Research Article

Accurate and efficient computations with Wronskian matrices of Bernstein and related bases †

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Summary

In this paper we provide a bidiagonal decomposition of the Wronskian matrices of Bernstein bases of polynomials and other related bases such as the Bernstein basis of negative degree or the negative binomial basis. The mentioned bidiagonal decompositions are used to achieve algebraic computations with high relative accuracy for these Wronskian matrices. The numerical experiments illustrate the accuracy obtained using the proposed decomposition when computing inverse matrices, eigenvalues or singular values and the solution of some related linear systems.

Keywords: Accurate computations, collocation matrices, Wronskian matrices, bidiagonal decompositions, Bernstein bases, negative binomial basis

1 Introduction

Bernstein polynomials have many useful properties and consequently, enjoy a great practical relevance not only in the field of computer-aided geometric design (CAGD), but also in many other fields of mathematics (see [13], [14], [3], [4] and the references therein). The Bernstein basis of polynomials allows the definition of Bézier curves and surfaces that can be used to approximate any curve or surface to a high degree of accuracy. Therefore, the Bernstein basis is the polynomial basis most used in computer-aided geometric design (CAGD) (see [13], [14]). In fact, Bernstein bases on a compact interval are totally positive on their natural domain and have optimal shape preserving [7] and stability [15] properties.

However, Bernstein bases also have numerous and important applications aside from CAGD. For instance, Bernstein bases have been considered in Galerkin methods and collocation methods for the resolution of elliptic and hyperbolic partial differential equations (cf. [3, 4]). They are also useful for applications in optimal control theory (cf. [34]), and in

†This work was partially supported through the Spanish research grant PGC2018-096321-B-I00 (MCIU/AEI) and by Gobierno de Aragón (E41_20R).

stochastic dynamics (cf. [21]). Moreover, in the modeling of chemical reactions, Bézier curves can be used to represent the most probable reaction path in high dimensional configuration space (cf. [5]). In addition, Bernstein polynomials play a fundamental role in approximation theory as they allow to prove the Weierstrass approximation theorem (see [2], [14]).

In [19], Bernstein bases of a negative degree are introduced. These bases are formed by rational functions sharing many properties of their polynomial counterpart. For instance, they form a partition of unity, satisfy Descartes' Rule of Signs and possess recurrence relations as well as two-term formulas for differentiation and degree elevation. Furthermore, negative degree Bernstein bases are also totally positive on their natural domain (see [27]). In contrast with polynomial Bernstein bases, Bernstein bases of negative degree can exactly represent arbitrary functions which are analytic in a neighborhood of zero and uniformly approximate all continuous functions that vanish at minus infinity.

The binomial distribution is frequently used to model the number of successes in a sample of size n. The binomial functions coincide with the Bernstein polynomials of degree n. On the other hand, the negative binomial distribution is an appropriate model to treat those processes in which a certain trial is repeated until a certain number of favorable results are achieved for the first time (see [19, 26]). An $(n+1)$-dimensional negative binomial basis can be obtained by multiplying the polynomials of a n-degree Bernstein basis by a linear factor.

It is well known that many fundamental problems in interpolation and approximation require linear algebra computations related to collocation matrices. Wronskian matrices arise when solving Hermite interpolation problems, in particular Taylor interpolation problems. In CAGD, the resolution of systems of equations with Wronskian matrices is also important for the definition of bases with good properties in interactive curve design (cf. [6]). Furthermore, in other applications of matrix theory, for example in spectral theory, Wronskian matrices of fundamental solution sets to linear differential equations play a relevant role (cf. [20]).

Despite the nice properties of Bernstein polynomials, the corresponding collocation or Wronskian matrices are ill-conditioned. This fact can produce substantial errors when numerically performing algebraic computations with these matrices.

The accurate computation with structured classes of matrices is an important issue in numerical linear algebra. This subject is receiving increasing attention in the recent years (cf. [10]). It usually requires finding an adequate parameterization of the matrices, adapted to their structure. In this case, the parameterizations of the matrices are given by their bidiagonal factorization. The bidiagonal factorization of a nonsingular totally positive (TP) matrix A is the starting point to compute with high relative accuracy (HRA) its eigenvalues and singular values, the matrix A^{-1} and even the solution of $Ax = b$ for vectors b with alternating signs. In fact, if we achieve the computation of this factorization with high relative accuracy (HRA), then we can apply the algorithms presented in [22–24] to solve with HRA the aforementioned algebraic problems. In [30], it was shown that, using a bidiagonal decomposition of the TP collocation matrices of the polynomial Bernstein basis, many algebraic computations related to these matrices can be performed with high relative accuracy (HRA). In [26], it was proved that negative binomial bases are strictly totally positive on $(0, 1)$. Moreover, using the bidiagonal decomposition of the collocation matrix of the Bernstein bases, the corresponding bidiagonal factorization of the collocation matrix of negative binomial bases was deduced. On the other hand, as far as the authors know, the bidiagonal factorization of the collocation matrix of Bernstein bases of negative degree has not been provided in the literature and will be presented in Section 3.

In [28], the bidiagonal decomposition of the Wronskian matrix of the monomial basis of the space of polynomials of a given degree and the bidiagonal factorization of the Wronskian matrix of the basis of exponential polynomials were obtained. Furthermore, in [29] a procedure to accurately compute the bidiagonal decomposition of collocation and Wronskian matrices of the wide family of Jacobi polynomials is proposed. The obtained results are used to get accurate computations using collocation and Wronskian matrices of well-known types of Jacobi polynomials.

In this paper we consider the Wronskian matrices of Bernstein polynomials and other related bases, including interesting bases such as the Bernstein basis of negative degree (see [19]) or the negative binomial basis. An initial difficulty is that some our Wronskian matrices are not TP. Nevertheless, as it will be seen in Section 4, they are shown to be closely related to totally positive matrices whose bidiagonal decomposition can be computed with HRA and so, the algorithms of references [22–24] can be applied to them and from the corresponding results, the results related to the original Wronskian matrices can be obtained.

We now describe the layout of the paper. Section 2 presents basic definitions and results that will be used in the paper. Section 3 provides the bidiagonal decomposition of the collocation matrix of the general class of functions related with the Bernstein basis. As a particular case, the bidiagonal decomposition of the collocation matrix of Bernstein bases of negative degree is obtained. Section 4 deals with the accurate computations with the corresponding Wronskian matrices. We obtain a bidiagonal factorization of these matrices, we characterize when they are TP and we show the algebraic computations that can be performed with HRA in the cases of the Bernstein basis, the Bernstein basis of negative degree and the negative binomial basis. Section 5 presents the complexity of the algorithms in the numerical experiments and confirm the accuracy of the proposed methods for the computation of all eigenvalues, all singular values, the inverses and the solution of some systems of linear equations. Our experiments use matrices whose condition numbers considerably increase with their dimension. Due to this ill conditioning, traditional methods do not achieve accurate solutions when solving the mentioned algebraic problems, in contrast to our proposed methods.

2 Notations and preliminary results

We are going to use the following generalization of combinatorial numbers. Given $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad \binom{\alpha}{\alpha-n} := \binom{\alpha}{n}.$$

Given a basis $(u_0, \dots, u_n)$ of a space of functions defined on a real interval I, the corresponding *collocation matrix* at the sequence $x_1 < \cdots < x_{n+1}$ on I is

$$M_{n+1,x_1,\dots,x_{n+1}} := \big(u_{j-1}(x_i)\big)_{1\le i,j\le n+1}.$$

If the functions are n-times continuously differentiable at $x \in I$, the *Wronskian matrix* at x is

$$W(u_0, \dots, u_n)(x) := (u_{j-1}^{(i-1)}(x))_{i,j=1,\dots,n+1}.$$

where $u^{(i)}(x)$ denotes the i-th derivative of u at x.

A matrix is totally positive (TP) if all its minors are nonnegative. A matrix is strictly totally positive (STP) if all its minors are positive. Some references with many applications of TP matrices are [1, 12, 33].

By Theorem 4.2 and the arguments of p.116 of [18], we have the following result.

Theorem 1. *A nonsingular TP matrix* $\mathrm{A} = (\mathrm{a}_{\mathrm{i,j}})_{1\le \mathrm{i,j}\le \mathrm{n+1}}$ *admits a factorization of the form*

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n, \tag{1}$$

where $\mathrm{F_i}$ *and* $\mathrm{G_i}$ *are the TP, lower and upper triangular bidiagonal matrices given by*

$$F_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \mathrm{m}_{\mathrm{i+1,1}} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \mathrm{m}_{\mathrm{n+1,n+1-i}} & 1 \end{pmatrix}, \quad G_i^T = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \widetilde{\mathrm{m}}_{\mathrm{i+1,1}} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \widetilde{\mathrm{m}}_{\mathrm{n+1,n+1-i}} & 1 \end{pmatrix}, \tag{2}$$

and $\mathrm{D} = \mathrm{diag}\,(\mathrm{p}_{1,1}, \ldots, \mathrm{p}_{\mathrm{n+1,n+1}})$ *has positive diagonal entries. If, in addition, the entries* m_{ij}, $\widetilde{\mathrm{m}}_{\mathrm{ij}}$ *satisfy*

$$m_{ij} = 0 \quad \Rightarrow \quad m_{hj} = 0, \quad \forall\, h > i, \quad \textit{and} \quad \widetilde{m}_{ij} = 0 \quad \Rightarrow \quad \widetilde{m}_{ik} = 0, \quad \forall\, k > j,$$

then the decomposition (1) is unique. The diagonal entries $\mathrm{p}_{\mathrm{i,i}}$ *of* D *are the diagonal pivots of the Neville elimination of* A *and the elements* $\mathrm{m}_{\mathrm{i,j}}$, $\widetilde{\mathrm{m}}_{\mathrm{i,j}}$ *are nonnegative and coincide with the multipliers of the Neville elimination of* A *and* A^{T}*, respectively.*

In [24], the bidiagonal factorization (1) of an $(n+1)\times(n+1)$ nonsingular and TP matrix A is represented by means of a matrix $BD(A) = (BD(A)_{i,j})_{1\le i,j\le n+1}$ such that

$$BD(A)_{i,j} := \begin{cases} m_{i,j}, & \text{if } i > j, \\ p_{i,i}, & \text{if } i = j, \\ \widetilde{m}_{j,i}, & \text{if } i < j. \end{cases} \tag{3}$$

Remark 1. *By Theorem 4.3 of* [18]*, if* $\mathrm{m}_{\mathrm{i,j}} > 0$, $\widetilde{\mathrm{m}}_{\mathrm{i,j}} > 0$, $1 \le \mathrm{j} < \mathrm{i} \le \mathrm{n}+1$, *and* $\mathrm{p}_{\mathrm{i,i}} > 0$, $1 \le \mathrm{i} \le \mathrm{n}+1$, *then* A *is STP.*

The following result can be easily checked and will be useful in next sections.

