Stability of pseudo-Kähler manifolds and cohomological decomposition

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Abstract. We consider compact complex manifolds endowed with a pseudo-Kähler structure and study their stability under deformations. It is known that if the Bott-Chern number $b^{1,1}_{BC}(X_t)$ is constant along a deformation X_t whose central fiber X_0 is pseudo-Kähler, then X_t also admits a pseudo-Kähler structure, at least for sufficiently small t. Here we find another condition for stability related to the cohomological decomposition of complex manifolds.

Keywords: complex manifold, pseudo-Kähler structure, cohomological decomposition, deformation

1 Introduction

Complex geometry deals with the study of complex manifolds, namely, spaces that locally look like $\mathbb{C}^n$ and whose changes of charts are biholomorphic. Any complex manifold X of complex dimension n is, in particular, a differentiable manifold M of real dimension $2n$. Therefore, if one endows M with a certain (real) geometric structure, it is natural to wonder how this structure interacts with X, the complex counterpart of M.

By the well-known Newlander-Nirenberg theorem, any complex manifold X can be equivalently seen as a pair (M, J), where M is an even-dimensional differentiable manifold and J is a *complex structure* on M. If M is equipped with a pseudo-Riemannian metric g, then J and g are said to be *compatible* when

$$g(JX, JY) = g(X, Y), \qquad \forall\, X, Y \in \mathfrak{X}(M), \tag{1}$$

where $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on M. This compatibility condition allows to define the *fundamental* 2*-form* of g as

$$F(X, Y) = g(JX, Y), \qquad \forall\, X, Y \in \mathfrak{X}(M). \tag{2}$$

Note that this 2-form is in one-to-one correspondence with g. Furthermore, it satisfies $F^n \neq 0$.

When g is positive-definite the manifold (M, J, g) is said to be Hermitian. Imposing different conditions to F several special metrics arise, such as the strong Kähler with torsion (SKT) or the balanced metrics. When $dF = 0$, i.e. the metric is Kähler, it is well-known that the compact manifold M satisfies strong topological conditions. If we no longer require the positive definiteness of the metric g but preserve the condition $dF = 0$, the metric g is called *pseudo-Kähler*. The pair (J, g) is also known as a pseudo-Kähler structure on M. Notice that F provides a symplectic form on M, both in the indefinite and in the positive-definite case.

In this work, we are interested in compact complex manifolds (M, J) with $\dim_{\mathbb{R}} M = 2n$ endowed with a pseudo-Kähler metric g. It is worth to observe [1] that the compatibility condition (1) is then equivalent to J being parallel with respect to the Levi-Civita connection of g, i.e., $\nabla J = 0$. Moreover, the signature of g is precisely $(2k, 2n - 2k)$, where $k = n$ corresponds to the Kähler case. There are many compact pseudo-Kähler manifolds with no Kähler metrics, the simplest example being the compact complex surface known as the Kodaira-Thurston manifold [28].

Pseudo-Kähler structures have been broadly studied in the literature, both independently and in relation with other geometric structures (see for instance [5], [7], [9], [13], [14], [26], [30] or [31], among others). However, the stability of these structures under small holomorphic deformations of the complex manifold has only recently been analized. In [21], it was shown that if X is a compact pseudo-Kähler manifold, then a sufficiently small deformation of X does not necessarily admit a pseudo-Kähler metric. This contrasts with the positive-definite case, already studied in 1960 by Kodaira and Spencer [18]. Trying to understand the differences between these two situations, the authors find in [21] some conditions that guarantee the stability of pseudo-Kähler structures. In this paper we continue working in this line, providing a new condition under which the existence of pseudo-Kähler metrics is, eventually, also preserved under small deformations of the complex structure.

This paper is organized as follows.

In Section 2 we recall the basic notions about complex manifolds and present some previous results about the stability of pseudo-Kähler structures.

The notion of cohomologically pseudo-Kähler manifold is introduced in Section 3, as a natural condition satisfied by any compact manifold with a pseudo-Kähler structure. Then, we prove that if a small deformation of a compact pseudo-Kähler manifold is C^∞-full, then the resulting deformed manifolds are cohomologically pseudo-Kähler for sufficiently small values of the deformation parameter (see Theorem 1 for the precise formulation of this stability result). The C^∞-fullness condition was defined and studied by Li and Zhang in [22] in the context of cohomological decomposition of symplectic manifolds.

In Section 4 we focus on solvmanifolds and show that, under a certain assumption on these spaces, the cohomologically pseudo-Kähler condition guarantees the existence of a pseudo-Kähler metric. We finally provide some examples to illustrate our result.

2 Bott-Chern cohomology and pseudo-Kähler stability

In this section we introduce the basic notions that will be used throughout the paper and recall some previous results. This will also serve us to fix the notation.

Let J be an almost complex structure on a $2n$-dimensional differentiable manifold M, namely, $J \in \mathrm{End}\big(\mathfrak{X}(M)\big)$ such that $J^2 = -id$. Let us observe that J can be equivalently defined on the space $\Omega^1(M)$ of smooth 1-forms on M, taking

$$(J\alpha)(V) = \alpha(JV),$$

for every $\alpha \in \Omega^1(M)$ and $V \in \mathfrak{X}(M)$. Extending J by $\mathbb{C}$-linearity to the complexified space $\Omega^1_{\mathbb{C}}(M) = \Omega^1(M) \otimes \mathbb{C}$, one obtains a decomposition

$$\Omega^1_{\mathbb{C}}(M) = \Omega^{1,0}(M,J) \oplus \Omega^{0,1}(M,J),$$

where $\Omega^{1,0}(M,J) = \big\{\alpha \in \Omega^1_{\mathbb{C}}(M) \mid J\alpha = i\,\alpha\big\}$ and $\Omega^{0,1}(M,J) = \{\alpha \in \Omega^1_{\mathbb{C}}(M) \mid J\alpha = -i\,\alpha\}$. Note that these spaces are conjugates of each other and that the previous decomposition depends on J. If one similarly extends J to the complexified space of k-forms, the following bigraduation arises

$$\Omega^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M,J).$$

Let us recall that the space $\Omega^*(M)$ of smooth forms on M is a differential graded algebra endowed with a product $\wedge$ and a differential d. Consequently, also is $\Omega^*_{\mathbb{C}}(M)$. The almost complex structure J defined on M is integrable, that is, (M,J) is a complex manifold, if and only if $d : \Omega^{p,q}(M,J) \to \Omega^{p+q+1}_{\mathbb{C}}(M)$ decomposes as

$$d = \partial + \bar\partial,$$

where $\partial : \Omega^{p,q}(M,J) \to \Omega^{p+1,q}(M,J)$ and $\bar\partial : \Omega^{p,q}(M,J) \to \Omega^{p,q+1}(M,J)$. Since $d^2 = 0$, one easily obtains the following equalities:

$$\partial^2 = 0, \qquad \bar\partial\partial = -\partial\bar\partial, \qquad \bar\partial^2 = 0.$$

