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Discrete Hölder spaces and their characterization via semigroups associated with the discrete Laplacian and kernel estimates

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Abstract. In this paper, we characterize the discrete Hölder spaces by means of the heat and Poisson semigroups associated with the discrete Laplacian. These characterizations allow us to get regularity properties of fractional powers of the discrete Laplacian and the Bessel potentials along these spaces and also in the discrete Zygmund spaces in a more direct way than using the pointwise definition of the spaces. To obtain our results, it has been crucial to get boundedness properties of the heat and Poisson kernels and their derivatives in both space and time variables. We believe that these estimates are also of independent interest.

1. Introduction

Classical Hölder spaces $C^{\alpha}(\mathbb{R}^n)$, $\alpha > 0$, $\alpha \notin \mathbb{N}$ (also denoted by $C^{k,\beta}(\mathbb{R}^n)$ or $C^{k+\beta}(\mathbb{R}^n)$, being $k+\beta = \alpha$, $k \in \mathbb{N}_0$, and $0 < \beta < 1$) are classes of smooth functions that are very important in partial differential equations, harmonic analysis and function theory. When $0 < \alpha < 1$, they are defined as the set of (bounded) functions f such that

$$|f(x+z) - f(x)| \leq C|z|^{\alpha} \quad x, z \in \mathbb{R}^n. \tag{1.1}$$

These spaces are in between of the space of bounded continuous functions, $\mathcal{C}^0(\mathbb{R}^n)$, and the one of bounded differentiable functions with bounded continuous derivative, $\mathcal{C}^1(\mathbb{R}^n)$. These spaces are usually called either Lipschitz or Hölder classes. For $\alpha = 1$, the natural space was introduced by Zygmund [39, Chapter II] and it is the set of continuous and bounded functions f such that

$$|f(x+z) + f(x-z) - 2f(x)| \leq C|z|, \quad x, z \in \mathbb{R}^n.$$

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This space is commonly known as the Zygmund's space, and we shall denote it by Z. It can be shown that if we denote by Lip the space of functions satisfying (1.1) for $\alpha = 1$, then $\mathcal{C}^1(\mathbb{R}^n) \subsetneq \text{Lip} \subsetneq Z$, see [17]. Given $\alpha > 1$, $C^\alpha(\mathbb{R}^n)$ is the set of functions such that all the derivatives of order less or equal than $[\alpha]$ are continuous and bounded and the derivatives of order $[\alpha]$ belong to $C^{\alpha-[\alpha]}(\mathbb{R}^n)$.

In the 1960s, Stein and Taibleson, see [28,33–35], characterized bounded Hölder functions via some integral estimates of the Poisson semigroup, $\{e^{-y\sqrt{-\Delta}}\}_{y>0}$, and of the Gauss semigroup, $\{e^{\tau\Delta}\}_{\tau>0}$. The advantage of this kind of results is that the semigroup descriptions allow to obtain regularity results in these spaces in a more direct way, avoiding the long, tedious and sometimes cumbersome computations that are needed when the pointwise expressions are handled. The works of Taibleson and Stein raise the question of analysing Hölder spaces adapted to different "Laplacians" and to find their pointwise and semigroup characterizations.

In [12], Lipschitz spaces adapted to the Ornstein–Uhlenbeck operator, $\mathcal{O} = -\frac{1}{2}\Delta + x\cdot\nabla$, were defined by means of its Poisson semigroup, $\{e^{-y\sqrt{\mathcal{O}}}\}_{y>0}$, and in [20] a pointwise characterization was obtained for $0 < \alpha < 1$.

In the case of Schrödinger operators, $-\Delta + V$ on $\mathbb{R}^n$, $n \geq 3$, where V satisfies a reverse Hölder inequality for some $q > n/2$, adapted Lipschitz classes were pointwise defined in [6] for $0 < \alpha < 1$. In [22], the authors characterized these spaces by means of the Poisson semigroup $\{e^{-y\sqrt{-\Delta+V}}\}_{y>0}$ and they got boundedness of fractional powers of $-\Delta + V$ in these spaces for $0 < \alpha < 1$. Recently, in [10] it was extended the pointwise and semigroup (heat and Poisson) characterizations to the range $0 < \alpha \leq 2 - n/q$. In addition, the authors used those semigroups definitions to get regularity results regarding fractional operators related to $-\Delta + V$. Moreover, in the particular case of the Hermite operator, $-\Delta + |x|^2$, in [9] the authors got, for every $\alpha > 0$, a characterization by means of the heat and Poisson semigroups of the adapted Hölder spaces defined in [31] and also of the adapted parabolic Hölder spaces introduced in [9].

Regarding the heat operator, $\partial_t - \Delta$, in [32] the parabolic Hölder spaces introduced by Krylov, see [18], were characterized by means of the Poisson semigroup $\{e^{-y\sqrt{\partial_t-\Delta}}\}_{y>0}$ and the authors used this semigroup characterization to show regularity properties for fractional powers $(\partial_t - \Delta_x)^{\pm\sigma}$.

In [4], it is proved that, in a general metric measure space (M, d, μ) where μ is doubling, if L is an operator such that the heat semigroup $\{e^{tL}\}_{t>0}$ is conservative, i.e. $e^{tL}1 = 1$, and the associated heat kernel satisfies Gaussian bounds and a Lipschitz condition in the spatial variable, then the Hölder spaces adapted to L (defined by increments) can be characterized by means of the heat semigroup, for $0 < \alpha < 1$.

In this paper, we shall deal with the discrete Hölder spaces, $C^\alpha(\mathbb{Z})$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, whose definition we are going to recall in the following lines, and also we will introduce new discrete Zygmund classes, Z_α, $\alpha \in \mathbb{N}$. Our first aim is to prove semigroup characterizations of these spaces by using the heat and Poisson semigroups associated with the discrete Laplacian, $-\Delta_d$. The heat kernel associated with the discrete

Laplacian neither has Gaussian control at zero, see [14,23], nor satisfies a Lipschitz condition, see Remark 2.5. Therefore, our results are not covered by the ones in [4] and the kernels have not the same kind of good estimates and homogeneity properties than in the works in the literature we cited above. These are the first main difficulties we have faced in this problem, and we have been able to sort them out by obtaining new estimates for the kernels and their derivatives, see Sect. 2.

For $f : \mathbb{Z} \to \mathbb{R}$, consider the discrete derivatives "from the right" and "from the left",

$$\delta_{\text{right}} f(n) := f(n) - f(n+1), \quad \delta_{\text{left}} f(n) := f(n) - f(n-1).$$

Observe that $\delta_{\text{right}}\delta_{\text{left}} f = \delta_{\text{left}}\delta_{\text{right}} f$ and this implies that every combination of these operators is not affected by the order when they are applied. For more properties of these operators see [1,2].

Now, we recall the definition of discrete Hölder spaces introduced in [8]. For $0 < \alpha < 1$,

$$C^{\alpha}(\mathbb{Z}) := \left\{ f : \mathbb{Z} \to \mathbb{R} : \sup_{n \neq m} \frac{|f(n) - f(m)|}{|n-m|^{\alpha}} < \infty \right\}.$$

In general, for $\alpha = k + \beta > 0$, where $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $0 < \beta < 1$,

$$C^{\alpha}(\mathbb{Z}) := \left\{ f : \mathbb{Z} \to \mathbb{R} : \sup_{n \neq m} \frac{|\delta^{l,s}_{\text{right/left}} f(n) - \delta^{l,s}_{\text{right/left}} f(m)|}{|n-m|^{\beta}} < \infty, \right.$$
$$\left. \forall l, s \in \mathbb{N}_0 \text{ s.t. } l + s = k \right\},$$

where $\delta^{l,s}_{\text{right/left}} := \delta^{l}_{\text{right}} \delta^{s}_{\text{left}}$ (or any other combination of these operators such that in the end we apply l times δ_{right} and s times δ_{left}), and $\delta^{0}_{\text{right}} f = \delta^{0}_{\text{left}} f = f$.

Observe that $\ell^{\infty}(\mathbb{Z})$ functions are trivially in $C^{\alpha}(\mathbb{Z})$. Furthermore, we will prove, see Lemma 3.1, that for every $f \in C^{\alpha}(\mathbb{Z})$, $\alpha > 0$, there exists $C > 0$ such that

$$|f(n)| \le C(1 + |n|^{\alpha}), \quad n \in \mathbb{Z}.$$

Moreover, when $\alpha \in \mathbb{N}$, we define the *discrete Zygmund classes*, Z_{α}, as

$$Z_1 := \left\{ f : \mathbb{Z} \to \mathbb{R} : \frac{f}{1 + |\cdot|} \in \ell^{\infty}(\mathbb{Z}) \quad \text{and} \right.$$
$$\left. \sup_{n \neq 0} \frac{\|f(\cdot + n) + f(\cdot - n) - 2f(\cdot)\|_{\infty}}{|n|} < \infty \right\}$$

and for $\alpha \in \mathbb{N} \setminus \{1\}$,

$$Z_{\alpha} := \left\{ f : \mathbb{Z} \to \mathbb{R} : \frac{f}{1 + |\cdot|^{\alpha}} \in \ell^{\infty}(\mathbb{Z}) \quad \text{and } \delta^{l,s}_{\text{right/left}} f \in Z_1, \ \ l + s = \alpha - 1. \right\}$$

Both definitions of $C^\alpha(\mathbb{Z})$ and Z_α involve pointwise estimates of the functions. Our first aim will be to get their characterizations by means of semigroups.

Let Δ_d denote the discrete Laplacian on $\mathbb{Z}$, that is, for each $f:\mathbb{Z}\to\mathbb{R}$,

$$(\Delta_d f)(n) := f(n+1) - 2f(n) + f(n-1), \quad n\in\mathbb{Z}.$$

The solution of the discrete heat problem

$$\begin{cases} \partial_t u(t,n) - \Delta_d u(t,n) = 0, & n\in\mathbb{Z},\ t\geq 0,\\ u(0,n) = f(n), & n\in\mathbb{Z}, \end{cases} \tag{1.2}$$

is given by the convolution $u(t,n) = e^{t\Delta_d}f(n) := \sum_{j\in\mathbb{Z}} G(t,n-j)f(j) = \sum_{j\in\mathbb{Z}} G(t,j)f(n-j)$, where the discrete heat kernel is

$$G(t,n) = e^{-2t}I_n(2t), \quad n\in\mathbb{Z},\ t>0,$$

being I_n the modified Bessel function of the first kind and order $n\in\mathbb{Z}$, see Sect. 2 for more details.

It seems that H. Bateman in [5] was the first author dealing with the solution of (1.2). Moreover, he studied a broad set of differential-difference equations (heat and wave equations), whose solutions are given in terms of special functions: the Bessel function J_n, the Bessel function of imaginary argument I_n, the Hermite polynomial H_n and the exponential function. In the past years, many mathematicians have been working in this discrete heat setting. For example, in [15,16], the author studies large time behaviour for $e^{t\Delta_d}f$ in $\ell^p(\mathbb{Z})$ spaces by using the semidiscrete Fourier transform. In [7], the authors do a deep harmonic analysis study of this problem. In [21], the authors study the spectrum of Δ_d on $\ell^p(\mathbb{Z})$, the associated wave problem, and holomorphic properties of $e^{z\Delta_d}f$. In [27] the author proves that the solution of (1.2) behaves asymptotically as the mean of the initial value, and in [3] the authors study large time behaviour in $\ell^p(\mathbb{Z})$ for the solutions of (1.2) with a non-homogeneous linear forcing term.

On the other hand, the solution of the discrete Poisson problem

$$\begin{cases} \partial_y^2 v(y,n) - \Delta_d v(y,n) = 0, & n\in\mathbb{Z},\ y\geq 0,\\ v(0,n) = f(n), & n\in\mathbb{Z}, \end{cases}$$

is denoted by $v(y,n) = e^{-y\sqrt{-\Delta_d}}f(n)$, $y>0$, $n\in\mathbb{Z}$. Moreover, Bochner's subordination formula (see [38, Chapter IX, Section 11]) allows us to write

$$\begin{aligned} e^{-y\sqrt{-\Delta_d}}f(n) &= \frac{y}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{y^2}{4t}}}{t^{3/2}} e^{t\Delta_d}f(n)\,\mathrm{d}t\\ &= \sum_{j\in\mathbb{Z}}\left(\frac{y}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{y^2}{4t}}}{t^{3/2}} G(t,j)\,\mathrm{d}t\right) f(n-j)\\ &=: \sum_{j\in\mathbb{Z}} P(y,j)f(n-j), \quad y>0,\ n\in\mathbb{Z}. \end{aligned}$$

To the best of our knowledge, an explicit expression of the Poisson kernel $P(y, j)$ is not known. However, by using the subordination formula and our new estimates for the heat kernel, we will be able to obtain useful bounds for $P(y, j)$, see Sect. 2.

Now, we consider the following spaces associated with the discrete Laplacian. Let $\alpha > 0$. We define

$$\Lambda_H^\alpha := \Big\{ f : \mathbb{Z} \to \mathbb{R} \,:\, \frac{f}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z}) \text{ and } \exists\, C_\alpha > 0 \text{ such that}$$
$$\|\partial_t^k e^{t\Delta_d} f\|_\infty \le C_\alpha t^{-k+\alpha/2}, k = [\alpha/2]+1,\ t > 0 \Big\}.$$
$$\Lambda_P^\alpha := \Big\{ f : \mathbb{Z} \to \mathbb{C} \,:\, \sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty \text{ and } \exists\, \widetilde{C_\alpha} > 0 \text{ such that}$$
$$\|\partial_y^l e^{-y\sqrt{-\Delta_d}} f\|_\infty \le \widetilde{C_\alpha} y^{-l+\alpha}, l = [\alpha]+1,\ y > 0 \Big\}.$$

The condition on the functions $\frac{f}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$, $\alpha > 0$, will be enough to have the heat semigroup and its derivatives well defined. However, for the case of the Poisson semigroup, we need a more restrictive condition, $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$, see Sect. 2 for the details.

Now, we present our main results. The first main theorem we prove is the characterization, for every $\alpha > 0$, of the pointwise spaces $C^\alpha(\mathbb{Z})$ and Z_α, by means of the heat and Poisson semigroups.

Theorem 1.1. (A) *Let* $\alpha > 0$.

(A.1) *If* $\alpha \notin \mathbb{N}$, *then* $\Lambda_H^\alpha = C^\alpha(\mathbb{Z})$.

(A.2) *If* $\alpha \in \mathbb{N}$, *then* $\Lambda_H^\alpha = Z_\alpha$.

(B) *Let* $f : \mathbb{Z} \to \mathbb{R}$ *such that* $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$.

(B.1) *For every* $\alpha > 0$, $\alpha \notin \mathbb{N}$,

$$f \in C^\alpha(\mathbb{Z}) \iff f \in \Lambda_H^\alpha \iff f \in \Lambda_P^\alpha.$$

(B.2) *For every* $\alpha \in \mathbb{N}$,

$$f \in Z_\alpha \iff f \in \Lambda_H^\alpha \iff f \in \Lambda_P^\alpha.$$

To prove previous theorem, some estimates about the discrete heat and Poisson kernels and their derivatives are crucial (see Lemmas 2.3, 2.4, 2.6, 2.8, 2.9). These results complement, extend and improve some of the ones obtained in [3,15,16]. We believe that ours results are also of independent interest because we give general pointwise and ℓ^1-estimates for the difference and derivatives of any order of the heat and Poisson discrete kernels.

Once the semigroup characterization is obtained, we have been able to get regularity results for fractional operators related to Δ_d, such as Bessel potentials, $(I - \Delta_d)^{-\beta}$, $\beta > 0$, and the fractional powers $(-\Delta_d)^{\pm\beta}$. For the definitions of these operators, see Sect. 4.

Theorem 1.2. *Let* $\alpha, \beta > 0$.

(i) *If* $f \in \Lambda_H^{\alpha}$, *then* $(I - \Delta_d)^{-\beta/2} f \in \Lambda_H^{\alpha+\beta}$.

(ii) *If* $f \in \ell^{\infty}(\mathbb{Z})$, *then* $(I - \Delta_d)^{-\beta/2} f \in \Lambda_H^{\beta}$.

To define the fractional powers of Δ_d, it is necessary to define auxiliary spaces of sequences ℓ_γ, $\gamma > -1/2$, $\gamma \neq 0$,

$$\ell_\gamma := \left\{ u : \mathbb{Z} \to \mathbb{R} : \sum_{m \in \mathbb{Z}} \frac{|u(m)|}{(1+|m|)^{1+2\gamma}} < \infty \right\}.$$

These spaces are the discrete analogue of the spaces needed in the case of the Laplacian in $\mathbb{R}^n$, see [26], and they were introduced in [8] for $\gamma \in (-1/2, 1)$, $\gamma \neq 0$ and, for sequences $f \in \ell_\gamma$, the authors got a pointwise convolution expression for $(-\Delta_d)^\gamma f$. We consider these spaces ℓ_γ for any $\gamma > -1/2$, $\gamma \neq 0$, and we are able to complete the results in [8] and extend the pointwise expression of $(-\Delta_d)^\gamma f$, for $\gamma \geq 1$, see Lemma 4.1 and Remark 4.2.