Lemma 1. *Let* $\mathrm{d}_1, \ldots, \mathrm{d}_{\mathrm{n+1}}$ *be real values and* A *an* $(\mathrm{n}+1)\times(\mathrm{n}+1)$ *TP matrix whose bidiagonal factorization* (1) *is*

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n.$$

Then, the bidiagonal factorization (1) *of* $\tilde{\mathrm{A}} := \mathrm{A}\Delta$ *with* $\Delta = \mathit{diag}(\mathrm{d}_1, \ldots, \mathrm{d}_{\mathrm{n+1}})$ *is*

$$\widetilde{A} = F_n F_{n-1} \cdots F_1 \widetilde{D} \widetilde{G}_1 \cdots \widetilde{G}_{n-1} \widetilde{G}_n,$$

where $\widetilde{\mathrm{D}} = \mathrm{diag}\,(\mathrm{d}_1\mathrm{p}_{1,1}, \mathrm{d}_2\mathrm{p}_{2,2}, \ldots, \mathrm{d}_{\mathrm{n+1}}\mathrm{p}_{\mathrm{n+1,n+1}})$ *and* $\widetilde{\mathrm{G}}_{\mathrm{i}}$, $\mathrm{i} = 1, \ldots, \mathrm{n}$, *are the upper triangular bidiagonal matrices described in* (2) *whose off-diagonal entries are*

$$\widetilde{\mathrm{r}}_{\mathrm{i,j}} = \frac{\mathrm{d}_{\mathrm{i}}}{\mathrm{d}_{\mathrm{i}-1}} \widetilde{\mathrm{m}}_{\mathrm{i,j}}, \quad 1 \le \mathrm{j} < \mathrm{i} \le \mathrm{n}+1.$$

Proof. *Taking into account that* $G_i \Delta = \Delta \widetilde{G_i}$, $i = 1, \ldots, n$, *the result follows.* $\square$

We say that a real x is computed with high relative accuracy (HRA) whenever the computed value $\tilde{x}$ satisfies

$$\frac{\|x - \tilde{x}\|}{\|x\|} < Ku,$$

where u is the unit round-off and $K > 0$ is a constant independent of the arithmetic precision. Clearly, HRA implies that the relative errors in the computations have the same order as the machine precision. It is well known that a sufficient condition to assure that an algorithm can be computed with HRA is the no inaccurate cancellation (NIC) condition and it is satisfied if it only evaluates products, quotients, sums of numbers of the same sign, subtractions of numbers of opposite sign or subtraction of initial data (cf. [10], [11], [24]).

If the bidiagonal factorization (1) of a nonsingular and TP matrix A can be computed with HRA, then the computation of its eigenvalues and singular values, the computation of A^{-1} and even the resolution of $Ax = b$ for vectors b with alternating signs can be also computed with HRA using the algorithms provided in [22].

In [25], we can find algorithms for computing the bidiagonal decomposition (1) of the collocation matrices of a general class of bases $(u_0^n, \ldots, u_n^n)$ with

$$u_i^n(x) := \binom{n}{i} f^i(x) g^{n-i}(x), \quad x \in [a, b], \quad i = 0, \ldots, n,$$

where $f, g : I \to \mathbb{R}$ are functions such that $f(x) \neq 0$, $g(x) \neq 0$ for all $x \in (a, b)$ and f/g is strictly increasing. These bases are of interesest in CAGD and also in Approximation Theory. In particular, Theorem 2 of [25] proves that their collocation matrices,

$$M_{n+1,x_1,\ldots,x_{n+1}} := \left(\tbinom{n}{j-1} f^{j-1}(x_i) g^{n-j+1}(x_i) \right)_{1 \leq i,j \leq n+1},$$

are STP at $x_1 < \cdots < x_{n+1}$ on (a, b). Moreover, Theorem 3 of [25] deduces their bidiagonal factorization (1). Using this factorization, in [25] accurate computations with collocation matrices of bases with algebraic, trigonometric, or hyperbolic polynomials are illustrated.

It can be checked that, following the proof of Theorem 3 of [25], and Lemma 1, the bidiagonal factorization (1) of systems $(u_0^\alpha, \ldots, u_n^\alpha)$ with $\alpha \in \mathbb{R}$ and

$$u_i^\alpha(x) := f^i(x) g^{\alpha-i}(x), \quad i = 0, \ldots, n, \tag{4}$$

can be obtained. The following result describes this factorization.

Theorem 2. *The collocation matrix* $M_{n+1,x_1,\ldots,x_{n+1}}$ *of the basis* (4) *at* $x_1 < \cdots < x_{n+1}$ *in its domain admits the following factorization*

$$M_{n+1,x_1,\ldots,x_{n+1}} = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n,$$

where the entries $m_{i,j}, \tilde{m}_{i,j}$ *and* $p_{i,i}$ *of* F_i *and* G_i, $i = 1, \ldots, n$, *and* D *are given by*

$$\begin{aligned}
m_{i,j} &= \frac{g^{\alpha-j+1}(x_i)g(x_{i-j})}{g^{\alpha-j+2}(x_{i-1})} \frac{\prod_{k=1}^{j-1}\left(f(x_i)g(x_{i-k}) - f(x_{i-k})g(x_i)\right)}{\prod_{k=2}^{j}\left(f(x_{i-1})g(x_{i-k}) - f(x_{i-k})g(x_{i-1})\right)}, \\
\widetilde{m}_{i,j} &= \frac{f(x_j)}{g(x_j)}, \quad 1 \le j < i \le n+1, \\
p_{i,i} &= \frac{g^{\alpha-i+1}(x_i)}{\prod_{k=1}^{i-1} g(x_k)} \prod_{k=1}^{i-1}\left(f(x_i)g(x_k) - f(x_k)g(x_i)\right), \quad 1 \le i \le n+1.
\end{aligned}$$

Let us denote by $\mathbf{P}^n$ the space of polynomials of degree less than or equal to n and $(p_0, \ldots, p_n)$ the monomial basis of $\mathbf{P}^n$ i.e.

$$p_i(x) := x^i, \quad i = 0, \ldots, n. \tag{5}$$

The following result will be used in the sequel and restates Corollary 1 of [28], providing the bidiagonal factorization (1) of the Wronskian matrix $W(p_0, \ldots, p_n)(x)$, $x \in \mathbb{R}$.

Proposition 1. *Let* $(p_0, \ldots, p_n)$ *be the monomial basis given in* (5). *For any* $x \in \mathbb{R}$, *the Wronskian matrix* $W(p_0, \ldots, p_n)(x)$ *is nonsingular and can be factorized as follows,*

$$W(p_0, \ldots, p_n)(x) = D G_{1,n} \cdots G_{n-1,n-1} G_{n,n}, \tag{6}$$

where $D = diag\{0!, 1!, \ldots, n!\}$ *and* $G_{i,n}$, $i = 1, \ldots, n$, *are the upper triangular bidiagonal matrices in* (2) *with*

$$\widetilde{m}_{k,k-i} = x, \quad i+1 \le k \le n+1. \tag{7}$$

Moreover, if $x > 0$ *then* $W(p_0, \ldots, p_n)(x)$ *is nonsingular and TP, its bidiagonal decomposition* (1) *is given by* (6) *and* (7) *and it can be computed with HRA.*

In [28], using this result, accurate computations with Wronskian matrices of monomial bases are achieved.

In the following sections we shall obtain the bidiagonal factorization (1) of collocation and Wronskian matrices associated to a general class of functions that includes, as particular cases, polynomial Bernstein bases, negative binomial bases or Bernstein bases of negative degree. For all considered cases, we are going to achieve algebraic computations with HRA.

3 Bidiagonal decomposition of the collocation matrix of a general class of functions

Let us consider the system of functions $(f_0^\alpha, \ldots, f_n^\alpha)$, $\alpha \in \mathbb{R}$, defined by

$$f_i^\alpha(x) := x^i(1-x)^{\alpha-i}, \quad i = 0, \ldots, n, \tag{8}$$

on their natural domain. Let us observe that the Bernstein basis of the space $\mathbf{P}^n$, given by $(B_0^n, \ldots, B_n^n)$ and

$$B_i^n(x) := \binom{n}{i} x^i \, (1-x)^{n-i}, \quad i = 0, \ldots, n, \tag{9}$$

is $(c_0 f_0^n, \ldots, c_n f_n^n)$ with $c_i = \binom{n}{i}$, $i = 0, \ldots, n$. Moreover, there are other interesting bases which can be obtained by scaling the systems (8). For example, if $\alpha = -n$ and $c_i = \binom{-n}{i} = (-1)^i \binom{n+i-1}{i}$, $i = 0, \ldots, n$, then $(c_0 f_0^{-n}, \ldots, c_n f_n^{-n})$ is the Bernstein basis of negative degree $(B_0^{-n}, \ldots, B_n^{-n})$ with

$$B_i^{-n}(x) := \binom{-n}{i} x^i (1-x)^{-n-i} = \binom{n+i-1}{i} (-x)^i (1-x)^{-n-i}, \quad i = 0, \ldots, n, \tag{10}$$

(cf. [19]).

On the other hand, if $\alpha = n+1$ and $c_i = \binom{n}{i}$, $i = 0, \ldots, n$, then $(c_0 f_0^{n+1}, \ldots, c_n f_n^{n+1})$ is the negative binomial basis $(b_0^{n+1}, \ldots, b_n^{n+1})$ with

$$b_i^{n+1}(x) := \binom{n}{i} x^i (1-x)^{n-i+1}, \quad i = 0, \ldots, n, \tag{11}$$

(cf. [19], [26]).

