k}}B_\infty^k\right).
$$

Therefore, for every $\lambda > 0$,

$$
\int_{F_0} f^\circ_{(\lambda(K\cap F_0))^\circ}(x)dx \leqslant \int_{\mathbb{R}^k} f^\circ_{(\lambda\sqrt{\frac{n}{k}}B_\infty^k)^\circ}(x)dx.
$$

Assume now that $K \subseteq \mathbb{R}^n$ is a (not necessarily symmetric) convex body in John's position, F_h is a k-dimensional affine subspace at distance h from the origin, and the closest point to the origin in F_h belongs to the relative interior to $K \cap F_h$. We will identify F_h with $\mathbb{R}^k$ and the closest point in F_h to the origin in $\mathbb{R}^n$ with the origin in F_h (identified with $\mathbb{R}^k$). As before, we have that, for every $\lambda > 0$,

$$
\int_{F_h} f^\circ_{(\lambda(K\cap F_h))^\circ}(x)dx = (2\pi)^k \int_0^\infty e^{-t}\gamma_k\left(\frac{t\lambda}{\sqrt{2\pi}}(K\cap F_h)\right) dt.
$$

We will follow the notation in Section 3.1. By Lemma 7.1, we have that for any $\alpha \in \mathbb{R}$ and any $\beta \leqslant 0$,

$$
\begin{aligned}
\sqrt{\frac{n+h^2}{n}} \int_0^\infty \frac{e^{-\frac{(r-\alpha\sqrt{n+1})^2}{2}}}{\sqrt{2\pi}} e^{\beta r\sqrt{n+1}}\gamma_k\left(\frac{r}{\sqrt{n}}(C\cap F_h)\right) dr \leqslant \\
\int_0^\infty \frac{e^{-\frac{(r-\alpha d_1)^2}{2}}}{\sqrt{2\pi}} e^{\beta r\sqrt{k+1}}\gamma_k\left(r\sqrt{k+1}\Delta_k\right) dr,
\end{aligned}
$$

where $d_1 = \frac{1}{\sqrt{k+1}}\sum_{j\in J}\delta_j\|P_H v_j\|_2$. Integrating with respect to $\alpha \in \mathbb{R}$, we see that for any $\beta \leqslant 0$,

$$
\sqrt{\frac{n+h^2}{n(n+1)}} \int_0^\infty e^{\beta r\sqrt{n+1}}\gamma_k\left(\frac{r}{\sqrt{n}}(C\cap F_h)\right) dr \leqslant \frac{1}{d_1}\int_0^\infty e^{\beta r\sqrt{k+1}}\gamma_k\left(r\sqrt{k+1}\Delta_k\right) dr.
$$

Equivalently, changing variables $u = \sqrt{\frac{n}{2\pi}}r$, for any $\beta \leqslant 0$,

$$
\sqrt{\frac{n+h^2}{n+1}} \int_0^\infty e^{\frac{\beta u\sqrt{n(n+1)}}{\sqrt{2\pi}}}\gamma_k\left(\frac{u}{\sqrt{2\pi}}(C\cap F_h)\right) du \leqslant \frac{1}{d_1\sqrt{k+1}}\int_0^\infty e^{\frac{\beta u}{\sqrt{2\pi}}}\gamma_k\left(\frac{u}{\sqrt{2\pi}}\Delta_k\right) du.
$$

For any $\lambda > 0$, take $\beta = -\frac{1}{\lambda}\sqrt{\frac{2\pi}{n(n+1)}}$ to obtain

$$
\sqrt{\frac{n+h^2}{n+1}} \int_0^\infty e^{-\frac{u}{\lambda}}\gamma_k\left(\frac{u}{\sqrt{2\pi}}(C\cap F_h)\right) du \leqslant \frac{1}{d_1\sqrt{k+1}}\int_0^\infty e^{\frac{-u}{\lambda\sqrt{n(n+1)}}}\gamma_k\left(\frac{u}{\sqrt{2\pi}}\Delta_k\right) du,
$$