The following theorems were proved in [8, Theorems 1.5 and 1.6] for nonzero powers between $-1/2$ and 1 in the discrete Hölder classes depending on $\alpha > 0$ (observe that, when $\alpha \in \mathbb{N}$, the spaces considered in [8] are not the discrete Zygmund classes). In [8], the authors obtained their results by using the pointwise definition of the fractional powers of the Laplacian. Our results cover all nonzero powers bigger than $-1/2$, and our proofs will be more direct and systematic.

Theorem 1.3. (Schauder estimates) *Let* $\alpha > 0$ *and* $0 < \beta < 1/2$.

(i) *If* $f \in \Lambda_H^{\alpha} \cap \ell_{-\beta}$, *then* $(-\Delta_d)^{-\beta} f \in \Lambda_H^{\alpha+2\beta}$.

(ii) *If* $f \in \ell^{\infty}(\mathbb{Z}) \cap \ell_{-\beta}$, *then* $(-\Delta_d)^{-\beta} f \in \Lambda_H^{2\beta}$.

Since the operator $-\Delta_d$ consists of second order differences, $\underbrace{(-\Delta_d) \circ \cdots \circ (-\Delta_d)}_{m \text{ times}} f$, $m \in \mathbb{N}$, is well defined for any $f : \mathbb{Z} \to \mathbb{R}$. Thus, the fractional powers of $-\Delta$ of order $\beta > 1$ can be defined as

$$\underbrace{(-\Delta_d) \circ \cdots \circ (-\Delta_d)}_{[\beta] \text{ times}} ((-\Delta_d)^{\beta-[\beta]} f), \quad \text{for } f \in \ell_{\beta-[\beta]},$$

where $(-\Delta_d)^0 f = f$. However, we will use the definition of $(-\Delta_d)^\beta f$, $\beta > 1$, given by formula (4.1) because it is valid for $f \in \ell_\beta$, which is a larger class of functions than $\ell_{\beta-[\beta]}$. In fact, for $\beta \in \mathbb{N}$ and $f \in \ell_\beta$, we have that $(-\Delta_d)^\beta f = \underbrace{(-\Delta_d) \circ \cdots \circ (-\Delta_d)}_{\beta \text{ times}} f$, see Remark 4.2.

Theorem 1.4. (Hölder estimates) *Let* $\alpha, \beta > 0$ *such that* $0 < 2\beta < \alpha$.

(i) *If* $f \in \Lambda_H^{\alpha} \cap \ell_\beta$, *then* $(-\Delta_d)^\beta f \in \Lambda_H^{\alpha-2\beta}$.

(ii) *If* $f \in \Lambda_H^{\alpha}$ *and* $\beta \in \mathbb{N}$, *then* $\underbrace{(-\Delta_d) \circ \cdots \circ (-\Delta_d)}_{\beta \text{ times}} f \in \Lambda_H^{\alpha-2\beta}$.

Discrete Hölder classes can be defined in the mesh of step $h > 0$, $\mathbb{Z}_h := \{nh : n \in \mathbb{Z}\}$, see [8]. All our results also hold in this setting, and the proofs can be obtained following step-by-step procedures that we are presenting in this paper. However, for simplicity we have written the results when $h = 1$. Moreover, doing a tedious work component to component, one can repeat the results in the multidimensional case $\mathbb{Z}_h^N$.

The paper is organized as follows. In Sect. 2, we prove all the results concerning pointwise and norm estimates of the discrete heat and Poisson kernels and semigroups. In Sect. 3, we prove Theorem 1.1 and all the properties related to these spaces. Finally, in Sect. 4 we prove the results regarding the applications and Theorems 1.2, 1.3 and 1.4.

Throughout this article, C and c always denote positive constants that can change in each occurrence.

2. Discrete Gaussian and Poisson semigroups

2.1. Bessel functions

2.1.1. Some known results

Along the paper, next estimates for the Euler's gamma function will be applied in some results. Recall that for every $\alpha, z \in \mathbb{C}$,

$$\frac{\Gamma(z+\alpha)}{\Gamma(z)} = z^\alpha (1 + \frac{\alpha(\alpha+1)}{2z} + O(|z|^{-2})), \quad |z| \to \infty,$$

whenever $z \neq 0, -1, -2, \dots$ and $z \neq -\alpha, -\alpha - 1, \dots$, see [36, Eq. (1)]. In particular,

$$\frac{\Gamma(z+\alpha)}{\Gamma(z)} = z^\alpha \left(1 + O\left(\frac{1}{|z|}\right)\right), \quad z \in \mathbb{C}_+, \ \mathfrak{Re}\, \alpha > 0.$$

We denote by I_n the modified Bessel function of the first kind and order $n \in \mathbb{Z}$, given by

$$I_n(t) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+n+1)} \left(\frac{t}{2}\right)^{2m+n}, \quad n \in \mathbb{N}_0, \ t \in \mathbb{C},$$

and $I_{-n} = I_n$ for $n \in \mathbb{N}$.

Now, we give some known properties about Bessel functions I_n which can be found in [19, Chapter 5] and [37], and we will use along the paper. They satisfy that $I_0(0) = 1$, $I_n(0) = 0$ for $n \neq 0$, and $I_n(t) \geq 0$ for $n \in \mathbb{Z}$ and $t \geq 0$. Also, the function I_n has the semigroup property (also called Neumann's identity) for the convolution on $\mathbb{Z}$, that is,

$$I_n(t+s) = \sum_{m \in \mathbb{Z}} I_m(t) I_{n-m}(s) = \sum_{m \in \mathbb{Z}} I_m(t) I_{m-n}(s), \quad t, s \geq 0,$$

see [11, Chapter II], and it satisfies the following differential-difference equation:

$$\frac{\partial}{\partial t} I_n(t) = \frac{1}{2}\Big(I_{n-1}(t) + I_{n+1}(t) \Big), \quad t \in \mathbb{C}. \tag{2.1}$$

Furthermore, for each $n \in \mathbb{Z}$ and $N \in \mathbb{N}_0$

$$I_n(t) = \frac{e^t}{\sqrt{2\pi t}} \left(\sum_{k=0}^{N} \frac{(-1)^k a_{n,k}}{(2t)^k} + O\left(\frac{1}{t^{N+1}} \right) \right), \quad |\arg t| < \pi/2, \tag{2.2}$$

with $a_{n,0} = 1$ and for $k \geq 1$ $a_{n,k} = \frac{(4n^2-1)(4n^2-3)\cdots(4n^2-(2k-1)^2)}{k!2^{2k}}$, see [19, (5.11.10)]. The previous big "o" function satisfies $\left| O\left(\frac{1}{t^{N+1}} \right) \right| \leq \frac{C_{n,N}}{t^{N+1}}$, being $C_{n,N}$ a positive constant depending on n, N. In particular, see [19], we have that

$$I_n(t) = C \frac{e^t}{t^{1/2}} + R_n(t), \tag{2.3}$$

where $|R_n(t)| \leq C_n e^t t^{-3/2}$, for $t \to \infty$.

The generating function of the Bessel function I_n is given by

$$e^{\frac{t(x+x^{-1})}{2}} = \sum_{n \in \mathbb{Z}} x^n I_n(t), \qquad x \neq 0,\ t \in \mathbb{C}. \tag{2.4}$$

From the generating function (2.4), it was proved in [3, Theorem 3.3] that, for every $k \in \mathbb{N}_0$,

$$\sum_{n \in \mathbb{Z}} n^{2k} I_n(t) = e^t p_k(t), \qquad \sum_{n \in \mathbb{Z}} n^{2k+1} I_n(t) = 0, \quad t > 0, \tag{2.5}$$

where each $p_k(t)$ is a polynomial of degree k with positive coefficients, $p_0(t) = 1$, and $p_k(0) = 0$ for all $k \in \mathbb{N}$.

The following identities will be useful to define fractional powers of the discrete Laplacian:

$$\int_0^\infty \frac{e^{-ct} I_n(ct)}{t^{\gamma+1}}\, \mathrm{d}t = \frac{(2c)^\gamma}{\sqrt{\pi}} \frac{\Gamma(1/2+\gamma)\Gamma(n-\gamma)}{\Gamma(n+1+\gamma)}, \quad c > 0,\ -1/2 < \gamma < n, \tag{2.6}$$

see [25, Section 2.15.3, formula 3, p.305], and

$$I_n(t) = \frac{e^t}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} e^{-2t\sin^2\theta/2}\, \mathrm{d}\theta, \tag{2.7}$$

see [7, Proof of Proposition 1].

2.1.2. A new important property of Bessel functions

The Bessel function I_n has the following useful integral representation:

$$I_n(t) = \frac{t^n}{\sqrt{\pi}2^n\Gamma(n+1/2)}\int_{-1}^{1} e^{-ts}(1-s^2)^{n-1/2}\,\mathrm{d}s, \quad n \in \mathbb{N}_0,\ t \geq 0, \quad (2.8)$$

that we generalize in the following lemma.

Lemma 2.1. *Let $n \in \mathbb{N}_0$. Then, for all $j \in \mathbb{N}$ such that $n - j \in \mathbb{N}_0$ one can write*

$$I_n(t) = \frac{(-1)^j t^{n-j}}{\sqrt{\pi}2^{n-j}\Gamma(n+1/2-j)}\int_{-1}^{1} e^{-ts} s \frac{Q_{j-1}(s,t)}{t^{j-1}}(1-s^2)^{n-1/2-j}\,\mathrm{d}s, \quad (2.9)$$

with $Q_{j-1}(s,t) = \sum_{k=0}^{j-1} c_{j-1-k,j-1}(st)^k$.

Proof. By (2.8), it follows easily integrating by parts that (2.9) holds for $j = 1$ and $Q_0(s,t) = 1$, where we have differentiated $(1-s^2)^{n-1/2}$ and integrated e^{-st}. Doing the same procedure, differentiating $(1-s^2)^{n-1/2-j}$, integrating $e^{-st}sQ_{j-1}(s,t)$ and denoting

$$Q_j(s,t) := -t^2 e^{ts}\left(\int e^{-wt} w Q_{j-1}(w,t)\,\mathrm{d}w\right)\Big|_s \quad (2.10)$$

one gets $Q_1(s,t) = st+1$, $Q_2(s,t) = s^2t^2+3st+3$, which satisfy (2.9) for $j = 2, 3$. Thus, by iterating the previous arguments we get, for $j \geq 3$, $n-(j+1) \in \mathbb{N}_0$, that

$$I_n(t) = \frac{(-1)^j t^{n-(j+1)}}{\sqrt{\pi}2^{n-(j+1)}\Gamma(n-1/2-j)}\int_{-1}^{1} \frac{s}{t^{j-2}}\left(\int e^{-wt} w Q_{j-1}(w,t)\,\mathrm{d}w\right)\Big|_s (1-s^2)^{n-1/2-(j+1)}\,\mathrm{d}s$$

$$= \frac{(-1)^{j+1} t^{n-(j+1)}}{\sqrt{\pi}2^{n-(j+1)}\Gamma(n-1/2-j)}\int_{-1}^{1} \frac{e^{-ts}s}{t^j}(-t^2e^{ts})\left(\int e^{-wt} w Q_{j-1}(w,t)\,\mathrm{d}w\right)\Big|_s (1-s^2)^{n-1/2-(j+1)}\,\mathrm{d}s$$

$$= \frac{(-1)^{j+1} t^{n-(j+1)}}{\sqrt{\pi}2^{n-(j+1)}\Gamma(n-1/2-j)}\int_{-1}^{1} \frac{e^{-ts}s}{t^j} Q_j(s,t)(1-s^2)^{n-1/2-(j+1)}\,\mathrm{d}s.$$

Moreover, if for some $j \in \mathbb{N}$ we can write $Q_{j-1}(s,t) := \sum_{k=0}^{j-1} c_{j-1-k,j-1}(st)^k$ for certain coefficients $c_{j-1-k,j-1} \geq 0$, then by [24, Section 1.3.2, formula 6, p.137] it follows that

$$Q_j(s,t) = -e^{st}\sum_{k=1}^{j} c_{j-k,j-1}t^{k+1}\left(\int e^{-wt} w^k\,\mathrm{d}w\right)\Big|_s$$

$$= \sum_{k=1}^{j} c_{j-k,j-1}k!\sum_{m=0}^{k}\frac{(st)^{k-m}}{(k-m)!} = \sum_{k=1}^{j} c_{j-k,j-1}k!\sum_{m=0}^{k}\frac{(st)^m}{m!}$$

$$
\begin{aligned}
&= \sum_{k=1}^{j} c_{j-k,j-1} k! + \sum_{m=1}^{j} \frac{(st)^m}{m!} \sum_{k=m}^{j} c_{j-k,j-1} k! \\
&= \sum_{m=0}^{j} \left(\sum_{k=\max\{1,m\}}^{j} \frac{k!}{m!} c_{j-k,j-1} \right) (sz)^m \\
&:= \sum_{m=0}^{j} c_{j-m,j} (sz)^m, \qquad (2.11)
\end{aligned}
$$

and the proof is over. □

Remark 2.2. Note that, by (2.11), if $Q_j(s,t) = \sum_{k=0}^{j} c_{j-k,j} (st)^k$, being $c_{j-k,j} = \sum_{m=\max\{1,k\}}^{j} \frac{m!}{k!} c_{j-m,j-1}$, it follows that $c_{0,j} = \frac{j!}{j!} c_{0,j-1} = \frac{(j-1)!}{(j-1)!} c_{0,j-2} = \ldots = c_{0,0} = 1$, and also $c_{j,j} = c_{j-1,j}$, for all $j \in \mathbb{N}$.

Also, note that since (2.10) holds, then

$$
\frac{t Q_j(s,t) - \frac{\mathrm{d}}{\mathrm{d}s} Q_j(s,t)}{t^2} = s Q_{j-1}(s,t),
$$

and therefore a few calculations give

$$
c_{k,j} = c_{k,j-1} + c_{k-1,j}(j-k+1), \quad k = 1, \ldots, j-1. \qquad (2.12)
$$

Note that by the recurrence formula (2.12), one gets

$$
\begin{aligned}
c_{1,j} &= c_{1,j-1} + j c_{0,j} = c_{1,j-1} + j = c_{1,j-2} + (j-1) + j = \cdots \\
&= c_{0,1} + 2 + \cdots + j = \frac{j(j+1)}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
c_{2,j} &= c_{2,j-1} + (j-1) c_{1,j} = c_{2,j-1} + \frac{(j-1)j(j+1)}{2} = \cdots \\
&= c_{2,2} + \frac{2 \cdot 3 \cdot 4}{2} + \cdots + \frac{(j-1)j(j+1)}{2} = \frac{1}{2 \cdot 4}(j-1)j(j+1)(j+2),
\end{aligned}
$$

where we have applied $c_{2,2} = c_{1,2} = \frac{2 \cdot 3}{2}$. In general, it follows by induction that

$$
c_{k,j} = \frac{1}{\prod_{v=1}^{k}(2v)} (j-k+1) \cdots (j+k), \quad k = 1, \ldots, j-1. \qquad (2.13)
$$

2.2. Discrete heat kernel

As we have said, $G(t,n) = e^{-2t} I_n(2t)$ is the fundamental solution of the heat problem on $\mathbb{Z}$, (1.2) (it is a straightforward consequence of (2.1)). In the following, we present some key properties for this heat kernel.

From the theory of confluent hypergeometric functions, see [19, Section 9.11], we have

$$\int_0^1 e^{-4ts}s^{\gamma-\alpha-1}(1-s)^{\alpha-1}\,\mathrm{d}s$$
$$=\Gamma(\gamma-\alpha)e^{-4t}\sum_{k=0}^{\infty}\frac{(4t)^k}{k!}\frac{\Gamma(\alpha+k)}{\Gamma(\gamma+k)},\quad \mathfrak{Re}\,\gamma>\mathfrak{Re}\,\alpha>0 \tag{2.14}$$

and therefore by (2.8) and a change of variable, one gets, for $n\in\mathbb{N}_0$, and $t\geq 0$,

$$\begin{aligned}G(t,n)&=\frac{t^n4^n}{\sqrt{\pi}\Gamma(n+1/2)}\int_0^1 e^{-4ts}s^{n-1/2}(1-s)^{n-1/2}\,\mathrm{d}s\\&=\frac{1}{\sqrt{4\pi t}\Gamma(n+1/2)}\int_0^{4t}e^{-u}u^{n-1/2}\left(1-\frac{u}{4t}\right)^{n-1/2}\mathrm{d}u\\&=\frac{t^n4^n}{\sqrt{\pi}}e^{-4t}\sum_{k=0}^{\infty}\frac{(4t)^k}{k!}\frac{\Gamma(n+1/2+k)}{\Gamma(2n+1+k)}.\end{aligned}\tag{2.15}$$

In the following, two lemmata we prove new pointwise estimates for the difference of any order of $G(t,n)$.