Using Theorem 2, with $f(x) = x$ and $g(x) = 1-x$, it can be checked that the collocation matrix

$$M_{n+1,x_1,\ldots,x_{n+1}} := \left(x_i^{j-1} (1-x_i)^{\alpha-j+1} \right)_{1 \le i,j \le n+1},$$

admits the following factorization:

$$M_{n+1,x_1,\ldots,x_{n+1}} = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n, \tag{12}$$

where F_i and G_i, $i = 1, \ldots, n$, are the lower and upper triangular bidiagonal matrices described in (2) and $D = \operatorname{diag}(p_{1,1}, \ldots, p_{n+1,n+1})$. The entries $m_{i,j}, \widetilde{m}_{i,j}$ and $p_{i,i}$ are given by

$$m_{i,j} = \frac{(1-x_i)^{\alpha-j+1}(1-x_{i-j})}{(1-x_{i-1})^{\alpha-j+2}} \frac{\prod_{k=1}^{j-1}\left(x_i - x_{i-k}\right)}{\prod_{k=2}^{j}\left(x_{i-1} - x_{i-k}\right)}, \quad 1 \le j < i \le n+1,$$
$$\widetilde{m}_{i,j} = \frac{x_j}{1-x_j}, \; 1 \le j < i \le n+1, \quad p_{i,i} = (1-x_i)^{\alpha-i+1} \prod_{k=1}^{i-1} \frac{x_i - x_k}{1-x_k}, \quad 1 \le i \le n+1.$$

Then, analyzing the sign of $m_{i,j}, \widetilde{m}_{i,j}$ and $p_{i,i}$ and using Remark 1, it can be easily deduced that $M_{n+1,x_1,\ldots,x_{n+1}}$ is STP for any $\alpha \in \mathbb{R}$ and any sequence of parameters such that $0 < x_1 < \cdots < x_{n+1} < 1$.

Using Lemma 1 and the decomposition (12) of the collocation matrix of $(f_0^\alpha, \ldots, f_n^\alpha)$, the bidiagonal factorization (1) of the collocation matrices of any system $(c_0 f_0^\alpha, \ldots, c_n f_n^\alpha)$, $c_i \in \mathbb{R}$, $i = 0, \ldots, n$, can be obtained.

In particular, the bidiagonal factorization (1) of the collocation matrices of Bernstein polynomial bases and negative binomial bases can be deduced. By means of this factorization, accurate computations with these matrices have been already achieved (see [25], [26], [30] and the references therein).

Furthermore, we can also deduce that the collocation matrix of the Bernstein basis of degree $-n$ satisfies

$$\left(B_{j-1}^{-n}(x_i)\right)_{1 \le i,j \le n+1} = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n,$$

and the entries $m_{i,j}, \widetilde{m}_{i,j}$ and $p_{i,i}$ of F_i, G_i, $i = 1, \ldots, n$, and D, respectively, are given by

$$m_{i,j} = \frac{(1-x_i)^{-n-j+1}(1-x_{i-j})}{(1-x_{i-1})^{-n-j+2}} \frac{\prod_{k=1}^{j-1}\left(x_i - x_{i-k}\right)}{\prod_{k=2}^{j}\left(x_{i-1} - x_{i-k}\right)}, \quad 1 \le j < i \le n+1,$$
$$\widetilde{m}_{i,j} = -\frac{n+i-2}{i-1} \frac{x_j}{1-x_j}, \quad 1 \le j < i \le n+1,$$
$$p_{i,i} = (-1)^{i-1} \binom{n+i-2}{i-1} (1-x_i)^{-n-i+1} \prod_{k=1}^{i-1} \frac{x_i - x_k}{1-x_k}, \quad 1 \le i \le n+1. \tag{13}$$

Analyzing the sign of the entries in (13), we can deduce that the collocation matrix of the Bernstein basis of negative degree defined in (10) is TP for $x_{n+1} < \cdots < x_1 < 0$.

4 Accurate computations with Wronskian matrices of a general class of functions including Bernstein polynomials

In the following results we analyze the total positivity of the Wronskian matrices of the systems $(f_0^\alpha, \ldots, f_n^\alpha)$, $\alpha \in \mathbb{R}$, with

$$f_i^\alpha(x) := x^i(1-x)^{\alpha-i}, \quad x<1, \quad i=0,\ldots,n, \tag{14}$$

through their bidiagonal decomposition (1). First, we prove some auxiliary results.

Lemma 2. *For given* $\alpha, \mathrm{t} \in \mathbb{R}$ *and* $\mathrm{n} \in \mathbb{N}$, *let* $\mathrm{L_{k,n}} = (\mathrm{l_{i,j}^{(k,n)}})_{1\le \mathrm{j,i} \le \mathrm{n}+1}$, $\mathrm{k}=1,\ldots,\mathrm{n}$, *be the* $(\mathrm{n}+1)\times(\mathrm{n}+1)$ *lower triangular bidiagonal matrix with unit diagonal entries, such that*

$$l_{i,i-1}^{(k,n)} = 0, \quad i=2,\ldots,k, \qquad l_{i,i-1}^{(k,n)} = (\alpha+2-i)t, \quad i=k+1,\ldots,n+1.$$

Then, $\mathrm{L_n} := \mathrm{L_{n,n}} \cdots \mathrm{L_{1,n}}$, *is a lower triangular matrix and*

$$L_n = (l_{i,j}^{(n)})_{1\le i,j\le n+1}, \quad l_{i,j}^{(n)} = \frac{(i-1)!}{(j-1)!}\binom{\alpha+1-j}{\alpha+1-i} t^{i-j}, \quad 1\le j\le i\le n+1. \tag{15}$$

Proof. *Clearly,* L_n *is a lower triangular matrix since it is the product of lower triangular bidiagonal matrices. Let us prove* (15) *by induction on* n. *For* $n=1$,

$$L_1 = L_{1,1} = \begin{pmatrix} 1 & \\ \alpha t & 1 \end{pmatrix},$$

and (15) *clearly holds. Now, let us suppose that* (15) *holds for* $n\ge 1$ *and consider the* $(n+2)\times(n+2)$ *product matrix*

$$L_{n+1} := L_{n+1,n+1} L_{n,n+1} \cdots L_{1,n+1}.$$

It can be checked that $\tilde{L}_{n+1} := L_{n+1,n+1} \cdots L_{2,n+1}$ *satisfies* $\tilde{L}_{n+1} = (\tilde{l}_{i,j}^{(n+1)})_{1\le i,j\le n+2}$, *with* $\tilde{l}_{i,1}^{(n+1)} = \delta_{i,1}$, $\tilde{l}_{1,i}^{(n+1)} = \delta_{1,i}$ *and the submatrix of* $\tilde{L}_{n+1}$ *containing rows and columns of places* $\{2,\ldots,n+2\}$, *denoted by* $\tilde{L}_{n+1}[2,\ldots,n+2]$, *satisfies* $\tilde{L}_{n+1}[2,\ldots,n+2] = L_{n,n}\cdots L_{1,n}$.

Therefore, applying the induction hypothesis to $\tilde{L}_{n+1}[2,\dots,n+2]$*, we can deduce from* (15) *that the entries of* $\tilde{L}_{n+1}$ *satisfy the following equalities*

$$\tilde{l}_{i,j}^{(n+1)} := \frac{(i-2)!}{(j-2)!}\binom{\alpha+2-j}{\alpha+2-i}t^{i-j}, \quad 2 \le j \le i \le n+2. \tag{16}$$

Moreover, we can write

$$L_{n+1} = \tilde{L}_{n+1}L_{1,n+1} = \tilde{L}_{n+1}\begin{pmatrix} 1 & & & \\ \alpha t & 1 & & \\ & \ddots & \ddots & \\ & & (\alpha-n)t & 1 \end{pmatrix}. \tag{17}$$

Now, taking into account equalities (16), (17) *and the fact that*

$$\binom{\alpha+2-j}{\alpha+2-i} + \frac{\alpha+2-j}{j-1}\binom{\alpha+1-j}{\alpha+2-i} = \frac{i-1}{j-1}\binom{\alpha+2-j}{\alpha+2-i},$$

we deduce that $L_{n+1} = (l_{i,j}^{(n+1)})_{1\le i,j\le n+2}$ *satisfies*

$$\begin{aligned} l_{i,j}^{(n+1)} &= \tilde{l}_{i,j}^{(n+1)} + \tilde{l}_{i,j+1}^{(n+1)}(\alpha+2-j)\,t \\ &= \frac{(i-2)!}{(j-2)!}\binom{\alpha+2-j}{\alpha+2-i}t^{i-j} + \frac{(i-2)!}{(j-1)!}\binom{\alpha+1-j}{\alpha+2-i}(\alpha+2-j)t^{i-j} \\ &= \frac{(i-2)!}{(j-2)!}\left(\binom{\alpha+2-j}{\alpha+2-i} + \frac{\alpha+2-j}{j-1}\binom{\alpha+1-j}{\alpha+2-i}\right)t^{i-j} = \frac{(i-1)!}{(j-1)!}\binom{\alpha+2-j}{\alpha+2-i}t^{i-j}, \end{aligned}$$

for $1 \le j \le i \le n+2$*. Consequently,* (15) *holds for all* $n \in \mathbb{N}$*.* □

Lemma 3. *For a given* $\mathrm{t} \in \mathbb{R}$ *and* $\mathrm{n} \in \mathbb{N}$*, let* $\mathrm{U_{k,n}} = (\mathrm{u_{i,j}^{(k,n)}})_{1\le \mathrm{j,i} \le \mathrm{n+1}}$*,* $\mathrm{k} = 1,\dots,\mathrm{n}$*, be the* $(\mathrm{n}+1)\times(\mathrm{n}+1)$*, upper triangular bidiagonal matrix with unit diagonal entries, such that*

$$u_{i-1,i}^{(k,n)} := 0, \quad i = 2,\dots,k, \qquad u_{i-1,i}^{(k,n)} := t, \quad i = k+1,\dots,n+1.$$

Then, $\mathrm{U_n} := \mathrm{U_{1,n}} \cdots \mathrm{U_{n,n}}$*, is an upper triangular matrix and*

$$U_n = (u_{i,j}^{(n)})_{1\le i,j\le n+1}, \quad u_{i,j}^{(n)} = \binom{j-1}{i-1}t^{j-i}, \quad 1 \le i \le j \le n+1. \tag{18}$$

Proof. *Clearly, U_n is an upper triangular matrix since it is the product of upper triangular bidiagonal matrices. Taking into account formula* (6) *of Proposition 1, we can deduce that $U_n = diag\{0!, 1!, \dots, n!\}^{-1} W(p_0, \dots, p_n)(x)$ where $p_j(t) := t^j$, $j = 0, \dots, n$. Then,*

$$U_n = \left(\frac{1}{(i-1)!} (p_{j-1}(t))^{(i-1)} \right)_{i,j=1,\dots,n+1}.$$

Finally, taking into account that

$$\frac{1}{i!}(p_j(t))^{(i)} = \binom{j}{i} t^{j-i}, \quad 0 \le i \le j \le n,$$

equalities (18) *are immediately obtained.* □

Now, using Lemma 2 and Lemma 3, we can derive the bidiagonal decomposition (1) of the Wronskian matrix of a system (14).