or, equivalently, changing variables $u=\lambda v$ in the integral on the left-hand side and $u=\lambda\sqrt{n(n+1)}v$ in the integral on the right-hand side and renaming v as u,

$$\sqrt{\frac{n+h^2}{n+1}}\int_0^\infty e^{-u}\gamma_k\left(\frac{u\lambda}{\sqrt{2\pi}}(C\cap F_h)\right)du\leqslant\frac{\sqrt{n(n+1)}}{d_1 s\sqrt{k+1}}\int_0^\infty e^{-u}\gamma_k\left(\frac{u\lambda\sqrt{n(n+1)}}{\sqrt{2\pi}}\Delta_k\right)du.$$

Since $\sqrt{k(k+1)}\Delta_k=S_k$, we see that for every $\lambda>0$,

$$\int_0^\infty e^{-u}\gamma_k\left(\frac{u\lambda}{\sqrt{2\pi}}(C\cap F_h)\right)du\leqslant\frac{(n+1)\sqrt{n}}{d_1\sqrt{(k+1)(n+h^2)}}\int_0^\infty e^{-u}\gamma_k\left(\frac{u\lambda}{\sqrt{2\pi}}\sqrt{\frac{n(n+1)}{k(k+1)}}S_k\right)du.$$

Consequently, for any $\lambda>0$, taking polars with respect to the closest point in F_h to the origin which we assumed to belong to the relative interior of $K\cap F_h$,

$$\begin{aligned}\int_{F_h}f^\circ_{(\lambda(K\cap F_h))^\circ}(x)dx&\leqslant\int_{F_h}f^\circ_{(\lambda(C\cap F_h))^\circ}(x)dx\\&\leqslant\frac{\sqrt{n}(n+1)}{d_1\sqrt{(k+1)(n+h^2)}}\int_{\mathbb{R}^k}f^\circ_{\left(\lambda\sqrt{\frac{n(n+1)}{k(k+1)}}S_k\right)^\circ}(x)dx.\end{aligned}$$

for every $j\in J$, denote κ_j as in (3.6) and s_j as in (3.7). Then, by (3.6), we have that

$$d_1\sqrt{k+1}=\sum_{j\in J}\kappa_j s_j\geqslant\sum_{j\in J}\kappa_j=\sum_{j=1}^m=k+1.$$

Thus, for every $\lambda>0$,

$$\int_{F_h}f^\circ_{(\lambda(K\cap F_h))^\circ}(x)dx\leqslant\frac{n+1}{k+1}\sqrt{\frac{n}{n+h^2}}\int_{\mathbb{R}^k}f^\circ_{\left(\lambda\sqrt{\frac{n(n+1)}{k(k+1)}}S_k\right)^\circ}(x)dx.$$

□

9. The Wills functional of projections of convex bodies in Löwner's position

In this section we will give the proof of Theorem 1.6.

Proof of Theorem 1.6. Let K be a centrally symmetric convex body in John's position and let $F_0\in G_{n,k}$ be a k-dimensional linear subspace. We follow the notation in Section 3.2. By (3.11), we have that

$$(K\cap F_0)^\circ\supseteq(C_0\cap F_0)^\circ=\mathrm{conv}\{\pm P_{F_0}u_j,\,j\in J_0\}=\mathrm{conv}\{\pm\|P_{F_0}u_j\|_2v_j,\,j\in J_0\}.$$

