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Accurate computations with Wronskian matrices

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Abstract In this paper we provide algorithms for computing the bidiagonal decomposition of the Wronskian matrices of the monomial basis of polynomials and of the basis of exponential polynomials. It is also shown that these algorithms can be used to perform accurately some algebraic computations with these Wronskian matrices, such as the calculation of their inverses, their eigenvalues or their singular values and the solutions of some linear systems. Numerical experiments illustrate the results.

Keywords Accurate computations · Wronskian matrices · Bidiagonal decompositions

1 Introduction

The accuracy of the calculations is a desirable goal in Computational Mathematics. Let us recall that an algorithm can be performed with high relative accuracy (HRA) if it does not include subtractions of numbers having the same sign (except of the initial data if they are exact), that is, if it only includes products, divisions, additions of numbers of the same sign and subtractions of the initial data having the same sign provided that they are not affected by errors (cf. [5]). For some structured classes of matrices such algorithms have been found through an adequate parameterization of the matrix. In particular, this has been achieved for some subclasses of totally positive (TP) matrices. In [11] it was shown that, given the bidiagonal factorization of a nonsingular TP matrix A with HRA, we can compute with HRA

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its eigenvalues and singular values, the matrix A^{-1} and even the solution of $Ax = b$ for vectors b with alternating signs. Among the subclasses of TP matrices for which the bidiagonal factorization has been obtained with HRA (cf. [3], [4], [13], [14]), there are many examples of collocation matrices $(u_{j-1}(t_i))_{1\le i,j\le n+1}$ of systems $(u_0,\dots,u_n)$ of functions defined on a real subset I ($t_1 < t_2 < \cdots < t_{n+1}$ in I). However, up to now, there are no examples of accurate computations for matrices involving derivatives of the basis functions. This paper presents some examples of Wronskian matrices for which many algebraic computations can be performed accurately. These Wronskian matrices come from applications in computer aided geometric design (CAGD) and they can also arise in Hermite interpolation problems, in particular in Taylor interpolation problems.

The paper is organized as follows. In Section 2, we provide basic concepts and tools. In particular we recall the Neville elimination procedure and the bidiagonal factorization of a nonsingular TP matrix. This factorization provides the adequate parameterization to derive the accurate algorithms with these matrices. Section 3 shows that the bidiagonal factorization of the Wronskian matrices of the monomial basis of polynomials can be performed with HRA. In Section 4 we first prove that Wronskian matrices of the basis of exponential polynomials on positive real numbers are strictly totally positive. We also provide the bidiagonal factorization of these matrices. The computation with HRA of this factorization should require the evaluation with HRA of the involved exponential functions. Although this cannot be guaranteed, numerical experiments show an accuracy similar to the obtained for the monomial basis. Finally, Section 5 includes numerical experiments showing the accuracy of the presented methods for the computation of all eigenvalues, all singular values, the inverses and the solution of linear systems.

2 Notations and auxiliary results

As usual, given an n-times continuously differentiable function f and x in its parameter domain, $f'(x)$ denotes the first derivative of f at x and, for any $i \le n$, $f^{(i)}(x)$ denotes the i-th derivative of f at x. Let us recall that for a given basis $(u_0,\dots,u_n)$ of a space of n-times continuously differentiable functions, defined on a real interval I and $x \in I$, the *Wronskian matrix* at x is defined by

$$W(u_0,\dots,u_n)(x) := (u_{j-1}^{(i-1)}(x))_{i,j=1,\dots,n+1}.$$

A matrix is totally positive: TP (respectively, strictly totally positive: STP) if all its minors are nonnegative (respectively, positive). Two recent books on these matrices are [6] and [16], where many applications of these matrices are presented, as well as in [1].

Neville elimination is an alternative procedure to Gaussian elimination and has been used to characterize TP and STP matrices. Given a nonsingular matrix $A = (a_{i,j})_{1\le i,j\le n+1}$, Neville elimination computes a matrix sequence

$$A^{(1)} := A \to A^{(2)} \to \cdots \to A^{(n+1)} = U,$$

such that, for $1 \le k \le n$, $A^{(k+1)} = (a_{i,j}^{(k+1)})_{1\le i,j\le n+1}$ has zeros below its main diagonal in the first k columns and is computed from $A^{(k)} = (a_{i,j}^{(k)})_{1\le i,j\le n+1}$ by:

$$a_{i,j}^{(k+1)} := \begin{cases} a_{i,j}^{(k)}, & \text{if } 1 \le i \le k, \\ a_{i,j}^{(k)} - \frac{a_{i,k}^{(k)}}{a_{i-1,k}^{(k)}} a_{i-1,j}^{(k)}, & \text{if } k+1 \le i, j \le n+1 \text{ and } a_{i-1,k}^{(k)} \neq 0, \\ a_{i,j}^{(k)}, & \text{if } \quad k+1 \le i \le n+1 \text{ and } a_{i-1,k}^{(k)} = 0. \end{cases}$$