Theorem 3. *Let* $\mathrm{n} \in \mathbb{N}$, $\alpha \in \mathbb{R}$ *and* $(\mathrm{f}_0^\alpha, \dots, \mathrm{f}_\mathrm{n}^\alpha)$ *the system defined in* (14)*. The Wronskian matrix* $\mathrm{W} := \mathrm{W}(\mathrm{f}_0^\alpha, \dots, \mathrm{f}_\mathrm{n}^\alpha)(\mathrm{x})$ *admits a factorization of the form*

$$W = L_{n,n} L_{n-1,n} \cdots L_{1,n} D U_{1,n} \cdots U_{n-1,n} U_{n,n}, \tag{19}$$

where $\mathrm{L}_{\mathrm{k,n}} = (\mathrm{l}_{\mathrm{i,j}}^{(\mathrm{k,n})})_{1 \le \mathrm{j,i} \le \mathrm{n}+1}$, $\mathrm{k} = 1, \dots, \mathrm{n}$, *are the lower triangular bidiagonal matrices with unit diagonal entries, such that*

$$l_{i,i-1}^{(k,n)} = 0, \quad i = 2, \dots, k, \qquad l_{i,i-1}^{(k,n)} = (\alpha + 2 - i) \frac{-1}{1-x}, \quad i = k+1, \dots, n+1, \tag{20}$$

$\mathrm{U}_{\mathrm{k,n}} = (\mathrm{u}_{\mathrm{i,j}}^{(\mathrm{k,n})})_{1 \le \mathrm{j,i} \le \mathrm{n}+1}$, $\mathrm{k} = 1, \dots, \mathrm{n}$, *are the upper triangular bidiagonal matrices with unit diagonal entries, such that*

$$u_{i-1,i}^{(k,n)} := 0, \quad i = 2, \dots, k, \qquad u_{i-1,i}^{(k,n)} = \frac{x}{1-x}, \quad i = k+1, \dots, n+1, \tag{21}$$

and D *is the diagonal matrix* $\mathrm{D} = \mathrm{diag}\,(\mathrm{d}_1, \dots, \mathrm{d}_{\mathrm{n}+1})$ *with*

$$d_i = (i-1)!(1-x)^{\alpha+2-2i}, \quad i = 1, \dots, n+1. \tag{22}$$

Proof. *Let us observe that, by considering $t=-1/(1-x)$ in Lemma 2, we deduce that $L_n := L_{n,n}L_{n-1,n}\cdots L_{1,n}$ is a lower triangular matrix satisfying*

$$L_n = (l_{i,j}^{(n)})_{1\le i,j\le n+1}, \quad l_{i,j}^{(n)} = \frac{(i-1)!}{(j-1)!}\binom{\alpha+1-j}{i-j}\left(\frac{-1}{1-x}\right)^{i-j}, \quad 1\le j\le i\le n+1. \tag{23}$$

On the other hand, using Lemma 3 with $t=x/(1-x)$, we conclude that $U_n := U_{1,n}\cdots U_{n,n}$ is an upper triangular matrix with

$$U_n = (u_{i,j}^{(n)})_{1\le i,j\le n+1}, \quad u_{i,j}^{(n)} = \binom{j-1}{i-1}\left(\frac{x}{1-x}\right)^{j-i}, \quad 1\le i\le j\le n+1. \tag{24}$$

In order to prove the result, taking into account (19), (22), (23) *and* (24), *we have to check that*

$$(f_{j-1}^{\alpha})^{(i-1)}(x) = (i-1)!\left(\sum_{k=1}^{min\{i,j\}}(-1)^{i-k}\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k}\right)(1-x)^{\alpha+2-i-j}, \tag{25}$$

for $1\le i,j\le n+1$. Let us prove (25) *by induction on i. Let $i=1$, then*

$$\sum_{k=1}^{1}(-1)^{1-k}\binom{\alpha+1-k}{1-k}\binom{j-1}{k-1}x^{j-k}(1-x)^{\alpha-1-j+2} = x^{j-1}(1-x)^{\alpha+1-j} = f_{j-1}^{\alpha}(x),$$

for $j=1,\ldots,n+1$, and (25) *follows. Now, let us assume that* (25) *holds for $i\ge 1$. Then, for any $i\le j\le n+1$, we have*

$$\begin{aligned}
(f_{j-1}^{\alpha})^{(i)}(x) &= \left((f_{j-1}^{\alpha})^{(i-1)}\right)' \\
&= \left((i-1)!\sum_{k=1}^{i}(-1)^{i-k}\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k}(1-x)^{\alpha+2-i-j}\right)' \\
&= (i-1)!(1-x)^{\alpha+1-i-j}\left(\sum_{k=1}^{i}(-1)^{i-k+1}(\alpha+2-i-j)\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k}\right. \\
&\quad \left. +\,(1-x)\sum_{k=1}^{i}(-1)^{i-k}(j-k)\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k-1}\right) \\
&= (i-1)!(1-x)^{\alpha+1-i-j}\left(\sum_{k=1}^{i}(-1)^{i-k+1}(\alpha+2-i-j)\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k}\right. \\
&\quad +\sum_{k=2}^{i+1}(-1)^{i-k+1}(j-k+1)\binom{\alpha+2-k}{i-k+1}\binom{j-1}{k-2}x^{j-k} \\
&\quad \left. +\sum_{k=1}^{i}(-1)^{i-k+1}(j-k)\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k}\right) \\
&= (i-1)!(1-x)^{\alpha+1-i-j}\left(\sum_{k=1}^{i+1}(-1)^{i-k+1}c_k x^{j-k}\right),
\end{aligned}$$

where

$$
\begin{aligned}
c_1 &= (\alpha+1-i)\binom{\alpha}{i-1}\binom{j-1}{0} = i\binom{\alpha}{i},\\
c_k &= (\alpha+2-i-k)\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1} + (j-k+1)\binom{\alpha+2-k}{i-k+1}\binom{j-1}{k-2}\\
&= i\binom{\alpha+1-k}{i+1-k}\binom{j-1}{k-1}, \quad k=2,\ldots,i\\
c_{i+1} &= (j-i)\binom{\alpha+1-i}{0}\binom{j-1}{i-1} = i\binom{j-1}{i}.
\end{aligned}
$$

Then, we can write

$$
(f_{j-1}^{\alpha})^{(i)}(x) = i!\left(\sum_{k=1}^{i+1}(-1)^{i-k+1}\binom{\alpha+1-k}{i+1-k}\binom{j-1}{k-1}x^{j-k}\right)(1-x)^{\alpha+1-i-j},
$$

and check that (25) *holds for* $j=i,\ldots,n+1$. *Now, for* $1 \le j < i$, *we can follow a similar reasoning,*

$$
\begin{aligned}
(f_{j-1}^{\alpha})^{(i)}(x) &= \left((f_{j-1}^{\alpha})^{(i-1)}(x)\right)'\\
&= \left((i-1)!\sum_{k=1}^{j}(-1)^{i-k}\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k}(1-x)^{\alpha+2-i-j}\right)'\\
&= (i-1)!(1-x)^{\alpha+1-i-j}\left(\sum_{k=1}^{j}(-1)^{i-k+1}(\alpha+2-i-j)\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k}\right.\\
&\quad\left.+(1-x)\sum_{k=1}^{j-1}(-1)^{i-k}(j-k)\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k-1}\right)\\
&= (i-1)!(1-x)^{\alpha+1-i-j}\left(\sum_{k=1}^{j}(-1)^{i-k+1}(\alpha+2-i-j)\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k}\right.\\
&\quad+\sum_{k=2}^{j}(-1)^{i-k+1}(j-k+1)\binom{\alpha+2-k}{i-k+1}\binom{j-1}{k-2}x^{j-k}\\
&\quad\left.+\sum_{k=1}^{j}(-1)^{i-k+1}(j-k)\binom{\alpha+1-k}{i-k}\binom{j-1}{k-1}x^{j-k}\right)\\
&= (i-1)!(1-x)^{\alpha+1-i-j}\left(\sum_{k=1}^{j}(-1)^{i-k+1}c_k x^{j-k}\right),
\end{aligned}
$$

and, again, $c_k = i\binom{\alpha+1-k}{i+1-k}\binom{j-1}{k-1}$, $k=1,\ldots,j$. *Then, we can write*

$$
(f_{j-1}^{\alpha})^{(i)}(x) = i!\left(\sum_{k=1}^{j}(-1)^{i-k+1}\binom{\alpha+1-k}{i+1-k}\binom{j-1}{k-1}x^{j-k}\right)(1-x)^{\alpha+1-i-j},
$$

for $j=i,\ldots,n+1$, *and* (25) *also follows for* $j=1,\ldots,i-1$. $\square$

Theorem 1 and the analysis of the sign of the entries (20), (21) and (22) provides the following characterization of the total positivity of $W(f_0^\alpha,\dots,f_n^\alpha)(x)$.

