

# The distribution of COVID-19 mortality

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## ABSTRACT

We estimate the distribution of COVID-19 mortality (measured as daily deaths) from the start of the pandemic until July 31st, 2022, for six European countries and the USA. We use the Pareto, the stretched exponential, the log-normal and the log-logistic distributions as well as mixtures of the log-normal and log-logistic distributions. The main results are that the Pareto does not describe well the data and that mixture distributions tend to offer a very good fit to the data. We also compute Value-at-Risk measures as well as mortality probabilities with our estimates. We also discuss the implications of our results and findings from the point of view of public health planning and modelling.

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## 1. Introduction

The distribution of mortality due to infectious diseases has begun to gather more interest from researchers. Having a better understanding of the distribution of the mortality for infectious diseases is important because public health and government officials can better plan for epidemics and pandemics by making suitable preparations, such as stockpiling personal protective equipment or planning for case surges at hospitals (Campolieti, 2021). In addition, having a better understanding of the distribution mortality can also be informative to public health and government officials who are deciding whether to lift or maintain public health measures (Vasconcelos et al., 2020, 2021). Some of the studies examining mortality from infectious diseases have focused on the “tail risks” of mortality and have also emphasized that the distribution of mortality can be heavy- or fat-tailed. A heavy- or fat-tailed distribution would mean that the probability of extreme mortality events would be much larger than expected based on a thin-tailed distributions, such as the normal. For example, Cirillo and Taleb (2020) considered the distribution of mortality from epidemics and pandemics during history and found evidence for a generalized Pareto distribution. Corral (2021) re-examined Cirillo and Taleb's data and found that there were other distributions that are plausible fits to their mortality data, such as a log-normal. Campolieti (2021) looked at influenza mortality for the U.S. from 1900 to 2018. Campolieti (2021) considered two measures of influenza mortality: 1) influenza mortality including

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deaths from pneumonia; and 2) influenza mortality excluding deaths from pneumonia. Campolieti (2021) found that the distribution of influenza mortality including pneumonia was heavy-tailed, but found that a Beta or exponential distribution could not be ruled out.<sup>1</sup> However, when he considered influenza mortality excluding deaths from pneumonia, he could rule out the thinner tailed Beta and exponential distributions. Overall, while there is evidence of heavy- or fat-tailed distributions, such as the power-law (Pareto), in the distribution of mortality for infectious diseases there is also evidence for alternative distributions. Consequently, there is some debate about what distribution best fits the distribution of mortality for infectious diseases. This is not like what has been observed in other settings, e.g., in physics, economics and computer science, where a power-law as well as other distributions could be plausible fits to the data (Clauset et al., 2009; Laherrère & Sornette, 1998; Mitzenmacher, 2004). Consequently, it is also important to use an empirical framework where one can consider alternative distributions and determine which is the most suitable for the mortality data being studied.

Our interest is also on the distribution of mortality from infectious diseases, but we focus on COVID-19 (SARS-CoV-2) mortality during the pandemic (up until the end of July 2022). There have been some papers studying the distribution of morbidity for COVID-19 during the early phases of the pandemic (Beare & Toda, 2020; Blasius, 2020; Chan et al., 2021; Komarova et al., 2020; Singer, 2020; Vazquez, 2020). Beare and Toda (2020); Blasius (2020); Komarova et al. (2020); Singer (2020); Vazquez (2020); Ahundjanov et al. (2022) and some have found evidence of a power-law distribution in the number of COVID-19 cases. Chan et al. (2021) found a negative binomial was the best fitting count regression model to COVID-19 case counts. There have been some papers that have studied COVID-19 mortality in Europe as well as other parts of the world. Xenikos and Asimakopoulou (2021) studied COVID-19 mortality for several European countries up until March 2020 and found that the power-law distribution fit these data well. Vasconcelos et al. (2020) studied COVID-19 fatalities with data from China, France, Germany, Iran, Italy, South Korea, Spain and Brazil. They described these countries as being at the middle or late stage of the epidemic, which they believed to be the case in the Fall of 2020.<sup>2</sup> They used the Richards growth model, which allows for both exponential and sub-exponential regimes in mortality dynamics, to evaluate intervention strategies. In a related paper, Vasconcelos et al. (2021) also studied the distribution of COVID-19 mortality for the period up to July 2020 in Canada, Germany, Italy, the Netherlands and the United Kingdom. They found that the power-law distribution fit the distribution of COVID-19 mortality during what they considered the final phase of the pandemic. These papers have generally considered the fit of the distributions visually. However, the disadvantage of this visual approach is that it is difficult to rule out competing distributions that provide a plausible fit to the data as well. Consequently, as pointed out by Corral (2021) there could be alternative heavy- or fat-tailed distributions that offer a plausible fit the data. This point was also made in Clauset et al. (2009) when considering the fit of the power-law distribution versus other alternative distributions.

We examine the distribution of COVID-19 mortality during the first two and a half years of the pandemic for France, Germany, Italy, Spain, Sweden, the United Kingdom and the United States. Our paper makes several contributions to the literature. First, we update the data to include much more of the pandemic, not just the first few months of the first year. This provides us with much longer window on the mortality that has occurred during the pandemic. Second, we consider a number of alternative fat- and heavy-tailed distributions, such as the power-law, stretched exponential, log-normal, mixtures of log-normals, log-logistic and mixtures of log-logistic. Third, we assess the fit of the distributions to the data using goodness-of-fit tests, such as the Kolmogorov-Smirnov, Cramér-von Mises and Anderson-Darling, as well as information criteria, unlike the existing literature which has primarily relied on visual checks. Using these tests and statistical information criteria means that we are better able to assess the fit of the distributions we consider to the data and determine the most plausible distribution. Fourth, we also consider the fit of distributions to the data below the upper tail, unlike some papers that have only considered the fit of power-law distributions to the upper tail of the distribution. Fifth, we also consider the distribution of mortality for specific years (and phases) of the pandemic, to assess whether the distribution of COVID-19 mortality changes over time.

In Section 2 we describe our data and the distributions we consider for mortality. Section 3 presents our findings. We end the paper with a summary of our main findings and discussion of their implications for public policy and future research.

## 2. Data and methods

We consider data on daily COVID-19 mortality (measured as the number of daily deaths) from the following countries: France; Germany; Italy; Spain; Sweden; the United Kingdom; and the United States of America. We obtained our data on daily deaths from <https://ourworldindata.org/covid-cases> and it includes data from the start of pandemic (February 1st, 2020) until July 31st, 2022. We consider the distribution for the whole period and also for the individual years (2020, 2021, 2022) in the pandemic as public health measures that are in place vary over time as does the dominant strain of the virus in circulation (Challen et al., 2021; Abdool Karim & de Oliveira, 2021; Several Authors, 2021; Abdool Karim & Abdool Karim, 2021; Callaway, 2022; Walker, J., Grubaugh, N. D., Gonsalves, G., Pitzer, V. and Rizvi, Z., 2022) and this might have an effect on mortality.

<sup>1</sup> Many individuals who have influenza can also develop pneumonia and in the more serious cases death results from the pneumonia, which will be listed as the cause of death on the death certificate. While measuring influenza mortality including deaths from pneumonia is the convention in most of the literature, Doshi (2008) argued that measuring influenza mortality without the deaths from pneumonia was a more reliable measure of mortality.

<sup>2</sup> Of course the pandemic is still continuing as new variants emerge and cases surge around the world.

We briefly describe next the distributions we fit to the daily mortality data. We let  $x$  denote our measure of mortality, which is the number of daily deaths. The first distribution we will consider is the well-known power-law distribution (or Pareto distribution):

$$f_P(x; \alpha, x_{\min}) = \frac{\alpha - 1}{x_{\min}} \left( \frac{x}{x_{\min}} \right)^{-\alpha} \quad (1)$$

where  $\alpha$  is the power-law exponent and  $x_{\min}$ , which is such that  $x \geq x_{\min} > 0$ , is the lower bound on power-law behavior.

The second distribution we consider for mortality,  $x > 0$ , is the Weibull or stretched exponential (STEXP) distribution

$$f_{\text{STEXP}}(x; \gamma, \eta) = \frac{\gamma}{\eta} \left( \frac{x}{\eta} \right)^{\gamma-1} \exp \left( - \left( \frac{x}{\eta} \right)^{\gamma} \right) \quad (2)$$

where  $\gamma > 0$  is a shape parameter and  $\eta > 0$  is a scale parameter. Weibull or stretched exponential distributions have often been found to fit skewed and heavy-tailed data (Jiang et al., 2013; Laherrère & Sornette, 1998).

The third distribution in our study is the well-known log-normal distribution

$$f_{\text{LN}}(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma x} \exp \left( - \frac{(\ln(x) - \mu)^2}{2\sigma^2} \right) \quad (3)$$

where  $\mu \in \mathbb{R}$  is the mean of  $\ln(x)$  and  $\sigma > 0$  is its standard deviation according to this distribution. Log-normal distributions have been found to fit many economic and physical phenomena with heavy tails (Mitzenmacher, 2004).