The element $p_{i,j} := a_{i,j}^{(j)}$, $1 \le j \le i \le n+1$, is called the (i,j) pivot and, in particular, $p_{i,i}$ is a diagonal pivot of the Neville elimination of A. If all the pivots are nonzero then Neville elimination can be carried out without row exchanges. In this case, by Lemma 2.6 of [7],

$$p_{i,1} = a_{i,1}, \quad 1 \le i \le n+1,$$
$$p_{i,j} = \frac{\det A[i-j+1,\ldots,i|1,\ldots,j]}{\det A[i-j+1,\ldots,i-1|1,\ldots,j-1]}, \quad 1 < j \le i \le n+1, \tag{1}$$

where, given increasing sequences of integers α and β, $A[\alpha|\beta]$ denotes the submatrix of A containing rows of places α and columns of places β. Moreover,

$$m_{i,j} := \begin{cases} a_{i,j}^{(j)}/a_{i-1,j}^{(j)} = p_{i,j}/p_{i-1,j}, & \text{if } a_{i-1,j}^{(j)} \ne 0, \\ 0, & \text{if } a_{i-1,j}^{(j)} = 0, \end{cases} \quad 1 \le j < i \le n+1, \tag{2}$$

is called the (i,j) multiplier of the Neville elimination of A.

By Theorem 4.2 and the arguments of p.116 of [9], a nonsingular TP matrix $A = (a_{i,j})_{1\le i,j\le n+1}$ admits a factorization of the form

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n, \tag{3}$$

where F_i and G_i are the TP, lower and upper triangular bidiagonal matrices given by

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & m_{i+1,1} & 1 & & \\ & & & & m_{i+2,2} & 1 & \\ & & & & & \ddots & \ddots \\ & & & & & m_{n+1,n+1-i} & 1 \end{pmatrix}, \quad G_i^T = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \widetilde{m}_{i+1,1} & 1 & & \\ & & & & \widetilde{m}_{i+2,2} & 1 & \\ & & & & & \ddots & \ddots \\ & & & & & \widetilde{m}_{n+1,n+1-i} & 1 \end{pmatrix}, \tag{4}$$

and $D = \mathrm{diag}(p_{1,1}, p_{2,2}, \ldots, p_{n+1,n+1})$ has positive diagonal entries. If, in addition, the entries m_{ij}, $\widetilde{m}_{ij}$ satisfy

$$m_{ij} = 0 \Rightarrow m_{hj} = 0 \quad \forall h > i$$

and

$$\widetilde{m}_{ij} = 0 \Rightarrow \widetilde{m}_{ik} = 0 \quad \forall k > j,$$

then the decomposition (3) is unique. The diagonal entries $p_{i,i}$ of D are the diagonal pivots of the Neville elimination of A and the elements $m_{i,j}$ and $\tilde{m}_{i,j}$ are the multipliers of the Neville elimination of A and A^T, respectively. We shall denote the bidiagonal decomposition (3) of a TP matrix A by $BD(A)$ (see [10]). Given $BD(A)$, using the results in [7–9], a bidiagonal decomposition of A^{-1} can be computed as

$$A^{-1} = \tilde{G}_1 \tilde{G}_2 \cdots \tilde{G}_n D^{-1} \tilde{F}_n \cdots \tilde{F}_2 \tilde{F}_1, \tag{5}$$

where $\tilde{F}_i$ and $\tilde{G}_i$, $i = 1,\ldots,n$, are the lower and upper triangular bidiagonal matrices of the form of F_i and G_i, respectively, but replacing the off-diagonal entries $\{m_{i+1,1},\ldots,m_{n+1,n+1-i}\}$ and $\{\tilde{m}_{i+1,1},\ldots,\tilde{m}_{n+1,n+1-i}\}$ by $\{-m_{i+1,i},\ldots,-m_{n+1,i}\}$ and $\{-\tilde{m}_{i+1,i},\ldots,-\tilde{m}_{n+1,i}\}$ respectively. From Theorem 4.1 of [7] and p. 116 of [9], a given matrix $A = (a_{i,j})_{1\le i,j\le n+1}$ is STP if and only if the Neville elimination of A and A^T can be performed without row exchanges, all the multipliers of the Neville elimination of A and A^T are positive and all the diagonal pivots of the Neville elimination of A are positive.