Lemma 2.3. *Let $l\in\mathbb{N}_0$, and $n\in\mathbb{Z}$, then*

$$|\delta^l_{\text{right}}G(t,n)|\leq\frac{C_n}{t^{[(l+1)/2]+1/2}},\quad t>0.$$

Proof. Note that for $l=0$ the result follows by (2.2) taking $N=0$. If $l\in\mathbb{N}$, take $N=[(l+1)/2]$, then

$$\begin{aligned}\delta^l_{\text{right}}G(n,t)&=\frac{1}{2\sqrt{\pi t}}\sum_{j=0}^{l}\binom{l}{j}(-1)^j\left(\sum_{k=0}^{N}\frac{(-1)^ka_{n+j,k}}{(4t)^k}+O\left(\frac{1}{t^{N+1}}\right)\right)\\&=\frac{1}{2\sqrt{\pi t}}\sum_{k=0}^{N}\frac{(-1)^k}{(4t)^k}\sum_{j=0}^{l}\binom{l}{j}(-1)^ja_{n+j,k}+O\left(\frac{1}{t^{N+3/2}}\right).\end{aligned}$$

Note that for $k=0,\ldots,N$, $a_{n+j,k}$ is a polynomial in j of order $2k$, so we can write $a_{n+j,k}=\sum_{p=0}^{2k}\gamma_{p,n,k}\ j(j-1)\cdots(j-p+1)$, being $\gamma_{p,n,k}$ real coefficients (for $p=0$ we have the constant term $\gamma_{0,n,k}$). Then,

$$\begin{aligned}\sum_{j=0}^{l}\binom{l}{j}(-1)^ja_{n+j,k}&=\sum_{j=0}^{l}\binom{l}{j}(-1)^j\sum_{p=0}^{\min\{2k,j\}}\gamma_{p,n,k}j(j-1)\cdots(j-p+1)\\&=\sum_{p=0}^{\min\{2k,l\}}\gamma_{p,n,k}\sum_{j=p}^{l}\binom{l}{j}(-1)^jj(j-1)\cdots(j-p+1)\\&=\sum_{p=0}^{\min\{2k,l\}}\beta_{p,n,k,l}\sum_{j=0}^{l-p}\binom{l-p}{j}(-1)^j,\end{aligned}$$

with $\beta_{p,n,k,l}$ real coefficients. Since $k = 0, \ldots, N$, with $N = [(l+1)/2]$, it is not difficult to see that previous expression is null whenever $k = 0, \ldots, N-1$, and it is not null when $k = N$. Therefore,

$$\delta^l_{\text{right}} G(n,t) = \frac{C_n}{t^{1/2+[(l+1)/2]}} + O\left(\frac{1}{t^{3/2+[(l+1)/2]}}\right),$$

and the result follows. □

Lemma 2.4. *Let $l \in \mathbb{N}$, and $n \in \mathbb{N}_0$. Then,*

$$|\delta^l_{\text{right}} G(t,n)| \le \frac{C_l}{t^{l/2}} \sum_{u=0}^{[l/2]} \left(\frac{(n+1/2)^2}{t}\right)^{l/2-u} G(t, n+l-2u)$$
$$+ C_l G(t,n) \sum_{u=[l/2]+1}^{l-1} \frac{1}{t^u},$$

being C_l a positive constant which is independent on t and n.

Proof. Let $n \in \mathbb{N}_0$, $t \ge 0$. Note that by Lemma 2.1 we have

$$\begin{aligned}\delta^l_{\text{right}} I_n(t) &= \sum_{j=0}^{l} \binom{l}{j} (-1)^j I_{n+j}(t) \\ &= \frac{t^n}{\sqrt{\pi} 2^n \Gamma(n+1/2)} \int_{-1}^{1} e^{-ts} (1-s^2)^{n-1/2} \left(1 + \sum_{j=1}^{l} \binom{l}{j} \frac{s Q_{j-1}(s,t)}{t^{j-1}}\right) \mathrm{d}s.\end{aligned}$$

Now, by Remark 2.2, we write

$$\begin{aligned}1 + \sum_{j=1}^{l} \binom{l}{j} \frac{s Q_{j-1}(s,t)}{t^{j-1}} &= 1 + \sum_{j=1}^{l} \binom{l}{j} \sum_{k=0}^{j-1} c_{j-k-1,j-1} s^{k+1} t^{k+1-j} \\ &= 1 + \sum_{j=1}^{l} \binom{l}{j} \sum_{u=0}^{j-1} c_{u,j-1} \frac{s^{j-u}}{t^u} \\ &= 1 + \sum_{j=1}^{l} \binom{l}{j} s^j + \sum_{u=1}^{l-1} \frac{1}{t^u} \sum_{j=u+1}^{l} \binom{l}{j} c_{u,j-1} s^{j-u} \\ &= (s+1)^l + \sum_{u=1}^{l-1} \frac{s}{t^u} \sum_{k=0}^{l-u-1} \binom{l}{u+k+1} c_{u,k+u} s^k.\end{aligned}$$

Observe that $c_{1,k+1} = \frac{(k+1)(k+2)}{2}$ and therefore

$$\sum_{k=0}^{l-2} \binom{l}{k+2} c_{1,k+1} s^k = l(l-1) \sum_{k=0}^{l-2} \binom{l-2}{k} s^k = l(l-1)(1+s)^{l-2}.$$

Now consider the case $u = 2, \dots, l-1$. Taking into account (2.13), we have

$$\sum_{k=0}^{l-u-1} \binom{l}{u+k+1} c_{u,k+u} s^k = \frac{1}{\prod_{v=1}^{u}(2v)} l(l-1)\cdots(l-u) \sum_{k=0}^{l-u-1} \binom{l-u-1}{k} (k+u+2)\cdots(k+2u) s^k.$$

Since $(k+u+2)\cdots(k+2u)$ is a polynomial in k of order $u-1$, we can write $(k+u+2)\cdots(k+2u) = \sum_{p=0}^{u-1} b_{p,u} k(k-1)\dots(k-p+1)$, being $b_{p,u}$ real coefficients (for $p=0$, we have the constant term $b_{0,u}$). Then,

$$\begin{aligned}
&\sum_{k=0}^{l-u-1} \binom{l-u-1}{k} (k+u+2)\cdots(k+2u) s^k \\
&= \sum_{k=0}^{l-u-1} \binom{l-u-1}{k} \sum_{p=0}^{\min\{u-1,k\}} b_{p,u} k(k-1)\dots(k-p+1) s^k \\
&= \sum_{p=0}^{\min\{u-1,l-u-1\}} b_{p,u} \sum_{k=p}^{l-u-1} \binom{l-u-1}{k} k(k-1)\dots(k-p+1) s^k \\
&= \sum_{p=0}^{\min\{u-1,l-u-1\}} b_{p,u} (l-u-1)\cdots(l-u-p) \sum_{k=p}^{l-u-1} \binom{l-u-1-p}{k-p} s^k \\
&= \sum_{p=1}^{\min\{u,l-u\}} d_{p,u,l} s^{p-1} (s+1)^{l-u-p},
\end{aligned}$$

with $d_{p,u,l} \in \mathbb{R}$. Therefore, we have that

$$\delta^l_{\text{right}} I_n(t) = \frac{t^n}{\sqrt{\pi} 2^n \Gamma(n+1/2)} \int_{-1}^{1} e^{-ts} (1-s^2)^{n-1/2} \Bigg((s+1)^l + \sum_{u=1}^{l-1} \frac{1}{t^u} \sum_{p=1}^{\min\{u,l-u\}} d_{p,u,l} s^p (s+1)^{l-u-p} \Bigg) \mathrm{d}s.$$

Taking into account that $|s| \le 1$ for $s \in [-1,1]$, by a change of variable we have

$$\begin{aligned}
&|\delta^l_{\text{right}} G(t,n)| \\
&\le C_l \frac{t^n 4^n}{\sqrt{\pi}\Gamma(n+1/2)} \int_0^1 e^{-4ts} s^{n-1/2} (1-s)^{n-1/2} \Bigg(s^l + \sum_{u=1}^{l-1} \frac{1}{t^u} \sum_{p=1}^{\min\{u,l-u\}} s^{l-u-p} \Bigg) \mathrm{d}s \\
&\le C_l \sum_{u=0}^{[l/2]} \frac{t^{n-u} 4^n}{\sqrt{\pi}\Gamma(n+1/2)} \int_0^1 e^{-4ts} s^{n-1/2+l-2u} (1-s)^{n-1/2} \,\mathrm{d}s \\
&\quad + C_l \sum_{u=[l/2]+1}^{l-1} \frac{t^{n-u} 4^n}{\sqrt{\pi}\Gamma(n+1/2)} \int_0^1 e^{-4ts} s^{n-1/2} (1-s)^{n-1/2} \,\mathrm{d}s
\end{aligned}$$

$$\leq C_l \sum_{u=0}^{[l/2]} \frac{t^{n-u}4^n\Gamma(n+l-2u+1/2)}{\sqrt{\pi}\Gamma(n+1/2)} e^{-4t} \sum_{k=0}^{\infty} \frac{(4t)^k\Gamma(n+1/2+k)}{k!\Gamma(2n+l+1-2u+k)}$$
$$+ C_l G(t,n) \sum_{u=[l/2]+1}^{l-1} \frac{1}{t^u},$$

where in the last inequality we have used (2.14). Now take $u = 0, \dots, [l/2]$, and note that $m := l-2u$ is positive. An easy computation shows that

$$\frac{\Gamma(n+1/2+k)}{\Gamma(2n+l+1-2u+k)} \leq C_l \frac{\Gamma(n+m+1/2+k)}{\Gamma(2n+2m+1+k)},$$

for all $n, k \in \mathbb{N}_0$. Also, $\frac{\Gamma(n+l-2u+1/2)}{\Gamma(n+1/2)} \leq C_l(n+1/2)^m$. Then, by (2.15) we have

$$|\delta^l_{\text{right}} G(t,n)| \leq C_l \sum_{u=0}^{[l/2]} \frac{t^{n-u}4^n(n+1/2)^{l-2u}}{\sqrt{\pi}} e^{-4t} \sum_{k=0}^{\infty} \frac{(4t)^k\Gamma(n+l-2u+1/2+k)}{k!\Gamma(2(n+l-2u)+1+k)}$$
$$+C_l \sum_{u=[l/2]+1}^{l-1} \frac{G(t,n)}{t^u}$$
$$\leq \frac{C_l}{t^{l/2}} \sum_{u=0}^{[l/2]} \left(\frac{(n+1/2)^2}{t}\right)^{l/2-u} G(t,n+l-2u) + C_l G(t,n) \sum_{u=[l/2]+1}^{l-1} \frac{1}{t^u}.$$

□

Remark 2.5. Let $l, n \in \mathbb{N}$, and $t > 1$. From the previous lemma, if $1 \leq n \leq \sqrt{t} - 1/2$, then

$$|\delta^l_{\text{right}} G(t,n)| \leq C \frac{G(t,n)}{t^{l/2}},$$

and if $n \geq \sqrt{t} - 1/2$, then

$$|\delta^l_{\text{right}} G(t,n)| \leq C \frac{G(t,n)(n+1/2)^l}{t^l}.$$

In particular, the previous bounds are also valid when $t \in (0,1]$. Since $G(t,j)$ is decreasing for $j \in \mathbb{N}_0$, then $|\delta^l_{\text{right}} G(t,n)| \leq CG(t,n)$, $G(t,n) \leq \frac{G(t,n)}{t^{l/2}}$ and $G(t,n) \leq C\frac{G(t,n)(n+1/2)^l}{t^l}$ for all $t \in (0,1]$ and $n \in \mathbb{N}$.

Moreover, by using Lemma 2.1 with $n+1$ and $j = 1$, we have that

$$\delta_{\text{right}} G(t,n) = \frac{e^{-2t}t^n}{\sqrt{\pi}\Gamma(n+1/2)} \int_{-1}^{1} e^{-2ts}(1+s)(1-s^2)^{n-1/2} dx$$
$$\geq \frac{1}{2} \frac{e^{-2t}t^n}{\sqrt{\pi}\Gamma(n+1/2)} \int_{-1}^{1} e^{-2ts}(1-s^2)^{n+1/2} dx = \frac{n+1/2}{2t} G(t,n+1).$$

Therefore, since the kernel $G(t,n)$ does not satisfy a Gaussian control, it will not fulfil a Lipschitz condition as in [4].

The following result shows decay rates for the ℓ^1-norm of the differences of any order. The first difference was proved in [3, Theorem 4.3]. One should keep in mind that $\sum_{j\in\mathbb{Z}} G(t,j) = 1$.

Lemma 2.6. *Let* $l \in \mathbb{N}$ *and* $t > 0$, *then*

$$\|\delta^l_{\text{right}} G(t,\cdot)\|_1 := \sum_{j\in\mathbb{Z}} |\delta^l_{\text{right}} G(t,j)| \le C \min\{1, \frac{1}{t^{l/2}}\}.$$

Proof. Let $t > 0$. If $t \in (0,1]$, then it is clear that $\sum_{j\in\mathbb{Z}} |\delta^l_{\text{right}} G(t,j)| \le C$. Now let $t > 1$. Then,

$$\sum_{j\in\mathbb{Z}} |\delta^l_{\text{right}} G(t,j)| = \left(\sum_{|j|\le l} + \sum_{|j|>l}\right) |\delta^l_{\text{right}} G(t,j)| := I + II.$$

On the one hand, by Lemma 2.3 we have that

$$I \le \frac{C}{t^{1/2+[(l+1)/2]}} \le \frac{C}{t^{1/2+l/2}} \le \frac{C}{t^{l/2}}.$$

On the other hand, since $|\delta^l_{\text{right}} G(t,j)| = |\delta^l_{\text{right}} G(t,|j|-l)|$ for $j \le -l$, we have that

$$II = \sum_{|j|>l} |\delta^l_{\text{right}} G(t,j)| \le 2\sum_{j\ge 1} |\delta^l_{\text{right}} G(t,j)| \le 2\left(\sum_{1\le j\le\sqrt{t}-1/2} + \sum_{\substack{j>\sqrt{t}-1/2\\ j\ge 1}}\right) |\delta^l_{\text{right}} G(t,j)|$$
$$=: II.1 + II.2.$$

Let now k be the least natural number such that $2k \ge l$. Then, by Lemma 2.4 and Remark 2.5 we have

$$II.1 \le \frac{C}{t^{l/2}} \sum_{1\le j\le\sqrt{t}-1/2} G(t,j) \le \frac{C}{t^{l/2}},$$

and

$$II.2 \le \frac{C}{t^l} \sum_{\substack{j>\sqrt{t}-1/2\\ j\ge 1}} G(t,j)(|j|+1/2)^l$$
$$\le \frac{C}{t^{l/2+k}} \sum_{\substack{j>\sqrt{t}-1/2\\ j\ge 1}} G(t,j) j^{2k} \le \frac{C p_k(2t)}{t^{l/2+k}} \le \frac{C}{t^{l/2}}.$$

□

Remark 2.7. Note that the ℓ^1-norm of the even differences $2l$ of $G(t,n)$ can be seen as the ℓ^1-norm of the derivative or order l in time. In this case, our result coincides with the ones proved in [15,16].

2.3. Discrete Poisson kernel

The discrete Poisson kernel is defined as

$$P(y,j):=\frac{y}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{y^2}{4t}}}{t^{3/2}}G(t,j)\,\mathrm{d}t,\quad j\in\mathbb{Z},\quad y>0.$$

Since we do not have an explicit expression in terms of known functions for $P(y,j)$, we will take advantage of the subordination formula and the properties we have proved for the heat kernel, $G(t,j)$, to get some estimates of the Poisson kernel.

Lemma 2.8. *Let $y,c>0$ and $l\in\mathbb{N}_0$. The following estimates hold:*

(i) $\left|\partial_y^l P(y,j)\right|\leq \frac{C}{y^l(1+|j|)},\quad j\in\mathbb{Z}.$
(ii) $\left|\partial_y^l P(y,j)\right|\leq \frac{Cy}{y^l|j|^2},\quad j\neq 0.$
(iii) $\left|\partial_y^l \delta_{\text{right}} P(y,j)\right|\leq \frac{C}{y^{l+2}},\quad j\in\mathbb{Z}.$
(iv) $\left|\partial_y^l \delta_{\text{right}} P(y,j)\right|\leq \frac{C}{y^l|j|^2},\quad j\neq 0$

Proof. First, we prove epigraphs (i) and (ii). Observe that, from [3, Lemma 4.1 (i)] we have that, for $0<t\leq |j|^2$, $|G(t,j)|\leq C\frac{t}{|j|^3}$, and for $t>0$, $G(t,j)\leq G(t,0)\leq \frac{C}{t^{1/2}}$, for $j\in\mathbb{Z}$. Since for every $l\in\mathbb{N}$, $y,c,t>0$

$$\left|\partial_y^l\left(\frac{ye^{-\frac{cy^2}{t}}}{t^{3/2}}\right)\right|\leq C\frac{e^{-\frac{cy^2}{t}}}{t^{l/2+1}}\leq \frac{C}{y^{l-1}}\frac{y^{l-1}e^{-\frac{cy^2}{t}}}{t^{\frac{l-1}{2}}t^{3/2}}\leq \frac{Cy}{y^l}\frac{e^{-\frac{cy^2}{t}}}{t^{3/2}},\tag{2.16}$$

and $\int_0^\infty y\frac{e^{-\frac{cy^2}{t}}}{t^{3/2}}\,\mathrm{d}t=C<\infty$, it follows that, for $j\neq 0$,

$$\begin{aligned}\left|\partial_y^l P(y,j)\right| &\leq \frac{C}{y^l}\left(\int_0^{|j|^2} y\frac{e^{-\frac{cy^2}{t}}}{t^{3/2}}\frac{t}{|j|^3}\mathrm{d}t+\int_{|j|^2}^\infty y\frac{e^{-\frac{cy^2}{t}}}{t^{3/2}}\frac{1}{t^{1/2}}\mathrm{d}t\right)\\ &\leq C\frac{C}{y^l|j|}\int_0^\infty y\frac{e^{-\frac{cy^2}{t}}}{t^{3/2}}\mathrm{d}t\leq \frac{C}{y^l(1+|j|)},\end{aligned}$$

and

$$\left|\partial_y^l P(y,j)\right|\leq \frac{Cy}{y^l}\left(\int_0^{|j|^2}\frac{1}{t^{3/2}}\frac{t}{|j|^3}\mathrm{d}t+\int_{|j|^2}^\infty\frac{1}{t^2}\mathrm{d}t\right)\leq C\frac{y}{y^l|j|^2}.$$

The case $j=0$ in part (i) follows from (2.16) and the fact that $G(t,0)\leq C$ for $t>0$.