We will also consider 2-mixtures and 3-mixtures (McLachlan & Peel, 2003) of log-normal distributions, like in Kwong and Nadarajah (2019); Băncescu et al. (2019); Su (2020); Puente-Ajovín et al. (2020a,b); Campolieti and Ramos (2021):

$$f_{2\text{LN}}(x; \mu_1, \sigma_1, \mu_2, \sigma_2, p_1) = p_1 f_{\text{LN}}(x; \mu_1, \sigma_1) + (1 - p_1) f_{\text{LN}}(x; \mu_2, \sigma_2) \quad (4)$$

where  $0 \leq p_1, 1 - p_1 \leq 1$ , and

$$f_{3\text{LN}}(x; \mu_1, \sigma_1, \mu_2, \sigma_2, \mu_3, \sigma_3, p_1, p_2) = p_1 f_{\text{LN}}(x; \mu_1, \sigma_1) + p_2 f_{\text{LN}}(x; \mu_2, \sigma_2) + (1 - p_1 - p_2) f_{\text{LN}}(x; \mu_3, \sigma_3) \quad (5)$$

where  $0 \leq p_1, p_2, 1 - p_1 - p_2 \leq 1$ .

The log-logistic distribution has been used to model mortality data (Muse et al., 2021), so we also consider it:

$$f_{\text{LL}}(x; \mu, \sigma) = \frac{\exp \left( - \frac{\ln(x) - \mu}{\sigma} \right)}{x\sigma \left( 1 + \exp \left( - \frac{\ln(x) - \mu}{\sigma} \right) \right)^2} \quad (6)$$

where  $\mu \in \mathbb{R}$  is a location parameter and  $\sigma > 0$  is a shape parameter. Like the log-normal distribution, we also use 2- and 3-mixtures of log-logistic distributions (Puente-Ajovín et al., 2020a):

$$f_{2\text{LL}}(x; \mu_1, \sigma_1, \mu_2, \sigma_2, p_1) = p_1 f_{\text{LL}}(x; \mu_1, \sigma_1) + (1 - p_1) f_{\text{LL}}(x; \mu_2, \sigma_2) \quad (7)$$

where  $0 \leq p_1, 1 - p_1 \leq 1$ , and

$$f_{3\text{LL}}(x; \mu_1, \sigma_1, \mu_2, \sigma_2, \mu_3, \sigma_3, p_1, p_2) = p_1 f_{\text{LL}}(x; \mu_1, \sigma_1) + p_2 f_{\text{LL}}(x; \mu_2, \sigma_2) + (1 - p_1 - p_2) f_{\text{LL}}(x; \mu_3, \sigma_3) \quad (8)$$

where  $0 \leq p_1, p_2, 1 - p_1 - p_2 \leq 1$ .

We estimate all these distributions with maximum likelihood (ML) estimation, and compute the standard errors (SE) of the estimates according to Efron and Hinkley (1978); McCullough and Vinod (2003).<sup>3</sup>

We assess the fit of the estimated distributions to the data with goodness-of-fit test statistics. In particular, we use the Kolmogorov–Smirnov (KS), Cramér–von Mises (CM) and Anderson–Darling (AD) tests. We bootstrap the  $p$ -values for the test statistics because we estimate the parameters of the distributions, so our  $p$ -values are more conservative because they incorporate the uncertainty about the parameters than the analytical  $p$ -values, which assume the parameters are fixed (not estimated).

We also use three well-known information criteria to perform model selection. As we increase the number of parameters we can improve the maximum of the log-likelihood function. The information criteria recognize this and incorporate a penalty term, which penalizes over-parameterized models. We consider the following information criteria:

<sup>3</sup> We obtain our estimates using the command `mle` in MATLAB®, with usage shown in a file of supplementary material.

- The Akaike Information Criterion (AIC) [Akaike \(1974\)](#); [Burnham and Anderson \(2002, 2004\)](#), defined as

$$AIC = 2k - 2\ln L^*$$

where  $k$  is the number of parameters of the distribution and  $\ln L^*$  is the corresponding (maximum) log-likelihood. The minimum value of AIC corresponds (asymptotically) to the minimum value of the Kullback–Leibler divergence, so a model with the lowest AIC is selected from among the competitors.

- The Bayesian or Schwarz Information Criterion (BIC) [Burnham and Anderson \(2002, 2004\)](#); [Schwarz \(1978\)](#), defined as

$$BIC = k\ln(n) - 2\ln L^*$$

where  $k$  is the number of parameters of the distribution,  $n$  the sample size and  $\ln L^*$  is as before. The BIC penalizes more heavily the number of parameters used than does the AIC. The model with the lowest BIC is selected according to this criterion.

- The Hannan–Quinn Information Criterion (HQC) [Burnham and Anderson \(2002, 2004\)](#); [Hannan and Quinn \(1979\)](#), defined as

$$HQC = 2k\ln(\ln(n)) - 2\ln L^*$$

where  $k$  is the number of parameters of the distribution,  $n$  the sample size and  $\ln L^*$  is as before. The HQC implements an intermediate penalization of the number of parameters when compared to the AIC and BIC. The model with the lowest HQC is selected according to this criterion.

Likewise, for the samples that take into account the data for individual years, we will consider two information criteria more adapted to reduced sample sizes (they take into account corrections for small sample sizes). They are:

- The corrected Akaike Information Criterion like in [Akhundjanov et al. \(2017\)](#):

$$AIC_c = 2k - 2\ln L^* + \frac{2(k+1)(k+2)}{n-k-2}$$

where  $k$  is the number of estimated parameters,  $n$  is the sample size and  $\ln L^*$  is the maximum log-likelihood of the corresponding model ([Hurvich & Tsai, 1989](#)).

- The adjusted Bayesian or Schwarz Information Criterion:

$$BIC_a = k\ln\left(\frac{n+2}{24}\right) - 2\ln L^*$$

where  $k$ ,  $n$  and  $\ln L^*$  are as before, see [Schlove \(1987\)](#) for details. Also, the minimum values of each of these criteria point out to the selected model according to it.

Finally, we describe two measures of risk that we will use in [Subsection 3.3](#) to provide some assessment about the probability of extreme mortality. They are:

- The Value-at-Risk (VaR) measure for the upper tail, to the level of significance  $\alpha$  is the number  $VaR_\alpha$  satisfying

**Table 1**  
Descriptive statistics of the pooled samples.

	Number of Observations	Mean	SD	Skewness	Kurtosis	Min	Max
France	836	182.343301	215.734102	2.49969992	10.9769701	1	1438
Germany	844	170.616114	212.359863	2.27145146	9.02510462	1	1244
Italy	891	193.17284	202.771747	1.30693711	4.01375786	1	993
Spain	563	198.293073	207.947094	1.80859547	7.8383609	1	1623
Sweden	423	46.3687943	62.0282092	3.18970358	17.1423396	1	474
UK	808	222.861386	295.517366	2.33697674	8.99371869	1	1820
USA	882	1168.03175	966.110268	1.12954673	3.69432338	1	4411

$$\text{cdf}(VaR_{\alpha}) = 1 - \alpha$$

where cdf is the cumulative distribution function the  $VaR_{\alpha}$  is computed respect to it, see, e.g., [Jorion \(2006\)](#) and references therein.

- The Tail-Value-at-Risk (TVaR) measure for the upper tail, is defined for a given  $VaR_{\alpha}$  as

$$TVaR_{\alpha} = \frac{1}{\alpha} \int_{VaR_{\alpha}}^{\infty} xf(x) dx$$

where  $f$  is the probability density function the  $TVaR_{\alpha}$  is computed respect to it, see, e.g., [Rachev et al. \(2007\)](#) and references therein.

The Value-at-Risk measures are often used in financial modelling for risk management and they provide the extreme quantiles of the distribution. The expression for the Value-at-Risk measure will depend on the distribution being considered. We consider the VaR measures at  $\alpha = 0.01, 0.05, 0.10$ , which correspond with the 99th, 95th and 90th percentiles of the distribution.

### 3. Results

#### 3.1. Pooled analysis

The first part of our analysis focuses on the pooled data sets, i.e., the data covering the period from February 1st, 2020 to July 31st, 2022 for the previously mentioned seven countries. We offer the descriptive statistics for these data in [Table 1](#). As the support for several of the different distributions we consider require positive values, the zeros have been removed from the data.

The descriptive statistics indicate that there is a great deal of dispersion in the mortality data, although the extent of this dispersion does differ across countries we consider. The daily deaths data we consider also exhibit a great of skewness, which suggests that the distribution of mortality is a skewed one. In addition, all of the kurtosis measures exceed 3, so the mortality distribution is not thin-tailed.

We provide maximum likelihood estimates for all the distributions we estimate in a supplemental file (an Excel spreadsheet), to keep the number of tables in the text of the paper to a manageable number.<sup>4</sup> We are able to obtain estimates for all of the distributions we consider, except for the 3LL distribution for Sweden.