Since the function $d(\cdot,(C_0\cap F_0)^\circ)^2$ is convex, for any $x\in F_0$ and any $\{\theta_j\}_{j\in J_0}\subseteq\mathbb{R}$ such that $x=\sum_{j\in J_0}\delta_j^0\theta_jv_j^0$, we have that

$$\begin{aligned}d(x,(K\cap F_0)^\circ)^2&\leqslant d(x,(C_0\cap F_0)^\circ)^2=d\left(\sum_{j\in J_0}\frac{\delta_j^0}{k}k\theta_jv_j^0,(C_0\cap F_0)^\circ\right)^2\\&\leqslant\frac{1}{k}\sum_{j\in J_0}\delta_j^0d\left(k\theta_jv_j^0,(C_0\cap F_0)^\circ\right)^2\\&\leqslant\frac{1}{k}\sum_{j\in J_0}\delta_j^0d\left(k\theta_jv_j^0,\left[-\|P_{F_0}u_j\|_2v_j^0,\|P_{F_0}u_j\|_2v_j^0\right]\right)^2\end{aligned}$$

$$= \sum_{j\in J_0}\delta_j^0 d\left(\sqrt{k}\theta_j v_j^0,\left[-\frac{\|P_{F_0}u_j\|_2}{\sqrt{k}}v_j^0,\frac{\|P_{F_0}u_j\|_2}{\sqrt{k}}v_j^0\right]\right)^2.$$

For every $j\in J_0$, we set

$$f_j(t)=e^{-\pi d\left(\sqrt{k}tv_j^0,\left[-\frac{\|P_{F_0}u_j\|_2}{\sqrt{k}}v_j^0,\frac{\|P_{F_0}u_j\|_2}{\sqrt{k}}v_j^0\right]\right)^2},\quad \forall t\in\mathbb{R}.$$

Moreover, for any $x\in F_0$ and any $\{\theta_j\}_{j\in J_0}\subseteq\mathbb{R}$ such that $x=\sum_{j\in J_0}\delta_j^0\theta_j v_j^0$, we have

$$\prod_{j\in J_0} f_j^{\delta_j^0}(\theta_j)\leqslant e^{-\pi d(x,(K\cap F_0)^\circ)^2}.$$

Therefore, by (3.9) and the reverse Brascamp–Lieb inequality (Theorem 2.1),

$$\begin{aligned}\mathcal{W}((K\cap F_0)^\circ) &= \int_{F_0} e^{-\pi d(x,(K\cap F_0)^\circ)^2}dx \geqslant \prod_{j\in J_0}\left(\int_{\mathbb{R}} f_j(t)dt\right)^{\delta_j^0}\\ &= \prod_{j\in J_0}\left(\frac{1}{\sqrt{k}}\int_{\mathbb{R}} e^{-\pi d\left(tv_j^0,\left[-\frac{\|P_{F_0}u_j\|_2}{\sqrt{k}}v_j^0,\frac{\|P_{F_0}u_j\|_2}{\sqrt{k}}v_j^0\right]\right)^2}dt\right)^{\delta_j^0}\\ &= \frac{1}{k^{k/2}}\prod_{j\in J_0}\mathcal{W}\left(\left[-\frac{\|P_{F_0}u_j\|_2}{\sqrt{k}}v_j^0,\frac{\|P_{F_0}u_j\|_2}{\sqrt{k}}v_j^0\right]\right)^{\delta_j^0}\\ &= \frac{1}{k^{k/2}}\prod_{j\in J_0}\left(1+\frac{2\|P_{F_0}u_j\|_2}{\sqrt{k}}\right)^{\delta_j^0}\geqslant\frac{1}{k^{k/2}}.\end{aligned}$$

□

10. Sections of convex bodies in minimal surface area position

In this section we are going to prove Theorem 1.7. Let us start assuming that K is a centrally symmetric polytope in minimal surface area position and $F_0\in G_{n,k}$. By an approximation argument, the inequalities we obtain will also be true for any centrally symmetric convex body in minimal surface area position. We will follow the notation introduced in Section 3.3.