Let us recall that a real value x is obtained with high relative accuracy (HRA) if the relative error of the computed value $\tilde{x}$ satisfies

$$\frac{\|x-\tilde{x}\|}{\|x\|} < Ku,$$

where K is a positive constant independent of the arithmetic precision and u is the unit round-off. HRA implies that the relative errors of the computations are of the order of the machine precision. So, performing an algorithm with HRA is a very desirable goal. An algorithm can be computed with HRA when it only uses products, quotients, sums of numbers of the same sign or subtraction of initial data (cf. [5], [10]).

In [11] it was shown that if $BD(A)$, the bidiagonal factorization (3) of a nonsingular TP matrix A, is computed with HRA then we can also compute with HRA its eigenvalues and singular values, the matrix A^{-1} and even the solution of $Ax = b$ for vectors b with alternating signs.

In the following sections we shall obtain the bidiagonal factorization (3) of Wronskian matrices associated with some bases with applications in CAGD, analyzing whether it can be computed with HRA.

3 Wronskian matrices of monomial bases

The monomial basis of the space $\mathbf{P}^n$ of polynomials of degree less than or equal to n is $(m_0,\dots,m_n)$ with

$$m_i(x) := x^i, \quad i = 0,\dots,n. \tag{6}$$

Given $x_0 \in \mathbb{R}$, we can define a Taylor basis $(n_0,\dots,n_n)$ of $\mathbf{P}^n$ by

$$n_i(x) := \frac{(x-x_0)^i}{i!}, \quad i = 0,\dots,n. \tag{7}$$

It can be checked that

$$(m_0,\dots,m_n) = (n_0,\dots,n_n)W,$$

where $W := W(m_0,\dots,m_n)(x_0)$. Equivalently, we can also write

$$(n_0,\dots,n_n) = (m_0,\dots,m_n)W^{-1}.$$

In this section we are going to obtain the bidiagonal factorization (3) of W and W^{-1} and see that they can be computed with HRA. First let us prove the following auxiliary result.

Lemma 1 *Given $i, j \in \mathbb{N}$, then*

$$\frac{1}{i!}m_j^{(i)}(x) = \frac{1}{(i-1)!}m_{j-1}^{(i-1)}(x) + \frac{x}{i!}m_{j-1}^{(i)}(x), \quad x \in \mathbb{R}. \tag{8}$$

Proof Let us prove the result by induction on i. For $i = 1$ and $j \in \mathbb{N}$, taking into account that $m_j'(x) = (x m_{j-1}(x))'$, we have

$$m_j'(x) = m_{j-1}(x) + x m_{j-1}'(x), \quad x \in \mathbb{R},$$

and so formula (8) holds. Let us now suppose that (8) holds for $i > 1$ and $j \in \mathbb{N}$. Then we have

$$\begin{aligned}\frac{1}{i!} m_j^{(i+1)}(x) &= \left(\frac{1}{(i-1)!} m_{j-1}^{(i-1)}(x) + \frac{x}{i!} m_{j-1}^{(i)}(x) \right)' \\ &= \frac{i+1}{i!} m_{j-1}^{(i)}(x) + \frac{x}{i!} m_{j-1}^{(i+1)}(x), \quad x \in \mathbb{R},\end{aligned}$$

and we can deduce that, for $j \in \mathbb{N}$,

$$\frac{1}{(i+1)!} m_j^{(i+1)}(x) = \frac{1}{i!} m_{j-1}^{(i)}(x) + \frac{x}{(i+1)!} m_{j-1}^{(i+1)}(x), \quad x \in \mathbb{R}.$$

□

For a given $x \in \mathbb{R}$, $k, n \in \mathbb{N}$ with $k \leq n$, let $U_{k,n} = (u_{i,j})_{1 \leq i,j \leq n+1}$ be the upper triangular bidiagonal matrix with unit diagonal entries and such that

$$u_{i,i+1} := 0, \quad i = 1, \ldots, k-1, \quad u_{i,i+1} := x, \quad i = k, \ldots, n. \tag{9}$$

In the following result we obtain an explicit expression of the entries of the product matrix $U_{1,n} \cdots U_{n,n}$.