Corollary 1. *Given $\alpha\in\mathbb{R}$, let $(\mathrm{f}_0^\alpha,\dots,\mathrm{f}_\mathrm{n}^\alpha)$ be the system defined in (14). The Wronskian matrix $\mathrm{W}(\mathrm{f}_0^\alpha,\dots,\mathrm{f}_\mathrm{n}^\alpha)(\mathrm{x})$ is TP if and only if $\alpha\le 0$ and $0\le \mathrm{x}<1$.*

Proof. *Let us observe that the coefficient $x/(1-x)$ in (21) is nonnegative for $0\le x<1$. For $0\le x<1$, the diagonal entries in (22) are also nonnegative. Finally, the nonnegativity of the coefficients (20) is satisfied if and only if $\alpha+2-i\le 0$ for $i=2,\dots,n+1$ that is, $\alpha\le 0$.* □

Example 1. *Let us illustrate with some examples the bidiagonal factorization (19), described by (20), (21) and (22), of the Wronskian matrix of $(\mathrm{f}_0^\alpha,\dots,\mathrm{f}_\mathrm{n}^\alpha)$ For the particular case $\mathrm{n}=2$ and $\alpha=\mathrm{n}$, the Wronskian matrix of the system $((1-\mathrm{x})^2,\mathrm{x}(1-\mathrm{x}),\mathrm{x}^2)$ can be decomposed as follows*

$$W(f_0^2,f_1^2,f_2^2)(x)=\begin{pmatrix}1&0&0\\0&1&0\\0&\frac{-1}{1-\mathrm{x}}&1\end{pmatrix}\begin{pmatrix}1&0&0\\\frac{-2}{1-\mathrm{x}}&1&0\\0&\frac{-1}{1-\mathrm{x}}&1\end{pmatrix}\begin{pmatrix}(1-\mathrm{x})^2&0&0\\0&1&0\\0&0&\frac{2}{(1-\mathrm{x})^2}\end{pmatrix}\begin{pmatrix}1&\frac{\mathrm{x}}{1-\mathrm{x}}&0\\0&1&\frac{\mathrm{x}}{1-\mathrm{x}}\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&\frac{\mathrm{x}}{1-\mathrm{x}}\\0&0&1\end{pmatrix}.$$

Clearly, $\mathrm{W}(\mathrm{f}_0^2,\mathrm{f}_1^2,\mathrm{f}_2^2)(\mathrm{x})$ is not a TP matrix at any $\mathrm{x}\in\mathbb{R}$.

For the particular case $\mathrm{n}=2$ and $\alpha=-\mathrm{n}$, the Wronskian matrix of the system $(1/(1-\mathrm{x})^2,\mathrm{x}/(1-\mathrm{x})^3,\mathrm{x}^2/(1-\mathrm{x})^4)$ can be decomposed as follows

$$W(f_0^{-2},f_1^{-2},f_2^{-2})(x)=\begin{pmatrix}1&0&0\\0&1&0\\0&\frac{3}{1-\mathrm{x}}&1\end{pmatrix}\begin{pmatrix}1&0&0\\\frac{2}{1-\mathrm{x}}&1&0\\0&\frac{3}{1-\mathrm{x}}&1\end{pmatrix}\begin{pmatrix}\frac{1}{(1-\mathrm{x})^2}&0&0\\0&\frac{1}{(1-\mathrm{x})^4}&0\\0&0&\frac{2}{(1-\mathrm{x})^6}\end{pmatrix}\begin{pmatrix}1&\frac{\mathrm{x}}{1-\mathrm{x}}&0\\0&1&\frac{\mathrm{x}}{1-\mathrm{x}}\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&\frac{\mathrm{x}}{1-\mathrm{x}}\\0&0&1\end{pmatrix}.$$

Clearly, the Wronskian matrix $\mathrm{W}(\mathrm{f}_0^{-2},\mathrm{f}_1^{-2},\mathrm{f}_2^{-2})(\mathrm{x})$ is TP for $\mathrm{x}\in(0,1)$.

For the particular case $\mathrm{n}=2$ and $\alpha=-5/2$, the Wronskian matrix of the system $(1/(1-\mathrm{x})^{5/2},\mathrm{x}/(1-\mathrm{x})^{7/2},\mathrm{x}^2/(1-\mathrm{x})^{9/2})$ can be decomposed as follows

$$W(f_0^{-5/2},f_1^{-5/2},f_2^{-5/2})(x)=\begin{pmatrix}1&0&0\\0&1&0\\0&\frac{7/2}{1-\mathrm{x}}&1\end{pmatrix}\begin{pmatrix}1&0&0\\\frac{5/2}{1-\mathrm{x}}&1&0\\0&\frac{7/2}{1-\mathrm{x}}&1\end{pmatrix}\begin{pmatrix}\mathrm{d}_1&0&0\\0&\mathrm{d}_2&0\\0&0&\mathrm{d}_3\end{pmatrix}\begin{pmatrix}1&\frac{\mathrm{x}}{1-\mathrm{x}}&0\\0&1&\frac{\mathrm{x}}{1-\mathrm{x}}\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&\frac{\mathrm{x}}{1-\mathrm{x}}\\0&0&1\end{pmatrix},$$

where $\mathrm{d}_1 = (1-\mathrm{x})^{-5/2}$, $\mathrm{d}_2 = (1-\mathrm{x})^{-9/2}$ *and* $\mathrm{d}_3 = 2(1-\mathrm{x})^{-13/2}$. *Clearly,* $\mathrm{W}(\mathrm{f}_0^{-5/2}, \mathrm{f}_1^{-5/2}, \mathrm{f}_2^{-5/2})(\mathrm{x})$ *is TP for* $\mathrm{x} \in (0,1)$.

Now, using Lemma 1 and taking into account that the bidiagonal decomposition of the Wronskian matrix of the polynomial basis $(f_0^n, \dots, f_n^n)$, provided by Theorem 3 with $\alpha = n$, can be extended for all $x \neq 1$, we can derive the bidiagonal factorization of the Wronskian matrix of the Bernstein basis (9) using that

$$W(B_0^n, \dots, B_n^n)(x) = W(f_0^n, \dots, f_n^n)(x)\Delta, \quad \Delta := \mathrm{diag}\left(\binom{n}{i-1}\right)_{1\leq i\leq n+1}$$

and the identity $\binom{n}{i-1}/\binom{n}{i-2} = (n+2-i)/(i-1), \quad i = 2, \dots, n+1$.

Theorem 4. *Let* $\mathrm{n} \in \mathbb{N}$ *and* $(\mathrm{B}_0^\mathrm{n}, \dots, \mathrm{B}_\mathrm{n}^\mathrm{n})$ *the Bernstein basis of* $\mathbf{P}^\mathrm{n}$ *defined in* (9). *For a given* $\mathrm{x} \in \mathbb{R}$, $\mathrm{x} \neq 1$, *the Wronskian matrix* $\mathrm{W} := \mathrm{W}(\mathrm{B}_0^\mathrm{n}, \dots, \mathrm{B}_\mathrm{n}^\mathrm{n})(\mathrm{x})$ *admits a factorization of the form*

$$W = L_{n,n}L_{n-1,n}\cdots L_{1,n}DU_{1,n}\cdots U_{n-1,n}U_{n,n}, \tag{26}$$

where $\mathrm{L}_{\mathrm{k,n}} = (\mathrm{l}_{\mathrm{i,j}}^{(\mathrm{k,n})})_{1\leq \mathrm{j,i}\leq \mathrm{n}+1}$, $\mathrm{k} = 1, \dots, \mathrm{n}$, *are the lower triangular bidiagonal matrices, with unit diagonal entries, such that*

$$l_{i,i-1}^{(k,n)} = 0, \quad i = 2, \dots, k, \qquad l_{i,i-1}^{(k,n)} = (n+2-i)\frac{-1}{1-x}, \quad i = k+1, \dots, n+1, \tag{27}$$

$\mathrm{U}_{\mathrm{k,n}} = (\mathrm{u}_{\mathrm{i,j}}^{(\mathrm{k,n})})_{1\leq \mathrm{j,i}\leq \mathrm{n}+1}$, $\mathrm{k} = 1, \dots, \mathrm{n}$, *are the upper triangular bidiagonal matrices, with unit diagonal entries, such that*

$$u_{i-1,i}^{(k,n)} := 0, \quad i = 2, \dots, k, \qquad u_{i-1,i}^{(k,n)} = \left(\frac{n+2-i}{i-1}\right)\frac{x}{1-x}, \quad i = k+1, \dots, n+1, \tag{28}$$

and D *is the diagonal matrix* $\mathrm{D} = \mathrm{diag}\,(\mathrm{d}_1, \dots, \mathrm{d}_{\mathrm{n}+1})$ *with*

$$d_i = \binom{n}{i-1}(i-1)!(1-x)^{n+2-2i}, \quad i = 1, \dots, n+1. \tag{29}$$

Example 2. *Let us illustrate the bidiagonal factorization* (26), *described by* (27), (28) *and* (29), *of the Wronskian matrix of the Bernstein polynomial basis. For the particular case* $\mathrm{n} = 2$, *the Wronskian matrix of* $((1-\mathrm{x})^2, 2(1-\mathrm{x})\mathrm{x}, \mathrm{x}^2)$

can be decomposed as follows

$$W(B_0^2,B_1^2,B_2^2)(x)=\begin{pmatrix}1&0&0\\0&1&0\\0&\frac{-1}{1-\mathrm{x}}&1\end{pmatrix}\begin{pmatrix}1&0&0\\\frac{-2}{1-\mathrm{x}}&1&0\\0&\frac{-1}{1-\mathrm{x}}&1\end{pmatrix}\begin{pmatrix}(1-\mathrm{x})^2&0&0\\0&2&0\\0&0&\frac{2}{(1-\mathrm{x})^2}\end{pmatrix}\begin{pmatrix}1&\frac{2\mathrm{x}}{1-\mathrm{x}}&0\\0&1&\frac{\mathrm{x}/2}{1-\mathrm{x}}\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&\frac{\mathrm{x}/2}{1-\mathrm{x}}\\0&0&1\end{pmatrix}.$$

Let us observe, that from Theorem 4, it can be deduced that the bidiagonal factorization (1) of the $(n+1)\times(n+1)$ dimensional Wronskian matrix W of the Bernstein basis of $\mathbf{P}^n$ can be represented by means of the $(n+1)\times(n+1)$ matrix $BD(W)=(BD(W)_{i,j})_{1\le i,j\le n+1}$ such that

$$BD(W)_{i,j}:=\begin{cases}(n+2-i)\frac{-1}{1-x}, & \text{if } i>j,\\ \binom{n}{i-1}(i-1)!(1-x)^{n+2-2i}, & \text{if } i=j,\\ \left(\frac{n+2-j}{j-1}\right)\frac{x}{1-x}, & \text{if } i<j.\end{cases} \tag{30}$$

Let us observe that, analyzing the sign of the entries of (30), we can deduce that the Wronskian matrix of the Bernstein basis of $\mathbf{P}^n$ is not TP at any $x\in\mathbb{R}$. However, the following result shows that the solution of several algebraic problems related to these matrices can be obtained with HRA using the bidiagonal decomposition (26).