These differential operators allow to define some specific cohomologies of complex manifolds such as Dolbeault and Bott-Chern, whose cohomology groups are respectively given by:

$$H^{\bullet,\bullet}_{\bar\partial}(M,J) := \frac{\ker\,\bar\partial}{\mathrm{im}\,\bar\partial} \qquad \text{and} \qquad H^{\bullet,\bullet}_{\mathrm{BC}}(M,J) := \frac{\ker\,\partial \cap \ker\,\bar\partial}{\mathrm{im}\,\partial\bar\partial}.$$

A *holomorphic family* of compact complex manifolds is a proper holomorphic submersion $\pi\colon \mathcal{X} \longrightarrow B$ between two complex manifolds $\mathcal{X}$ and B [19]. This implies that the fibres $X_t = \pi^{-1}(t)$ are compact complex manifolds of the same dimension. In fact, a classical result of Ehresmann [11] states that any such family is locally $\mathcal{C}^\infty$ trivial (globally, if B is contractible), which means that the $\mathcal{C}^\infty$ manifold underlying each fibre X_t is the same for every $t \in B$. Consequently, any holomorphic family $\{X_t\}_{t\in B}$ can be equivalently seen as a fixed differentiable manifold M endowed with a holomorphic family of complex structures $\{J_t\}_{t\in B}$, i.e., for each $t \in B$ one has $X_t = (M, J_t)$.

In general, one can consider that B is an open ball centered at the origin 0 in $\mathbb{C}^m$. We will also deal with differentiable families $\{J_t\}_{t\in B}$ of complex structures, in which case B will be an open ball around $0 \in \mathbb{R}^m$.

A property $\mathcal{P}$ is said to be *stable* if the following condition holds: given any holomorphic (or differentiable) family $\{X_t\}_{t\in B}$ of compact complex manifolds and any $t_0 \in B$, if the fibre $X = X_{t_0}$ satisfies $\mathcal{P}$, then X_t also satisfies $\mathcal{P}$ for every $t \in B$ sufficiently close to t_0. In our case, the property $\mathcal{P}$ is the existence of a pseudo-Kähler metric on $X = (M, J)$, namely, a pseudo-Riemannian metric g compatible with J in the sense of (1) whose fundamental 2-form F defined by (2) is closed. Note that F has type $(1, 1)$ with respect to the bigraduation induced by J.

In [21, Proposition 2.1], it is shown that the pseudo-Kähler property is in general not stable under holomorphic deformations. However, the authors prove that the existence of pseudo-Kähler metrics is guaranteed under additional assumptions:

Proposition 1. *[21, Proposition 2.5] Let $\{X_t\}_{t\in(-\varepsilon,\varepsilon)}$ be a differentiable family of deformations of a compact pseudo-Kähler manifold $X = X_0$, where $\varepsilon > 0$. If the upper-semi-continuous function $t \mapsto \dim H^{1,1}_{BC}(X_t)$ is constant, then X_t admits a pseudo-Kähler metric for every t close enough to* 0.

This result explains why the pseudo-Kähler property is unstable in complex dimension ≥ 3. However, one can prove that pseudo-Kähler compact complex surfaces are stable, as an application of the previous proposition and a result of Teleman [27] (see [21, Theorem 2.13]).

3 Cohomological decomposition and stability

In this section we prove a stability result for compact pseudo-Kähler manifolds under a condition based on the cohomological decomposition property.

Given a compact almost-complex manifold (M, J), we consider the spaces

$$\Omega^{\pm}_J(M) = \{\alpha \in \Omega^2(M) \mid J\alpha = \pm\alpha\}$$

defined by the real 2-forms α on M which are J-invariant, resp. J-anti-invariant. As already observed in [22], the relation between these two spaces and the bigraduation induced by J is given by

$$\Omega^{+}_J(M) = \Omega^{1,1}(M,J)_{\mathbb{R}} := \Omega^{1,1}(M,J) \cap \Omega^2(M),$$
$$\Omega^{-}_J(M) = \Omega^{(2,0),(0,2)}(M,J)_{\mathbb{R}} := \big(\Omega^{2,0}(M,J) \oplus \Omega^{0,2}(M,J)\big) \cap \Omega^2(M).$$

Let us also denote $\mathcal{Z}^{+}_J(M) \subset \Omega^{+}_J(M)$, resp. $\mathcal{Z}^{-}_J(M) \subset \Omega^{-}_J(M)$, the space of closed real 2-forms that are J-invariant, resp. J-anti-invariant. Li and Zhang considered in [22] the subspaces $H^+_J(M)$ and $H^-_J(M)$ of the second de Rham cohomology group $H^2_{\mathrm{dR}}(M;\mathbb{R})$ given by

$$H^{\pm}_J(M) := \left\{ \boldsymbol{a} = [\alpha] \in H^2_{\mathrm{dR}}(M;\mathbb{R}) \mid \alpha \in \mathcal{Z}^{\pm}_J(M) \right\}.$$

Recall that the almost-complex structure J is said to be $\mathcal{C}^\infty$*-pure* if

$$H^+_J(M) \cap H^-_J(M) = \{0\}, \tag{3}$$

and it is called $\mathcal{C}^\infty$*-full* if

$$H^+_J(M) + H^-_J(M) = H^2_{\mathrm{dR}}(M;\mathbb{R}). \tag{4}$$

If both properties are satisfied, then we have the decomposition

$$H^2_{\mathrm{dR}}(M;\mathbb{R}) = H^+_J(M) \oplus H^-_J(M), \tag{5}$$

and the almost-complex structure J is called $\mathcal{C}^\infty$*-pure-and-full*. Drăghici, Li and Zhang proved in [10] that every compact 4-dimensional almost-complex manifold is $\mathcal{C}^\infty$-pure-and-full.