Now, we prove epigraphs (iii) and (iv). Since $G(t,j)=G(t-j)$ for $j\in\mathbb{N}$, and $G(t,j+1)\leq G(t,j)$ for $j\in\mathbb{N}_0$, from Lemma 2.4 we get that

$$|\delta_{\text{right}}G(t,j)|\leq C\frac{j+1/2}{t}G(t,j+1)\leq C\frac{j+1/2}{t}G(t,j),\quad \text{for } j\in\mathbb{N}_0$$

and

$$|\delta_{\text{right}} G(t, j)| = |G(t, |j|) - G(t, |j| - 1)| \leq C \frac{|j| + 1/2}{t} G(t, |j|), \text{ for } j \leq -1.$$

Therefore, we have that

$$|\delta_{\text{right}} G(t, j)| \leq C \frac{(|j| + 1/2)}{t} G(t, |j|), \quad \text{for every } j \in \mathbb{Z} \text{ and } t > 0.$$

Also, we have for $t > 0$ and $j \neq 0$,

$$|\delta_{\text{right}} G(t, j)| \leq C \begin{cases} \frac{|j|}{t^{3/2}}, & \text{if } |j|^2 \leq t, \\ \frac{t}{|j|^4}, & \text{if } |j|^2 \geq t, \end{cases}$$

see [3, Lemma 4.1 (ii)]. Thus, by using (2.16) and the estimates above we get that, for $j \in \mathbb{Z}$,

$$\begin{aligned} \left|\partial_y^l \delta_{\text{right}} P(y, j)\right| &\leq \frac{C}{y^l} \int_0^\infty y \frac{e^{-\frac{cy^2}{t}}}{t^{3/2}} \frac{|j| + 1/2}{t} G(t, j)\, \mathrm{d}t \\ &\leq \frac{C(|j| + 1/2)}{y^{l+2}} \int_0^\infty y \frac{e^{-\frac{cy^2}{t}}}{t^{3/2}} \frac{y^2}{t} G(t, j)\, \mathrm{d}t \leq \frac{C(|j| + 1/2)}{y^{l+2}} \int_0^\infty y \frac{e^{-\frac{cy^2}{t}}}{t^{3/2}} G(t, j)\, \mathrm{d}t \\ &\leq \frac{C}{y^{l+2}}, \end{aligned}$$

where in the last inequality we have proceeded as in epigraph (i), and for $j \neq 0$

$$\begin{aligned} \left|\partial_y^l \delta_{\text{right}} P(y, j)\right| &\leq \frac{C}{y^l} \left(\int_0^{|j|^2} y \frac{e^{-\frac{cy^2}{t}}}{t^{3/2}} \frac{t}{|j|^4} \mathrm{d}t + \int_{|j|^2}^\infty y \frac{e^{-\frac{cy^2}{t}}}{t^{3/2}} \frac{|j|}{t^{3/2}} \mathrm{d}t \right) \\ &\leq \frac{C}{y^l |j|^2} \int_0^\infty y \frac{e^{-\frac{cy^2}{t}}}{t^{3/2}} \mathrm{d}t = \frac{C}{y^l |j|^2}. \end{aligned}$$

□

The previous lemma also holds when we substitute $P(y, j)$ by $\int_0^\infty y \frac{e^{-\frac{cy^2}{t}}}{t^{3/2}} \mathrm{d}t$, being $c > 0$ an arbitrary constant.

From Lemma 2.6, we can get the ℓ^1-norm estimates for the Poisson semigroup.

Lemma 2.9. *Let* $y > 0$.

(1) $\|\partial_y^m P(y, \cdot)\|_1 = \sum_{j \in \mathbb{Z}} |\partial_y^m P(y, j)| \leq C y^{-m}$, *for every* $m \in \mathbb{N}_0$.

(2) $\|\delta_{\text{right}}^l P(y, \cdot)\|_1 = \sum_{j \in \mathbb{Z}} |\delta_{\text{right}}^l P(y, j)| \leq C y^{-l}$, *for every* $l \in \mathbb{N}_0$.

Proof. Let $y > 0$ and $m \in \mathbb{N}_0$. By (2.16) we have that

$$\sum_{j\in\mathbb{Z}} |\partial_y^m P(y,j)| \le \frac{C}{y^m}\int_0^\infty y\frac{e^{-\frac{cy^2}{t}}}{t^{3/2}}\sum_{j\in\mathbb{Z}} G(t,j)\mathrm{d}t = \frac{C}{y^m}.$$

On the other hand, Lemma 2.6 implies that

$$\sum_{j\in\mathbb{Z}} |\delta^l_{\text{right}} P(y,j)| \le \frac{y}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{y^2}{4t}}}{t^{3/2}}\sum_{j\in\mathbb{Z}}|\delta^l_{\text{right}} G(t,j)|\,\mathrm{d}t \le Cy\int_0^\infty \frac{e^{-\frac{y^2}{4t}}}{t^{3/2+l/2}}\mathrm{d}t \le \frac{C}{y^l}.$$

□

Remark 2.10. Taking into account that $P(y, j)$ satisfies the Poisson equation, the results in Lemma 2.9 imply that

$$\|\delta^2_{\text{right}} P(y,\cdot)\|_1 = \|\partial_y^2 P(y,\cdot)\|_1 \le Cy^{-2}, \quad y>0.$$

2.4. Heat and Poisson semigroups

The following lemma contains crucial observations to get our results.

Lemma 2.11.

- *Let $f : \mathbb{Z} \to \mathbb{R}$ be a function such that the semigroup $e^{t\Delta_d} f$ is well defined for every $t > 0$. Then, $\delta_{\text{right}} e^{t\Delta_d} f$ and $\partial_t^l e^{t\Delta_d} f$, $l \in \mathbb{N}$, are well defined. Moreover,*

$$\delta_{\text{right}} e^{t\Delta_d} f(n) = \sum_{j\in\mathbb{Z}} (\delta_{\text{right}} G(t,n-j)) f(j) = \sum_{j\in\mathbb{Z}} G(t,j)\delta_{\text{right}} f(n-j),$$
$$\text{(analogously for } \delta_{\text{left}})$$

and, for $t = t_1 + t_2$, where $t, t_1, t_2 > 0$,

$$\begin{aligned}\partial_t e^{t\Delta_d} f(n)|_{t=t_1+t_2} &= \sum_{j\in\mathbb{Z}} \partial_{t_1} G(t_1,j) e^{t_2\Delta_d} f(n-j)\\ &= \sum_{j\in\mathbb{Z}} G(t_1,j)\partial_{t_2} e^{t_2\Delta_d} f(n-j).\end{aligned}$$

- *Let $f : \mathbb{Z} \to \mathbb{R}$ be a function such that $e^{-y\sqrt{-\Delta_d}} f$ is well defined for every $y > 0$. Then, $\delta_{\text{right}} e^{-y\sqrt{-\Delta_d}} f$ is well defined. In addition, if*

$$\sum_{j\in\mathbb{Z}} y\left(\int_0^\infty \frac{e^{-\frac{cy^2}{t}}}{t^{3/2}} G(t,j)\right)|f(n-j)| < \infty, \quad y>0,\ n\in\mathbb{Z} \text{ and } c>0,$$

then $\partial_y^l e^{-y\sqrt{-\Delta_d}} f$ is well defined for every $l \in \mathbb{N}$ and the properties above-stated for the discrete heat semigroup are also fulfilled for the Poisson semigroup.

Proof. Suppose that $f : \mathbb{Z} \to \mathbb{R}$ is a function such that $e^{t\Delta_d} f$ is well defined for every $t > 0$. Then, it is clear that $\delta_{\text{right}} e^{t\Delta_d} f$ and $\partial_t^l e^{t\Delta_d} f$ are also well defined for every $l \in \mathbb{N}$. Moreover,

$$\begin{aligned}\delta_{\text{right}} e^{t\Delta_d} f(n) &= e^{t\Delta_d} f(n) - e^{t\Delta_d} f(n+1) \\ &= \sum_{j\in\mathbb{Z}} G(t, n-j) f(j) - \sum_{j\in\mathbb{Z}} G(t, n+1-j) f(j) \\ &= \sum_{j\in\mathbb{Z}} (\delta_{\text{right}} G(t, n-j)) f(j)\end{aligned}$$

and, by performing a change of variables, we get that

$$\begin{aligned}\delta_{\text{right}} e^{t\Delta_d} f(n) &= \sum_{j\in\mathbb{Z}} G(t, j) f(n-j) - \sum_{j\in\mathbb{Z}} G(t, j) f(n+1-j) \\ &= \sum_{j\in\mathbb{Z}} G(t, j) \delta_{\text{right}} f(n-j).\end{aligned}$$

On the other hand, for $t = t_1 + t_2$, where $t, t_1, t_2 > 0$, the semigroup property gives

$$e^{t\Delta_d} f(n) = e^{t_1\Delta_d}(e^{t_2\Delta_d} f)(n) = \sum_{j\in\mathbb{Z}} G(t_1, j) e^{t_2\Delta_d} f(n-j).$$

Furthermore, since

$$\partial_t e^{t\Delta_d} f(n)|_{t=t_1+t_2} = \partial_{t_1} e^{(t_1+t_2)\Delta_d} f(n) = \partial_{t_2} e^{(t_1+t_2)\Delta_d} f(n),$$

we obtain that

$$\partial_t e^{t\Delta_d} f(n)|_{t=t_1+t_2} = \sum_{j\in\mathbb{Z}} \partial_{t_1} G(t_1, j) e^{t_2\Delta_d} f(n-j) = \sum_{j\in\mathbb{Z}} G(t_1, j) \partial_{t_2} e^{t_2\Delta_d} f(n-j).$$

Now assume that $f : \mathbb{Z} \to \mathbb{R}$ is a function such that $e^{-y\sqrt{-\Delta_d}} f$ is well defined for every $y > 0$. Then, it is clear that $\delta_{\text{right}} e^{-y\sqrt{-\Delta_d}} f$ is well defined. Also, if

$$\sum_{j\in\mathbb{Z}} y \left(\int_0^\infty \frac{e^{-\frac{cy^2}{t}}}{t^{3/2}} G(t, j) \right) |f(n-j)| < \infty$$

for each $y > 0$, $n \in \mathbb{Z}$, and $c > 0$, then, by (2.16), $\partial_y^l e^{-y\sqrt{-\Delta_d}} f$ is well defined for every $l \in \mathbb{N}$. The remaining properties can be obtained analogously to the heat kernel case. □

Next, we study functions for which the semigroups are well defined.

Lemma 2.12. *Let* $f : \mathbb{Z} \to \mathbb{R}$.

A. *Suppose that for certain* $\alpha > 0$, $\frac{|f|}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$. *Then,*

(i) *For every* $t > 0$, $e^{t\Delta_d} f$ *is well defined and*

$$|e^{t\Delta_d} f(n)| \le C(1 + |n|^\alpha + t^{\alpha/2}), \quad n \in \mathbb{Z}.$$

(ii) *For every* $l \in \mathbb{N}$, *and* $t > 0$,

$$|\delta^l_{\text{right}} e^{t\Delta_d} f(n)| \le C \left((1 + |n|^\alpha) \min\left\{1, \frac{1}{t^{l/2}}\right\} + t^{\alpha/2 - l/2} \right), \quad n \in \mathbb{Z}.$$

(iii) $\lim_{t\to 0} e^{t\Delta_d} f(n) = f(n)$, *for every* $n \in \mathbb{Z}$.

B. *Suppose that* f *satisfies* $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$. *Then,* $e^{-y\sqrt{-\Delta_d}} f$ *is well defined for every* $y > 0$ *and* $\lim_{y\to 0} e^{-y\sqrt{-\Delta_d}} f(n) = f(n)$, *for every* $n \in \mathbb{Z}$.

Proof. We start proving A.(i). Let $t > 0$, $n \in \mathbb{Z}$ and m be the smallest natural number such that $2m \ge \alpha$. By using (2.5), we have that

$$\begin{aligned}
|e^{t\Delta_d} f(n)| &\le C \sum_{j\in\mathbb{Z}} G(t, j)(1 + |n - j|^\alpha) \le C \sum_{j\in\mathbb{Z}} G(t, j)(1 + |n|^\alpha + |j|^\alpha) \\
&\le C \left(1 + |n|^\alpha + \sum_{|j|\le\sqrt{t}} G(t, j)|j|^\alpha + \sum_{|j|>\sqrt{t}} G(t, j)|j|^\alpha \min\left\{\frac{|j|}{\sqrt{t}}, |j|\right\}^{2m-\alpha} \right) \\
&= C \left(1 + |n|^\alpha + t^{\alpha/2} + p_m(2t) \min\left\{\frac{1}{t^{m-\alpha/2}}, 1\right\} \right) \le C(1 + |n|^\alpha + t^{\alpha/2}).
\end{aligned}$$

In the last inequality, we have used that $|p_m(2t)| \le C$, if $0 < t < 1$, and $|p_m(2t)| \le Ct^m$ whenever $t > 1$.

Next, we prove (ii). Let $t > 0$, $n \in \mathbb{Z}$ and m be the smallest natural number such that $2m \ge l + \alpha$. Then, since $\sum_{j\in\mathbb{Z}} |\delta^l_{\text{right}} G(t, j)| \le C \min\{1, \frac{1}{t^{l/2}}\}$ (see Lemma 2.6), we have that

$$\begin{aligned}
|\delta^l_{\text{right}} e^{t\Delta_d} f(n)| &\le C \sum_{j\in\mathbb{Z}} |\delta^l_{\text{right}} G(t, j)|(1 + |n|^\alpha + |j|^\alpha) \\
&\le C(1 + |n|^\alpha) \sum_{|j|\le l} |\delta^l_{\text{right}} G(t, j)| + C \sum_{|j|>l} |\delta^l_{\text{right}} G(t, j)|(1 + |n|^\alpha + |j|^\alpha) \\
&\le C((1 + |n|^\alpha) \min\left\{1, \frac{1}{t^{l/2}}\right\} + C \sum_{|j|>l} |\delta^l_{\text{right}} G(t, j)||j|^\alpha.
\end{aligned}$$

Recall that if $j < -l$, one can write $|\delta^l_{\text{right}} G(t, j)| = |\delta^l_{\text{right}} G(t, |j| - l)|$, with $|j| - l \ge 1$, so by Remark 2.5 and Lemma 2.6, it follows that

$$\begin{aligned}
&\sum_{|j|>l} |\delta^l_{\text{right}} G(t, j)||j|^\alpha \\
&\quad\le \sum_{j\ge 1} |\delta^l_{\text{right}} G(t, j)|(|j + l|^\alpha + |j|^\alpha) \le C \sum_{j\ge 1} |\delta^l_{\text{right}} G(t, j)||j|^\alpha \\
&\quad\le C \left(t^{\alpha/2} \sum_{1\le j\le\sqrt{t}} |\delta^l_{\text{right}} G(t, j)| + \frac{1}{t^l} \sum_{j>\sqrt{t}\ge 1} G(t, j)|j|^{\alpha+l} + \sum_{\substack{j>\sqrt{t} \\ 0<t<1}} G(t, j)|j|^\alpha \right) \\
&\quad\le C \left(t^{\alpha/2 - l/2} + p_m(2t) \min\left\{\frac{1}{t^{l+m-\alpha/2-l/2}}, 1\right\} \right) \le C(1 + t^{\alpha/2 - l/2}).
\end{aligned}$$

In the last inequality, we have used that $|p_m(2t)| \leq C$, if $0 < t < 1$, and $|p_m(2t)| \leq Ct^m$ whenever $t > 1$.