Proposition 1 *For a given $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let*

$$U_n := U_{1,n} \cdots U_{n,n},$$

where $U_{k,n}$, $k = 1, \ldots, n$, is the upper triangular bidiagonal matrix with unit diagonal entries satisfying (9). *Then $U_n = (u_{i,j})_{1 \leq i,j \leq n+1}$ is an upper triangular matrix and*

$$u_{i,j} = \frac{1}{(i-1)!} m_{j-1}^{(i-1)}(x), \quad 1 \leq i, j \leq n+1. \tag{10}$$

Proof Clearly, U_n is an upper triangular matrix since it is the product of upper triangular bidiagonal matrices. Let us now prove (10) by induction on n. For $n = 1$,

$$U_1 = U_{1,1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

and (10) clearly holds. Let us now suppose that (10) holds for $n \geq 1$. Then

$$U_{n+1} := U_{1,n+1} \cdots U_{n+1,n+1} = U_{1,n+1} \tilde{U}_{n+1},$$

where $\tilde{U}_{n+1} := U_{2,n+1} \cdots U_{n+1,n+1}$ satisfies $\tilde{U}_{n+1} = (\tilde{u}_{i,j})_{1 \leq i,j \leq n+2}$ with $\tilde{u}_{i,1} = \tilde{u}_{1,i} = \delta_{1,i}$, that is, $\delta_{1,1} = 1$ and $\delta_{1,i} = 0$ for $i = 2, \ldots, n+2$, and $\tilde{U}_{n+1}[2, \ldots, n+2 | 2, \ldots, n+2] = U_{1,n} \cdots U_{n,n}$. Then we have that

$$\tilde{u}_{i,j} = \frac{1}{(i-2)!} m_{j-2}^{(i-2)}(x), \quad 2 \leq i, j \leq n+2.$$

Now taking into account that

$$U_{n+1} = U_{1,n+1} \tilde{U}_{n+1} = \begin{pmatrix} 1 & x & & \\ & \ddots & \ddots & \\ & & 1 & x \\ & & & 1 \end{pmatrix} \tilde{U}_{n+1},$$

and using Lemma 1, we deduce that $U_{n+1} = (u_{i,j})_{1\le i,j\le n+2}$ satisfies

$$\begin{aligned} u_{i,j} &= \tilde{u}_{i,j} + x\tilde{u}_{i+1,j} = \frac{1}{(i-2)!} m_{j-2}^{(i-2)}(x) + \frac{x}{(i-1)!} m_{j-2}^{(i-1)}(x) \\ &= \frac{1}{(i-1)!} m_{j-1}^{(i-1)}(x), \quad 1 \le i,j \le n+2. \end{aligned}$$

□

Let us observe that for $x > 0$ the matrices $U_{k,n}$, $k = 1,\dots,n$, are TP. Then, as a direct consequence of the previous result and taking into account that, by Theorem 3.1 of [1], the product of TP matrices is TP, we can derive the following result providing a bidiagonal factorization of the Wronskian matrix of the monomial basis (6).

Corollary 1 *Let $n \in \mathbb{N}$ and $(m_0,\dots,m_n)$ be the monomial basis given in* (6). *Then for any $x \in \mathbb{R}$,*

$$W := W(m_0,\dots,m_n)(x) := \begin{pmatrix} 0! & & & \\ & 1! & & \\ & & \ddots & \\ & & & n! \end{pmatrix} U_{1,n} U_{2,n} \cdots U_{n,n}, \tag{11}$$

where $U_{k,n}$, $k = 1,\dots,n$, is the upper triangular bidiagonal matrix with unit diagonal entries satisfying (9). *Moreover, if $x > 0$ then $W(m_0,\dots,m_n)(x)$ is TP.*

Let us observe that (11) is the bidiagonal factorization (3) of the upper triangular, nonsingular and TP Wronskian matrix $W = W(m_0,\dots,m_n)(x)$, $x > 0$, where F_i and G_i are the TP, lower and upper triangular bidiagonal matrices in (4). Clearly $BD(W)$ can be computed with HRA and, consequently, using the bidiagonal factorization (5), W^{-1} can also be computed with HRA as stated in the following result.

Proposition 2 *Let W be the Wronskian matrix at x_0 of the monomial basis of the space of polynomials $\mathbf{P}^n$. Then W^{-1} can be computed with HRA.*

Furthermore, Section 5 will show accurate results obtained when computing the eigenvalues, singular values, the inverse and the solutions of some linear systems associated with the Wronskian matrices of monomial bases, using the bidiagonal factorization (11) and the algorithms presented in [11] and [12].

Finally, in the following example, we illustrate the bidiagonal factorization (11) of the Wronskian matrix of a basis of monomials.