Here we focus on the integrable case, that is, $X = (M, J)$ is a compact complex manifold, and we will denote by $H^{\pm}(X)$ the subgroups $H^{\pm}_J(M)$ if there is no confusion from the context. If X is a compact Kähler manifold (or, more generally, if X satisfies the $\partial\bar\partial$-Lemma [8]) then the $\mathcal{C}^\infty$-pure-and-full property is satisfied (see [3, 10, 22]). Indeed, in such case the subgroups $H^{\pm}(X)$ are precisely the (real) Dolbeault cohomology groups, i.e.,

$$\begin{aligned} H^+(X) &= H^{1,1}_{\bar\partial}(X) \cap H^2_{\mathrm{dR}}(M;\mathbb{R}), \\ H^-(X) &= \left(H^{2,0}_{\bar\partial}(X) \oplus H^{0,2}_{\bar\partial}(X)\right) \cap H^2_{\mathrm{dR}}(M;\mathbb{R}). \end{aligned}$$

In addition, since the $\partial\bar\partial$-property is stable by small deformations of the complex structure J [4, 29], any sufficiently small holomorphic deformation X_t of a compact $\partial\bar\partial$-manifold $X = (M, J)$ is also $\mathcal{C}^\infty$-pure-and-full.

However, in complex dimension greater than or equal to 3, there are small deformations of $\mathcal{C}^\infty$-pure-and-full manifolds that are neither pure nor full [3]. Furthermore, there exist small deformations of $\mathcal{C}^\infty$-pure-and-full manifolds along which the $\mathcal{C}^\infty$-full property is lost while the $\mathcal{C}^\infty$-pure property is preserved, and vice versa (see [20, Propositions 3.1 and 3.3]).

Thus, for arbitrary dimensions it seems a difficult problem to understand which compact complex manifolds satisfy the "Hodge type" decomposition (5).

Let X be a compact pseudo-Kähler manifold of complex dimension n with pseudo-Kähler 2-form F. Since the real form F is closed and J-invariant, it defines a class $[F] \in H^+(X)$. Clearly, the de Rham cohomology classes $[F]^k$ are non-zero for every $1 \leq k \leq n$ because F is non-degenerate. In particular:

Lemma 1. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. If X admits a pseudo-Kähler metric, then $h^+(X) := \dim H^+(X) \geq 1$. Moreover, there exists a cohomology class $\boldsymbol{a} \in H^+(X)$ satisfying $\boldsymbol{a}^n \neq 0$.*

We introduce the following notion, similarly to the symplectic case:

Definition 1. *A compact complex manifold X, of complex dimension n, is said to be* cohomologically pseudo-Kähler *if there exists a class $\boldsymbol{a} \in H^+(X)$ such that $\boldsymbol{a}^n \neq 0$.*

In the following result we will prove that any sufficiently small deformation of a compact pseudo-Kähler manifold is cohomologically pseudo-Kähler if the deformation satisfies the $\mathcal{C}^\infty$-full property (4), together with an additional condition that we next explain.

Let X be a $\mathcal{C}^\infty$-full compact complex manifold, and let F be any closed real 2-form on X. We define

$$\Delta(X,F) = \{\gamma \in \Omega^1(X) \mid d\gamma = F - (\alpha+\beta),\ \text{for some } \alpha \in \mathcal{Z}^+(X),\ \beta \in \mathcal{Z}^-(X)\}.$$

Note that $\Delta(X,F)$ is non-empty due to the $\mathcal{C}^\infty$-full property. Indeed,

$$[F] \in H^2_{\mathrm{dR}}(X;\mathbb{R}) = H^+(X) + H^-(X),$$

so there exists a cohomology class $\boldsymbol{a} \in H^+(X)$ and a cohomology class $\boldsymbol{b} \in H^-(X)$ such that $[F] = \boldsymbol{a} + \boldsymbol{b}$. This implies the existence of a representative α in the class $\boldsymbol{a}$ and a representative β in the class $\boldsymbol{b}$ satisfying

$$F = \alpha + \beta + d\gamma,$$

for some real 1-form γ on X, hence $\gamma \in \Delta(X,F)$.

Now, let $\{X_t\}_{t\in(-\varepsilon,\varepsilon)}$ be a differentiable family of compact complex manifolds such that $X_0 = X$. If X_t is $\mathcal{C}^\infty$-full, then one has $\Delta(X_t,F) \neq \emptyset$. Thus, there is a real 1-form γ_t such that $d\gamma_t = F - (\alpha_t + \beta_t)$ for some $\alpha_t \in \mathcal{Z}^+(X_t)$ and $\beta_t \in \mathcal{Z}^-(X_t)$. Recall that $\mathcal{Z}^\pm(X_t)$ depend on the complex structure of X_t. Moreover, the exterior derivative d decomposes as $d = \partial_t + \bar\partial_t$ on the complex manifold X_t.

Theorem 1. *Let X be a compact complex manifold endowed with a pseudo-Kähler form F, and let $\{X_t\}_{t\in(-\varepsilon,\varepsilon)}$ be a differentiable family of deformations of $X_0 = X$, where $\varepsilon > 0$. If the complex manifold X_t is $\mathcal{C}^\infty$-full and there exists $\gamma_t \in \Delta(X_t,F)$ satisfying $\partial_t\bar\partial_t\gamma_t = 0$, for every $t \neq 0$, then X_t is cohomologically pseudo-Kähler for any sufficiently small t.*

Proof. Let M be the differentiable manifold underlying the family $\{X_t\}_{t\in(-\varepsilon,\varepsilon)}$. Denote by $\pi_t^{p,q}\colon \Omega^*_{\mathbb{C}}(M) \to \Omega^{p,q}(X_t)$ the projection of the space of complexified forms on M onto the space of (p,q)-forms on X_t. Let F be a pseudo-Kähler metric on X. Viewing F in $\Omega^2_{\mathbb{C}}(M)$, one can write

$$F = F_t^+ + F_t^- \ \in \Omega^+(X_t) \oplus \Omega^-(X_t), \tag{6}$$

where $F_t^+ = \pi_t^{1,1}(F) \in \Omega^{1,1}(X_t)_{\mathbb{R}}$ and $F_t^- = \pi_t^{2,0}(F) + \pi_t^{0,2}(F) \in \Omega^{(2,0),(0,2)}(X_t)_{\mathbb{R}}$ since F is real. Moreover, $F = F_0^+$ because F has bidegree $(1,1)$ with respect to J_0 for being a pseudo-Kähler metric on X. The above decomposition is unique and the family $\{F_t^+\}_t$ is smooth in t; however, F_t^+ may not be a closed form.