Corollary 2. *Let* $\mathrm{W}:=\mathrm{W}(\mathrm{B}_0^{\mathrm{n}},\dots,\mathrm{B}_{\mathrm{n}}^{\mathrm{n}})(\mathrm{x})$ *be the Wronskian matrix of the Bernstein basis defined in* (9) *and* J *the diagonal matrix* $\mathrm{J}:=\mathit{diag}((-1)^{\mathrm{i}-1})_{1\le\mathrm{i}\le\mathrm{n}+1}$. *Then, for any* $\mathrm{x}<0$,

$$W_J:=JWJ \tag{31}$$

is an STP matrix and its bidiagonal factorization (1) *can be computed with HRA. Consequently, the computation of the eigenvalues, singular values of* W, *the matrix* W^{-1}, *as well as the solution* $\mathrm{c}=(\mathrm{c}_1,\dots,\mathrm{c}_{\mathrm{n}+1})^{\mathrm{T}}$ *of linear systems* $\mathrm{Wc}=\mathrm{b}$, *where the entries of* $\mathrm{b}=(\mathrm{b}_1,\dots,\mathrm{b}_{\mathrm{n}+1})^{\mathrm{T}}$ *have the same sign, can be performed with HRA.*

Proof. *Using Theorem 4 and the fact that J^2 is the identity matrix, by (26) we can write*

$$W_J=(JL_{n,n}J)\cdots(JL_{1,n}J)(JDJ)(JU_{1,n}J)\cdots(JU_{n,n}J), \tag{32}$$

which gives its bidiagonal factorization (1)*. Now, it can be easily checked that the multipliers and diagonal pivots of the bidiagonal factorization* (32) *of W_J are positive if*

$$\frac{1}{1-x} > 0, \quad \frac{-x}{1-x} > 0, \quad 1-x > 0.$$

Therefore, by Remark 1, W_J is STP and its bidiagonal decomposition (32) *can be computed with HRA at any $x < 0$. This fact guarantees the computation with HRA of the eigenvalues and singular values of W_J, the inverse matrix W_J^{-1} and the solution of the linear systems $W_J c = d$, where $d = (d_1, \ldots, d_{n+1})^T$ has alternating signs (see Section 3 of* [11]*).*

Let us observe that, since J is a unitary matrix, the eigenvalues and singular values of W coincide with those of W_J and therefore, using the bidiagonal decomposition (32) *of W_J, their computation for $x < 0$ can be performed with HRA.*

For the accurate computation of W^{-1}, we can take into account that

$$W^{-1} = JW_J^{-1}J. \tag{33}$$

Since, for $x < 0$, $W_J^{-1} = (\tilde{w}_{i,j})_{1\le i,j\le +1}$ can be computed with HRA and, by (33)*, the inverse of the Wronskian matrix W satisfies $W^{-1} = ((-1)^{i+j}\tilde{w}_{i,j})_{1\le i,j\le +1}$, we can also accurately compute W^{-1} by means of a suitable change of sign of the accurate computed entries of W_J^{-1}.*

Finally, if we have a linear system of equations $Wc = b$, where the elements of $b = (b_1, \ldots, b_{n+1})^T$ have the same sign, we can compute with HRA the solution $d \in \mathbb{R}^{n+1}$ of $W_J d = Jb$ and, consequently, the solution $c \in \mathbb{R}^{n+1}$ of the initial system since $c = Jd$. □

Now, the following result describes the bidiagonal factorization (1) of Bernstein bases of negative degree (10). This decomposition can be deduced from Theorem 3 with $\alpha = -n$ and Lemma 1, taking into account that

$$W(B_0^{-n}, \ldots, B_n^{-n})(x) = W(f_0^{-n}, \ldots, f_n^{-n})(x)\Delta, \tag{34}$$

with $\Delta := \operatorname{diag}\left((-1)^{i-1}\binom{n+i-2}{i-1}\right)_{1\le i\le n+1}$ and the identity $\binom{n+i-2}{i-1}/\binom{n+i-3}{i-2} = (n+i-2)/(i-1)$, $i = 2, \ldots, n+1$.

Theorem 5. *Let* $\mathrm{n} \in \mathbb{N}$ *and* $(\mathrm{B}_0^{-\mathrm{n}}, \ldots, \mathrm{B}_\mathrm{n}^{-\mathrm{n}})$ *the Bernstein basis of degree* $-\mathrm{n}$*, defined in* (10)*. For a given* $\mathrm{x} \in \mathbb{R}$*,* $\mathrm{x} \neq 1$*, the Wronskian matrix* $\mathrm{W} := \mathrm{W}(\mathrm{B}_0^{-\mathrm{n}}, \ldots, \mathrm{B}_\mathrm{n}^{-\mathrm{n}})(\mathrm{x})$ *admits a factorization of the form*

$$W = L_{n,n} L_{n-1,n} \cdots L_{1,n} D U_{1,n} \cdots U_{n-1,n} U_{n,n}, \tag{35}$$

where $\mathrm{L}_{\mathrm{k,n}} = (\mathrm{l}_{\mathrm{i,j}}^{(\mathrm{k,n})})_{1 \le \mathrm{j,i} \le \mathrm{n}+1}$, $\mathrm{k} = 1, \ldots, \mathrm{n}$*, are the lower triangular bidiagonal matrices, with unit diagonal entries, such that*

$$l_{i,i-1}^{(k,n)} = 0, \quad i = 2, \ldots, k, \qquad l_{i,i-1}^{(k,n)} = (n+i-2)\frac{1}{1-x}, \quad i = k+1, \ldots, n+1, \tag{36}$$

$\mathrm{U}_{\mathrm{k,n}} = (\mathrm{u}_{\mathrm{i,j}}^{(\mathrm{k,n})})_{1 \le \mathrm{j,i} \le \mathrm{n}+1}$, $\mathrm{k} = 1, \ldots, \mathrm{n}$*, are the upper triangular bidiagonal matrices, with unit diagonal entries, such that*

$$u_{i-1,i}^{(k,n)} := 0, \quad i = 2, \ldots, k, \qquad u_{i-1,i}^{(k,n)} = -\binom{n+i-2}{i-1}\frac{x}{1-x}, \quad i = k+1, \ldots, n+1, \tag{37}$$

and D *is the diagonal matrix* $\mathrm{D} = \mathrm{diag}\,(\mathrm{d}_1, \ldots, \mathrm{d}_{\mathrm{n}+1})$ *with*

$$d_i = (-1)^{i-1}\frac{(n+i-2)!}{(n-1)!}(1-x)^{-n+2-2i}, \quad i = 1, \ldots, n+1. \tag{38}$$

Example 3. *Let us illustrate the bidiagonal factorization* (35)*, described by* (36)*,* (37) *and* (38)*, of the Wronskian matrix of Bernstein bases of negative degree. For the particular case* $\mathrm{n} = 2$*, the Wronskian matrix of* $(1/(1-\mathrm{x})^2, -2\mathrm{x}/(1-\mathrm{x})^3, 3\mathrm{x}^2/(1-\mathrm{x})^4)$ *can be decomposed as follows*

$$W(B_0^{-2}, B_1^{-2}, B_2^{-2})(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{3}{1-\mathrm{x}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{1-\mathrm{x}} & 1 & 0 \\ 0 & \frac{3}{1-\mathrm{x}} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{(1-\mathrm{x})^2} & 0 & 0 \\ 0 & \frac{-2}{(1-\mathrm{x})^4} & 0 \\ 0 & 0 & \frac{3!}{(1-\mathrm{x})^6} \end{pmatrix} \begin{pmatrix} 1 & \frac{-2\mathrm{x}}{1-\mathrm{x}} & 0 \\ 0 & 1 & \frac{-3/2\mathrm{x}}{1-\mathrm{x}} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-3/2\mathrm{x}}{1-\mathrm{x}} \\ 0 & 0 & 1 \end{pmatrix},$$

Now, from Theorem 5, the bidiagonal factorization (1) of the $(n+1) \times (n+1)$ dimensional Wronskian matrix W of the Bernstein basis of negative degree can be represented by means of the $(n+1) \times (n+1)$ matrix $BD(W) =$

$(BD(W)_{i,j})_{1\le i,j\le n+1}$, such that

$$BD(W)_{i,j} := \begin{cases} (n+i-2)\frac{1}{1-x}, & \text{if } i>j, \\ (-1)^{i-1}\binom{n+i-2}{i-1}(i-1)!(1-x)^{-n+2-2i}, & \text{if } i=j, \\ -\left(\frac{n+j-2}{j-1}\right)\frac{x}{1-x}, & \text{if } i<j. \end{cases} \tag{39}$$

Let us observe that the off-diagonal entries of the matrix $BD(W)$ in (39) are all positive only if $x<0$. On the other hand, the diagonal entries of $BD(W)$ have alternating sign. Then, taking into account the sign of these entries, it can be deduced that $W(B_0^{-n},\ldots,B_n^{-n})(x)$ is not TP at any $x\in\mathbb{R}$. Nevertheless, the following result shows that the bidiagonal decomposition (35) provides accurate computations with these matrices.