Example 1 For the particular case $n = 3$, the bidiagonal factorization of the Wronskian matrix of the basis (m_0,m_1,m_2,m_3) at $x \in \mathbb{R}$ is

$$W(m_0,m_1,m_2,m_3)(x) = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&6 \end{pmatrix} \begin{pmatrix} 1&x&0&0 \\ 0&1&x&0 \\ 0&0&1&x \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0 \\ 0&1&x&0 \\ 0&0&1&x \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&x \\ 0&0&0&1 \end{pmatrix}.$$

4 Bidiagonal factorization of the Wronskian matrix of a basis of exponential polynomials

Given $\lambda_0,\dots,\lambda_n$ and $x\in\mathbb{R}$, let us consider the basis $(u_0,\dots,u_n)$ of exponential polynomials defined on $\mathbb{R}$ by

$$u_i(x) := e^{\lambda_i x}, \quad i=0,\dots,n. \tag{12}$$

The following result proves that, if $0<\lambda_0<\lambda_1<\cdots<\lambda_n$, the Wronskian matrix of the basis (12),

$$W(u_0,\dots,u_n)(x) = (\lambda_{j-1}^{i-1} e^{\lambda_{j-1}x})_{i,j=1,\dots,n+1}, \tag{13}$$

is STP for any $x\in\mathbb{R}$.

Theorem 1 *Let* $0<\lambda_0<\cdots<\lambda_n$ *and the basis* (12) *of exponential polynomials. For any* $x\in\mathbb{R}$*, the corresponding Wronskian matrix* (13) *is STP and*

$$\det W(u_0,\dots,u_n)(x) = \prod_{k=0}^{n} e^{\lambda_k x} \prod_{0\le k<\ell\le n} (\lambda_\ell-\lambda_k). \tag{14}$$

Proof The matrix $D := \operatorname{diag}\left(e^{\lambda_0 x},\dots,e^{\lambda_n x}\right)$ is nonsingular and TP since $e^{\lambda_k x}>0$, for all $k=0,\dots,n$. It can be easily checked that

$$W(u_0,\dots,u_n)(x) = V_{n,\lambda_0,\dots,\lambda_n} D,$$

where $V_{n,\lambda_0,\dots,\lambda_n} := \left(\lambda_{j-1}^{i-1}\right)_{1\le i,j\le n+1}$ is the $(n+1)\times(n+1)$ Vandermonde matrix corresponding to the values λ_i, $i=0,\dots,n$. Using that $0<\lambda_0<\cdots<\lambda_n$, we deduce that $V_{n,x_0,\dots,x_n}$ is STP (see [2]). Taking into account that, by Theorem 3.1 of [1], the product of a STP matrix by a nonsingular, TP matrix is a STP matrix, we conclude that $W(u_0,\dots,u_n)(x)$ is STP. Since $\det W(u_0,\dots,u_n)(x) = \det V_{n,\lambda_0,\dots,\lambda_n} \det D$ we can write

$$\det V_{n,\lambda_0,\dots,\lambda_n} = \prod_{0\le k<\ell\le n} (\lambda_\ell-\lambda_k), \tag{15}$$

and deduce (14). □

In the following result we present the bidiagonal decomposition (3) of the Wronskian matrices (13) and their inverses.

Theorem 2 *Let* $0<\lambda_0<\cdots<\lambda_n$ *and the corresponding basis* (12) *of exponential polynomials. For a given* $x\in\mathbb{R}$, $W := W(u_0,\dots,u_n)(x)$ *admits a factorization of the form*

$$W = F_n F_{n-1}\cdots F_1 D G_1\cdots G_{n-1} G_n, \tag{16}$$

where F_i *and* G_i*,* $1\le i\le n$*, are the lower and upper triangular bidiagonal matrices given by* (4) *and* $D=\operatorname{diag}(p_{1,1},p_{2,2},\dots,p_{n+1,n+1})$*. The entries* $m_{i,j},\tilde m_{i,j}$ *and* $p_{i,i}$ *are given by*

$$m_{i,j} = \lambda_{j-1}, \quad \tilde m_{i,j} = e^{(\lambda_{i-1}-\lambda_{i-2})x} \prod_{k=2}^{j} \frac{(\lambda_{i-1}-\lambda_{i-k})}{(\lambda_{i-2}-\lambda_{i-k-1})}, \quad 1\le j<i\le n+1,$$

$$p_{i,i} = e^{\lambda_{i-1}x} \prod_{k=0}^{i-2} (\lambda_{i-1}-\lambda_k), \quad 1\le i\le n+1.$$

Proof By Theorem 1, the matrix W is STP and then the Neville elimination of W and W^T can be performed without row exchanges, leading to a factorization of type (3). The computation of the minors of W with initial consecutive columns and consecutive rows will allow us to determine the corresponding pivots $p_{i,j}$ and multipliers $m_{i,j}$.