Let J_0 and, for every $j\in J_0$, c_j, δ_j^0, and v_j^0 be as in (3.14). Let for every $j\in J_0$, $f_j:\mathbb{R}\to[0,\infty)$ be the function

$$f_j(t)=e^{-\pi d(tv_j^0,P_{\langle v_j^0\rangle}(K\cap F_0))^2}\quad\forall t\in\mathbb{R},$$

where $\langle v_j^0\rangle$ denotes the 1-dimensional subspace spanned by v_j^0. Notice that, for every $j\in J_0$,

$$\int_{\mathbb{R}} f_j(t)dt=\int_{\mathbb{R}} e^{-\pi d(tv_j^0,P_{\langle v_j^0\rangle}(K\cap F))^2}dt=\mathcal{W}(P_{\langle v_j^0\rangle}(K\cap F_0)).$$

For every $j\in J_0$, we have that

$$P_{\langle v_j^0\rangle}(K\cap F_0)\subseteq[-t_jh_K(u_j),t_jh_K(u_j)]\,v_j^0,$$

where t_j is defined as in (3.15). Then, for every $j\in J_0$,

$$\int_{\mathbb{R}} f_j(t)dt\leqslant\mathcal{W}\left([-t_jh_K(u_j),t_jh_K(u_j)]\,v_j\right)=\left(1+2t_jh_K(u_j)\right).$$

Therefore, by (3.14) and the Brascamp–Lieb inequality (Theorem 2.1),

$$\begin{aligned}\int_{F_0} e^{-\pi\sum_{j\in J_0}\delta_j^0 d\left(\langle x,v_j^0\rangle v_j^0, P_{\langle v_j^0\rangle}(C_0\cap F_0)\right)^2} &= \int_{F_0}\prod_{j\in J_0} f_j^{\delta_j^0}(\langle x,v_j^0\rangle)dx \\ \leqslant \prod_{j\in J_0}\left(\int_{\mathbb{R}} f_j(t)dt\right)^{\delta_j^0} &\leqslant \prod_{j\in J_0}(1+2t_jh_K(u_j))^{\delta_j^0}.\end{aligned}$$

By the arithmetic-geometric mean inequality and (3.13), we have

$$\begin{aligned}&\prod_{j\in J_0}(1+2t_jh_K(u_j))^{\frac{\delta_j^0}{k}} \leqslant \sum_{j\in J_0}\frac{\delta_j^0}{k}(1+2t_jh_K(u_j)) \\ &= 1+\frac{2}{k}\sum_{j\in J_0}\frac{n|F_j|\|P_{F_0}u_j\|_2h_K(u_j)}{|\partial K|} \leqslant 1+\frac{2n}{k|\partial K|}\sum_{j\in J_0}|F_j|h_K(u_j) \\ &\leqslant 1+\frac{2n}{k|\partial K|}\sum_{j=1}^m|F_j|h_K(u_j) = 1+2\frac{n^2|K|}{k|\partial K|}.\end{aligned}$$

Thus,

$$\prod_{j\in J_0}(1+2t_jh_K(u_j))^{\delta_j^0} \leqslant \left(1+2\frac{n^2|K|}{k|\partial K|}\right)^k = \mathcal{W}\left(\frac{n^2|K|}{k|\partial K|}B_\infty^k\right).$$

Let $x_0\in K\cap F_0$. For every $x\in F_0$ and every $j\in J_0$ we have

$$d\left(\langle x,v_j^0\rangle v_j^0, P_{\langle v_j^0\rangle}(K\cap F_0)\right)^2 \leqslant d\left(\langle x,v_j^0\rangle v_j^0, \langle x_0,v_j^0\rangle v_j^0\right)^2 = \langle x-x_0,v_j\rangle^2.$$

Thus, for every $x_0\in C_0\cap F_0$ and every $x\in F_0$,

$$\sum_{j\in J_0}\delta_j^0 d\left(\langle x,v_j^0\rangle v_j^0, P_{\langle v_j^0\rangle}(K\cap F_0)\right)^2 \leqslant \sum_{j\in J_0}\delta_j^0\langle x-x_0,v_j^0\rangle^2 = |x-x_0|^2.$$

Hence, for every $x\in F_0$,

$$\sum_{j\in J_0}\delta_j^0 d\left(\langle x,v_j^0\rangle v_j^0, P_{\langle v_j^0\rangle}(K\cap F_0)\right)^2 \leqslant d(x,K\cap F_0)^2.$$