Corollary 3. *Let* $\mathrm{W} := \mathrm{W}(\mathrm{B}_0^{-\mathrm{n}},\ldots,\mathrm{B}_\mathrm{n}^{-\mathrm{n}})(\mathrm{x})$ *be the Wronskian matrix of the Bernstein basis of degree* $-\mathrm{n}$ *defined in* (10) *and* J *the diagonal matrix* $\mathrm{J} := \mathit{diag}((-1)^{\mathrm{i}-1})_{1\le\mathrm{i}\le\mathrm{n}+1}$. *Then, for* $0<\mathrm{x}<1$,

$$\widetilde{W}_J := WJ \tag{40}$$

is an STP matrix and its bidiagonal factorization (1) *can be computed with HRA. Consequently, the computation of the singular values of* W, *the matrix* W^{-1}, *as well as the solution* $\mathrm{c}=(\mathrm{c}_1,\ldots,\mathrm{c}_{\mathrm{n}+1})^{\mathrm{T}}$ *of linear systems* $\mathrm{Wc}=\mathrm{b}$, *where the entries of* $\mathrm{b}=(\mathrm{b}_1,\ldots,\mathrm{b}_{\mathrm{n}+1})^{\mathrm{T}}$ *have alternating signs, can be performed with HRA.*

Proof. *Taking into account Theorem 5,* (34) *and Lemma 1, it can be easily checked that the multipliers and diagonal pivots of the bidiagonal factorization* (1) *of* $\widetilde{W}_J$ *are positive if*

$$\frac{1}{1-x}>0, \quad \frac{x}{1-x}>0, \quad 1-x>0,$$

that is, if $0<x<1$. *This fact guarantees, by Remark 1, that* $\widetilde{W}_J$ *is STP and the computation with HRA of its bidiagonal decomposition* (1) *and so, the computation with HRA of its eigenvalues and singular values, the inverse matrix* $\widetilde{W}_J^{-1}$ *and the solution of the linear systems* $\widetilde{W}_J c = b$, *where* $b=(b_0,\ldots,b_n)^T$ *has alternating signs (see Section 3 of* [11]*).*

On the other hand, since J *is a unitary matrix, the singular values of* $\widetilde{W}_J$ *coincide with those of* W *and so, their computation for* $0<x<1$ *can be performed with HRA. Similarly, taking into account that*

$$W^{-1} = J\widetilde{W}_J^{-1},$$

we can compute W^{-1} accurately. Finally, if we have a linear system of equations $Wc = b$, where the elements of $b = (b_1, \ldots, b_n)^T$ have alternating signs, we can solve with HRA the system $\widetilde{W}_J d = b$ and then obtain $c = Jd$. $\square$

Finally, using Lemma 1 and Theorem 3 with $\alpha = n+1$, we can derive the bidiagonal factorization of the Wronskian matrices of negative binomial bases (11), taking into account that

$$W(b_0^n, \ldots, b_n^n)(x) = W(f_0^{n+1}, \ldots, f_n^{n+1})(x)\Delta, \quad \Delta := \operatorname{diag}\left(\binom{n}{i-1}\right)_{1\le i\le n+1}$$

and the identity $\binom{n}{i-1}/\binom{n}{i-2} = (n+2-i)/(i-1)$, $i = 2, \ldots, n+1$.

Theorem 6. *Let* $\mathrm{n} \in \mathbb{N}$ *and* $(\mathrm{b}_0^{\mathrm{n}}, \ldots, \mathrm{b}_{\mathrm{n}}^{\mathrm{n}})$ *the negative binomial basis of* $\mathbf{P}^{\mathrm{n}}$ *defined in* (11). *For a given* $\mathrm{x} \in \mathbb{R}$, $\mathrm{x} \neq 1$, *the Wronskian matrix* $\mathrm{W} := \mathrm{W}(\mathrm{b}_0^{\mathrm{n}}, \ldots, \mathrm{b}_{\mathrm{n}}^{\mathrm{n}})(\mathrm{x})$ *admits a factorization of the form*

$$W = L_{n,n}L_{n-1,n}\cdots L_{1,n}DU_{1,n}\cdots U_{n-1,n}U_{n,n}, \tag{41}$$

where $\mathrm{L}_{\mathrm{k,n}} = (\mathrm{l}_{\mathrm{i,j}}^{(\mathrm{k,n})})_{1\le \mathrm{j,i}\le \mathrm{n}+1}$, $\mathrm{k} = 1, \ldots, \mathrm{n}$, *are the lower triangular bidiagonal matrices, with unit diagonal entries, such that*

$$l_{i,i-1}^{(k,n)} = 0, \quad i = 2, \ldots, k, \qquad l_{i,i-1}^{(k,n)} = (n+3-i)\frac{-1}{1-x}, \quad i = k+1, \ldots, n+1, \tag{42}$$

$\mathrm{U}_{\mathrm{k,n}} = (\mathrm{u}_{\mathrm{i,j}}^{(\mathrm{k,n})})_{1\le \mathrm{j,i}\le \mathrm{n}+1}$, $\mathrm{k} = 1, \ldots, \mathrm{n}$, *are the upper triangular bidiagonal matrices, with unit diagonal entries, such that*

$$u_{i-1,i}^{(k,n)} := 0, \quad i = 2, \ldots, k, \qquad u_{i-1,i}^{(k,n)} = \left(\frac{n+2-i}{i-1}\right)\frac{x}{1-x}, \quad i = k+1, \ldots, n+1, \tag{43}$$

and D *is the diagonal matrix* $\mathrm{D} = \operatorname{diag}(\mathrm{d}_1, \ldots, \mathrm{d}_{\mathrm{n}+1})$ *with*

$$d_i = \binom{n}{i-1}(i-1)!(1-x)^{n+3-2i}, \quad i = 1, \ldots, n+1. \tag{44}$$

Example 4. *Let us illustrate the bidiagonal factorization* (41), *described by* (42), (43) *and* (44), *of the Wronskian matrix of the negative binomial polynomial basis. For the particular case* $\mathrm{n} = 2$, *the Wronskian matrix of* $((1-$

$\mathrm{x})^3, 2(1-\mathrm{x})^2\mathrm{x}, (1-\mathrm{x})\mathrm{x}^2)$ *can be decomposed as follows*

$$W(b_0^2, b_1^2, b_2^2)(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-2}{1-\mathrm{x}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{-3}{1-\mathrm{x}} & 1 & 0 \\ 0 & \frac{-2}{1-\mathrm{x}} & 1 \end{pmatrix} \begin{pmatrix} (1-\mathrm{x})^3 & 0 & 0 \\ 0 & 2(1-\mathrm{x}) & 0 \\ 0 & 0 & \frac{2}{(1-\mathrm{x})} \end{pmatrix} \begin{pmatrix} 1 & \frac{2\mathrm{x}}{1-\mathrm{x}} & 0 \\ 0 & 1 & \frac{\mathrm{x}/2}{1-\mathrm{x}} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\mathrm{x}/2}{1-\mathrm{x}} \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, from Theorem 6, the bidiagonal factorization (1) of the Wronskian matrix W of the negative binomial basis of $\mathbf{P}^n$ defined in (11) can be represented by $BD(W) = (BD(W)_{i,j})_{1\le i,j\le n+1}$, such that

$$BD(W)_{i,j} := \begin{cases} (n+3-i)\frac{-1}{1-x}, & \text{if } i > j, \\ \binom{n}{i-1}(i-1)!(1-x)^{n+3-2i}, & \text{if } i = j, \\ \left(\frac{n+2-j}{j-1}\right)\frac{x}{1-x}, & \text{if } i < j. \end{cases} \tag{45}$$

Using formula (45), it can be deduced that the Wronskian matrix of the negative binomial basis (11) is not TP at any $x \in \mathbb{R}$. However, following the reasoning in the proof of Corollary 2, we can guarantee that the solution of several algebraic problems related to these Wronskian matrices can be computed with HRA.

Corollary 4. *Let* $\mathrm{W} := \mathrm{W}(\mathrm{b}_0^\mathrm{n}, \ldots, \mathrm{b}_\mathrm{n}^\mathrm{n})(\mathrm{x})$ *be the Wronskian matrix of the negative binomial basis defined in* (11) *and* $\mathrm{J} := \mathit{diag}((-1)^{\mathrm{i}-1})_{1\le \mathrm{i}\le \mathrm{n}+1}$. *Then, for any* $\mathrm{x} < 0$,

$$W_J := JWJ \tag{46}$$

is TP and its bidiagonal factorization (1) *can be computed with HRA. Consequently, the computation of the eigenvalues, singular values of* W, *the matrix* W^{-1}, *as well as the solution of linear systems* $\mathrm{Wx} = \mathrm{b}$, *where the entries of* $\mathrm{b} = (\mathrm{b}_1, \ldots, \mathrm{b}_{\mathrm{n}+1})^\mathrm{T}$ *have the same sign, can be performed with HRA.*

Section 5 will show accurate computations with the Wronskian matrices of Bernstein bases, Bernstein bases of negative degree and negative binomial bases obtained by using the bidiagonal decomposition (1) and the algorithms in [23].

5 Numerical experiments

Let us suppose that A is an $(n+1)\times(n+1)$ nonsingular, TP matrix, whose bidiagonal decomposition (1) is represented by means of the matrix $BD(A)$ given in (3). If $BD(A)$ can be computed with HRA, then the Matlab functions `TNEigenValues`, `TNSingularValues`, `TNInverseExpand` and `TNSolve` of the library `TNTool` in [23] take as input argument $BD(A)$ and compute with HRA the eigenvalues of A, the singular values of A, its inverse matrix A^{-1} and the solution of systems of linear equations $Ax = b$, for vectors b whose entries have alternating signs. The computational cost of the function `TNSolve` is $O(n^2)$ elementary operations. On the other hand, as it can be checked in page 303 of reference [32], the function `TNInverseExpand` has a computational cost of $O(n^2)$ and then improves the computational cost of the computation of the inverse matrix by solving linear systems with `TNSolve`, taking the columns of the identity matrix as data ($O(n^3)$). The computational cost of the other mentioned functions is $O(n^3)$.

For the Bernstein basis $(B_0^n, \ldots, B_n^n)$, $n \in \mathbb{N}$, using Theorem 4 and Corollary 2, we have implemented a Matlab function that computes $BD(W_J)$, where W_J is the scaled Wronskian matrix at $x < 0$, $W_J := JWJ$ described in (31).

For the Bernstein basis of negative degree $(B_0^{-n}, \ldots, B_n^{-n})$, $n \in \mathbb{N}$, considering Theorem 5 and Corollary 3, we have also implemented a Matlab function, which computes $BD(\widetilde{W}_J)$ for the matrix $\widetilde{W}_J := WJ$, obtained from its Wronskian matrix W at $0 < x < 1$ (see (40)).