By the hypothesis, for every $t \neq 0$ we can choose $\gamma_t \in \Delta(X_t, F)$ satisfying $\partial_t\bar{\partial}_t\gamma_t = 0$. Thus, there exist $\alpha_t \in \mathcal{Z}^+(X_t)$ and $\beta_t \in \mathcal{Z}^-(X_t)$ so that the 2-form F can be written as

$$F = \alpha_t + \beta_t + d\gamma_t, \tag{7}$$

with $\partial_t\bar{\partial}_t\gamma_t = 0$, for every $t \neq 0$. Notice that α_t is a (real) closed 2-form of bidegree $(1,1)$ with respect to the complex structure of X_t, but the family $\{\alpha_t\}_t$ may not be smooth in t.

Next we show that there exists a form τ_t of total degree $2n-1$ on X_t, where n is the complex dimension of X_t, such that

$$(F_t^+)^n = \alpha_t^n + d\tau_t, \qquad \text{for every } t \neq 0. \tag{8}$$

To prove it, we first notice that (6) and (7) give

$$F_t^+ = \alpha_t + \bar{\partial}_t\eta_t + \partial_t\overline{\eta_t}, \qquad t \neq 0,$$

where we use that the real 1-form γ_t can be written as $\gamma_t = \eta_t + \overline{\eta_t}$, for a $(1,0)$-form η_t on X_t. Since $\partial_t\bar{\partial}_t\gamma_t = 0$, we also have that $\partial_t\bar{\partial}_t\eta_t = 0 = \partial_t\bar{\partial}_t\overline{\eta_t}$.

The equality (8) comes directly from the following equalities:

$$\begin{aligned}
(F_t^+)^n &= \left(\alpha_t + \bar{\partial}_t\eta_t + \partial_t\overline{\eta_t}\right)^n \\
&= \sum_{r=0}^{n}\sum_{s=0}^{n-r}\binom{n}{r}\binom{n-r}{s}\alpha_t^{n-r-s}\wedge\left(\bar{\partial}_t\eta_t\right)^s\wedge\left(\partial_t\overline{\eta_t}\right)^r \\
&= \alpha_t^n + \sum_{s=1}^{n-r}\binom{n}{s}\alpha_t^{n-s}\wedge\left(\bar{\partial}_t\eta_t\right)^s \\
&\quad + \sum_{r=1}^{n}\sum_{s=0}^{n-r}\binom{n}{r}\binom{n-r}{s}\alpha_t^{n-r-s}\wedge\left(\bar{\partial}_t\eta_t\right)^s\wedge\left(\partial_t\overline{\eta_t}\right)^r \\
&= \alpha_t^n + \sum_{s=1}^{n-r}\binom{n}{s}\alpha_t^{n-s}\wedge\left(d\eta_t\right)^s \\
&\quad + \sum_{r=1}^{n}\sum_{s=0}^{n-r}\binom{n}{r}\binom{n-r}{s}\alpha_t^{n-r-s}\wedge\left(\bar{\partial}_t\eta_t\right)^s\wedge\left(d\overline{\eta_t}\right)^r \\
&= \alpha_t^n + d\left(\sum_{s=1}^{n-r}\binom{n}{s}\alpha_t^{n-s}\wedge\eta_t\wedge\left(d\eta_t\right)^{s-1}\right) \\
&\quad + d\left(\sum_{r=1}^{n}\sum_{s=0}^{n-r}\binom{n}{r}\binom{n-r}{s}\alpha_t^{n-r-s}\wedge\left(\bar{\partial}_t\eta_t\right)^s\wedge\overline{\eta_t}\wedge\left(d\overline{\eta_t}\right)^{r-1}\right).
\end{aligned}$$

In the fourth equality we are substituting $(\bar\partial_t\eta_t)^s$ by $(d\eta_t)^s$ and $(\partial_t\overline{\eta_t})^r$ by $(d\overline{\eta_t})^r$, as each summand has top bidegree (n,n). The last equality holds due to the hypothesis and the closedness of α_t.

The forms $(F_t^+)^n$ and α_t^n in (8) are both real $2n$-forms on the manifold M. By conjugation we have that $(F_t^+)^n = \alpha_t^n + d\overline{\tau_t}$, so we get

$$(F_t^+)^n = \alpha_t^n + d\rho_t, \qquad t \neq 0,$$

where $\rho_t = (\tau_t + \overline{\tau_t})/2$ is a real form on M of degree $2n-1$. By Stokes' theorem one has

$$\int_M (F_t^+)^n = \int_M \alpha_t^n, \qquad t \neq 0.$$

Since the left-hand-side of this equality is smooth in $t \in (-\varepsilon, \varepsilon)$ and non-zero for $t = 0$ (because $F_0 = F$ is the given pseudo-Kähler form), we have that $\int_M \alpha_t^n$ is non-zero for any t close enough to 0. This implies that α_t^n cannot be exact, so the cohomology class $\boldsymbol{a}_t = [\alpha_t]$ satisfies $\boldsymbol{a}_t^n = [\alpha_t^n] \neq 0$.

In conclusion, since α_t is J_t-invariant, one has $\boldsymbol{a}_t \in H^+(X_t)$ and the compact complex manifold $X_t = (M, J_t)$ is cohomologically pseudo-Kähler for any sufficiently small t. □

The next example provides a compact pseudo-Kähler manifold X_0 and a deformation X_t of it with no pseudo-Kähler structure for $t \neq 0$. The behaviour of this deformation was analized in [21] in relation to the Bott-Chern cohomology (see Proposition 1). We here show that the non stability of the pseudo-Kähler property can also be explained by the lost of the $\mathcal{C}^\infty$-full condition, in accord to Theorem 1.