Let $1 \leq j \leq i \leq n+1$. The k-th column of $M[i-j+1,\ldots,i|1,\ldots,j]$ has common factor $\lambda_{k-1}^{i-j} e^{\lambda_{k-1}x}$ and then

$$W[i-j+1,\ldots,i|1,\ldots,j] = V^T_{n,\lambda_0,\ldots,\lambda_{j-1}} D,$$

where $D := \operatorname{diag}\left(\lambda_0^{i-j} e^{\lambda_0 x},\ldots,\lambda_{j-1}^{i-j} e^{\lambda_{j-1}x}\right)$ and $V_{n,\lambda_0,\ldots,\lambda_{j-1}}$ is the $j\times j$ Vandermonde matrix corresponding to parameters $\lambda_0,\ldots,\lambda_{j-1}$. Using properties of determinants and (15), we can write

$$\det W[i-j+1,\ldots,i|1,\ldots,j] = \prod_{0\leq k<\ell\leq j-1} (\lambda_\ell - \lambda_k) \prod_{k=0}^{j-1} \lambda_k^{i-j} e^{\lambda_k x}. \tag{17}$$

By (1) and (17), the pivot $p_{i,j}$ of the Neville elimination of W satisfies

$$p_{i,j} = \frac{\det W[i-j+1,\ldots,i|1,\ldots,j]}{\det W[i-j+1,\ldots,i-1|1,\ldots,j-1]} = \lambda_{j-1}^{i-j} e^{\lambda_{j-1}x} \prod_{k=0}^{j-2} (\lambda_{j-1} - \lambda_k), \tag{18}$$

and, for the particular case $i = j$,

$$p_{i,i} = e^{\lambda_{i-1}x} \prod_{k=0}^{i-2} (\lambda_{i-1} - \lambda_k), \quad 1 \leq i \leq n+1. \tag{19}$$

Finally, using (2) and (18), the multipliers $m_{i,j}$ can be obtained by

$$m_{i,j} = \frac{p_{i,j}}{p_{i-1,j}} = \lambda_{j-1}, \quad 1 \leq j < i \leq n+1. \tag{20}$$

Now let us observe that each entry of the k-th row of W^T has common factor $e^{\lambda_{i-j+k-1}x}$. Then we have that

$$W^T[i-j+1,\ldots,i|1,\ldots,j] = D_1 V_{n,\lambda_{i-j},\ldots,\lambda_{i-1}},$$

where $D_1 := \operatorname{diag}\left(e^{\lambda_{i-j}x},\ldots,e^{\lambda_{i-1}x}\right)$ and $V_{n,\lambda_{i-j},\ldots,\lambda_{i-1}}$ is the $j\times j$ Vandermonde matrix corresponding to parameters $\lambda_{i-j},\ldots,\lambda_{i-1}$. Using properties of determinants and (15), we can write

$$\det W^T[i-j+1,\ldots,i|1,\ldots,j] = \prod_{k=i-j}^{i-1} e^{\lambda_k x} \prod_{i-j\leq k<\ell\leq i-1} (\lambda_\ell - \lambda_k). \tag{21}$$

By (1) and (21), we deduce that

$$\tilde{p}_{i,j} = \frac{\det W^T[i-j+1,\ldots,i|1,\ldots,j]}{\det W^T[i-j+1,\ldots,i-1|1,\ldots,j-1]} = e^{\lambda_{i-1}x} \prod_{k=i-j}^{i-2} (\lambda_{i-1} - \lambda_k). \tag{22}$$

Finally, using (2) and (22), we have

$$\tilde{m}_{i,j} = \frac{\tilde{p}_{i,j}}{\tilde{p}_{i-1,j}} = e^{(\lambda_{i-1}-\lambda_{i-2})x} \frac{\prod_{k=i-j}^{i-2}(\lambda_{i-1}-\lambda_k)}{\prod_{k=i-j-1}^{i-3}(\lambda_{i-2}-\lambda_k)} = e^{(\lambda_{i-1}-\lambda_{i-2})x} \prod_{k=2}^{j} \frac{(\lambda_{i-1}-\lambda_{i-k})}{(\lambda_{i-2}-\lambda_{i-k-1})}, \tag{23}$$

for $1 \leq j < i \leq n+1$. □

Let us observe that the computation with HRA of the bidiagonal decomposition (16) should require the evaluation with HRA of the involved exponential function. Although this cannot be guaranteed, Section 5 will show accurate results obtained when computing their eigenvalues, singular values, inverses or the solutions of some linear systems associated with these Wronskian matrices of non-polynomial bases.

We finish this section illustrating the bidiagonal factorization (16) of the Wronskian matrix of a basis of exponential polynomials.