Now, we prove (iii). Note that

$$|e^{t\Delta_d} f(n) - f(n)| = \left|\sum_{j\neq 0} G(t,j)(f(n-j) - f(n))\right| \leq C \sum_{j\neq 0} G(t,j)(1 + |n|^\alpha + |j|^\alpha).$$

On the one hand,

$$\sum_{j\neq 0} G(t,j)(1 + |n|^\alpha) = (1 + |n|^\alpha)(1 - G(t,0)) \to 0, \quad t \to 0^+,$$

since $G(t,0) \to 1$ as $t \to 0^+$. On the other hand,

$$\sum_{j\neq 0} G(t,j)|j|^\alpha \leq p_m(2t) \to 0, \quad t \to 0^+,$$

being now $2m \geq \alpha$. Then, the result follows.

Finally, we prove B. Let $|n| \leq A$, $A \in \mathbb{N}$. We can write

$$f = f\chi_{\{|j|\leq 2A\}} + f\chi_{\{|j|>2A\}} := f_1 + f_2.$$

Note that when $|j| > 2A$ one gets $|j| \leq 2|n-j|$. Then, by Lemma 2.8 (ii) we have

$$\begin{aligned}|e^{-y\sqrt{-\Delta_d}} f_2(n)| &\leq \sum_{|j|>2A} P(y, n-j)|f(j)| \leq C \sum_{|j|>2A} \frac{y}{|n-j|^2}|f(j)| \\ &\leq Cy \sum_{|j|>2A} \frac{|f(j)|}{|j|^2} \to 0,\end{aligned}$$

as $y \to 0$.

On the other hand, we have that $f_1 \in \ell^p(\mathbb{Z})$ for each $p \geq 1$, and $e^{t\Delta_d}$ is C_0-semigroup on $\ell^p(\mathbb{Z})$, see [7]. In particular, it is strongly continuous at the origin in $\ell^p(\mathbb{Z})$, and therefore pointwise, that is,

$$\lim_{y\to 0} e^{-y\sqrt{-\Delta_d}} f_1(n) = f_1(n) = f(n).$$

We conclude that $e^{-y\sqrt{-\Delta_d}} f$ is well defined for every $y > 0$ and $\lim_{y\to 0} e^{-y\sqrt{-\Delta_d}} f(n) = f(n)$, for every $n \in \mathbb{Z}$. □

Lemma 2.13. *Let* $f : \mathbb{Z} \to \mathbb{R}$.

1. *If* f *satisfies* $\frac{|f|}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$, *for certain* $\alpha > 0$, *then, for every* $n \in \mathbb{Z}$, $m := m_1 + m_2$, *with* $m_1, m_2 \in \mathbb{N}_0$ *and* $l \in \mathbb{N}_0$, *such that* $\frac{m}{2} + l > \alpha/2$, *we have that*

$$\partial_t^l \delta^{m_1,m_2}_{\text{right/left}} e^{t\Delta_d} f(n) \to 0, \quad \text{as } t \to \infty.$$

2. *If f satisfies $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$, then, for every $n \in \mathbb{Z}$, $m := m_1 + m_2$, with $m_1, m_2 \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$, such that $m + l \geq 1$, we have that*

$$\partial_y^l \delta_{\text{right/left}}^{m_1,m_2} e^{-y\sqrt{-\Delta_d}} f(n) \to 0, \quad \textit{as } y \to \infty.$$

Proof. Suppose that f satisfies $\frac{|f|}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$, for certain $\alpha > 0$, and let $n \in \mathbb{Z}$ and $m_1, m_2, l \in \mathbb{N}_0$ such that $\frac{m}{2} + l > \alpha/2$. There exists $n' \in \mathbb{Z}$ (n' is comparable to n) and $q = 2l + m_1 + m_2 \in \mathbb{N}$ with $q > \alpha$ such that

$$|\partial_t^l \delta_{\text{right/left}}^{m_1,m_2} e^{t\Delta_d} f(n)| = |\delta_{\text{right}}^q e^{t\Delta_d} f(n')|.$$

Then, it follows from Lemma 2.12 A (ii) that $\delta_{\text{right}}^q e^{t\Delta_d} f(n') \to 0, \quad t \to \infty.$

Now, we prove 2. Suppose that f satisfies $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$ and let $n \in \mathbb{Z}$ and $m_1, m_2, l \in \mathbb{N}_0$ such that $m + l \geq 1$.

Suppose first that $m = 0$, and $l \in \mathbb{N}$. By (2.16) we have that, for every $y > 0$ and $n \in \mathbb{Z}$,

$$|\partial_y^l e^{-y\sqrt{-\Delta_d}} f(n)| \leq \left(\sum_{|j|\leq\sqrt{y}} + \sum_{|j|\geq\sqrt{y}} \right) |\partial_y^l P(y, j)||f(n-j)| \mathrm{d}t =: A1 + B1.$$

Note that by Lemma 2.8 (i) we have that

$$A1 \leq \frac{C(1+\sqrt{y})}{y^l} \sum_{|j|\leq\sqrt{y}} \frac{1}{(1+|j|)^2} |f(n-j)| \to 0, \quad y \to \infty,$$

and by Lemma 2.8 (ii),

$$B1 \leq \frac{C}{y^{l-1}} \sum_{|j|\geq\sqrt{y}} \frac{1}{|j|^2} |f(n-j)| \to 0, \quad y \to \infty.$$

Secondly, since $\delta_{\text{right}}^2 e^{-y\sqrt{-\Delta_d}} f(n) = \Delta_d e^{-y\sqrt{-\Delta_d}} f(n+1) = \partial_y^2 e^{-y\sqrt{-\Delta_d}} f(n+1)$, the case when m is even follows from the previous one.

Finally, it remains to prove that, for $l \in \mathbb{N}_0$ and $n \in \mathbb{Z}$, $\partial_y^l \delta_{\text{right}} e^{-y\sqrt{-\Delta_d}} f(n) \to 0$, as $y \to \infty$. Let $l \in \mathbb{N}_0$, $n \in \mathbb{Z}$ and $y > 0$. We write

$$\begin{aligned} |\partial_y^l \delta_{\text{right}} e^{-y\sqrt{-\Delta_d}} f(n)| &\leq \left(\sum_{|j|\leq\sqrt{y}} + \sum_{|j|\geq\sqrt{y}} \right) |\partial_y^l \delta_{\text{right}} P(y, j)||f(n-j)| \mathrm{d}t \\ &= A2 + B2. \end{aligned}$$

On the one hand, by Lemma 2.8 (iii)

$$A2 \leq \frac{C}{y^{l+2}} \sum_{|j|\leq\sqrt{y}} |f(n-j)| \leq \frac{C(1+y)}{y^{l+2}} \sum_{|j|\leq\sqrt{y}} \frac{1}{1+|j|^2} |f(n-j)| \to 0, \quad y \to \infty,$$

and by Lemma 2.8 (iv)

$$B2 \leq \frac{C}{y^l} \sum_{|j| \geq \sqrt{y}} \frac{|f(n-j)|}{|j|^2} \to 0, \quad y \to \infty.$$

The case including δ_{left} is analogous. □

Lemma 2.14. 1. *Let* $f : \mathbb{Z} \to \mathbb{R}$ *satisfying* $\frac{|f|}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$, *for certain* $\alpha > 0$. *For every* $k, l \in \mathbb{N}$ *such that* $l > k \geq [\alpha/2] + 1$ *and* $t > 0$, *the following are equivalent:*

$$(i)\ \|\partial_t^k e^{t\Delta_d} f\|_\infty \leq Ct^{-k+\alpha/2}, \quad (ii)\ \|\partial_t^l e^{t\Delta_d} f\|_\infty \leq Ct^{-l+\alpha/2}.$$

2. *Let* $f : \mathbb{Z} \to \mathbb{R}$ *satisfying* $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$. *For every* $p, q \in \mathbb{N}$ *such that* $p > q \geq [\alpha] + 1$, $\alpha > 0$ *and* $y > 0$, *the following are equivalent:*

$$(i)\ \|\partial_y^q e^{-y\sqrt{-\Delta_d}} f\|_\infty \leq Cy^{-q+\alpha}, \quad (ii)\ \|\partial_y^p e^{-y\sqrt{-\Delta_d}} f\|_\infty \leq Cy^{-p+\alpha}.$$

Proof. We only do the proof for the heat kernel and the case $k = [\alpha/2] + 1$ and $l = k + 1$. The rest of the cases are analogous.

Suppose that f satisfies (i). Then, by the semigroup property and Lemma 2.6, we have that

$$\begin{aligned}
|\partial_t^l e^{t\Delta_d} f(n)| &= C \left| \sum_{j\in\mathbb{Z}} \partial_u G(u,j)|_{u=t/2} \partial_v^k e^{v\Delta_d} f(n-j)|_{v=t/2} \right| \\
&\leq C \|\partial_v^k e^{v\Delta_d} f|_{v=t/2}\|_\infty \sum_{j\in\mathbb{Z}} |\partial_u G_u(j)|_{u=t/2}| \\
&\leq Ct^{-k+\alpha/2} t^{-1} = Ct^{-l+\alpha/2}.
\end{aligned}$$

Conversely, suppose that (ii) holds. Since for each $n \in \mathbb{Z}$, $\partial_t^k e^{t\Delta_d} f(n) \to 0$ as $t \to \infty$, see Lemma 2.13, we have that

$$|\partial_t^k e^{t\Delta_d} f(n)| = \left| \int_t^\infty \partial_u^{k+1} e^{u\Delta_d} f(n) \mathrm{d}u \right| \leq Ct^{-k+\alpha/2}.$$

□

Lemma 2.15. *Let* $f : \mathbb{Z} \to \mathbb{R}$.

- *Suppose that* $f \in \Lambda_H^\alpha$, *for some* $\alpha > 0$. *Then, for every* $l \in \mathbb{N}_0$ *and* $m \in \{1, 2\}$ *such that* $\frac{m}{2} + l \geq [\alpha/2] + 1$, *we have that*

$$\|\partial_t^l \delta_{\text{right/left}}^{m_1,m_2} e^{t\Delta_d} f\|_\infty \leq Ct^{-(l+\frac{m}{2})+\frac{\alpha}{2}},$$
$$m_1, m_2 \in \mathbb{N}_0,\ m_1 + m_2 = m,\ t > 0.$$

- *Suppose that $f \in \Lambda_P^\alpha$, for some $\alpha > 0$. Then, for every $l \in \mathbb{N}_0$ and $m \in \{1, 2\}$ such that $m + l \geq [\alpha] + 1$, we have that*

$$\|\partial_y^l \delta_{\text{right/left}}^{m_1,m_2} e^{-y\sqrt{-\Delta_d}} f\|_\infty \leq C y^{-(l+m)+\alpha},$$
$$m_1, m_2 \in \mathbb{N}_0,\ m_1 + m_2 = m,\ y > 0.$$

Proof. We only do the proof for the heat semigroup. For the Poisson is completely analogous. Suppose that $f \in \Lambda_H^\alpha$, for some $\alpha > 0$. We consider first the case $l \geq [\alpha/2]+1, m \in \{1, 2\}$. From the semigroup property, Lemmas 2.6, 2.14 and Remark 2.7, we have that

$$\begin{aligned}
|\partial_t^l \delta_{\text{right/left}}^{m_1,m_2} e^{t\Delta_d} f(n)| &= C \left| \sum_{j\in\mathbb{Z}} \delta_{\text{right/left}}^{m_1,m_2} G(u, j)|_{u=t/2} \partial_v^l e^{v\Delta_d} f(n-j)|_{v=t/2} \right| \\
&\leq C \|\partial_v^l e^{v\Delta_d} f|_{v=t/2}\|_\infty \left| \sum_{j\in\mathbb{Z}} \delta_{\text{right/left}}^{m_1,m_2} G(u, j)|_{u=t/2} \right| \\
&\leq C t^{-l+\alpha/2} t^{-m/2}. \qquad (2.17)
\end{aligned}$$

Now assume that $l < [\alpha/2] + 1$ and $m \in \{1, 2\}$ so that $\frac{m}{2} + l \geq [\alpha/2] + 1$. Then, $l = [\alpha/2]$ and $m = 2$. Then, by using (2.17) with $[\alpha/2] + 1$ derivatives in the variable t and the fact that $\partial_t^{[\alpha/2]} \delta_{\text{right/left}}^{m_1,m_2} e^{t\Delta_d} f(n) \to 0$ as $t \to \infty$ for each $n \in \mathbb{Z}$ (see Lemma 2.13), we get that

$$|\partial_t^l \delta_{\text{right/left}}^{m_1,m_2} e^{t\Delta_d} f(n)| = \left| \int_t^\infty \partial_u^{[\alpha/2]+1} \delta_{\text{right/left}}^{m_1,m_2} e^{t\Delta_d} f(n) \mathrm{d}u \right| \leq C t^{-l+\alpha/2} t^{-m/2}.$$

□

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. For that aim, we need to prove some results that are important to understand the classes $C^\alpha(\mathbb{Z})$, Λ_H^α, and Λ_P^α.

Lemma 3.1. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, and $f \in C^\alpha(\mathbb{Z})$. Then, there exists a constant $C > 0$ such that*

$$|f(n)| \leq C(1 + |n|^\alpha), \quad n \in \mathbb{Z}.$$

Proof. Assume first that $0 < \alpha < 1$. Then, $|f(n)| \leq |f(n) - f(0)| + |f(0)| \leq C(1 + |n|^\alpha)$.

Now, assume that $1 < \alpha < 2$. By definition, this means that $\delta_{\text{right}} f, \delta_{\text{left}} f \in C^{\alpha-1}(\mathbb{Z})$ and, from the previous case, we have that

$$|\delta_{\text{right}} f(n)| \leq C(1 + |n|^{\alpha-1})$$

and the same inequality holds for $\delta_{\text{left}} f$.

Therefore, for $n \in \mathbb{N}$,

$$\begin{aligned}|f(n)| &\le |f(n) - f(n-1)| + \cdots + |f(1) - f(0)| + |f(0)| \\ &= \sum_{j=1}^{n} |\delta_{\text{left}} f(j)| + |f(0)| \\ &\le C\, n(1 + |n|^{\alpha-1}) + |f(0)| \le C(1 + |n|^{\alpha}).\end{aligned}$$

Similarly, for $n \in \mathbb{Z}_- = \mathbb{Z}\backslash\mathbb{N}_0$,

$$\begin{aligned}|f(n)| &\le |f(n) - f(n+1)| + \cdots + |f(-1) - f(0)| + |f(0)| \\ &= \sum_{j=n}^{-1} |\delta_{\text{right}} f(j)| + |f(0)| \\ &\le C|n|(1 + |n|^{\alpha-1}) + |f(0)| \le C(1 + |n|^{\alpha}).\end{aligned}$$

By iterating the previous arguments, we get the result for $\alpha > 2$. □

The following theorem was proved in [9, Theorem 4.1] for the Hermite operator and [10, Theorem 5.6] for general Schrödinger operators satisfying a reverse Hölder inequality. The proof for the discrete Lipschitz spaces is the same, so we omit the details.

Theorem 3.2. *Let* $\alpha > 0$ *and* $f : \mathbb{Z} \to \mathbb{R}$ *such that* $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$. *If* $f \in \Lambda_H^{\alpha}$, *then* $f \in \Lambda_P^{\alpha}$.

Theorem 3.3. *For* $0 < \alpha < 1$, $C^{\alpha}(\mathbb{Z}) = \Lambda_H^{\alpha} = \Lambda_P^{\alpha}$.

Proof. Let $f \in C^{\alpha}(\mathbb{Z})$, $0 < \alpha < 1$. From Lemma 3.1, we have that $\frac{|f|}{1+|\cdot|^{\alpha}} \in \ell^{\infty}(\mathbb{Z})$.
Since the total mass $\sum_{j\in\mathbb{Z}} G(t, j) = 1$, we can write

$$|\partial_t e^{t\Delta_d} f(n)| = \left|\sum_{j\in\mathbb{Z}} \partial_t G(t, n-j)(f(j) - f(n))\right| \le C \sum_{j\in\mathbb{Z}} |\partial_t G(t, j)||j|^{\alpha}.$$

Since $\partial_t G(t, j) = \Delta_d G(t, j) = \delta^2_{\text{right}} G(t, j-1)$, and $\delta^2_{\text{right}} G(t, j-1) = \delta^2_{\text{right}} G(t, |j|-1)$ for $j \le -1$, we can write for every $t > 0$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |\partial_t G(t, j)||j|^{\alpha} &= 2 \sum_{j\ge 1} |\delta^2_{\text{right}} G(t, j-1)||j|^{\alpha} \\ &= 2\Bigg(\sum_{1\le j\le\sqrt{t}} + \sum_{j>\sqrt{t}}\Bigg)|\delta^2_{\text{right}} G(t, j-1)||j|^{\alpha} \\ &\le C\Bigg(t^{-1+\alpha/2} + \sum_{j>\sqrt{t}} |\delta^2_{\text{right}} G(t, j-1)||j|^{\alpha}\Bigg),\end{aligned}$$

where in the last inequality we have applied Lemma 2.6. Assume first that $t \leq 1$. Then, $j > \sqrt{t}$ if and only if $j \geq 1$, so we have, by (2.5), that

$$\sum_{j\geq 1} |\delta^2_{\text{right}} G(t, j-1)||j|^\alpha \leq C \sum_{j\geq 0} G(t, j)(j+1)^2 \leq C(1 + p_1(2t)) \leq Ct^{-1+\alpha/2}.$$

Now assume that $t > 1$. If $j > \sqrt{t}$, then $j \geq 2$ and therefore, $j \leq 2(j-1)$. Thus, by using Lemma 2.4, the fact that $G(t, j)$ is decreasing in $j \in \mathbb{N}_0$ and (2.5), we get that

$$\begin{aligned}\sum_{j>\sqrt{t}} |\delta^2_{\text{right}} G(t, j-1)||j|^\alpha &\leq \frac{C}{t} \sum_{j>\sqrt{t}} G(t, j-1)\left(\frac{(j-1/2)^2}{t} + 1\right)|j|^\alpha \\ &\leq \frac{C}{t^{3-\alpha/2}} \sum_{j\geq 2} G(t, j-1)|j-1|^4 \leq C\frac{p_2(2t)}{t^{3-\alpha/2}} \\ &\leq Ct^{-1+\alpha/2}.\end{aligned}$$

Since for $0 < \alpha < 1$ a function such that $\frac{|f|}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$ also satisfies $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$, from Theorem 3.2 we know that $\Lambda^\alpha_H \subseteq \Lambda^\alpha_P$.