Consequently,

$$\begin{aligned}\mathcal{W}(K\cap F_0) &= \int_{F_0} e^{-\pi d(x,K\cap F_0)^2}dx \leqslant \int_{F_0} e^{-\pi\sum_{j\in J_0}\delta_j^0 d\left(\langle x,v_j^0\rangle v_j^0, P_{\langle v_j^0\rangle}(K\cap F_0)\right)^2}dx \\ &\leqslant \mathcal{W}\left(\frac{n^2|K|}{k|\partial K|}B_\infty^k\right),\end{aligned}$$

which proves (i).

Notice that for every $\lambda\geqslant 0$ we have that λK is in minimal surface area position. Therefore we can use (i) to get that for every $\lambda\geqslant 0$

$$\mathcal{W}(\lambda(K\cap F_0)) \leqslant \mathcal{W}\left(\lambda\frac{n^2}{k}\frac{|K|}{|\partial K|}B_\infty^k\right).$$

As explained in Section 2.6, the last one implies that $V_1(K\cap F_0)\leqslant V_1\left(\lambda\frac{n^2}{k}\frac{|K|}{|\partial K|}B_\infty^k\right)$ and $V_n((K\cap F_0)\leqslant V_n\left(\lambda\frac{n^2}{k}\frac{|K|}{|\partial K|}B_\infty^k\right)$, which are equivalent to (ii) and (iii), respectively.

Now, using (3.16), we observe that for every $x \in F_0$

$$\begin{aligned} h_{K\cap F_0}(x) &= \inf\left\{\sum_{j\in J_0}|\alpha_j| \,:\, x=\sum_{j\in J_0}\frac{\alpha_j}{h_K(u_j)}P_{F_0}u_j\right\} \\ &= \inf\left\{\sum_{j\in J_0}|\beta_j|t_j h_K(u_j) \,:\, x=\sum_{j\in J_0}\beta_j v_j^0\right\}, \end{aligned}$$

where $t_j = \frac{1}{\|P_{F_0}u_j\|_2} = \left(\frac{c_j}{\delta_j^0}\right)^{\frac{1}{2}}$ for all $j \in J_0$. For every $j \in J_0$, we define

$$f_j(t) := e^{-|t|t_j h_K(u_j)}, \quad t \in \mathbb{R}.$$

Then, if $x = \sum_{j\in J_0}\delta_j^0\theta_j v_j^0$ for some $\{\theta_j\}_{j\in J_0} \subseteq \mathbb{R}$, we have

$$\prod_{j\in J_0} f_j^{\delta_j^0}(\theta_j) = e^{-\sum_{j\in J_0}\delta_j|\theta_j|t_j h_K(u_j)} \leqslant e^{-h_{K\cap F_0}(x)}.$$

Therefore, by (3.14) and the reverse Brascamp–Lieb inequality (Theorem 2.1),

$$\begin{aligned} k!|(K\cap F_0)^\circ| &= \int_{F_0} e^{-h_{K\cap F_0}(x)}dx \geqslant \prod_{j\in J}\left(\int_{\mathbb{R}} e^{-|t|t_j h_K(u_j)}dt\right)^{\delta_j} \\ &= \frac{2^k}{\prod_{j\in J_0}(t_j h_K(u_j))^{\delta_j}}. \end{aligned}$$

By the arithmetic-geometric mean inequality and (3.13),

$$\begin{aligned} \prod_{j\in J_0}(t_j h_K(u_j))^{\frac{\delta_j^0}{k}} &\leqslant \sum_{j\in J_0}\frac{\delta_j^0}{k}t_j h_K(u_j) = \sum_{j\in J_0}\frac{n|F_j|\|P_{F_0}u_j\|_2 h_K(u_j)}{k|\partial K|} \\ &\leqslant \frac{n}{k|\partial K|}\sum_{j=1}^m |F_j|h_K(u_j) = \frac{n^2|K|}{k|\partial K|}. \end{aligned}$$

Taking into account that $|(B_\infty^k)^\circ| = |B_1^k| = \frac{2^k}{k!}$, we obtain

$$|(K\cap F_0)^\circ|^{1/k} \geqslant \frac{k|\partial K|}{n^2|K|}|(B_\infty^k)^\circ|^{1/k},$$

which gives us (iv).