Example 2 For the particular case $n = 2$, the bidiagonal factorization of the Wronskian matrix of the basis $(e^{\lambda_0 x}, e^{\lambda_1 x}, e^{\lambda_2 x})$ at $x \in \mathbb{R}$ is

$$W(e^{\lambda_0 x}, e^{\lambda_1 x}, e^{\lambda_2 x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \lambda_0 & 1 & 0 \\ 0 & \lambda_1 & 1 \end{pmatrix} \begin{pmatrix} p_{1,1} & 0 & 0 \\ 0 & p_{2,2} & 0 \\ 0 & 0 & p_{3,3} \end{pmatrix} \begin{pmatrix} 1 & e^{(\lambda_1-\lambda_0)x} & 0 \\ 0 & 1 & e^{(\lambda_2-\lambda_1)x}\frac{\lambda_2-\lambda_1}{\lambda_1-\lambda_0} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{(\lambda_2-\lambda_1)x} \\ 0 & 0 & 1 \end{pmatrix},$$

where $p_{1,1} = e^{\lambda_0 x}$, $p_{2,2} = e^{\lambda_1 x}(\lambda_1 - \lambda_0)$ and $p_{3,3} = e^{\lambda_2 x}(\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1)$.

5 Numerical experiments

When the bidiagonal factorization of a nonsingular totally positive matrix is obtained with HRA, using the Matlab libraries `TNInverseExpand`, `TNEigenvalues`, `TNSingularValues` and `TNSolve`, available in [12], the computation of its inverse matrix, its eigenvalues and singular values or the solutions of some linear systems can be also performed with HRA.

We have implemented the Matlab functions `TNBDWM` and `TNBDWE` providing the bidiagonal decomposition (3) of the Wronkian matrix at x of the $(n+1)$-dimensional monomial and exponential basis. Now we include some numerical experiments illustrating the high accuracy obtained when using these functions and the previous libraries. Due to the ill conditioning of these matrices, traditional methods do not achieve accurate solutions when solving the mentioned algebraic problems. The numerical experiments show this fact and confirm the accuracy of the obtained results even though for some cases we cannot guarantee that the bidiagonal factorization (3) can be computed with HRA. The software with the numerical experiments will be provided by the authors upon request.

5.1 Linear systems

Let U be an $(n+1)$-dimensional space of n-times continuously differentiable functions defined on a real interval $I \subseteq \mathbb{R}$ and $x_0 \in I$. Given real values $d_0, d_1, \dots, d_n$, the corresponding Taylor interpolant in U is the function $u \in U$ such that $u^{(k)}(x_0) = d_k$, $k = 0, \dots, n$. Given a basis $\mathbf{u} = (u_0, \dots, u_n)$ of U, the Taylor interpolant can be expressed as $u(x) = \sum_{i=0}^{n} c_i u_i(x)$, $x \in I$, where $\mathbf{c} = (c_0, \dots, c_n)^T$ is the solution of the linear system

$$W\mathbf{c} = \mathbf{d}, \tag{24}$$

with $W = W(u_0, \dots, u_n)(x_0)$ and $\mathbf{d} = (d_0, \dots, d_n)^T$. Then we have $u(x) = \mathbf{u}(x)^T \mathbf{c}$ where $\mathbf{c} = W^{-1}\mathbf{d}$.

We have solved some linear systems (24) by considering the bases of the previuos sections. We have obtained the solution of these systems using Mathematica with a precision of

100 digits and considered this solution exact. We have also computed with Matlab two approximations of this solution, the first one using `TNSolve` with the bidiagonal factorization proposed in this paper and the second one using the Matlab command \.

First, we have considered $x_0 = 50$ and the corresponding Wronskian matrices $\mathbf{W}_n$ of the monomial basis $(1, x, \ldots, x^n)$. Table 1 (left) illustrates the 2-norm condition number of these matrices using the Mathematica command Norm[A,2]·Norm[Inverse[A],2]. We have taken a vector $\mathbf{d}_n = ((-1)^{i+1} d_i)_{1 \le i \le n+1}$ where d_i is a random integer value. As we have mentioned in Section 3, the parameters of the bidiagonal decomposition (11) of $\mathbf{W}_n$ can be obtained with HRA and so, the solution of $\mathbf{W}_n \mathbf{c}_n = \mathbf{d}_n$ can be performed with HRA. The numerical experiments confirm this fact and the greater accuracy of using the bidiagonal decomposition (11) (see Table 1).