Now, we prove that $\Lambda^\alpha_P \subseteq C^\alpha(\mathbb{Z})$. Let $f \in \Lambda^\alpha_P$ and $n \neq m$ integer numbers. We assume without loss of generality that $m > n$. We fix $y = |n - m| > 0$. Then,

$$\begin{aligned}|f(n) - f(m)| &\leq |f(n) - e^{-y\sqrt{-\Delta_d}} f(n)| + |e^{-y\sqrt{-\Delta_d}} f(n) \\ &\quad - e^{-y\sqrt{-\Delta_d}} f(m)| + |e^{-y\sqrt{-\Delta_d}} f(m) - f(m)| \\ &= (I) + (II) + (III).\end{aligned}$$

From Lemma 2.12 B and the hypothesis, we have that

$$(I) = \left|\int_0^y \partial_u e^{-u\sqrt{-\Delta_d}} f(n)\, du\right| \leq C \int_0^y u^{-1+\alpha}\, du = Cy^\alpha = C|n-m|^\alpha.$$

The same computation works for (III).

On the other hand, by using Lemma 2.15, we get that

$$\begin{aligned}|e^{-y\sqrt{-\Delta_d}} f(n) - e^{-y\sqrt{-\Delta_d}} f(m)| &\leq |n-m| \sup_{n'\in[n,m-1]} \left|\delta_{\text{right}} e^{-y\sqrt{-\Delta_d}} f(n')\right| \\ &\leq C|n-m|y^{-1+\alpha} = C|n-m|^\alpha.\end{aligned}$$

We conclude that $f \in C^\alpha(\mathbb{Z})$. □

Theorem 3.4. *Let $0 < \alpha < 2$ and $f : \mathbb{Z} \to \mathbb{C}$ be a function such that $\frac{f}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$ and $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$. The following are equivalent:*

(1) $f \in \Lambda^\alpha_H$.
(2) $f \in \Lambda^\alpha_P$.

(3) *f satisfies*

$$\sup_{n\neq 0}\frac{\|f(\cdot+n)+f(\cdot-n)-2f(\cdot)\|_\infty}{|n|^\alpha}<\infty. \tag{3.1}$$

Proof. From Theorem 3.2, we know that (1) $\implies$ (2). Let $f\in\Lambda_P^\alpha$. If $0<\alpha<1$, then from Theorem 3.3 we have that

$$\begin{aligned}|f(n+m)+f(n-m)-2f(n)|&\le|f(n+m)-f(n)|+|f(n-m)-f(n)|\\&\le C|m|^\alpha,\quad n,m\in\mathbb{Z}.\end{aligned}$$

Now assume that $1\le\alpha<2$ and, without loss of generality, that $m\in\mathbb{N}$. Then, for $y=m$ and $n\in\mathbb{Z}$ we have

$$\begin{aligned}&|f(n+m)+f(n-m)-2f(n)|\\&\quad\le|f(n+m)-e^{-y\sqrt{-\Delta_d}}f(n+m)+f(n-m)-e^{-y\sqrt{-\Delta_d}}f(n-m)-2f(n)\\&\qquad+2e^{-y\sqrt{-\Delta_d}}f(n)|\\&\qquad+|e^{-y\sqrt{-\Delta_d}}f(n+m)+e^{-y\sqrt{-\Delta_d}}f(n-m)-2e^{-y\sqrt{-\Delta_d}}f(n)|=I+II.\end{aligned}$$

If $1<\alpha<2$, Lemmas 2.12B and 2.15 gives that

$$\begin{aligned}I&=\left|\int_0^y(\partial_ue^{-u\sqrt{-\Delta_d}}f(n+m)+\partial_ue^{-u\sqrt{-\Delta_d}}f(n-m)-2\partial_ue^{-u\sqrt{-\Delta_d}}f(n))\mathrm{d}u\right|\\&\le C\,m\int_0^y\Big(\sup_{n'\in[n,n+m-1]}|\delta_{\text{right}}\partial_ue^{-u\sqrt{-\Delta_d}}f(n')|\\&\quad+\sup_{n''\in[n-m,n-1]}|\delta_{\text{right}}\partial_ue^{-u\sqrt{-\Delta_d}}f(n'')|\Big)\mathrm{d}u\\&\le C\,m\int_0^y u^{-2+\alpha}\mathrm{d}u=Cm^\alpha.\end{aligned}$$

If $\alpha=1$, by using that $\partial_ue^{-u\sqrt{-\Delta_d}}f(n)=-\int_u^y\partial_w^2e^{-w\sqrt{-\Delta_d}}f(n)\mathrm{d}w+\partial_ye^{-y\sqrt{-\Delta_d}}f(n)$, we have that

$$\begin{aligned}I&\le C\int_0^y\int_u^y w^{-1}\mathrm{d}w\mathrm{d}u+\Big|\int_0^y(\partial_ye^{-y\sqrt{-\Delta_d}}f(n+m)+\partial_ye^{-y\sqrt{-\Delta_d}}f(n-m)\\&\quad-2\partial_ye^{-y\sqrt{-\Delta_d}}f(n))\mathrm{d}u\Big|\\&\le C\left(y\log(y)-\int_0^y\log(u)\mathrm{d}u\right)+|y||m|(\sup_{n'\in[n,n+m-1]}|\delta_{\text{right}}\partial_ye^{-y\sqrt{-\Delta_d}}f(n')|\\&\quad+\sup_{n''\in[n-m,n-1]}|\delta_{\text{right}}\partial_ye^{-y\sqrt{-\Delta_d}}f(n'')|)\\&\le C(y+y\,m\,y^{-1})=C\,m.\end{aligned}$$

On the other hand, we have that

$$
\begin{aligned}
II &= |(e^{-y\sqrt{-\Delta_d}} f(n+m) - e^{-y\sqrt{-\Delta_d}} f(n+m-1)) + \cdots + (e^{-y\sqrt{-\Delta_d}} f(n+1) \\
&\quad - e^{-y\sqrt{-\Delta_d}} f(n)) \\
&\quad - (e^{-y\sqrt{-\Delta_d}} f(n) - e^{-y\sqrt{-\Delta_d}} f(n-1)) - \cdots - (e^{-y\sqrt{-\Delta_d}} f(n-m+1) \\
&\quad - e^{-y\sqrt{-\Delta_d}} f(n-m))| \\
&= \left| \sum_{j=1}^{m} (\delta_{\text{right}} e^{-y\sqrt{-\Delta_d}} f(n-j) - \delta_{\text{right}} e^{-y\sqrt{-\Delta_d}} f(n+j-1)) \right| \\
&\leq \sum_{j=1}^{m} |2j-1| \left| \sup_{n' \in [n-j, n+j-2]} \delta_{\text{right}} (\delta_{\text{right}} e^{-y\sqrt{-\Delta_d}} f(n')) \right| \leq C m^{\alpha}.
\end{aligned}
$$

Finally, we prove (3) $\Longrightarrow$ (1). Suppose that f satisfies (3.1). Since $G(t, j) = G(t, -j)$, $j \in \mathbb{N}$, and $\partial_t e^{t\Delta_s} 1 = 0$, we have for $t > 0$ that

$$
\begin{aligned}
|\partial_t e^{t\Delta_d} f(n)| &= \left| \frac{1}{2} \sum_{j \in \mathbb{Z}} \partial_t G(t, j)(f(n-j) + f(n+j) - 2f(n)) \right| \\
&\leq C \sum_{j \in \mathbb{Z}} |\partial_t G(t, j)||j|^{\alpha}.
\end{aligned}
$$

The rest of the argument follows as in the proof of Theorem 3.3. □

Remark 3.5. Notice that in the previous theorem, the assumption $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$ is only needed in the implications in which Λ_P^{α} appears. It can be proved that (1) $\Longleftrightarrow$ (3) only assuming that the function satisfies $\frac{f}{1+|\cdot|^{\alpha}} \in \ell^{\infty}(\mathbb{Z})$.

Theorem 3.6. *Let $\alpha > 1$ and $f : \mathbb{Z} \to \mathbb{R}$. Then, $f \in \Lambda_H^{\alpha}$ if, and only if $\delta_{\text{right}} f \in \Lambda_H^{\alpha-1}$.*

Proof. Suppose that $f \in \Lambda_H^{\alpha}$ and let $k = [\alpha/2] + 1$.

We prove first that $\frac{|\delta_{\text{right}} f|}{1+|\cdot|^{\alpha-1}} \in \ell^{\infty}(\mathbb{Z})$. Take $n \neq 0$. From Lemma 2.12 A (iii), we have that

$$
\begin{aligned}
|\delta_{\text{right}} f(n)| &\leq \sup_{0<t<|n|^2} |e^{t\Delta_d} \delta_{\text{right}} f(n)| \\
&\leq \sup_{0<t<|n|^2} |e^{t\Delta_d} \delta_{\text{right}} f(n) - e^{|n|^2\Delta_d} \delta_{\text{right}} f(n)| \\
&\quad + |e^{|n|^2\Delta_d} \delta_{\text{right}} f(n)| = A + B.
\end{aligned}
$$

Regarding B, by using Lemma 2.12 A (ii) we get that

$$
|B| = |\delta_{\text{right}} e^{|n|^2\Delta_d} f(n)| \leq C(1 + |n|^{\alpha-1}).
$$

To deal with A, we have to distinguish cases. If $1 < \alpha < 2$, then

$$
\begin{aligned}
|A| &= \sup_{0<t<|n|^2} \left| \int_t^{|n|^2} \partial_u \delta_{\text{right}} e^{u\Delta_d} f(n) \mathrm{d}u \right| \\
&\leq C \sup_{0<t<|n|^2} (|n|^{-1+\alpha} + t^{-1/2+\alpha/2}) \leq C(1+|n|^{\alpha-1}).
\end{aligned}
$$

Now consider the case $2 \leq \alpha < 4, \alpha \neq 3$. Then, $[\alpha/2]+1 = 2$ and from Lemma 2.15 we have that

$$
\begin{aligned}
|A| = &\sup_{0<t<|n|^2} \left| \int_t^{|n|^2} \left(\int_u^{|n|^2} \partial_w^2 \delta_{\text{right}} e^{w\Delta_d} f(n) \mathrm{d}w + \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n) \Big|_{v=|n|^2} \right) \mathrm{d}u \right| \\
&\leq C \sup_{0<t<|n|^2} \left(\int_t^{|n|^2} \int_u^{|n|^2} w^{-5/2+\alpha/2} \mathrm{d}w \mathrm{d}u + (|n|^2 - t) \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n) \Big|_{v=|n|^2} \right) \\
&\leq C \sup_{0<t<|n|^2} \left(\int_t^{|n|^2} (|n|^{-3+\alpha} - u^{-3/2+\alpha/2}) \mathrm{d}u + (|n|^2 - t) \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n) \Big|_{v=|n|^2} \right) \\
&\leq C \sup_{0<t<|n|^2} \Big(|n|^{-3+\alpha} (|n|^2 - t) + (|n|^{-1+\alpha} - t^{-1/2+\alpha/2}) \\
&\quad + (|n|^2 - t) \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n) \Big|_{v=|n|^2} \Big) \\
&\leq C|n|^{\alpha-1} + C|n|^2 \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n) \Big|_{v=|n|^2}.
\end{aligned} \tag{3.2}
$$

Now, we use Lemma 2.12 A (ii) to get

$$
\left| \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n) \Big|_{v=|n|^2} \right| = |\delta_{\text{right}}^3 e^{|n|^2\Delta_d} f(n-1)| \leq C \frac{1+|n|^\alpha}{n^3}.
$$

Therefore, $|A| \leq C(1+|n|^{\alpha-1})$.

In general, if α is not an odd number we can proceed as in (3.2), but writing $[\alpha/2]+1$ integrals, such that inside the inner integral will be $\partial_w^{[\alpha/2]+1} \delta_{\text{right}} e^{w\Delta_d} f(n)$.

If α is odd, we have to proceed similarly, but now it will appear some logarithms in the integrals. We do the case $\alpha = 3$ to illustrate the computation, but the rest of the cases are analogous.

$$
\begin{aligned}
|A| = &\sup_{0<t<|n|^2} \left| \int_t^{|n|^2} \left(\int_u^{|n|^2} \partial_w^2 \delta_{\text{right}} e^{w\Delta_d} f(n) \mathrm{d}w + \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n) \Big|_{v=|n|^2} \right) \mathrm{d}u \right| \\
&\leq \sup_{0<t<|n|^2} \left(\int_t^{|n|^2} \int_u^{|n|^2} w^{-1} \mathrm{d}w \mathrm{d}u + (|n|^2 - t) \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n) \Big|_{v=|n|^2} \right) \\
&\leq C \sup_{0<t<|n|^2} \left(\int_t^{|n|^2} (\log(|n|^2) - \log u) \mathrm{d}u + |n|^2 \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n) \Big|_{v=|n|^2} \right) \\
&= C \sup_{0<t<|n|^2} [\log |n|^2 (|n|^2 - t) - (|n|^2 \log |n|^2) + |n|^2
\end{aligned}
$$

$$+ t\log t - t + |n|^2 \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n)\Big|_{v=|n|^2}]$$
$$\leq C|n|^2 + C|n|^2 \partial_v \delta_{\text{right}} e^{v\Delta_d} f(n)\Big|_{v=|n|^2} \leq C(1+|n|^2).$$

Now, we prove the condition on the semigroup. Lemma 2.15 implies that

$$\|\partial_t^k \delta_{\text{right}} e^{t\Delta_d} f\|_\infty \leq C t^{-(k+1/2)+\alpha/2} = C t^{-k+\frac{\alpha-1}{2}}.$$

Since $\partial_t^k \delta_{\text{right}} e^{t\Delta_d} f = \partial_t^k e^{t\Delta_d}(\delta_{\text{right}} f)$, see Lemma 2.11, from Lemma 2.14 we get that $\delta_{\text{right}} f \in \Lambda_H^{\alpha-1}$.

Assume now that $\delta_{\text{right}} f \in \Lambda_H^{\alpha-1}$. By definition, we have that $\frac{|\delta_{\text{right}} f|}{(1+|\cdot|^{\alpha-1})} \in \ell^\infty(\mathbb{Z})$. Thus, the proof of Lemma 3.1 gives that $\frac{|f|}{(1+|\cdot|^{\alpha})} \in \ell^\infty(\mathbb{Z})$.

Let $k = [(\alpha-1)/2] + 1$. From Lemma 2.15 we have that

$$\|\partial_t^k \delta_{\text{left}} e^{t\Delta_d}(\delta_{\text{right}} f)\|_\infty \leq C t^{-(k+1/2)+\frac{\alpha-1}{2}} = C t^{-(k+1)+\frac{\alpha}{2}}.$$

Since $\partial_t^k \delta_{\text{left}} e^{t\Delta_d}(\delta_{\text{right}} f) = \partial_t^k \delta_{\text{left}} \delta_{\text{right}} e^{t\Delta_d} f$, we have that

$$\|\partial_t^k \Delta_d e^{t\Delta_d} f\|_\infty \leq C t^{-(k+1)+\frac{\alpha}{2}}.$$

Therefore, (1.2) gives that $\|\partial_t^{k+1} e^{t\Delta_d} f\|_\infty \leq C t^{-(k+1)+\alpha/2}$, so from Lemma 2.14 we conclude that $f \in \Lambda_H^\alpha$. □

Theorem 3.7. *Let* $\alpha > 1$ *and* $f : \mathbb{Z} \to \mathbb{R}$. *If* $f \in \Lambda_P^\alpha$, *then* $\delta_{\text{right}} f \in \Lambda_P^{\alpha-1}$.