Finally, for the negative binomial basis $(b_0^{n+1}, \ldots, b_n^{n+1})$, using Theorem 6 and Corollary 4, we have also implemented a Matlab function for computing $BD(W_J)$ for the matrix $W_J := JWJ$ obtained from its Wronskian matrix W at $x < 0$ (see (46)).

Observe that, in all cases, the computational complexity in the computation of the entries $m_{i,j}$, $\tilde{m}_{i,j}$, $1 \leq j < i \leq n + 1$, is $O(n^2)$ and in the computation of $p_{i,i}$, $1 \leq i \leq n + 1$, is $O(n)$.

In the numerical experimentation, we have considered different $(n+1) \times (n+1)$ Wronskian matrices corresponding to Bernstein bases, Bernstein bases of negative degree and negative binomial bases. The numerical results illustrate the accuracy of the computations for dimensions $n + 1 = 10, 15, 20, 25$. The authors will provide upon request the software with the implementation of the above mentioned routines.

The 2-norm condition number of the considered Wronskian matrices has been obtained by means of the Mathematica command `Norm[A,2]`$\cdot$ `Norm[Inverse[A],2]` and is shown in Table 1. We can clearly observe that the condition numbers significantly increase with the dimension of the matrices. This explains that traditional methods do not obtain accurate solutions when solving the aforementioned algebraic problems. In contrast, the numerical results will illustrate the high accuracy obtained when using the bidiagonal decompositions deduced in this paper with the Matlab functions available in [23].

The eigenvalues and singular values of the considered Wronskian matrices have been computed with the Matlab functions `TNEigenValues` and `TNSingularValues`, respectively, taking as argument the matrix representation (3) of the corresponding deduced bidiagonal decomposition (1). Additionally, they have also been obtained by using the Matlab commands `eig` and `svd`, respectively. To analyze the accuracy of the approximations, the eigenvalues and singular values of the considered matrices have been calculated in Mathematica using a 100-digit arithmetic. The

Table 1: Condition number of Wronskian matrices of Bernstein bases at $x_0 = -1$, Wronskian matrices of Bernstein bases of negative degree at $x_0 = 1/7$ and Wronskian matrices of negative binomial bases at $x_0 = -2$.

	Bernstein bases	Bernstein bases of negative degree	Negative binomial bases
$\mathbf{n+1}$	$\kappa_2(\mathbf{W})$	$\kappa_2(\mathbf{W})$	$\kappa_2(\mathbf{W})$
10	1.3×10^{9}	1.7×10^{15}	1.4×10^{11}
15	1.3×10^{16}	4.3×10^{23}	1.0×10^{18}
20	1.6×10^{21}	2.6×10^{31}	2.3×10^{23}
25	9.9×10^{25}	3.4×10^{37}	3.3×10^{26}

values provided by Mathematica have been considered as the exact solution of the algebraic problem and the relative error e of each approximation has been computed as $e := |a - \tilde{a}|/|a|$, where a denotes the eigenvalue or singular value computed with Mathematica and $\tilde{a}$ the eigenvalue or singular value computed with Matlab.

In Tables 2 and 3, the relative errors of the approximation to the lowest eigenvalue and the lowest singular value of the considered matrices are shown. We can observe that our methods provide very accurate results in contrast to the not accurate results provided by the Matlab commands `eig` and `svd`.

Table 2: Relative errors when computing the lowest eigenvalue of the Wronskian matrices of Bernstein bases at $x_0 = -1$ (left) and negative binomial bases at $x_0 = -2$ (right).

$\mathbf{n+1}$	$\texttt{eig}(\mathbf{W})$	$\texttt{TNEV}(\mathbf{BD}(\mathbf{W_J}))$	$\texttt{eig}(\mathbf{W})$	$\texttt{TNEV}(\mathbf{BD}(\mathbf{W_J}))$
10	3.0×10^{-10}	6.9×10^{-16}	1.9×10^{-6}	8.0×10^{-16}
15	1.9×10^{-4}	9.9×10^{-17}	3.1×10^{-7}	1.0×10^{-17}
20	2.8×10^{1}	4.4×10^{-16}	1.9×10^{4}	6.5×10^{-17}
25	8.8×10^{6}	4.9×10^{-16}	1.5×10^{8}	5.2×10^{-16}

Table 3: Relative errors when computing the lowest singular value of Wronskian matrices of Bernstein bases at $x_0 = -1$ (left), Bernstein bases of negative degree at $x_0 = 1/7$ (middle) and negative binomial bases at $x_0 = -2$ (right).

$\mathbf{n+1}$	$\texttt{svd}(\mathbf{W})$	$\texttt{TNSV}(\mathbf{BD}(\mathbf{W_J}))$	$\texttt{svd}(\mathbf{W})$	$\texttt{TNSV}(\mathbf{BD}(\widetilde{\mathbf{W}}_\mathbf{J}))$	$\texttt{svd}(\mathbf{W})$	$\texttt{TNSV}(\mathbf{BD}(\mathbf{W_J}))$
10	2.4×10^{-8}	3.6×10^{-19}	2.1×10^{-2}	1.2×10^{-15}	3.0×10^{-6}	1.2×10^{-15}
15	1.5×10^{-1}	3.0×10^{-16}	7.8×10^{3}	1.3×10^{-15}	5.4×10^{-7}	5.8×10^{-16}
20	3.2×10^{3}	5.2×10^{-16}	1.6×10^{7}	1.1×10^{-15}	8.8×10^{2}	8.6×10^{-16}
25	9.1×10^{5}	1.0×10^{-16}	5.6×10^{13}	4.3×10^{-15}	3.7×10^{9}	5.7×10^{-16}

On the other hand, two approximations to the inverse matrix of the considered Wronskian matrices have also been calculated with Matlab. One of them, has been calculated using the function `TNInverseExpand`, with the corresponding matrix representation of the bidiagonal decomposition (1) as argument, and the other one, using the Matlab command `inv`. To look over the errors we have compared both Matlab approximations with the inverse matrix A^{-1} computed by Mathematica using 100-digit arithmetic, taking into account the formula $e = \|A^{-1} - \widetilde{A}^{-1}\|_2 / \|A^{-1}\|_2$

for the corresponding relative error. The obtained relative errors are shown in Table 4. Observe that the relative errors achieved through the bidiagonal decompositions obtained in this paper are much smaller than those obtained with the Matlab command `inv`.

Table 4: Relative errors when computing the inverse of Wronskian matrices of Bernstein bases at $x_0 = -1$ (left), Bernstein bases of negative degree at $x_0 = 1/7$ (middle) and negative binomial bases at $x_0 = -2$ (right).

$\mathbf{n+1}$	$\texttt{inv}(\mathbf{W})$	$\texttt{TNIE}(\mathbf{BD}(\mathbf{W_J}))$	$\texttt{inv}(\mathbf{W})$	$\texttt{TNIE}(\mathbf{BD}(\widetilde{\mathbf{W}}_\mathbf{J}))$	$\texttt{inv}(\mathbf{W})$	$\texttt{TNIE}(\mathbf{BD}(\mathbf{W_J}))$
10	6.6×10^{-11}	3.2×10^{-17}	6.8×10^{-11}	8.2×10^{-15}	1.5×10^{-8}	3.7×10^{-17}
15	1.3×10^{-6}	3.6×10^{-17}	1.2×10^{-6}	1.5×10^{-15}	2.2×10^{-2}	6.9×10^{-17}
20	3×10^{-1}	3.8×10^{-17}	6.5×10^{-2}	1.8×10^{-15}	2.3	1.8×10^{-16}
25	1.2	3.5×10^{-17}	5.7×10^{-1}	2.4×10^{-15}	1.0	1.3×10^{-16}

At last, given random nonnegative integer values d_i, $i = 1, \dots, n+1$, we have considered linear systems $Wc = d$ where, in the case of Bernstein bases and negative binomial bases, $d = ((-1)^{i+1} d_i)_{1 \le i \le n+1}$ and, in the case of Bernstein bases of negative degree, $d = (d_i)_{1 \le i \le n+1}$. We have computed in Matlab two approximations of the vector solution. An approximation has been computed by using the proposed bidiagonal decomposition (1) with the function `TNSolve`, and the other using the Matlab command \. We have also calculated the solution of the mentioned linear systems using 100-digit arithmetic in Mathematica. The vector provided by Mathematica has been considered as the exact solution c. Then, we have computed in Mathematica the relative error of the computed approximation $\tilde{c}$, taking into account the formula $e = \|c - \tilde{c}\|_2 / \|c\|_2$.

In Table 5, the relative errors when solving the aforementioned linear systems for different values of n are shown. Notice that the proposed methods preserve the accuracy, which does not considerably increases with the dimension of the system in contrast with the results obtained with the Matlab command \.

Table 5: Relative errors when solving $\mathbf{Wc} = \mathbf{d}$ with Wronskian matrices of Bernstein bases at $x_0 = -1$ (left), Bernstein bases of negative degree at $x_0 = 1/7$ (middle) and negative binomial bases at $x_0 = -2$ (right).

$\mathbf{n+1}$	$\mathbf{W} \setminus \mathbf{d}$	$\texttt{TNS}\left(\mathbf{BD}(\mathbf{W_J}), \mathbf{d}\right)$	$\mathbf{W} \setminus \mathbf{d}$	$\texttt{TNS}(\mathbf{BD}(\widetilde{\mathbf{W}}_\mathbf{J}), \mathbf{d})$	$\mathbf{W} \setminus \mathbf{d}$	$\texttt{TNS}(\mathbf{BD}(\mathbf{W_J}), \mathbf{d})$
10	6.7×10^{-14}	1.4×10^{-16}	8.9×10^{-11}	9.2×10^{-16}	1.5×10^{-8}	4.4×10^{-17}
15	3.1×10^{-11}	1.5×10^{-16}	1.6×10^{-6}	1.6×10^{-16}	2.2×10^{-2}	5.9×10^{-17}
20	1.9×10^{-10}	3.7×10^{-15}	8.4×10^{-2}	2.0×10^{-15}	2.0	3.1×10^{-17}
25	1.3×10^{-8}	1.5×10^{-15}	7.1×10^{-1}	2.6×10^{-15}	1.0	7.7×10^{-17}

Acknowledgement

We thank the anonymous referees for their helpful comments and suggestions, which have improved this paper.

Conflict of interest

This study does not have any conflicts to disclose.

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