The goodness-of-fit tests are presented in [Table 2](#) and we provide a summary of the results of those tests in [Table 3](#).

We say that a distribution is “Non-rejected” if all of the three tests yield non-rejection; “Mixed” if one test or two yield(s) rejection, and “Rejected” if all of the three tests yield rejection or if the distribution cannot be estimated. For the Pareto distribution, the tests always reject it.<sup>5</sup> For the stretched exponential, we have non-rejected in 5 occasions and rejected in the other countries. For the log-normal distribution, we soundly reject it in all 7 countries. Using a mixture of two log-normals dramatically improves the fit to the data, as we do not reject this distribution for all of the 7 countries we consider. We observe similar findings, namely that we cannot reject the distribution when we take 3-mixtures of log-normals (3LN). For the log-logistic distribution, like the log-normal we reject it in all 7 countries. Taking the 2LL (a mixture of two log-logistics) improves the fit to the data notably as well by the corresponding numbers, as we obtain full 7 non-rejections. When we use a mixture of 3 log-logistic distributions, the fit is not as good as the 2LL distribution, as we obtain 4 non-rejections, 2 mixed instance and 1

<sup>4</sup> This file is available at <https://doi.org/10.7910/DVN/HWHKSG>.

<sup>5</sup> Please recall that we are considering the full data sets and not only the upper tails.

**Table 2**Outcomes of the KS, CM and AD tests in the format  $p$ -value (statistic). Non-rejections at the 5% level are marked in bold.

	Pareto			STEXP			LN		
	KS	CM	AD	KS	CM	AD	KS	CM	AD
France	0 (0.369)	0.000000000000000111	0 (191.)	<b>0.487</b> (0.029)	<b>0.505</b> (0.118)	<b>0.378</b> (0.961)	0.000881 (0.0683)	0.0052 (0.862)	0.0015 (5.59)
Germany	0 (0.277)	0 (27.2)	0 (137.)	0.0107 (0.0559)	0.0255 (0.577)	0.0114 (3.76)	0.00000179 (0.0912)	0.00000216 (2.35)	0.000000743 (13.4)
Italy	0 (0.344)	0 (39.3)	0 (193.)	0.0069 (0.0567)	0.017 (0.649)	0.0085 (4.02)	0.00156 (0.0636)	0.00156 (1.09)	0.000123 (7.93)
Spain	0 (0.35)	0 (23.2)	0 (112.)	<b>0.225</b> (0.0441)	<b>0.257</b> (0.206)	<b>0.103</b> (1.91)	0.00000342 (0.109)	0.00000471 (2.19)	0.00000067 (14.4)
Sweden	0 (0.251)	0 (9.55)	0 (53.1)	<b>0.138</b> (0.0563)	<b>0.378</b> (0.154)	<b>0.209</b> (1.38)	0.00601 (0.083)	0.0177 (0.641)	0.00883 (3.99)
UK	0 (0.321)	0.000000000000000333	0 (157.)	<b>0.238</b> (0.0364)	<b>0.324</b> (0.174)	<b>0.212</b> (1.37)	0.0000174 (0.0853)	0.00155 (1.09)	0.000951 (6.02)
USA	0 (0.451)	0 (57.6)	0 (271.)	<b>0.172</b> (0.0375)	<b>0.324</b> (0.174)	<b>0.0812</b> (2.1)	0.00000000113 (0.11)	0.00000000372 (4.05)	0 (25.9)
	2LN			3LN			LL		
	KS	CM	AD	KS	CM	AD	KS	CM	AD
France	<b>0.555 (0.0276)</b>	<b>0.386 (0.151)</b>	<b>0.388</b> (0.944)	<b>0.992</b> (0.0151)	<b>0.983</b> (0.0279)	<b>0.982</b> (0.226)	0.0422 (0.0482)	0.0401 (0.499)	0.00736 (4.15)
Germany	<b>0.491 (0.0288)</b>	<b>0.678</b> (0.0824)	<b>0.565</b> (0.693)	<b>0.992 (0.015)</b>	<b>0.994</b> (0.0226)	<b>0.904</b> (0.341)	0.000345 (0.0719)	0.0000862 (1.63)	0.00000197 (12.1)
Italy	<b>0.0578</b> (0.0448)	<b>0.111 (0.331)</b>	<b>0.0803</b> (2.11)	<b>0.885</b> (0.0196)	<b>0.964</b> (0.0333)	<b>0.983</b> (0.222)	0.00117 (0.0649)	0.00769 (0.791)	0.000226 (7.36)
Spain	<b>0.177 (0.0465)</b>	<b>0.115 (0.325)</b>	<b>0.0826</b> (2.08)	<b>0.984</b> (0.0195)	<b>0.977</b> (0.0298)	<b>0.902</b> (0.344)	0.00193 (0.0787)	0.0056 (0.849)	0.0000404 (8.99)
Sweden	<b>0.744 (0.0331)</b>	<b>0.92 (0.0424)</b>	<b>0.688</b> (0.558)	<b>0.722</b> (0.0338)	<b>0.946</b> (0.0375)	<b>0.749</b> (0.498)	0.0336 (0.0697)	0.0316 (0.54)	0.00843 (4.03)
UK	<b>0.491 (0.0294)</b>	<b>0.723</b> (0.0747)	<b>0.679</b> (0.568)	<b>0.927</b> (0.0193)	<b>0.927</b> (0.0412)	<b>0.881</b> (0.367)	0.00699 (0.0594)	0.0213 (0.609)	0.00292 (4.99)
USA	<b>0.143 (0.0388)</b>	<b>0.199 (0.242)</b>	<b>0.112 (1.85)</b>	<b>0.115</b> (0.0404)	<b>0.194 (0.246)</b>	<b>0.118 (1.81)</b>	0.000247 (0.0717)	0.0194 (0.625)	0.000024 (9.49)
	2LL			3LL					
	KS	CM	AD	KS	CM	AD			
France	<b>0.881 (0.0204)</b>	<b>0.839 (0.056)</b>	<b>0.666 (0.581)</b>	<b>0.951 (0.018)</b>	<b>0.968 (0.0322)</b>	<b>0.77 (0.477)</b>			
Germany	<b>0.376 (0.0315)</b>	<b>0.551 (0.107)</b>	<b>0.489 (0.789)</b>	<b>0.887 (0.0201)</b>	<b>0.952 (0.0361)</b>	<b>0.769 (0.479)</b>			
Italy	<b>0.484 (0.0282)</b>	<b>0.257 (0.206)</b>	<b>0.101 (1.93)</b>	<b>0.767 (0.0224)</b>	<b>0.893 (0.0473)</b>	<b>0.878 (0.37)</b>			
Spain	<b>0.315 (0.0406)</b>	<b>0.171 (0.265)</b>	<b>0.074 (2.17)</b>	<b>0.889 (0.0245)</b>	<b>0.873 (0.0505)</b>	<b>0.712 (0.535)</b>			
Sweden	<b>0.7 (0.0344)</b>	<b>0.344 (0.166)</b>	<b>0.735 (0.511)</b>	NA	NA	NA			
UK	<b>0.422 (0.0311)</b>	<b>0.626 (0.092)</b>	<b>0.528 (0.737)</b>	<b>0.977 (0.0168)</b>	<b>0.969 (0.0321)</b>	<b>0.941 (0.296)</b>			
USA	<b>0.127 (0.0397)</b>	<b>0.126 (0.311)</b>	0.0445 (2.59)	<b>0.451 (0.0291)</b>	0.0144 (0.679)	0.498 (0.777)			

**Table 3**

Summary of the results of the goodness-of-fit tests KS, CM and AD for the pooled samples.

	Distribution eq.	Non-rejected	Mixed	Rejected	Total
Pareto	(1)	0	0	7	7
STEXP	(2)	5	0	2	7
LN	(3)	0	0	7	7
2LN	(4)	7	0	0	7
3LN	(5)	7	0	0	7
LL	(6)	0	0	7	7
2LL	(7)	7	0	0	7
3LL	(8)	4	2	1	7

rejection (estimation non-obtained). One result that we observe is that the mixture distributions (both log-normal and log-logistic) tend to fit the data much better than the non-mixture distributions' counterparts. Consequently, the 2LN, 3LN and 2LL (and 3LL to a lesser extent) are excellent distributions in terms of goodness-of-fit.

We present the detailed information criteria in Table 4, where the selected model according to each criterion is marked in bold. We summarize the results from the information criteria in Table 5. The information criteria do not offer much support to the Pareto, log-normal, log-logistic and mixture of two log-logistic distributions. The AIC and HQC information criteria tend to favour the mixture of log-normal distributions, but the results for the BIC are a little more varied. While the BIC does select the

**Table 4**

Maximum log-likelihoods, AIC, BIC and HQC information criteria. Selected models according to each criterion are marked in bold.