Proof. Let $k = [\alpha]$. Suppose that $f \in \Lambda_P^\alpha$. Then, $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$ and Lemma 2.15 implies that

$$\|\partial_y^k \delta_{\text{right}} e^{-y\sqrt{-\Delta_d}} f\|_\infty \leq C y^{-(k+1)+\alpha} = C y^{-k+\alpha-1}.$$

It is clear that $\sum_{j\in\mathbb{Z}} \frac{|\delta_{\text{right}} f(j)|}{1+|j|^2} < \infty$. Moreover, since $\partial_y^k \delta_{\text{right}} e^{-y\sqrt{-\Delta_d}} f = \partial_y^k e^{-y\sqrt{-\Delta_d}}(\delta_{\text{right}} f)$, see Lemma 2.11, from Lemma 2.14 we get that $\delta_{\text{right}} f \in \Lambda_P^{\alpha-1}$. □

Remark 3.8. Since $\delta_{\text{left}} f(n) = -\delta_{\text{right}} f(n-1), n \in \mathbb{Z}$, it is clear that Theorems 3.6, 3.7 hold for δ_{left}.

Finally, we can prove Theorem 1.1.

Theorem 1.1. (A) *Let* $\alpha > 0$.

(A.1) *If* $\alpha \notin \mathbb{N}$, *then* $\Lambda_H^\alpha = C^\alpha(\mathbb{Z})$.

(A.2) *If* $\alpha \in \mathbb{N}$, *then* $\Lambda_H^\alpha = Z_\alpha$.

(B) *Let* $f : \mathbb{Z} \to \mathbb{R}$ *such that* $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$.

(B.1) *For every* $\alpha > 0$, $\alpha \notin \mathbb{N}$,

$$f \in C^\alpha(\mathbb{Z}) \iff f \in \Lambda_H^\alpha \iff f \in \Lambda_P^\alpha.$$

(B.2) *For every* $\alpha \in \mathbb{N}$,

$$f \in Z_\alpha \iff f \in \Lambda_H^\alpha \iff f \in \Lambda_P^\alpha.$$

Proof. We prove first (A.1). In Theorem 3.3, we have proved the result for $0 < \alpha < 1$. Let $k < \alpha < k+1$, for certain $k \in \mathbb{N}$. Assume first that $f \in \Lambda_H^\alpha$. Then, by applying k times Theorem 3.6 we get that $\delta_{\text{right/left}}^{l,s} f \in \Lambda_H^{\alpha-k}$, $l+s=k$, and from Theorem 3.3 and the definition of $C^{\alpha-k}(\mathbb{Z})$ we get that

$$\sup_{n\neq m} \frac{|\delta_{\text{right/left}}^{l,s} f(n) - \delta_{\text{right/left}}^{l,s} f(m)|}{|n-m|^{\alpha-k}} < \infty, \quad \text{whenever } l+s=k,$$

so $f \in C^\alpha(\mathbb{Z})$.

Conversely, suppose that $f \in C^\alpha(\mathbb{Z})$. From Lemma 3.1 we know that $\frac{|f|}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$. Moreover, the definition of the space gives that $\delta_{\text{right/left}}^{l,s} f \in C^{\alpha-k}(\mathbb{Z})$, $l+s=k$. Therefore, Theorem 3.3 implies that $\delta_{\text{right/left}}^{l,s} f \in \Lambda_H^{\alpha-k}$, $l+s=k$. Applying k times Theorem 3.6, we conclude that $f \in \Lambda_H^\alpha$.

Regarding the proof of (A.2), we proceed as in the proof of (A.1), but now we use Theorem 3.4 (see Remark 3.5) instead of Theorem 3.3.

In virtue of Theorem 3.2 and (A.1), to establish (B) we only need to prove that $f \in \Lambda_P^\alpha \implies f \in C^\alpha(\mathbb{Z})$. Let $f \in \Lambda_P^\alpha$. Then, by applying k times Theorem 3.7 we get that $\delta_{\text{right/left}}^{l,s} f \in \Lambda_P^{\alpha-k}$, $l+s=k$, and from Theorem 3.3 and the definition of $C^{\alpha-k}(\mathbb{Z})$ we get that

$$\sup_{n\neq m} \frac{|\delta_{\text{right/left}}^{l,s} f(n) - \delta_{\text{right/left}}^{l,s} f(m)|}{|n-m|^{\alpha-k}} < \infty, \quad \text{whenever } l+s=k,$$

so $f \in C^\alpha(\mathbb{Z})$.

Regarding the proof of (B.2), we proceed as in the proof of (B.1), but now we use Theorem 3.4 instead of Theorem 3.3. □

4. Applications

In this section, we shall prove regularity results for fractional powers of the discrete Laplacian in the Lipschitz spaces defined through the heat semigroup. To this aim, we recall the definition of the fractional powers of the discrete Laplacian, by using the semigroup method, see [8,29,30]. For other works considering fractional powers of the discrete Laplacian, see for instance [13,21].

Let I denote the identity operator. For good enough functions, we define the following operators:

- The Bessel potential of order $\beta > 0$,

$$(I-\Delta_d)^{-\beta/2} f(n) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty e^{-\tau(I-\Delta_d)} f(n) \tau^{\beta/2} \frac{d\tau}{\tau}, \quad n \in \mathbb{Z}.$$

- The positive fractional power of the Laplacian,

$$(-\Delta_d)^{\beta} f(n) = \frac{1}{c_\beta} \int_0^\infty \left(e^{\tau\Delta_d} - I\right)^{[\beta]+1} f(n) \frac{\mathrm{d}\tau}{\tau^{1+\beta}}, \quad n \in \mathbb{Z}, \quad \beta > 0, \tag{4.1}$$

where $c_\beta = \int_0^\infty \left(e^{-\tau} - 1\right)^{[\beta]+1} \frac{\mathrm{d}\tau}{\tau^{1+\beta}}$.

- The negative fractional power of the Laplacian,

$$(-\Delta_d)^{-\beta} f(n) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{\tau\Delta_d} f(n) \frac{\mathrm{d}\tau}{\tau^{1-\beta}}, \quad n \in \mathbb{Z}, \quad 0 < \beta < 1/2.$$

The previous formulae come from the following Gamma formulae, see [8],

$$\lambda^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\lambda t} t^{\beta} \frac{\mathrm{d}t}{t}, \quad \text{and} \quad \lambda^{\beta} = \frac{1}{c_\beta} \int_0^\infty (e^{-\lambda t} - 1)^{[\beta]+1} \frac{\mathrm{d}t}{t^{1+\beta}}, \tag{4.2}$$

where $\beta > 0$ and λ is a complex number with $\mathfrak{Re}\,\lambda \geq 0$.

As shown in Theorem 1.2, Bessel potentials of order $\beta > 0$ are well defined for $f \in \Lambda_H^\alpha$, $\alpha > 0$. However, the fractional powers of the Laplacian, $(-\Delta_d)^{\pm\beta}$, are not well defined in general for Λ_H^α functions and an additional condition is needed. In [8], the authors assumed that the functions belongs to the space

$$\ell_{\pm\beta} := \left\{ u : \mathbb{Z} \to \mathbb{R} : \sum_{m\in\mathbb{Z}} \frac{|u(m)|}{(1+|m|)^{1\pm 2\beta}} < \infty \right\},$$

in order to define $(-\Delta_d)^{\pm\beta} f$, where $0 < \beta < 1$ in the case of the positive powers and $0 < \beta < 1/2$ for the negative ones. Note that such spaces are the analogues in the discrete setting of the ones considered in [26] for the Laplacian in $\mathbb{R}^n$. The choice of these spaces is justified since the discrete kernel in the pointwise formula

$$(-\Delta_d)^{\pm\beta} f(n) = \sum_{m\in\mathbb{Z}} K_{\pm\beta}(n-m) f(m), \quad n \in \mathbb{Z}, \tag{4.3}$$

satisfies $K_\beta(m) \sim \frac{1}{|m|^{1+2\beta}}$, whenever $0 < \beta < 1$ and $K_{-\beta}(m) \sim \frac{1}{|m|^{1-2\beta}}$, for $0 < \beta < 1/2$, see [8]. Observe that the negative powers of the Laplacian are only well defined for $0 < \beta < 1/2$, since the integral that defines it is not absolutely convergent for $\beta \geq 1/2$, see (2.3).

In this section, we also want to prove regularity results for positive powers which can be larger than 1. For that purpose, we extend the definition above of ℓ_β for any $\beta > 0$. Let $\beta > -1/2$ and $n \in \mathbb{Z}$, we define the discrete kernel

$$K_\beta(n) := \begin{cases} 0, & |n| - \beta - 1 \in \mathbb{N}_0, \\ \dfrac{(-1)^{|n|}\Gamma(2\beta+1)}{\Gamma(1+\beta+|n|)\Gamma(1+\beta-|n|)}, & \text{otherwise.} \end{cases} \tag{4.4}$$

Note that when $\beta \in \mathbb{N}_0$, then $K_\beta(n) = 0$ for all $|n| \geq \beta + 1$.

Lemma 4.1. *Let $f \in \ell_\beta$, $\beta > 0$. Then, $(-\Delta_d)^\beta f$ is well defined and*

$$(-\Delta_d)^\beta f(n) = \sum_{j\in\mathbb{Z}} K_\beta(j)(f(n-j) - f(n)), \quad n \in \mathbb{Z}.$$

Moreover, in that case,

$$|(-\Delta_d)^\beta f(n)| \le C \sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|n-j|^{1+2\beta}}, \quad n \in \mathbb{Z}.$$

Proof. First note that since $f \in \ell_\beta$, f has polynomial growth and then $e^{t\Delta_d} f$ is well defined. Let $k \in \mathbb{N}$ such that $k-1 \le \beta < k$ (so $k = [\beta]+1$). Then,

$$\begin{aligned}(e^{t\Delta_d} - I)^k f(n) &= \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} e^{tl\Delta_d} f(n) \\ &= \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} \Bigg(\sum_{j\in\mathbb{N}} G(lt, j)(f(n+j) + f(n-j)) \\ &\quad + G(lt, 0) f(n) \Bigg) + (-1)^k f(n).\end{aligned}$$

Since $-1 = \sum_{l=1}^{k} (-1)^l \binom{k}{l}$ and $G(lt, 0) - 1 = -2\sum_{j\in\mathbb{N}} G(lt, j)$, one obtains that

$$\begin{aligned}&(-1)^k f(n) \left(\sum_{l=1}^{k} (-1)^l \binom{k}{l} G(lt, 0) - \sum_{l=1}^{k} (-1)^l \binom{k}{l} \right) \\ &= (-1)^k f(n) \sum_{l=1}^{k} (-1)^l \binom{k}{l} (-2 \sum_{j\in\mathbb{N}} G(lt, j))\end{aligned}$$

and therefore

$$\begin{aligned}(e^{t\Delta_d} - I)^k f(n) &= \sum_{j\in\mathbb{N}} (f(n+j) + f(n-j) - 2f(n)) \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} G(lt, j) \\ &= \sum_{j\in\mathbb{Z}} (f(n-j) - f(n)) \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} G(lt, j) \\ &= \sum_{j\in\mathbb{Z}\setminus\{0\}} (f(n-j) - f(n)) \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} G(lt, j).\end{aligned}$$

Now, we denote

$$T(t, j) := \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} G(lt, j) = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} G(lt, j), \quad j \in \mathbb{Z} \setminus \{0\},$$

where in the last identity we have used that $G(0, j) = 0$ for $j \neq 0$.

From (2.7) and the fact that $\sum_{l=0}^{k} \int_{-\pi}^{\pi} \left| \binom{k}{l} e^{-ij\theta} e^{-4lt\sin^2\theta/2} \right| \,\mathrm{d}\theta < \infty$, we can apply Fubini's theorem to get that

$$T(t, j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\theta} (e^{-4t\sin^2\theta/2} - 1)^k \,\mathrm{d}\theta.$$

By (4.2), observe that for all $j \neq 0$

$$\begin{aligned} \int_0^\infty |T(t, j)| \frac{\mathrm{d}t}{t^{1+\beta}} &\leq C \int_{-\pi}^{\pi} \int_0^\infty (1 - e^{-4t\sin^2\theta/2})^k \frac{\mathrm{d}t}{t^{1+\beta}} \,\mathrm{d}\theta \\ &\leq C \int_{-\pi}^{\pi} (\sin^2\theta/2)^\beta \,\mathrm{d}\theta < \infty, \end{aligned}$$

and therefore

$$\begin{aligned} \frac{1}{c_\beta} \int_0^\infty T(t, j) \frac{\mathrm{d}t}{t^{1+\beta}} &= \frac{4^\beta}{2\pi} \int_{-\pi}^{\pi} e^{-ij\theta} (\sin^2\theta/2)^\beta \,\mathrm{d}\theta = \frac{4^\beta}{\pi} \int_{-\pi}^{0} \cos(j\theta)(\sin^2\theta/2)^\beta \,\mathrm{d}\theta \\ &= \frac{2}{\pi} 4^\beta \int_{-\pi/2}^{0} \cos(2j\theta)(\sin^2\theta)^\beta \,\mathrm{d}\theta \\ &= \frac{2}{\pi} 4^\beta (-1)^j \int_0^{\pi/2} \cos(2j\theta)\cos^{2\beta}\theta \,\mathrm{d}\theta \\ &= K_\beta(j), \end{aligned}$$

see [24, Section 2.5.12, formula (22)].

Finally, for $|j| \geq k$ we have by (2.6) that

$$\int_0^\infty |T(t, j)| \frac{\mathrm{d}t}{t^{1+\beta}} \leq C \sum_{l=1}^{k} \int_0^\infty G(lt, j) \frac{\mathrm{d}t}{t^{1+\beta}} \leq \frac{C}{1 + |j|^{1+2\beta}}.$$

Therefore, we have proved that $(-\Delta_d)^\beta f$ is well defined that

$$|(-\Delta_d)^\beta f(n)| \leq C \sum_{j\in\mathbb{Z}} |f(n-j) - f(n)| \frac{1}{1 + |j|^{1+2\beta}} \leq C \sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1 + |n-j|^{1+2\beta}},$$

and that $(-\Delta_d)^\beta f(n) = \sum_{j\in\mathbb{Z}} (f(n-j) - f(n)) K_\beta(j)$. □

Remark 4.2. Some observations are now in order:

- Note that if $\beta \in \mathbb{N}_0$, the definition of K_β [see (4.4)] implies that K_β is a sequence of compact support, so K_β belongs to $\ell^1(\mathbb{Z})$. Also, if $\beta > 0$ is not a natural number, then the proof above gives that $|K_\beta(j)| \leq \frac{C}{1+|j|^{1+2\beta}}$ for all $j \in \mathbb{Z}$. So $K_\beta \in \ell^1(\mathbb{Z})$ for all $\beta \geq 0$. Moreover, in the previous proof one also have that $K_\beta(0) = \frac{4^\beta}{2\pi} \int_{-\pi}^{\pi} (\sin^2\theta/2)^\beta \,\mathrm{d}\theta$, so $K_\beta(j)$ are the Fourier coefficients of the function $(2-z-1/z)^\beta = (4\sin^2\theta/2)^\beta$, $z=e^{i\theta}\in\mathbb{T}$. Taking $z=1$, we get

$$\sum_{j\in\mathbb{Z}} K_\beta(j) = 0,$$

so if $f\in\ell_\beta$, then

$$(-\Delta_d)^\beta f(n) = \sum_{j\in\mathbb{Z}} K_\beta(j) f(n-j).$$

- Lemma 4.1 extends and complements [8, Theorem 1.1 (i) and Theorem 1.3 (i)].
- When β is a natural number, the expression $(-\Delta_d)^\beta f(n) = \frac{1}{c_\beta}\int_0^\infty \left(e^{\tau\Delta_d}-I\right)^{[\beta]+1} f(n)\frac{d\tau}{\tau^{1+\beta}}$ given at the beginning of this section coincides with $\underbrace{(-\Delta_d)\circ\cdots\circ(-\Delta_d)}_{\beta\text{ times}} f$ whenever $f\in\ell_\beta$ (recall that any iteration of $\Delta_d f$ is defined for every sequence f).