Example 1. For each $t \in B = \{t \in \mathbb{C} \mid |t| < 1\}$, let X_t be the nilmanifold endowed with an invariant complex structure defined by the equations

$$d\omega_t^1 = d\omega_t^2 = 0, \qquad d\omega_t^3 = \omega_t^{1\bar 2} - t\,\omega_t^{2\bar 1}. \tag{9}$$

Here $\{\omega_t^1, \omega_t^2, \omega_t^3\}$ is a complex basis of invariant (1,0)-forms on X_t, and we write $\omega_t^{1\bar 2}$ instead of $\omega_t^1 \wedge \overline{\omega}_t^2$, and so on.

The invariant 2-form $F = i\,\omega_0^{1\bar 1} + \omega_0^{2\bar 3} + \omega_0^{\bar 2 3}$ defines a pseudo-Kähler structure on the manifold X_0, and $\{X_t\}_{t\in B}$ is a holomorphic deformation of X_0. In the proof of [21, Proposition 2.1], it is shown that X_t admits pseudo-Kähler metrics if and only if $t = 0$.

Due to the results in [20], the subspaces $H^\pm(X_t)$ can be directly computed from the structure equations (9) (see Section 4 for more details). Indeed, one can show that X_t is $\mathcal{C}^\infty$-full only for $t = 0$. More precisely, for the compact complex manifold $X = X_0$ we have the following subspaces:

$$H^+(X_0) = \langle\, [\, i\omega^{1\bar 1}\,],\, [\, i\omega^{2\bar 2}\,],\, [\,\omega^{2\bar 3} - \omega^{3\bar 2}\,],\, [\, i\,(\omega^{2\bar 3} + \omega^{3\bar 2})\,]\,\rangle,$$

and

$$H^-(X_0) = \langle\, [\,\omega^{12} + \omega^{\bar 1\bar 2}\,],\, [\, i\,(\omega^{12} - \omega^{\bar 1\bar 2})\,],\, [\,\omega^{13} + \omega^{\bar 1\bar 3}\,],\, [\, i\,(\omega^{13} - \omega^{\bar 1\bar 3})\,]\,\rangle.$$

The second Betti number of the manifold is $b_2 = 8$ and $H^+(X_0) \cap H^-(X_0) = \{0\}$, so the complex manifold X_0 is $\mathcal{C}^\infty$-pure-and-full. However, for any $t \in B - \{0\}$, the subspaces $H^\pm(X_t)$ are

$$H^+(X_t) = \langle\, [\, i\omega_t^{1\bar{1}}\,],\, [\, i\omega_t^{2\bar{2}}\,]\,\rangle, \qquad H^-(X_t) = \langle\, [\, \omega_t^{12} + \omega_t^{\bar{1}\bar{2}}\,],\, [\, i\,(\omega_t^{12} - \omega_t^{\bar{1}\bar{2}})\,]\,\rangle,$$

so the compact complex manifold X_t is not $\mathcal{C}^\infty$-full for $t \neq 0$.

4 Cohomologically pseudo-Kähler solvmanifolds

In this section we consider pseudo-Kähler structures on certain solvmanifolds $M = \Gamma\backslash G$ and study the behaviour of the pseudo-Kähler property along small deformations on such spaces. We recall that a solvmanifold is a compact quotient of a connected and simply connected solvable Lie group G by a discrete subgroup Γ of maximal rank (lattice).

Any $2n$-dimensional compact symplectic manifold M is cohomologically symplectic, i.e. it has a class $\boldsymbol{a} \in H^2_{\mathrm{dR}}(M; \mathbb{R})$ such that $\boldsymbol{a}^n \neq 0$. However, the converse is in general not true; for instance $\mathbb{C}P^2 \# \mathbb{C}P^2$ is cohomologically symplectic but not symplectic. This problem has been studied by several authors, as Geiges for $\mathbb{T}^2$-bundles over $\mathbb{T}^2$ [12], Ishida for real Bott manifolds [16], or Kasuya for solvmanifolds [17], among others. We next show that for compact solvmanifolds $M = \Gamma\backslash G$ satisfying the Mostow condition and endowed with an invariant complex structure, the cohomologically pseudo-Kähler condition implies the existence of a pseudo-Kähler metric.

Let $\mathrm{Ad}_G(G)$ and $\mathrm{Ad}_G(\Gamma)$ denote the subgroups of $\mathrm{GL}(\mathfrak{g})$ generated by e^{ad_Z} for all Z in the respective Lie algebras of G and Γ. The solvmanifold $M = \Gamma\backslash G$ is said to satisfy the *Mostow condition* if the real algebraic closures $\mathcal{A}(\mathrm{Ad}_G(G))$ and $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ are isomorphic. In that case, the de Rham cohomology $H^*_{\mathrm{dR}}(M; \mathbb{R})$ of the solvmanifold can be computed by means of the Chevalley-Eilenberg cohomology $H^*(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G [23], i.e.

$$H^k_{\mathrm{dR}}(M; \mathbb{R}) \cong H^k(\mathfrak{g}), \quad \text{for any } k. \tag{10}$$

Recall that for completely solvable Lie groups G, i.e. those for which the linear operators $\mathrm{ad}_Z \colon \mathfrak{g} \longrightarrow \mathfrak{g}$ have only real eigenvalues for every $Z \in \mathfrak{g}$, any lattice Γ of G satisfies the Mostow condition. In particular, the Mostow condition holds for nilmanifolds, as their Lie group G is nilpotent and thus ad_Z has only zero eigenvalues for every $Z \in \mathfrak{g}$. The isomorphism (10) for nilmanifolds and for completely solvable solvmanifolds was first proved respectively by Nomizu in [25] and by Hattori in [15].