	Pareto				STEXP				LN			
	log-likelihood	AIC	BIC	HQC	log-likelihood	AIC	BIC	HQC	log-likelihood	AIC	BIC	HQC
France	−5875.5617	11753.1234	11757.8521	11754.9362	−5178.129	10360.2579	<b>10369.7152</b>	<b>10363.8834</b>	−5213.827779	10431.6556	10441.1128	10435.281
Germany	−5608.9581	11219.9163	11224.6544	11221.7319	−5122.0353	10248.0705	10257.5469	10251.7017	−5171.520379	10347.0408	10356.5171	10350.6719
Italy	−6289.5024	12581.0047	12585.7971	12582.8363	−5568.6832	11141.3663	11150.951	11145.0295	−5609.124013	11222.248	11231.8327	11225.9112
Spain	−3955.3627	7912.72532	7917.0586	7914.41696	−3530.8476	7065.69526	7074.36182	7069.07853	−3602.468377	7208.93675	7217.60331	7212.32003
Sweden	−2168.3199	4338.6397	4342.68708	4340.23895	−2026.9666	4057.93321	4066.02796	4061.13171	−2030.99471	4065.98942	4074.08416	4069.18792
UK	−5658.6706	11319.3411	11324.0357	11321.1437	−5118.6431	10241.2861	<b>10250.6752</b>	10244.8913	−5144.254093	10292.5082	10301.8973	10296.1134
USA	−8333.1388	16668.2776	16673.0598	16670.1062	−7101.7227	14207.4455	<b>14217.0099</b>	14211.1027	−7274.694883	14553.3898	14562.9542	14557.047
	2LN				3LN				LL			
	log-likelihood	AIC	BIC	HQC	log-likelihood	AIC	BIC	HQC	log-likelihood	AIC	BIC	HQC
France	−5179.9519	10369.9038	10393.5469	10378.9675	−5167.3066	<b>10350.6132</b>	10388.4422	10365.1151	−5208.051092	10420.1022	10429.5594	10423.7277
Germany	−5094.5671	10199.1341	<b>10222.8249</b>	10208.212	−5085.7183	<b>10187.4366</b>	10225.3418	<b>10201.9612</b>	−5186.174077	10376.3482	10385.8245	10379.9793
Italy	−5550.2704	11110.5407	11134.5025	11119.6987	−5530.8845	<b>11077.7689</b>	<b>11116.1077</b>	<b>11092.4217</b>	−5624.497352	11252.9947	11262.5794	11256.6579
Spain	−3522.0045	7054.00893	7075.67533	7062.46712	−3505.6174	<b>7027.23489</b>	<b>7061.90113</b>	<b>7040.76798</b>	−3589.821143	7183.64229	7192.30885	7187.02556
Sweden	−2009.5867	<b>4029.17349</b>	<b>4049.41035</b>	<b>4037.16973</b>	−2007.3299	4030.65981	4063.03879	4043.4538	−2041.814057	4087.62811	4095.72286	4090.82661
UK	−5110.6934	10231.3869	10254.8597	10240.3998	−5104.335	<b>10224.67</b>	10262.2265	<b>10239.0908</b>	−5152.201611	10308.4032	10317.7923	10312.0084
USA	−7094.1865	14198.373	14222.284	14207.516	−7090.0371	14196.0743	14234.3318	14210.7031	−7190.57064	14385.1413	14394.7057	14388.7985
	2LL				3LL							
	log-likelihood	AIC	BIC	HQC	log-likelihood	AIC	BIC	HQC	log-likelihood	AIC	BIC	HQC
France	−5180.1747	10370.3495	10393.9926	10379.4132	−5177.1824	10370.3648	10408.1938	10384.8668				
Germany	−5102.608	10215.216	10238.9068	10224.2939	−5093.6982	10203.3965	10241.3017	10217.921				
Italy	−5556.8319	11123.6637	11147.6254	11132.8217	−5538.2384	11092.4768	11130.8156	11107.1296				
Spain	−3531.7871	7073.57422	7095.24062	7082.0324	−3512.4834	7040.96681	7075.63304	7054.4999				
Sweden	−2013.447	4036.89392	4057.13078	4044.89015	NA	NA	NA	NA				
UK	−5117.5725	10245.1451	10268.6179	10254.158	−5106.6951	10229.3903	10266.9468	10243.811				
USA	−7111.5255	14233.0511	14256.962	14242.1941	−7082.3261	<b>14180.6522</b>	14218.9098	<b>14195.281</b>				



**Table 5**

Summary of the results of the selection of the models by the information criteria AIC, BIC and HQC.

	Distribution eq.	Sel AIC	Sel BIC	Sel HQC
Pareto	(1)	0	0	0
STEXP	(2)	0	3	1
LN	(3)	0	0	0
2LN	(4)	1	2	1
3LN	(5)	5	2	4
LL	(6)	0	0	0
2LL	(7)	0	0	0
3LL	(8)	1	0	1
Total		7	7	7

mixture of log-normal distributions, the stretched exponential distribution is also selected in 3 countries. Overall, it seems that a mixture of log-normal distributions tends to fit the mortality data for this pandemic from its start quite well.

Thus we obtain the global conclusion that the best fitting distributions for daily COVID-19 deaths in all of the 7 countries we consider are the 2LN and the 3LN, i.e., the mixtures of 2- or 3- log-normal distributions. Also the stretched exponential (STEXP) distribution seems to be a good model if one wants to deal with a model with only two parameters. This conclusion is similar to one in Campolieti and Ramos (2021) for the distribution of strike size: the 2LN and 3LN perform extremely well. And there is also one difference with the conclusion of Campolieti and Ramos (2021): there, the Weibull or stretched exponential performed very poorly; meanwhile for daily deaths' data sets, the STEXP does offer a reasonable fit with as few as two parameters.

We also provide a graphical assessment of the fit for, our most preferred and plausible, models, i.e., the STEXP, 2LN and 3LN in Fig. 1 (for the log-coranks as a function of the log-daily deaths) and in Fig. 2 (for the log-ranks as a function of the log-daily deaths). In them, we see that the best graphical fit is obtained by far by the 3LN (black), although sometimes almost indistinguishable from the 2LN (green), while for the STEXP (red) there are some deviations, above all at the lower tail.

We have also computed the tail probabilities for the distributions we consider and present them in Table 6. We evaluate the distributions at a few different points to illustrate how the distributions differ both below and in the upper tail. We evaluate the distributions for each country at their respective mean, the mean plus one standard deviation, the mean plus two standard deviations and the mean plus three standard deviations.<sup>6</sup> The quantities we present are the probability that the value exceeds the levels we consider. Having estimates of probabilities means that government and public health planners could compute the probability of mortality or the expected mortality and use this information to determine the benefits of various public health interventions.

Table 6 shows that as the level of mortality increases the probability of observing that level decreases. It also illustrates the differences between distributions. In particular, the mixtures of log-normal distributions tend to place much less probability on a given level of mortality than the log-normal distribution. The smallest probability tends to be in the mixture of 3 log-normal distributions (3LN), which was also the best fitting distribution. We see a similar pattern when we compare the mixture of log-logistic distributions (2LL and 3LL) with the log-logistic distribution. We also find that the probabilities from the 3LN mixture also tend to smaller than the probability from the 3LL mixture. Also the 3LN can be estimated for all the countries we consider but the 3LL is not numerically feasible for Sweden. The information criteria also favour the mixtures of log-normal distributions.