Lemma 4.3. *Let $f:\mathbb{Z}\to\mathbb{R}$.*

- *If $f\in\ell_{-\beta}$, $0<\beta<1/2$, then for every $s>0$, $e^{s\Delta_d}f\in\ell_{-\beta}$.*
- *If $f\in\ell_\beta$, $\beta>0$, then for every $s>0$, $e^{s\Delta_d}f\in\ell_\beta$.*

Proof. Suppose that $f\in\ell_{-\beta}$, for some $0<\beta<1/2$ and let $s>0$. Then,

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\frac{|e^{s\Delta_d}f(m)|}{1+|m|^{1-2\beta}} &\le \sum_{m\in\mathbb{Z}}\frac{\sum_{j\in\mathbb{Z}}G(s,m-j)|f(j)|}{1+|m|^{1-2\beta}} = \frac{\sum_{j\in\mathbb{Z}}|f(j)|\sum_{u\in\mathbb{Z}}G(s,u)}{1+|j+u|^{1-2\beta}}\\ &\le \sum_{j\in\mathbb{N}}\left(\sum_{u=-\infty}^{-(j+1)}+\sum_{u=-j}^{-1}+\sum_{u=0}^{\infty}\right)\frac{G(s,u)}{1+|j+u|^{1-2\beta}}|f(j)|\\ &\quad+\sum_{j\in\mathbb{Z}_-}\sum_{u\in\mathbb{Z}}\frac{G(s,u)}{1+|j+u|^{1-2\beta}}|f(j)|+\sum_{u\in\mathbb{Z}}\frac{G(s,u)}{1+|u|^{1-2\beta}}|f(0)|.\end{aligned}$$

Observe that the last sum is clearly bounded. On the other hand,

$$\sum_{j\in\mathbb{N}}\sum_{u=0}^{\infty}\frac{G(s,u)}{1+|j+u|^{1-2\beta}}|f(j)| \le \sum_{j\in\mathbb{N}}\frac{|f(j)|}{1+|j|^{1-2\beta}}\sum_{u=0}^{\infty}G(s,u)\le C<\infty.$$

Now, we split the sum in $j\in\mathbb{N}$ into two, obtaining

$$\begin{aligned}&\sum_{j=1}^{[\sqrt{s}]+1}|f(j)|\sum_{u=j+1}^{\infty}\frac{G(s,u)}{1+(u-j)^{1-2\beta}}+\sum_{j=1}^{[\sqrt{s}]+1}|f(j)|\sum_{u=1}^{j}\frac{G(s,u)}{1+(j-u)^{1-2\beta}}\\ &\quad\le \sum_{j=1}^{[\sqrt{s}]+1}|f(j)|\sum_{u=1}^{\infty}G(s,u)\le C_s\end{aligned}$$

and, by using (2.5),

$$
\begin{aligned}
&\sum_{j=[\sqrt{s}]+1}^{\infty}\sum_{u=j+1}^{\infty}\frac{G(s,u)|f(j)|}{1+(u-j)^{1-2\beta}}+\sum_{j=[\sqrt{s}]+1}^{\infty}\sum_{u=1}^{[j/2]}\frac{G(s,u)|f(j)|}{1+(j-u)^{1-2\beta}}\\
&\quad+\sum_{j=[\sqrt{s}]+1}^{\infty}\sum_{u=[j/2]+1}^{j}\frac{G(s,u)|f(j)|}{1+(j-u)^{1-2\beta}}\\
&\quad\le \sum_{j=[\sqrt{s}]+1}^{\infty}|f(j)|\sum_{u=j+1}^{\infty}\frac{G(s,u)}{1+u^{1-2\beta}}(1+u^{1-2\beta})+\sum_{j=[\sqrt{s}]+1}^{\infty}\sum_{u=1}^{[j/2]}\frac{G(s,u)|f(j)|}{1+(j/2)^{1-2\beta}}\\
&\quad+\sum_{j=[\sqrt{s}]+1}^{\infty}\frac{|f(j)|}{\left(\frac{1}{2}\right)^{1-2\beta}+\left(\frac{j}{2}\right)^{1-2\beta}}\sum_{u=[j/2]+1}^{j}G(s,u)\left(\left(\frac{1}{2}\right)^{1-2\beta}+\left(\frac{j}{2}\right)^{1-2\beta}\right)\\
&\quad\le C\sum_{j=[\sqrt{s}]+1}^{\infty}\frac{|f(j)|}{1+j^{1-2\beta}}\sum_{u=1}^{\infty}G(s,u)(1+u^{1-2\beta})\le C(1+p_k(2s)),
\end{aligned}
$$

where k is the least natural number such that $1-2\beta<2k$. For the sum with $j\in\mathbb{Z}_-$, we can proceed similarly. We left the details to the interested reader.

Now assume that $f\in\ell_\beta$, for some $\beta>0$. Then, we can proceed in a completely analogous way as in the previous case, but now the power will be $1+2\beta$, instead of $1-2\beta$. □

Now, we prove our main results of this section.

Theorem 1.2. *Let* $\alpha,\beta>0$.

(i) *If* $f\in\Lambda_H^{\alpha}$, *then* $(I-\Delta_d)^{-\beta/2}f\in\Lambda_H^{\alpha+\beta}$.
(ii) *If* $f\in\ell^\infty(\mathbb{Z})$, *then* $(I-\Delta_d)^{-\beta/2}f\in\Lambda_H^{\beta}$.

Proof. Let $f\in\Lambda_H^\alpha$ and $\ell=[\frac{\alpha+\beta}{2}]+1$. From Lemma 2.12, we have that

$$
\begin{aligned}
|(I-\Delta)^{-\beta/2}f(n)|&\le C\int_0^\infty e^{-\tau}(1+|n|^\alpha+\tau^{\alpha/2})\tau^{\beta/2}\frac{d\tau}{\tau}\\
&\le C(1+|n|^\alpha)\int_0^\infty e^{-\tau}(1+\tau^{\alpha/2})\tau^{\beta/2}\frac{d\tau}{\tau}\le C(1+|n|^{\alpha+\beta}),\quad n\in\mathbb{Z}.
\end{aligned}
$$

Now, we prove the condition on the semigroup. By using again Lemma 2.12, we obtain that

$$
\begin{aligned}
|\partial_t e^{t\Delta_d}f(n)|&=|\Delta_d e^{t\Delta_d}f(n)|=|e^{t\Delta_d}f(n+1)+e^{t\Delta_d}f(n-1)-2e^{t\Delta_d}f(n)|\\
&\le C(1+|n|^\alpha+t^{\alpha/2}),\quad n\in\mathbb{Z},\quad t>0.
\end{aligned}
$$

Thus,

$$|\partial_t^2 e^{t\Delta_d} f(n)| = |\partial_t(\Delta_d e^{t\Delta_d} f(n))| = |\partial_t e^{t\Delta_d} f(n+1) + \partial_t e^{t\Delta_d} f(n-1) - 2\partial_t e^{t\Delta_d} f(n)|$$
$$\leq C(1+|n|^\alpha + t^{\alpha/2}), \quad n \in \mathbb{Z}, \quad t > 0.$$

By iterating the arguments, we have that $|\partial_t^\ell e^{t\Delta_d} f(n)| \leq C(1+|n|^\alpha + t^{\alpha/2}), \quad n \in \mathbb{Z}, \quad t > 0.$

Therefore, by introducing the derivatives inside the integral and by using Lemmas 2.11 and 2.14 we have that

$$\begin{aligned}
|\partial_y^\ell e^{y\Delta_d}((I-\Delta)^{-\beta/2} f)(n)| &= \left| \frac{1}{\Gamma(\beta/2)} \int_0^\infty e^{-\tau} \partial_y^\ell e^{y\Delta_d}(e^{\tau\Delta_d} f)(n) \tau^{\beta/2} \frac{d\tau}{\tau} \right| \\
&\leq C_\beta \int_0^\infty e^{-\tau} |(\partial_w^\ell e^{w\Delta_d} f(n)\Big|_{w=y+\tau})| \tau^{\beta/2} \frac{d\tau}{\tau} \\
&\leq C_\beta \int_0^\infty e^{-\tau} (y+\tau)^{-\ell+\alpha/2} \tau^{\beta/2} \frac{d\tau}{\tau} \\
&\overset{\frac{\tau}{y}=u}{\leq} C_\beta y^{\alpha/2+\beta/2-\ell} \int_0^\infty \frac{u^{\beta/2} e^{-yu}}{(1+u)^{\ell-\alpha/2}} \frac{du}{u} \\
&\leq C_\beta y^{\alpha/2+\beta/2-\ell}.
\end{aligned}$$

When $f \in \ell^\infty(\mathbb{Z})$, we proceed analogously by using that, for $\ell = [\beta/2]+1$,

$$\|\partial_u^\ell e^{u\Delta_d} f\|_\infty \leq \sup_{n\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |\partial_u^\ell G(u,j)||f(n-j)| \leq C \frac{\|f\|_\infty}{u^\ell}, \qquad u > 0,$$

see Lemma 2.6 and Remark 2.7. □

Theorem 1.3. (Schauder estimates) *Let* $\alpha > 0$ *and* $0 < \beta < 1/2$.

(i) *If* $f \in \Lambda_H^\alpha \cap \ell_{-\beta}$, *then* $(-\Delta_d)^{-\beta} f \in \Lambda_H^{\alpha+2\beta}$.

(ii) *If* $f \in \ell^\infty(\mathbb{Z}) \cap \ell_{-\beta}$, *then* $(-\Delta_d)^{-\beta} f \in \Lambda_H^{2\beta}$.

Proof. We shall prove that if $f \in \Lambda_H^\alpha \cap \ell_{-\beta}$, then

$$\frac{|(-\Delta)^{-\beta} f|}{1+|\cdot|^{\alpha+2\beta}} \in \ell^\infty(\mathbb{Z}).$$

Let $f \in \Lambda_H^\alpha \cap \ell_{-\beta}$. Since (4.3) holds, from we have that

$$\begin{aligned}
|(-\Delta)^{-\beta} f(n)| &\leq \sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|n-j|^{1-2\beta}} \\
&= \sum_{|n-j|>2|n|} \frac{|f(j)|}{1+|n-j|^{1-2\beta}} + \sum_{|n-j|\leq 2|n|} \frac{|f(j)|}{1+|n-j|^{1-2\beta}}.
\end{aligned}$$

Since $|n-j| \geq \frac{|j|}{2}$ when $|n-j| > 2|n|$, by using that $f \in \ell_{-\beta}$, we get that the first summand is bounded.

On the other hand, by using that $\frac{|f|}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$ and $|j| \leq 3|n|$ when $|n-j| \leq 2|n|$, we have that

$$\sum_{|n-j|\le 2|n|} \frac{|f(j)|}{1+|n-j|^{1-2\beta}} \le C(1+|n|^{\alpha}) \sum_{|n-j|\le 2|n|} \frac{1}{1+|n-j|^{1-2\beta}} \le C(1+|n|^{\alpha+2\beta}).$$

Following the same steps, it can be proved that for $f \in \ell^\infty(\mathbb{Z}) \cap \ell_{-\beta}$, we have that $\frac{|(-\Delta)^{-\beta} f|}{1+|\cdot|^{2\beta}} \in \ell^\infty(\mathbb{Z})$.

Let $n \in \mathbb{Z}$. From Lemma 4.3, we know that, for every $y > 0$, $e^{y\Delta_d} f \in \ell_{-\beta}$. Moreover, since $\partial_y e^{y\Delta_d} g(n) = \Delta_d e^{y\Delta_d} g(n)$ we can introduce the derivatives inside the integral and apply Fubini's theorem so that, for every $\ell \in \mathbb{N}$,

$$|\partial_y^\ell e^{y\Delta_d}((-\Delta)^{-\beta} f)(n)| = \left|\frac{1}{\Gamma(\beta)} \int_0^\infty \Delta_d^\ell e^{\tau\Delta_d}(e^{y\Delta_d} f)(n)\tau^\beta \frac{d\tau}{\tau}\right| < \infty.$$

The rest of the proof of $\|\partial_y^\ell e^{y\Delta_d}((-\Delta)^{-\beta} f)\|_\infty \le C y^{-\ell+\alpha+2\beta}$, $\ell = [\alpha+2\beta]+1$, follows the same steps as the corresponding proof on Theorem 1.2. □

Theorem 1.4. (Hölder estimates) *Let* $\alpha, \beta > 0$ *such that* $0 < 2\beta < \alpha$.

(i) *If* $f \in \Lambda_H^\alpha \cap \ell_\beta$, *then* $(-\Delta_d)^\beta f \in \Lambda_H^{\alpha-2\beta}$.
(ii) *If* $f \in \Lambda_H^\alpha$ *and* $\beta \in \mathbb{N}$, *then* $\underbrace{(-\Delta_d) \circ \cdots \circ (-\Delta_d)}_{\beta \text{ times}} f \in \Lambda_H^{\alpha-2\beta}$.

Proof. We prove first (i). Let $f \in \Lambda_H^\alpha \cap \ell_\beta$, $\alpha > 2\beta$. Then, by proceeding in a completely analogous way as in Theorem 1.3 and using Lemma 4.1, but now the power will be 2β, instead of -2β, we get that

$$\frac{|(-\Delta)^\beta f|}{1+|\cdot|^{\alpha-2\beta}} \in \ell^\infty(\mathbb{Z}).$$

Now, we prove the condition on the semigroup.

Let $n \in \mathbb{Z}$ and $\ell = [\beta]+1$. From Lemma 4.3, we know that, for every $y > 0$, $e^{y\Delta_d} f \in \ell_\beta$. Moreover, since $\partial_y e^{y\Delta_d} g(n) = \Delta_d e^{y\Delta_d} g(n)$ we can introduce the derivatives inside the integral and apply Fubini's theorem so that, for every $m \in \mathbb{N}$,

$$\begin{aligned}
&\left|\partial_y^m e^{y\Delta_d}((-\Delta_d)^\beta f)(n)\right| \\
&= \left|\frac{1}{c_\beta} \partial_y^m e^{y\Delta_d}\left(\int_0^\infty \int_{[0,t]^\ell} \partial_v^\ell e^{v\Delta_d}|_{v=s_1+\cdots+s_\ell} f(n) d(s_1,\ldots,s_\ell) \frac{dt}{t^{1+\beta}}\right)\right| \\
&= \left|C \int_0^\infty \left(\int_{[0,t]^\ell} \partial_v^{m+\ell} e^{v\Delta_d}|_{v=y+s_1+\cdots+s_\ell} f(n) d(s_1,\ldots,s_\ell)\right) \frac{dt}{t^{1+\beta}}\right| \\
&= \left|C \int_0^\infty \left(\int_{[0,t]^\ell} \Delta_d^{m+\ell} e^{v\Delta_d}|_{v=y+s_1+\cdots+s_\ell} f(n) d(s_1,\ldots,s_\ell)\right) \frac{dt}{t^{1+\beta}}\right| < \infty.
\end{aligned}$$

Let $m=\left[\frac{\alpha}{2}-\beta\right]+1$. Then, $m+\ell=\left[\frac{\alpha}{2}-\beta\right]+1+[\beta]+1>\alpha/2-\beta+\beta=\alpha/2$. As $m+\ell\in\mathbb{N}$ we get $m+\ell\geq[\alpha/2]+1$. Therefore, by using Lemma 2.14, we get that

$$\begin{aligned}
&\left|\partial_y^m e^{y\Delta_d}((-\Delta_d)^\beta f)(n)\right|\\
&\quad=\left|C\int_0^\infty\Big(\int_{[0,t]^\ell}\partial_\nu^{m+\ell}e^{\nu\Delta_d}|_{\nu=y+s_1+\cdots+s_\ell}f(n)d(s_1,\ldots,s_\ell)\Big)\frac{\mathrm{d}t}{t^{1+\beta}}\right|\\
&\quad\leq C\int_0^\infty\Big(\int_{[0,t]^\ell}(y+s_1+\ldots s_\ell)^{-(m+\ell)+\alpha/2}d(s_1,\ldots,s_\ell)\Big)\frac{\mathrm{d}t}{t^{1+\beta}}\\
&\quad=C\int_0^y(\ldots)\frac{\mathrm{d}t}{t^{1+\beta}}+C\int_y^\infty(\ldots)\frac{\mathrm{d}t}{t^{1+\beta}}=C\,[(I)+(II)].
\end{aligned}$$

Now, we shall estimate (I) and (II).

$$\begin{aligned}
(I)&=Cy^{-m+\alpha/2}\int_0^y\int_{[0,t/y]^\ell}(1+s_1+\ldots s_\ell)^{-(m+\ell)+\alpha/2}d(s_1,\ldots,s_\ell)\frac{\mathrm{d}t}{t^{1+\beta}}\\
&\leq C\,y^{-m+\alpha/2}\int_0^y\Big(\frac{t}{y}\Big)^\ell\frac{\mathrm{d}t}{t^{1+\beta}}=C\,y^{-m+\alpha/2-\beta}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(II)&\leq\int_y^\infty\sum_{j=0}^\ell\frac{C_j}{(y+jt)^{m-\alpha/2}}\frac{\mathrm{d}t}{t^{1+\beta}}=\sum_{j=0}^\ell\int_y^\infty\frac{C_j}{(y+jt)^{m-\alpha/2}}\frac{\mathrm{d}t}{t^{1+\beta}}\\
&\leq\sum_{j=0}^\ell C_j y^{-m+\alpha/2-\beta}.
\end{aligned}$$

The last inequality is obtained by observing that $y\leq y+jt\leq(1+\ell)t$ inside the integrals together with the discussion about the sign of $m-\alpha/2$.

Finally, we prove (ii). Assume that $\beta\in\mathbb{N}$ and $f\in\Lambda_H^\alpha$. Understanding now $(-\Delta_d)^\beta f$ as the β-times iteration of $(-\Delta_d)$, and taking into account that $-\Delta_d f(n)=\delta^2_{\text{right}}f(n-1)$, the result follows from applying 2β times Theorem 3.6. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest. No data sets have been generated or analysed in connection with this article.

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