Proposition 2. *Let $X = (M, J)$ be a compact complex manifold, where $M = \Gamma\backslash G$ is a solvmanifold satisfying the Mostow condition and J is an invariant complex structure on M. If X is cohomologically pseudo-Kähler, then there exists a pseudo-Kähler metric on X.*

Proof. Let us first recall that for the compact quotient manifold $M = \Gamma\backslash G$, the natural map $\iota\colon H^k(\mathfrak{g}) \longrightarrow H^k_{\mathrm{dR}}(M;\mathbb{R})$ is injective. Since the Mostow condition is satisfied, we have by (10) that the map ι is an isomorphism, with inverse map ι^{-1} given by *symmetrization*. This means that for any closed k-form α on $M = \Gamma\backslash G$ there is a left-invariant closed k-form $\widetilde{\alpha}$ on G which descends to a k-form on the quotient $M = \Gamma\backslash G$, also denoted by $\widetilde{\alpha}$, such that $[\alpha] = [\widetilde{\alpha}]$ in $H^k_{\mathrm{dR}}(M;\mathbb{R})$. That is, α is cohomologous to the invariant form $\widetilde{\alpha}$ obtained by the symmetrization process.

Since the complex structure J is invariant, we have $\widetilde{J\alpha} = J\widetilde{\alpha}$, which implies that the natural map

$$\iota|_{H^+_J(\mathfrak{g})}\colon H^+_J(\mathfrak{g}) \longrightarrow H^+(X) \tag{11}$$

is an isomorphism with inverse map given again by symmetrization. In fact, if α is a closed 2-form such that $J\alpha = \alpha$, then the 2-form $\widetilde{\alpha}$ is closed and satisfies $J\widetilde{\alpha} = \widetilde{J\alpha} = \widetilde{\alpha}$.

Now, since X is cohomologically pseudo-Kähler by hypothesis, there exists a class $\boldsymbol{a} \in H^+(X)$ satisfying $\boldsymbol{a}^n \neq 0$. Hence, there is a closed element $\widetilde{F} \in \bigwedge^2(\mathfrak{g}^*)$ such that $J\widetilde{F} = \widetilde{F}$ that represents the class $\boldsymbol{a}$, i.e. $[\widetilde{F}] = \boldsymbol{a}$. Since $[\widetilde{F}^n] = \boldsymbol{a}^n \neq 0$, we get that $\widetilde{F}^n \neq 0$ in $\bigwedge^{2n}(\mathfrak{g}^*)$. In conclusion, $\widetilde{F}$ defines an (invariant) pseudo-Kähler metric on X. □

In the following result we combine Theorem 1 and Proposition 2 to obtain a stability result for compact pseudo-Kähler solvmanifolds.

Proposition 3. *Let $X = (M, J, F)$ be a compact pseudo-Kähler solvmanifold $M = \Gamma\backslash G$ satisfying the Mostow condition and endowed with an invariant complex structure. Let $\{X_t = (M, J_t)\}_{t\in(-\varepsilon,\varepsilon)}$, with $\varepsilon > 0$, be a differentiable family of deformations of $X = X_0$ such that J_t is an invariant complex structure for every t. Suppose that X_t is $\mathcal{C}^\infty$-full for every $t \neq 0$ and $\partial_t\bar{\partial}_t(\mathfrak{g}^*_{\mathbb{C}}) = 0$, where $\mathfrak{g}$ denotes the Lie algebra of G. Then, X_t admits a pseudo-Kähler metric for any sufficiently small t.*

Proof. The same argument as for (11) proves that the natural map

$$\iota|_{H^-_J(\mathfrak{g})}\colon H^-_J(\mathfrak{g}) \longrightarrow H^-(X) \tag{12}$$

is an isomorphism for any invariant complex structure J, with inverse map given again by symmetrization. By the hypothesis, note that we have isomorphisms (11) and (12) for every J_t.

Let $\widetilde{F}$ be an invariant pseudo-Kähler metric on X, obtained from F as in the proof of Proposition 2. Then, $[\widetilde{F}] = [F] \in H^2_{\mathrm{dR}}(X;\mathbb{R})$ and by the $\mathcal{C}^\infty$-full property and the above isomorphisms, we get

$$[\widetilde{F}] \in H^2(\mathfrak{g}) = H^+_{J_t}(\mathfrak{g}) + H^-_{J_t}(\mathfrak{g}),$$

for every $t \neq 0$. Thus, there exist $\alpha_t \in \mathcal{Z}^+_{J_t}(\mathfrak{g})$ and $\beta_t \in \mathcal{Z}^-_{J_t}(\mathfrak{g})$ that allow us to write the 2-form $\widetilde{F}$ as $\widetilde{F} = \alpha_t + \beta_t + d\gamma_t$, for some left-invariant 1-form $\gamma_t \in \mathfrak{g}^*$. In fact, observe that the 1-forms in $\Delta(X_t, \widetilde{F})$ can be chosen to be left-invariant.

Since $\partial_t\bar\partial_t(\mathfrak{g}^*_{\mathbb{C}}) = 0$, our 1-form $\gamma_t \in \Delta(X_t, \widetilde{F})$ satisfies $\partial_t\bar\partial_t\gamma_t = 0$. Hence, Theorem 1 ensures that X_t is cohomologically pseudo-Kähler for any sufficiently small t, so X_t is pseudo-Kähler by Proposition 2. □

The following result provides a large class of examples where the condition $\partial\bar\partial(\mathfrak{g}^*_{\mathbb{C}}) = 0$ in the previous proposition holds.

Proposition 4. *Let $X = (M, J)$ be a compact complex manifold, where $M = \Gamma\backslash G$ is a solvmanifold satisfying the Mostow condition and J is an invariant complex structure on M. Denote by $\mathfrak{g}$ the Lie algebra of G. The condition $\partial\bar\partial(\mathfrak{g}^*_{\mathbb{C}}) = 0$ in Proposition 3 is satisfied in the following cases:*

(i) the complex structure J is abelian;
(ii) the Lie algebra $\mathfrak{g}$ is 2-step nilpotent;
(iii) J is a nilpotent complex structure on a 6-dimensional Lie algebra $\mathfrak{g}$.