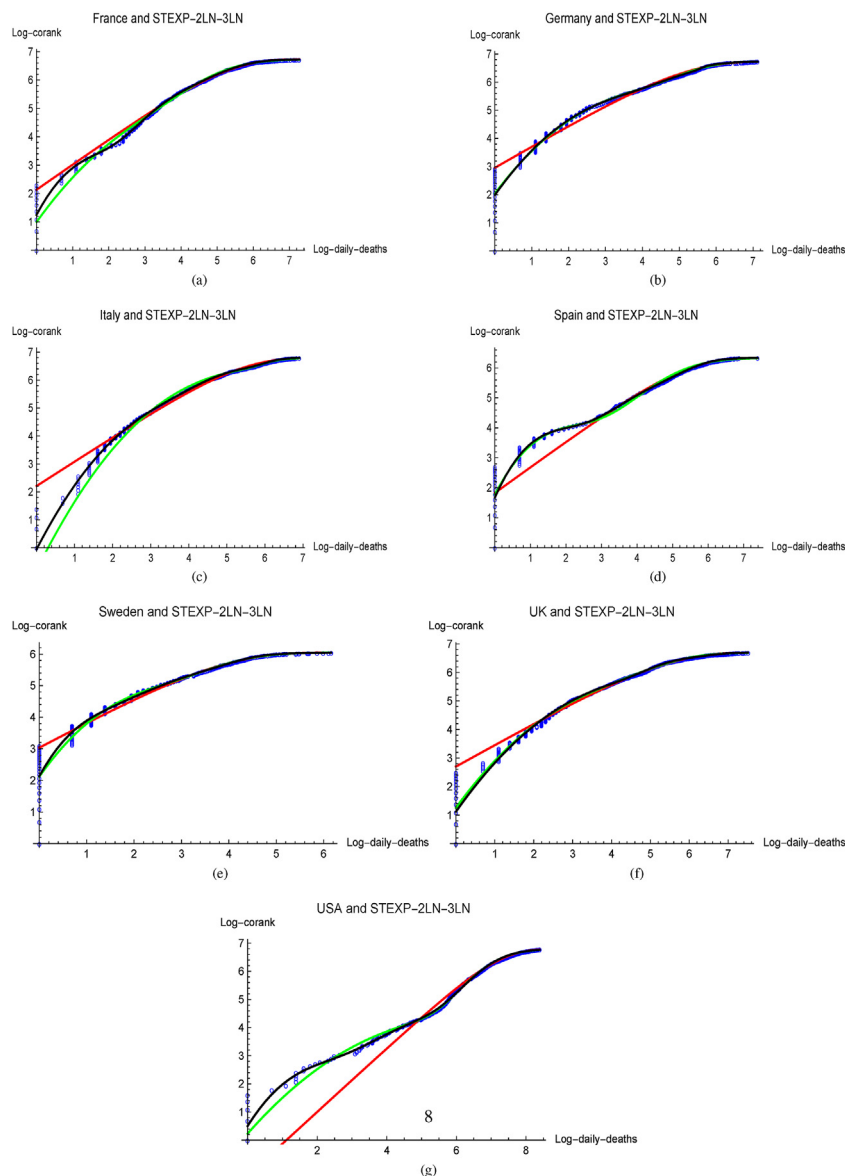
Another striking result is the comparison with the stretched exponential or Weibull and power-law distributions. The power-law distribution tends to place a much larger probability on all the mortality levels we evaluate across all countries. As our analysis shows, the power-law is rejected as a fit to the data and using it would tend to overestimate the probability of the tail risks of mortality from COVID-19. Even more striking, is how similar the probabilities for the stretched exponential distribution are to the mixture of 3 log-normal distributions for some of the countries we consider. This confirms our findings based on the goodness-of-fit and information criteria that the stretched exponential could be a plausible fit to the mortality data in many countries if one desired a more parsimonious distribution.

The best fitting distributions for the countries we consider are often mixtures distributions. These distributions have been shown to arise if a suitable stochastic process is specified and the Fokker-Planck equations are solved (Peña et al., 2022).<sup>7</sup> The Fokker-Planck (or forward Kolmogorov) equations also arise in stochastic models of epidemics (Allen, 2008). While these stochastic models are generally applied to study an initial outbreak of an epidemic in a small population (Brauer, 2017), our findings suggest that there could be a greater role for them when studying mortality during a pandemic and epidemic (and not only during the opening phase of a pandemic or epidemic).

<sup>6</sup> We use the mean and standard deviation for each country to evaluate these probabilities as there are some differences in both the level of deaths and their dispersion. Recall that we measure mortality as the number of daily deaths in each country.

<sup>7</sup> The same is also true for the stretched exponential as well as the other distributions we consider. In other words, these distributions can also be obtained as solutions to the Fokker-Planck equations.



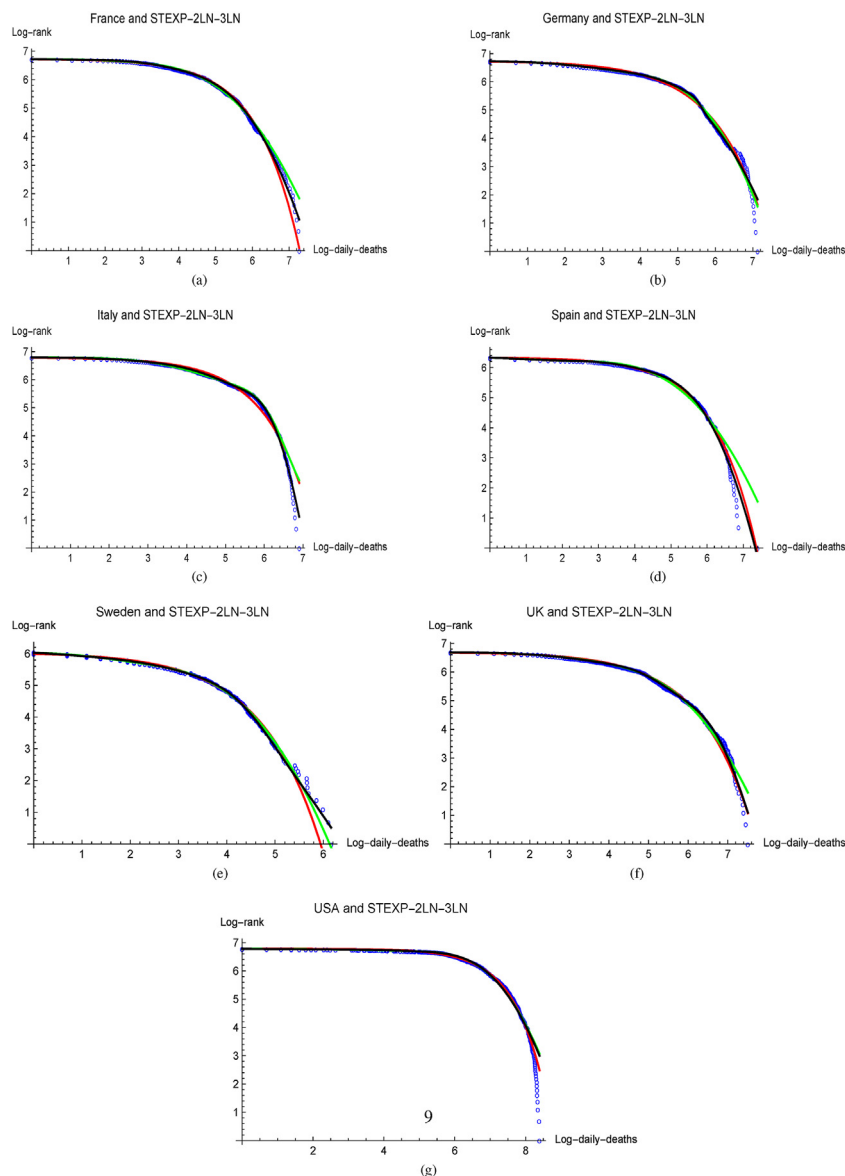


**Fig. 1.** Log-corank plots for the whole samples of log-new-deaths, using the STEXP (red), 2LN (green) and 3LN (black) distributions and the empirical data (blue).

### 3.2. Analysis by year

As we indicated earlier, the COVID-19 pandemic has gone through different phases as public health interventions have changed and the dominant strain of the virus has evolved. So we also estimate the distribution of COVID-19 mortality for each individual year in the pandemic, i.e., 2020, 2021 and 2022. We present the descriptive statistics for each country by year in Table 7. We can see in Table 7 that the number of observations varies by year, as mortality has declined during 2022 and some of the distributions we estimate require positive support, i.e., mortality greater than 0. As in the pooled data for 2020 to 2022, we observe a great deal of dispersion, skewness and kurtosis, indicating again skewed and heavy-tailed data.

We have re-estimated the same statistical distributions using data for the individual years in the pandemic. The maximum likelihood estimates, along with their standard errors, are presented in the supplemental Excel file. Unfortunately, we encounter more numerical problems with the data for individual years and we cannot obtain estimates for all the distributions we consider. We can obtain estimates for Pareto, log-normal and log-logistic distributions as well as mixtures of two log-logistic distributions for all countries and all years. We cannot obtain estimates for the following distributions: the stretched exponential for Sweden in 2021; mixture of two log-normals for Sweden in 2021 and the UK in 2022; the mixture of three



**Fig. 2.** Log-rank plots for the whole samples of log-new-deaths, using the STEXP (red), 2LN (green) and 3LN (black) distributions and the empirical data (blue).

log-normals for Spain and Sweden in 2021 and the UK in 2022; and the mixture of three log-logistic distributions for Sweden and UK in 2022.

As they are quite numerous, the goodness-of-fit tests, i.e., the KS, CM and AD tests, are presented in the supplementary Excel file. However, we do provide a summary of the results from these tests in [Table 8](#) (with the same terminology as that used for the pooled samples we discussed earlier).

The Pareto distribution is always strongly rejected (21 out of 21 times). For 2020, the STEXP, the LN and the LL are rejected 5 times out of 7, and non-rejected and mixed once. The 2LN, 3LN, 2LL and 3LL are non-rejected 7 times. For 2021, the STEXP is non-rejected 3 times and rejected 4, the LN and LL are non-rejected and mixed 3 times each and rejected once. The 2LN is non-rejected 6 times and rejected (non-estimated) once. The 3LN is similar to the 2LN, but 5 non-rejections and 2 rejected (non-estimated). The 2LL and 3LL achieve both 7 non-rejections. For 2022 the STEXP fits the data better, with 5 non-rejections, one mixed and one rejection. The LN and LL get 3 non-rejections, 1 mixed and 3 rejections. The 2LN and 3LN behave exactly as in 2021. Finally, the 2LL gets 6 non-rejections and one mixed, and the 3LL gets 5 non-rejections and 2 rejections (non-estimations). Overall, the mixture distributions tend to offer a good fit data.