Finally, from (3.15) observe that for every $t \geqslant 0$

$$t(K\cap F_0) = \{x\in F_0 \,:\, |\langle x, v_j^0\rangle| \leqslant t t_j h_K(u_j), \forall\, j \in J_0\}.$$

By (3.14) and the Brascamp–Lieb inequality (Theorem 2.1), we have that

$$\gamma_k(t(K\cap F_0)) = \int_{\mathbb{R}^n}\prod_{j\in J_0}\chi_{[-tt_jh_K(u_j),tt_jh_K(u_j)]}(\langle x, v_j^0\rangle)\frac{e^{-\sum_{j\in J_0}\frac{\delta_j^0\langle x,v_j^0\rangle^2}{2}}}{(2\pi)^{k/2}}dx$$

$$\leqslant \prod_{j\in J_0}\left(\int_{-tt_jh_K(u_j)}^{tt_jh_K(u_j)}\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}ds\right)^{\delta_j^0} = \prod_{j\in J_0}\gamma_1\left([-tt_jh_K(u_j), tt_jh_K(u_j)]\right)^{\delta_j^0}.$$

Since the 1-dimensional Gaussian measure is log-concave, we obtain that

$$\begin{aligned}\prod_{j\in J_0}\gamma_1\left([-tt_jh_K(u_j),tt_jh_K(u_j)]\right)^{\delta_j^0} &\leqslant \gamma_1\left(\left(\left(\sum_{j\in J_0}\frac{\delta_j^0tt_jh_K(u_j)}{k}\right)[-e_1,e_1]\right)^k\right.\\ &= \gamma_k\left(t\left(\sum_{j\in J_0}\frac{\delta_j^0t_jh_K(u_j)}{k}\right)B_\infty^k\right).\end{aligned}$$

Therefore, by (3.13),

$$\begin{aligned}\sum_{j\in J_0}\frac{\delta_j^0t_jh_K(u_j)}{k} &= \sum_{j\in J_0}\frac{n|F_j|\|P_{F_0}u_j\|_2h_K(u_j)}{k|\partial K|}\leqslant\sum_{j\in J_0}\frac{n|F_j|h_K(u_j)}{k|\partial K|}\\ &\leqslant \sum_{j=1}^m\frac{n|F_j|h_K(u_j)}{k|\partial K|}=\frac{n^2}{k}\frac{|K|}{|\partial K|}.\end{aligned}$$

Thus, for any $t\geq 0$,

$$\gamma_k(t(K\cap F_0))\leqslant\gamma_k\left(t\frac{n^2}{k}\frac{|K|}{|\partial K|}B_\infty^k\right).$$

Therefore,

$$w((K\cap F_0)^\circ)\geqslant w\left(\left(\frac{n^2}{k}\frac{|K|}{|\partial K|}B_\infty^k\right)^\circ\right)=\frac{k}{n^2}\frac{|\partial K|}{|K|}w((B_\infty^k)^\circ),$$

and we obtain (v).