Proof. For part *(i)*, let us simply recall that any abelian complex structure J satisfies $\partial(\mathfrak{g}^{1,0}) = 0$. In the case of *(ii)*, the result comes from the structure of 2-step nilpotent Lie algebras, as one has $\bar\partial(\mathfrak{g}^*_{\mathbb{C}}) \subset \ker\partial$. Let us then focus on *(iii)*, namely, 6-dimensional s-step nilpotent Lie algebras $\mathfrak{g}$ endowed with a nilpotent complex structure J. Note that it suffices to prove the result for $s \geq 3$. It is well-known (see for instance [20]) that for any such $(\mathfrak{g}, J)$ there is a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying equations of the form

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar1}, \quad d\omega^3 = \rho\,\omega^{12} + B\,\omega^{1\bar2} + c\,\omega^{2\bar1},$$

where $\rho \in \{0, 1\}$, $B \in \mathbb{C}$, $c \in \mathbb{R}^{\geq 0}$, with $(\rho, B, c) \neq (0, 0, 0)$. A direct calculation allows to check that $\partial\bar\partial\omega^k = 0$ for every $1 \leq k \leq 3$; hence $\partial\bar\partial(\mathfrak{g}^*_{\mathbb{C}}) = 0$. □

We note that the condition $\partial\bar\partial(\mathfrak{g}^*_{\mathbb{C}}) = 0$ is also satisfied for any complex parallelizable structure (indeed, $\bar\partial(\mathfrak{g}^{1,0}) = 0$); however, one cannot find any pseudo-Kähler structures in this setting. In more detail, let $X = \Gamma\backslash G$ be a compact holomorphically parallelizable solvmanifold, i.e. G is a simply-connected *complex* solvable Lie group and Γ is a lattice in G. Yamada proved in [30, Proposition 1.1] that if X satisfies the Mostow condition, then X has a pseudo-Kähler metric if and only if X is a complex torus, extending in this way the corresponding result for nilmanifolds proved in [6, Theorem 3.2]. As a consequence of Yamada's result and Proposition 2, one gets

Corollary 1. *Let X be a compact holomorphically parallelizable solvmanifold satisfying the Mostow condition. Then, X is cohomologically pseudo-Kähler if and only if X is a complex torus.*

In the following example we study a family of left-invariant complex structures in the conditions of Proposition 4 (iii). We apply Proposition 3 to guarantee the existence of a pseudo-Kähler structure along the deformation.

Example 2. Let us consider the differentiable family $\{X_t\}_{t\in(-1,1)}$ of compact complex nilmanifolds given in [21, Example 2.3], which are defined by the complex structure equations

$$d\omega_t^1 = 0, \quad d\omega_t^2 = \omega_t^{1\bar{1}}, \quad d\omega_t^3 = \omega_t^{12} + t\,\omega_t^{1\bar{2}}. \tag{13}$$

For $t = 0$, the manifold X_0 is pseudo-Kähler as, for instance, $F_0 = i\,\omega_0^{1\bar{3}} + i\,\omega_0^{3\bar{1}} + i\,\omega_0^{2\bar{2}}$ is a symplectic form of bidegree (1,1) with respect to the complex structure of X_0.

Note that all the complex structures above are nilpotent, so it follows from Proposition 4 *(iii)* that $\partial_t\bar{\partial}_t\gamma_t = 0$, for every invariant 1-form γ_t on X_t. We next show, following again the ideas in [20], that X_t is $\mathcal{C}^\infty$-full for any $t \in (-1,1)$.

From the structure equations (13), a direct calculation leads to

$$H^+(X_t) = \langle\, [\,\omega_t^{1\bar{2}} - \omega_t^{2\bar{1}}\,],\, [\,i\,(\omega_t^{1\bar{2}} + \omega_t^{2\bar{1}})\,],\, [\,i\,(\omega_t^{1\bar{3}} + \omega_t^{3\bar{1}}) + i\,(1+t)\,\omega_t^{2\bar{2}}\,]\,\rangle. \tag{14}$$

Let us observe that for $t = 0$ the subspace $H^-(X_0)$ is given by

$$H^-(X_0) = \langle\, [\,\omega_0^{13} + \omega_0^{\bar{1}\bar{3}}\,],\, [\,i\,(\omega_0^{13} - \omega_0^{\bar{1}\bar{3}})\,]\,\rangle,$$

whereas for $t \neq 0$ one has

$$H^-(X_t) = \langle\, [\,\omega_t^{13} + \omega_t^{\bar{1}\bar{3}}\,],\, [\,i\,(\omega_t^{13} - \omega_t^{\bar{1}\bar{3}})\,],\, [\,\omega_t^{12} + \omega_t^{\bar{1}\bar{2}}\,],\, [\,i\,(\omega_t^{12} - \omega_t^{\bar{1}\bar{2}})\,]\,\rangle.$$

Furthermore, the following cohomological relations hold:

$$[\,\omega_t^{12} + \omega_t^{\bar{1}\bar{2}}\,] = -t\,[\,(\omega_t^{1\bar{2}} - \omega_t^{2\bar{1}})\,], \qquad [\,i\,(\omega_t^{12} - \omega_t^{\bar{1}\bar{2}})\,] = -t\,[\,i\,(\omega_t^{1\bar{2}} + \omega_t^{2\bar{1}})\,].$$

Therefore, the intersection of $H^+(X_t)$ and $H^-(X_t)$ is

$$H^+(X_t) \cap H^-(X_t) = \langle\, [\,\omega_t^{12} + \omega_t^{\bar{1}\bar{2}}\,],\, [\,i\,(\omega_t^{12} - \omega_t^{\bar{1}\bar{2}})\,]\,\rangle, \quad \text{for } t \neq 0.$$

Consequently, X_t does not satisfy the $\mathcal{C}^\infty$-pure property (3) for $t \neq 0$.