We next present the results of the information criteria adapted to the fact that the sample sizes have been reduced by roughly a factor of 1/3 with respect to the pooled samples. This means that it is advisable to take into account small sample

**Table 6**

Tail probabilities according to each studied distribution.

	mean							
	Pareto	STEXP	LN	2LN	3LN	LL	2LL	3LL
France	0.31606255	0.34863387	0.30525187	0.32089169	0.3411608	0.31186051	0.34472997	0.33625588
Germany	0.29488146	0.32280997	0.28615532	0.34922596	0.35966193	0.30371819	0.35550773	0.36059518
Italy	0.3140743	0.3476258	0.30295675	0.36680513	0.35540579	0.31397848	0.37054551	0.35620018
Spain	0.31007712	0.34573259	0.31334237	0.32894629	0.37312887	0.33755142	0.33749006	0.37592595
Sweden	0.28078783	0.33031666	0.28492903	0.33339651	0.34066815	0.29419708	0.33953001	NA
UK	0.30070509	0.32099314	0.28115863	0.31526765	0.30360281	0.28885671	0.3115988	0.30042824
USA	0.34106256	0.38517074	0.35119957	0.36716717	0.36854081	0.3701242	0.36821743	0.3839662
	mean + sd							
	Pareto	STEXP	LN	2LN	3LN	LL	2LL	3LL
France	0.2659202	0.12079603	0.13809241	0.11692636	0.11580924	0.13834219	0.11376493	0.11885677
Germany	0.24333804	0.12426105	0.14568544	0.12061095	0.10573568	0.15743951	0.11742251	0.11496243
Italy	0.26819634	0.13702005	0.15124062	0.17959997	0.17504864	0.15842337	0.18320364	0.17569398
Spain	0.26455853	0.13936118	0.17408359	0.13658284	0.13416696	0.18092729	0.14403168	0.13087862
Sweden	0.21197564	0.11131605	0.12310546	0.10789573	0.09881418	0.13205383	0.10248625	NA
UK	0.24926418	0.11525157	0.13166554	0.11223873	0.12695494	0.13660275	0.10815132	0.1284276
USA	0.31114821	0.15149732	0.19892383	0.14215808	0.14043766	0.18661533	0.14677522	0.16884725
	mean+2sd							
	Pareto	STEXP	LN	2LN	3LN	LL	2LL	3LL
France	0.2416245	0.04461896	0.07921258	0.05414942	0.04621816	0.08281667	0.05098456	0.04790732
Germany	0.21912201	0.05435397	0.09296407	0.04828163	0.04662861	0.10527486	0.05153586	0.04199446
Italy	0.24487199	0.05723915	0.09209259	0.06358393	0.05303812	0.10138848	0.06761282	0.05454387
Spain	0.24142588	0.05995787	0.11535804	0.06945622	0.05041742	0.12006456	0.08062339	0.05471284
Sweden	0.18248496	0.0423855	0.07014555	0.04296315	0.03695734	0.0816632	0.0434993	NA
UK	0.2254839	0.04755102	0.07976113	0.05031117	0.05755425	0.08725115	0.05342479	0.05532531
USA	0.29394673	0.05615357	0.12870074	0.0627658	0.06055571	0.11356775	0.07390995	0.05561814
	mean+3sd							
	Pareto	STEXP	LN	2LN	3LN	LL	2LL	3LL
France	0.22604766	0.01712091	0.05113264	0.0288705	0.02100104	0.05705532	0.02852207	0.02492624
Germany	0.20380082	0.02550683	0.0658068	0.02198281	0.02351537	0.07872564	0.02807926	0.01943191
Italy	0.22965034	0.02477215	0.06210134	0.02446095	0.01292411	0.0729007	0.03178485	0.01892058
Spain	0.22633433	0.02683094	0.08345622	0.03981879	0.02070985	0.08853412	0.05233369	0.02796879
Sweden	0.16466428	0.01726778	0.04543203	0.01982448	0.01867619	0.05795708	0.02324238	NA
UK	0.21048298	0.02115436	0.05430026	0.02597492	0.02464286	0.06334967	0.03201495	0.02525215
USA	0.28205032	0.01995295	0.08994367	0.03089457	0.0289864	0.07741034	0.04366041	0.02124884

sizes in the information criteria, and therefore we will use the previously defined corrected AIC (AICc) and adjusted BIC (BICa). To the best of our knowledge, there exist no version of the HQC adapted or adjusted to small sample sizes, so we will use only the first two in what follows. The complete set of information criteria statistics are available in the supplementary Excel file, but we provide a summary of the results in [Table 9](#).

For 2020, the 2LN and 3LN are clearly selected 3 times each, and the 3LL once. For the 2021, information criteria the LN once, the 3LN twice and the mixtures of log-logistic distribution four times, with the 3LL chosen 4 times based on the AICc and 3 times by the BICa. For the 2022, the results are more varied as the information criteria select the STEXP, LN, 2LN, 2LL and 3LL once and the 3LN is selected twice.

The information criteria suggest that the distribution of COVID-19 mortality does vary somewhat over time. For 2020, our findings are much like those for the pooled sample, in that we find a great deal of support for the mixture of log-normal distributions, the 2LN and 3LN. While the 3LN distribution is selected twice in the 2021 data, the information criteria favour the 3LL distribution, i.e., the mixture of 3 log-logistic distributions. The findings for 2021 are thus not as clear as those for 2020, which had strong support for the mixture of log-normal distributions. The information criteria for 2022 are much more varied as they select candidates such as the stretched exponential, the 2LN, 3LN, 2LL and 3LL. The clearest result based on the information criteria is that the Pareto distribution is not selected in any country we study during any phase of the pandemic. This finding is quite striking as the Pareto distribution has been the focus of earlier research studying the distribution of COVID-19 mortality. In summary, our findings suggest that the distribution of COVID-19 mortality does appear to vary over time. This finding is perhaps not too surprising because, as we have noted earlier, new variants of the disease have emerged and public health measures used to control the spread of the disease have changed over time, e.g., mass vaccination programs. Moreover, it also appears that mixture distributions are quite useful for modelling COVID-19 mortality, although whether they are log-normal or log-logistic does vary across time. Overall, it seems a mixture of log-normal distributions fits best in 2020, but a mixture of log-logistic distributions may offer a better fit in 2021. The results for 2022 are much more

**Table 7**

Descriptive statistics of the samples by year.

	Number of Observations	Mean	SD	Skewness	Kurtosis	Min	Max
France 2020	284	228.982394	298.001571	1.92754664	6.60369801	1	1438
Germany 2020	286	115.660839	189.934893	2.66306535	10.8838146	1	1129
Italy 2020	314	236.273885	271.759951	0.89859981	2.39080637	1	993
Spain 2020	229	225.231441	250.295919	1.6745334	7.01018441	1	1623
Sweden 2020	183	49.0655738	61.693821	2.81903148	14.5198422	1	448
UK 2020	298	246.875839	284.031691	1.29908229	3.97845046	1	1224
USA 2020	306	1143.58824	784.311631	1.05986803	4.01468784	1	3890
France 2021	363	162.977961	165.708856	1.92122649	7.60220859	4	1008
Germany 2021	365	216.038356	253.466014	1.92837746	6.57264868	1	1244
Italy 2021	365	173.268493	168.341014	0.9344871	2.66086961	3	718
Spain 2021	250	154.272	178.264767	1.92635698	6.14570713	1	909
Sweden 2021	178	37.005618	67.3601919	3.89328007	20.4538802	1	474
UK 2021	364	206.505495	338.014454	2.70953901	9.95560464	1	1820
USA 2021	365	1301.74247	1049.94687	1.03864538	3.32824162	41	4411
France 2022	189	149.455026	122.723458	2.00460631	8.29244535	1	691
Germany 2022	193	166.150259	119.225728	0.86091171	4.56408862	1	668
Italy 2022	212	163.603774	103.238433	1.05927763	3.31518552	18	468
Spain 2022	84	255.869048	119.049426	0.96598906	3.69510797	58	617
Sweden 2022	62	65.2903226	38.1276489	0.69538239	3.04107916	6	176
UK 2022	146	214.623288	181.227392	2.68263516	13.7323439	1	1226
USA 2022	211	972.180095	1020.28844	1.28574119	3.59097296	1	4118

**Table 8**

Summary of the results of the goodness-of-fit tests KS, CM and AD for the samples by year.