Let us now assume that K is a (not necessarily symmetric) polytope in minimal surface area position and $F_0\in G_{n,k}$. Again, by approximation, the inequalities we obtain will be true for any convex body. By (2.5), we have that for any $x\in F_0$,

$$\begin{aligned}\|x\|_{\Pi^*K\cap F_0} &= \frac12\sum_{j=1}^m|F_j||\langle x,u_j\rangle|=\frac12\sum_{j\in J_0}|F_j|\|P_{F_0}u_j\|_2\left|\langle x,v_j^0\rangle\right|\\ &= \sum_{j\in J_0}\frac{|\partial K|\delta_j^0t_j}{2n}\left|\langle x,v_j^0\rangle\right|,\end{aligned}$$

where the vectors u_j are defined as in (3.12) and, for every $j\in J_0$, δ_j^0 and v_j^0 are defined as in (3.14) and t_j is defined as in (3.15). Therefore, by (3.14) and the Brascamp–Lieb inequality (Theorem 2.1),

$$\begin{aligned}k!|\Pi^*K\cap F_0| &= \int_{F_0}e^{-\|x\|_{\Pi^*K\cap F_0}}dx=\int_{F_0}e^{-\sum_{j\in J_0}\frac{|\partial K|\delta_j^0t_j}{2n}\left|\langle x,v_j^0\rangle\right|}dx\\ &= \int_{F_0}\prod_{j\in J_0}\left(e^{-\frac{|\partial K|t_j}{2n}\left|\langle x,v_j^0\rangle\right|}\right)^{\delta_j^0}dx\\ &\leqslant \prod_{j\in J_0}\left(\int_{\mathbb{R}}e^{-\frac{|\partial K|t_j}{2n}t}dt\right)^{\delta_j^0}=\prod_{j\in J_0}\left(\frac{4n}{|\partial K|t_j}\right)^{\delta_j^0}\\ &= \left(\frac{4n}{|\partial K|}\right)^k\prod_{j\in J_0}\left(\|P_{F_0}u_j\|\right)^{\delta_j^0}\leqslant\left(\frac{4n}{|\partial K|}\right)^k.\end{aligned}$$

To prove the remaining inequality, we start observing that $P_{F_0}\Pi K = (\Pi^* K \cap F_0)^\circ$. Then, for every $x \in F_0$,

$$\|x\|_{P_{F_0}\Pi K} = \inf \left\{ \max_{j\in J_0} |\tau_j| \,:\, x = \sum_{j\in J_0} \frac{|\partial K|\delta_j^0 t_j \tau_j}{2n} v_j^0 \right\}.$$

Any decomposition of x of the form $x = \sum_{j\in J_0} \delta_j^0 \theta_j v_j^0$ with $|\theta_j| \leqslant \frac{|\partial K| t_j}{2n}$ gives a decomposition of x of the form

$$x = \sum_{j\in J_0} \frac{|\partial K|\delta_j^0 t_j \tau_j}{2n} v_j^0 \quad \text{with} \quad \tau_j = \frac{2n\theta_j}{|\partial K| t_j}.$$

Since $\max_{j\in J_0} |\tau_j| \leqslant 1$, we get that the functions $h_j = \chi_{\left[-\frac{|\partial K|t_j}{2n}, \frac{|\partial K|t_j}{2n}\right]}$, $j \in J_0$, and the function $h = \chi_{P_{F_0}\Pi K}$ have the property that

$$h\left(\sum_{j\in J_0} \delta_j^0 \theta_j v_j^0\right) \geqslant \prod_{j\in J_0} h_j^{\delta_j^0}(\theta_j),$$

for every $\{\theta_j\}_{j\in J_0} \subseteq \mathbb{R}$. Hence, by (3.14) and the reverse Brascamp–Lieb inequality (Theorem 2.1),

$$\begin{aligned} |P_{F_0}\Pi K| &= \int_{F_0} h(x)dx \geqslant \prod_{j\in J_0} \left(\int_{\mathbb{R}} h_j(t)dt\right)^{\delta_j^0} = \prod_{j\in J_0} \left(\frac{|\partial K| t_j}{n}\right)^{\delta_j^0} \\ &= \left(\frac{|\partial K|}{n}\right)^k \prod_{j\in J_0} \left(\frac{1}{\|P_{F_0} u_j\|}\right)^{\delta_j^0} \geqslant \left(\frac{|\partial K|}{n}\right)^k. \end{aligned}$$

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