Since the second Betti number of the manifolds X_t equals 5, from the previous calculations one can check that

$$H^2_{\mathrm{dR}}(X_t;\mathbb{R}) = H^+(X_t) + H^-(X_t), \qquad t \in (-1,1);$$

that is to say, X_t has the $\mathcal{C}^\infty$-full property for every $t \in (-1,1)$. Hence, by Proposition 3, the manifold X_t admits a pseudo-Kähler metric for any sufficiently small t. Indeed, one can give explicit pseudo-Kähler metrics on X_t for every $t \in (-1,1)$. It is not difficult to prove that the space of invariant pseudo-Kähler forms F_t on X_t is given by

$$F_t = i\,a\,\omega_t^{1\bar{1}} + u\,\omega_t^{1\bar{2}} - \bar{u}\,\omega_t^{2\bar{1}} + i\,b\,\big(\omega_t^{1\bar{3}} + \omega_t^{3\bar{1}} + (1+t)\,\omega_t^{2\bar{2}}\big),$$

where $a, b \in \mathbb{R}$ with $b \neq 0$ and $u \in \mathbb{C}$. Then, it follows from (14) that the cohomology classes of the pseudo-Kähler forms F_t in $H^+(X_t)$ are given by

$$[F_t] = \Re(u)\,[\,\omega_t^{1\bar{2}} - \omega_t^{2\bar{1}}\,] + \Im(u)\,[\,i\,(\omega_t^{1\bar{2}} + \omega_t^{2\bar{1}})\,] + b\,[\,i\,(\omega_t^{1\bar{3}} + \omega_t^{3\bar{1}}) + i\,(1+t)\,\omega_t^{2\bar{2}}\,],$$

where $\Re(u)$ and $\Im(u)$ are the real and imaginary parts, respectively, of the complex coefficient u.

In the following example we consider a small deformation of the holomorphically parallelizable Nakamura manifold [24]. It was already studied in [21, Proposition 2.9] in relation to the variation of the Bott-Chern number $h^{1,1}_{\mathrm{BC}}$ along the deformation. We here analyze it with respect to the cohomological decomposition property.

Example 3. The first example of a non-toral compact holomorphically parallelizable pseudo-Kähler solvmanifold $X = \Gamma\backslash G$ was constructed by Yamada in [30, Theorem 2.1] (see [14] for an extension of this result). Let G be the simply-connected complex solvable Lie group given by the semi-direct product $G = \mathbb{C} \ltimes_{\varphi} \mathbb{C}^2$, where

$$\varphi(z_1) = \begin{pmatrix} e^{z_1} & 0 \\ 0 & e^{-z_1} \end{pmatrix}, \quad z_1 \in \mathbb{C}.$$

There is a lattice $\Gamma = \Gamma_1 \ltimes_{\varphi} \Gamma_{\mathbb{C}^2}$, with $\Gamma_1 = a\,\mathbb{Z} + 2\pi i\,\mathbb{Z}$ and $\Gamma_{\mathbb{C}^2}$ a lattice in $\mathbb{C}^2$. On the compact holomorphically parallelizable solvmanifold $X = \Gamma\backslash G$ there is a pseudo-Kähler structure F defined as follows [30].

Let (z_2, z_3) be the coordinates on $\mathbb{C}^2$. The following forms

$$\omega^1 = dz_1, \quad \omega^2 = e^{-z_1}\,dz_2, \quad \omega^3 = e^{z_1}\,dz_3,$$

constitute a basis of left-invariant forms of bidegree (1,0) on the complex Lie group G. We have

$$d\omega^1 = 0, \quad d\omega^2 = -\omega^1 \wedge \omega^2, \quad d\omega^3 = \omega^1 \wedge \omega^3.$$

Now, the (1,1)-form

$$F = i\,\omega^1 \wedge \omega^{\bar{1}} + e^{2i\,\mathrm{Im}\,z_1}\,\omega^2 \wedge \omega^{\bar{3}} + e^{-2i\,\mathrm{Im}\,z_1}\,\omega^{\bar{2}} \wedge \omega^3, \tag{15}$$

where the functions $e^{2i\,\mathrm{Im}\,z_1}$ and $e^{-2i\,\mathrm{Im}\,z_1}$ are Γ-invariant, defines a pseudo-Kähler structure on X since F is closed and non-degenerate.

Angella and Kasuya studied in [2, Section 4] the small deformations X_t of X. We are interested in the deformation given in *case (1)*, which is determined by $t\,\frac{\partial}{\partial z_1} \otimes d\bar{z}_1 \in H^{0,1}(X; T^{1,0}X)$. Note that this deformation defines a holomorphic family $\{X_t\}_{t\in B}$, where $B = \{t \in \mathbb{C} \mid |t| < 1\}$ and $X_0 = X$. An important result, proved in [2, Proposition 4.2], is that in *case (1)* the compact complex manifolds X_t satisfy the $\partial\bar{\partial}$-Lemma for $t \neq 0$, thus every X_t is $\mathcal{C}^\infty$-pure-and-full.

The complex structure J_t on X_t comes from a left-invariant complex structure on G. There is a basis of (1,0)-forms given by

$$\omega^1_t = dz_1 - t\,d\bar{z}_1, \quad \omega^2_t = e^{-z_1} dz_2, \quad \omega^3_t = e^{z_1} dz_3, \tag{16}$$

whose differentials satisfy

$$\begin{cases} d\omega^1_t = 0, \\ d\omega^2_t = -\frac{1}{1-|t|^2}\,\omega^{12}_t + \frac{t}{1-|t|^2}\,\omega^{2\bar{1}}_t, \\ d\omega^3_t = \frac{1}{1-|t|^2}\,\omega^{13}_t - \frac{t}{1-|t|^2}\,\omega^{3\bar{1}}_t. \end{cases}$$

Now, we study the condition on $\Delta(X_t, F)$ in the statement of Theorem 1. From (15) and (16), a direct calculation shows that

$$F = \frac{i}{1-|t|^2}\,\omega_t^1 \wedge \omega_t^{\bar{1}} + e^{2i\,\mathrm{Im}\, z_1}\,\omega_t^2 \wedge \omega_t^{\bar{3}} + e^{-2i\,\mathrm{Im}\, z_1}\,\omega_t^{\bar{2}} \wedge \omega_t^3,$$

so the form F is J_t-invariant for any $t \in B$. In other words, the cohomology class $[F]$ in the decomposition $[F] \in H^2_{\mathrm{dR}}(X_t;\mathbb{R}) = H^+(X_t) + H^-(X_t)$ can be written (in a unique way for $t \neq 0$ because X_t is $\mathcal{C}^\infty$-pure-and-full) as $[F] = [\alpha_t]+[\beta_t]$, where $\beta_t = 0$ and $\alpha_t = F$. Moreover, we can choose $\gamma_t = 0$ in $\Delta(X_t, F)$.

Notice that in this deformation the result is stronger that in Theorem 1 because X_t is a pseudo-Kähler manifold, with the same metric F for every $t \in B$.

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