2020		Non-Rejected	Mixed	Rejected	Total
	Pareto	0	0	7	7
	STEXP	1	1	5	7
	LN	1	1	5	7
	2LN	7	0	0	7
	3LN	7	0	0	7
	LL	1	1	5	7
	2LL	7	0	0	7
	3LL	7	0	0	7
2021		Non-Rejected	Mixed	Rejected	Total
	Pareto	0	0	7	7
	STEXP	3	0	4	7
	LN	3	3	1	7
	2LN	6	0	1	7
	3LN	5	0	2	7
	LL	3	3	1	7
	2LL	7	0	0	7
	3LL	7	0	0	7
2022		Non-Rejected	Mixed	Rejected	Total
	Pareto	0	0	7	7
	STEXP	5	1	1	7
	LN	3	1	3	7
	2LN	6	0	1	7
	3LN	5	0	2	7
	LL	5	1	1	7
	2LL	6	1	0	7
	3LL	5	0	2	7

indeterminate, but this also reflects that there is less complete data than in the other years we consider, which makes it more difficult to distinguish between alternative distributions (Clauset et al., 2009).

### 3.3. An assessment on prediction

The distribution of mortality can also be used to undertake some predictive exercises. As we are estimating the distribution of COVID-19 mortality, we do not forecast the level of mortality as we would with a regression or time series model, but present the VaR and TVaR measures we discussed earlier. Recall that the VaR and TVaR present the level of mortality in the upper tails of the distribution, i.e., 99th, 95th and 90<sup>th</sup> percentiles.

We compute the VaR measures using the most recent data from 2022. For each country and sample, we successively remove the last 30, 29, 28, ..., 2, 1 observations. This splits the data into “in-sample”, which is used for estimation, and “out-of-

**Table 9**  
Summary of the results of the AICc and BICa information criteria for the samples by year.

		Sel AICc	Sel BICa
2020	Pareto	0	0
	STEXP	0	0
	LN	0	0
	2LN	3	3
	3LN	3	3
	LL	0	0
	2LL	0	0
	3LL	1	1
	Total	7	7
2021	Pareto	0	0
	STEXP	0	0
	LN	1	1
	2LN	0	0
	3LN	2	2
	LL	0	0
	2LL	0	1
	3LL	4	3
	Total	7	7
2022	Pareto	0	0
	STEXP	1	1
	LN	1	1
	2LN	1	1
	3LN	2	2
	LL	0	0
	2LL	1	1
	3LL	1	1
	Total	7	7

sample” portions, with the out-of-sample portion corresponding (more or less) to the month of data, i.e., July 2022, which we use to compute the VaR measures based on the rolling in-sample data.

To assess whether the two partitions of the data come from the same distribution, we have performed two-sample Kolmogorov–Smirnov (KS), Cramér–von Mises (CM) and Anderson–Darling (AD) tests.<sup>8</sup> In general, the null hypothesis that the data come from the same distribution is non-rejected for France, Germany, Italy, Spain, Sweden, but we reject this null for the UK and the USA. This suggests that the COVID-19 mortality distribution for the UK and the USA for our out-of-sample data differs from the in-sample data, meanwhile for the other countries it is more stable. Another observation that emerges from these computations is that the COVID-19 mortality distribution is somewhat sensitive to the addition or removal of observations. This will be reflected in the VaR and TVaR we present.

We compute the VaR and TVaR for the mixture of three log-normal distributions (3LN) as we can always obtain estimates for this distribution in the period we consider for the VaR computations and the fit of the distribution to the data is good. Using the estimates for the 3LN, we have computed the  $VaR_\alpha$  and  $TVaR_\alpha$  for the levels of significance  $\alpha = 0.01$ ,  $\alpha = 0.05$  and  $\alpha = 0.10$  for all the rolling samples. The results are plotted in Figs. 3–5, respectively.

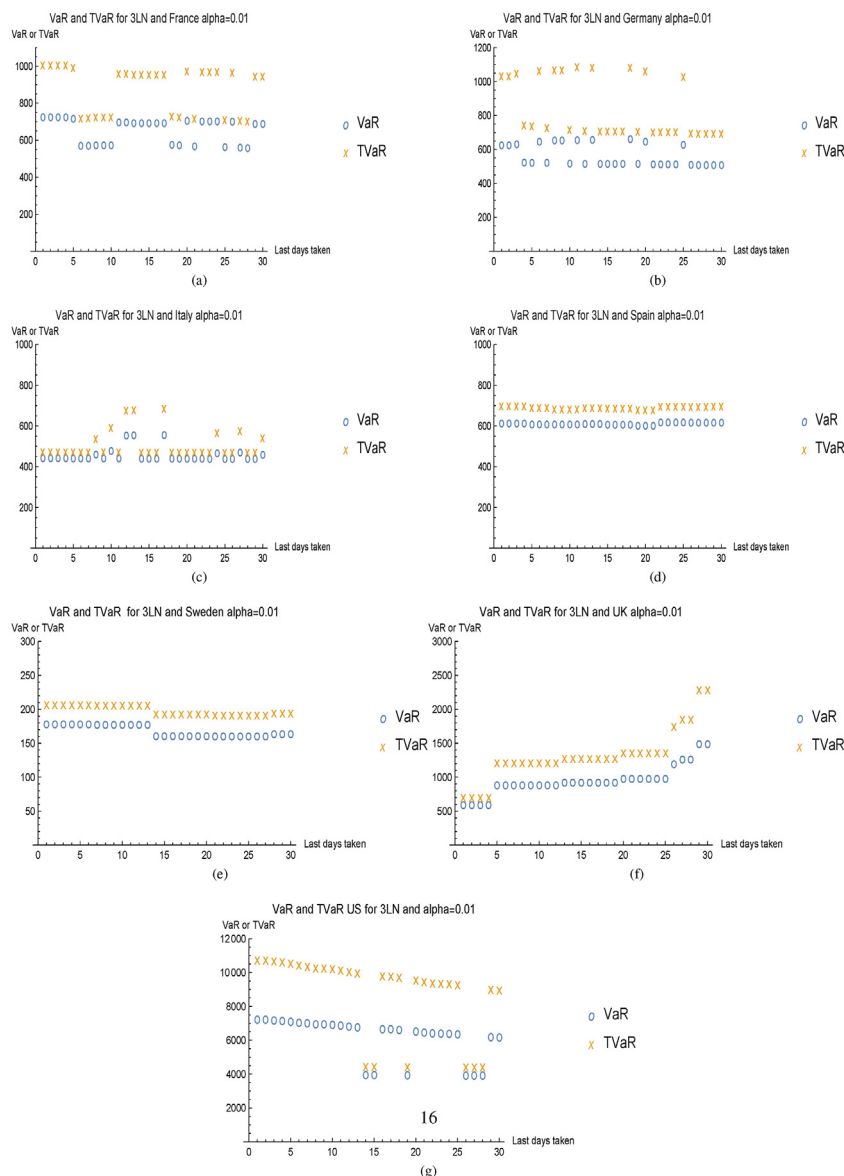
The main result out of these figures is that the VaR and TVaR remain quite stable for all the studied countries except the UK and the USA, but with the qualification that even one more or less observation can cause these measures to jump somewhat. Another result is that for the UK, both the VaR and TVaR increase during the out-of-sample period. This pattern might reflect our observation that the distribution for mortality in the in-sample and out-of-sample periods is likely different. For the USA we observe that the VaR and TVaR are decreasing during the out-of-sample period. As with the UK, this could reflect that distribution of mortality for the in-sample and out-of-sample periods is different for the USA.

In short, there is variability in the values of the extremes of COVID-19 mortality; for some countries this value remains more or less stable, but for other countries it is much more variable. It is not inconceivable that these patterns could interchange among themselves for a given country as the pandemic evolves. An important implication of the VaR and TVaR measures is that they could be used to obtain worst case scenarios for mortality, i.e., the tails of the distribution or, equivalently, providing the tail risks of mortality, and be used for risk management during a pandemic by public health and government officials.

#### 4. Concluding remarks and discussion

We studied the distribution of daily COVID-19 deaths for six European countries and the United States. While some earlier papers have found evidence for a power-law distribution in COVID-19 mortality, we soundly reject this distribution based on

<sup>8</sup> The results are available in the supplemental Excel file. There are missing results but are so because one of the samples is too small to perform the tests.

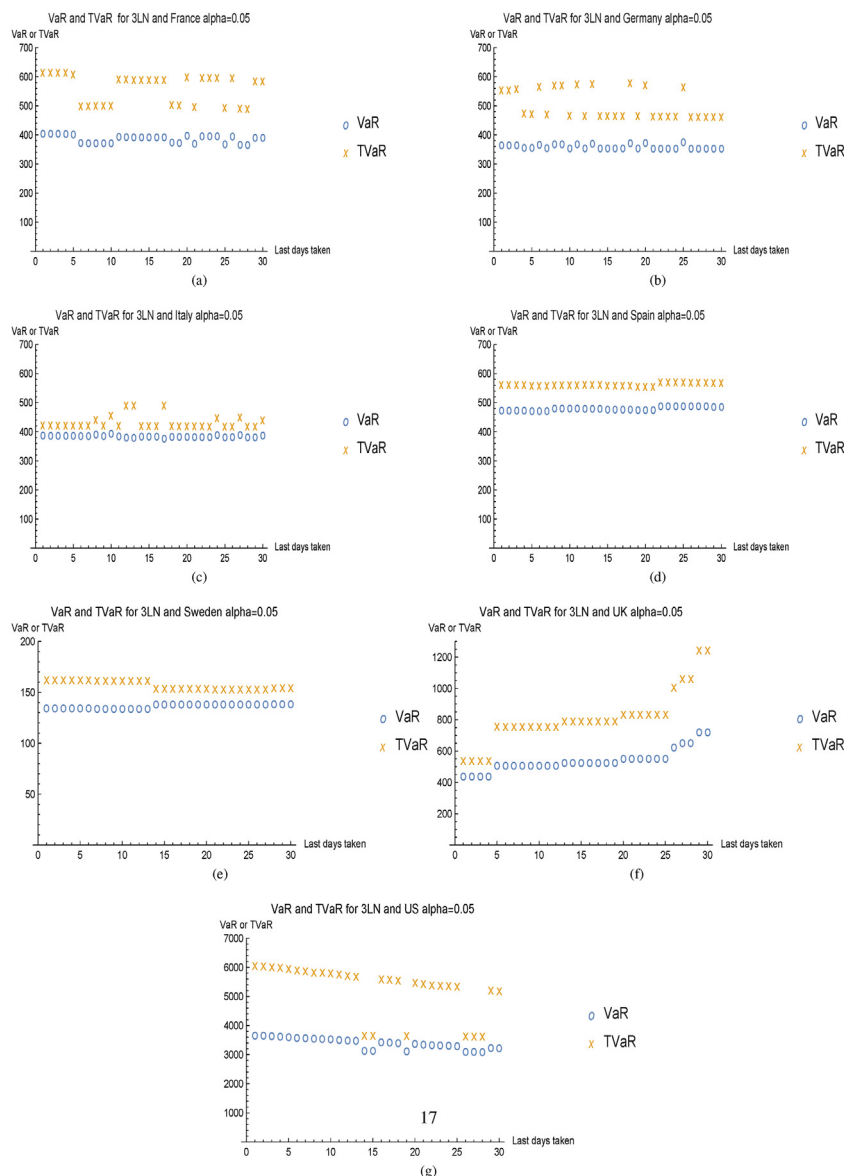


**Fig. 3.** Graphs of the VaR and TVaR for the value of  $\alpha = 0.01$  and the samples contained in 2022 where we remove the last 30, 29, 28, ..., 1 days of observations.

the goodness of fit tests and the information criteria. Our analysis shows the potential problems of using visual methods of assessing the fit of a distribution. Our results also point to a mixture of 2- or 3- log-normal distributions as a better fitting alternative to the data from the start of the pandemic to the end of July 2022. This confirms the observation in Corral (2021) about the potential for alternative distributions to fit the distribution of mortality from infectious diseases. Our analysis also includes the data below the upper tail and from three years of the pandemic, so the preferred mixture distributions can be used to assess the probability of differing levels of mortality. When we conduct our analysis by year, we find that distribution of COVID-19 mortality does change over time. This should not be too surprising as new variants have emerged and public health measures have changed. In particular, we find a mixture of log-normal distributions is best for 2020, but a mixture of log-logistic distributions for 2021. For 2022, our results are much more varied, but this could also reflect that the sample size for our data is much smaller and this can make it much more difficult to distinguish between alternative distributions.

We used Value-at-Risk measures to estimate the levels of mortality in the tails of the distribution for an out-of-sample period, i.e., July 2022. While there is some instability in the estimates for the USA and the UK, the estimates for the other countries are amenable to these sorts of computations and provide some indication of the tail risks of mortality during this phase of the pandemic.

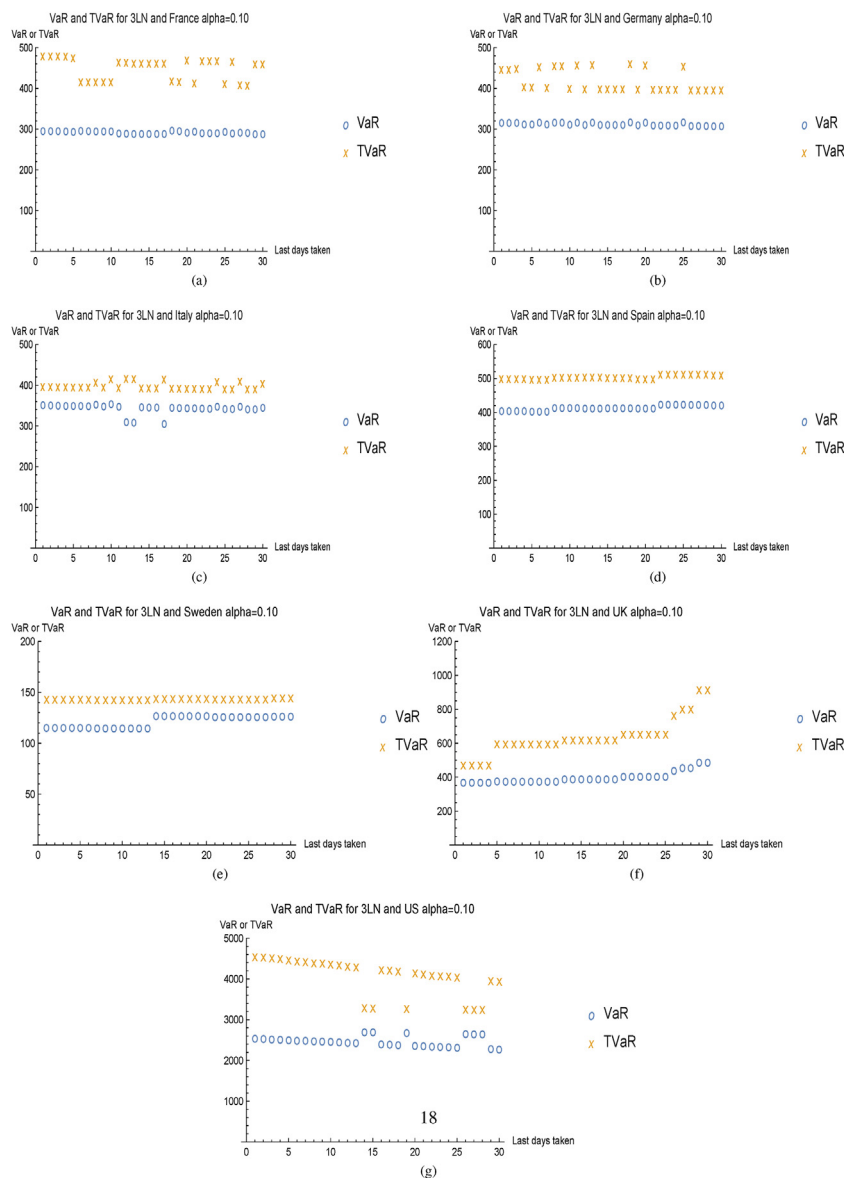
Our findings and results are of great interest to government and public health officials as they can use these distributions to assess the probability of elevated mortality as part of their pandemic planning. Our results show that the probabilities from



**Fig. 4.** Graphs of the VaR and TVaR for the value of  $\alpha = 0.05$  and the samples contained in 2022 where we remove the last 30, 29, 28, ..., 1 days of observations.

the most plausible distribution can differ substantially from some of the other distributions we consider, which are not a plausible fit to the data, such as the Pareto (power-law) distribution (which has been a focus of earlier research). Consequently, government and public health officials would have better estimates of the consequences of the initiatives that they are considering by using the distribution that best fits the data. In addition, our paper also presents Value-at-Risk measures as a tool for pandemic planning and risk management. While Value-at-Risk measures are often used in financial and insurance markets, our results show that these measures could also be used to assess the tail risks for mortality from infectious diseases. Value-at-Risk measures are thus an interesting area for future research on modelling mortality from infectious diseases and pandemic management to consider.





**Fig. 5.** Graphs of the VaR and TVaR for the value of  $\alpha = 0.10$  and the samples contained in 2022 where we remove the last 30, 29, 28, ..., 1 days of observations.

## Author contributions

Michele Campolieti: Conceptualization, data curation, formal analysis, investigation, methodology, software, supervision, validation, visualization, writing-original draft, writing-review & editing. Arturo Ramos: Conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, resources, software, validation, visualization, writing-original draft, writing-review & editing.

## Data availability statement

Data are fully and freely available at <https://ourworldindata.org/covid-cases>. Code files are available at <https://doi.org/10.7910/DVN/HWHKSG>.

A supplementary spreadsheet containing the MLE estimates, goodness of fit tests, information criteria statistics and other output from this paper are also available at <https://doi.org/10.7910/DVN/HWHKSG>.

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## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix A. Supplementary data

Supplementary data to this article can be found online at <https://doi.org/10.1016/j.idm.2022.11.003>.

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