Table 1 Condition number of Wronskian matrices of monomial bases at $x_0 = 50$ (left) and relative errors when solving $\mathbf{W}_n \mathbf{c}_n = \mathbf{d}_n$ with these matrices (middle and right).

n+1	$\kappa_2(\mathbf{W_n})$	$\mathbf{W_n} \setminus \mathbf{d_n}$	`TNsolve`
10	1.1×10^{25}	3.8102×10^{-14}	8.8082×10^{-17}
15	4.8×10^{36}	6.6581×10^{-12}	1.7749×10^{-16}
20	3.7×10^{47}	5.0996×10^{-9}	1.1459×10^{-16}
25	8.2×10^{57}	2.7182×10^{-7}	2.8366×10^{-16}

Now, for $x_0 = 1/2$, we have also considered Wronskian matrices $\mathbf{W}_n$ of exponential polynomial bases with $\lambda_i = i/(n+2)$, $i = 1, \ldots, n+1$. Table 2 (left) illustrates the 2-norm condition number of these matrices using the Mathematica command Norm[A,2]·Norm[Inverse[A],2]. We have also taken $\mathbf{d}_n = ((-1)^{i+1} d_i)_{1 \le i \le n+1}$, where d_i is a random integer value. The computation with HRA of the parameters of the bidiagonal factorization of $\mathbf{W}_n$ cannot be guaranteed. However, these numerical experiments show again the high accuracy in the computations when using `TNSolve` with the bidiagonal factorization (16) (see Table 2).

Table 2 Condition number of Wronskian matrices of exponential bases at $x_0 = 1/2$ and $\lambda_i = i/(n+2)$, $i = 1, \ldots, n+1$, (left) and relative errors when solving $\mathbf{W_n c_n} = \mathbf{d_n}$ with these matrices (middle and right).

n+1	$\kappa_2(\mathbf{W_n})$	$\mathbf{W_n} \setminus \mathbf{d_n}$	`TNsolve`
10	9.6×10^{7}	4.0424×10^{-11}	5.4201×10^{-16}
15	2.8×10^{12}	2.7929×10^{-7}	9.3188×10^{-17}
20	8.2×10^{16}	4.7662×10^{-3}	3.8596×10^{-16}
25	2.5×10^{21}	1.4272	2.5409×10^{-15}

5.2 Inverse matrix

In Section 4 of [15] the authors present the algorithm `TNInverseExpand`, which is an accurate and fast algorithm for computing the inverse of a nonsingular totally positive matrix A starting from $BD(A)$ and it has been included by P. Koev in his package TNTool [12].

We have used the Matlab function `TNInverseExpand` with the factorization proposed in this paper in order to compute the inverse of Wronskian matrices of the bases considered in the paper. We have also computed their approximations with the Matlab function `inv`. In order to determine the accuracy of the approximations, we have calculated the inverse of these Wronskian matrices by using Mathematica with a precision of 100 digits and computed the relative errors corresponding to the approximations, considering the inverse matrix provided by Mathematica as exact.

The approximation of the inverse of the Wronskian matrices obtained by means of `TNInverseExpand` is very accurate for all considered n, providing much more accurate results than those obtained by Matlab using the command `inv`. Tables 3 and 4 show the relative errors of the approximations to the inverse of the Wronskian matrices obtained with both methods.

Table 3 Relative errors when computing the inverses of Wronskian matrices of monomial bases at $x_0 = 50$.

n+1	`inv`	`TNInverseExpand`
10	5.5583×10^{-14}	8.8081×10^{-17}
15	2.8550×10^{-11}	1.7749×10^{-16}
20	1.0218×10^{-9}	1.1497×10^{-16}
25	8.3974×10^{-7}	1.1944×10^{-16}

Table 4 Relative errors when computing the inverses of Wronskian matrices of exponential bases at $x_0 = 1/2$ and $\lambda_i = i/(n+2)$, $i = 1, \ldots, n+1$.

n+1	`inv`	`TNInverseExpand`
10	4.0206×10^{-11}	4.0436×10^{-16}
15	2.8247×10^{-7}	3.5637×10^{-16}
20	4.8134×10^{-3}	4.0018×10^{-16}
25	1.4611	2.6557×10^{-15}

5.3 Eigenvalues and singular values

We have also used the bidiagonal decomposition proposed in this paper with the Matlab functions `TNEigenValues` and `TNSingularValues`, to compute the eigenvalues and the singular values, respectively, of the previous Wronskian matrices. We have also computed their approximations with the Matlab functions `eig` and `svd`, respectively. In order to determine the accuracy of the approximations, we have calculated the eigenvalues and singular values of previous Wronskian matrices by using Mathematica with a precision of 100 digits and computed the relative errors corresponding to the approximations, considering the eigenvalues and singular values provided by Mathematica as exact.

Let us consider the Wronskian matrices at $x = 0.3$ of monomial bases. Table 5 (left) illustrates the 2-norm condition number of these matrices using the Mathematica command Norm[A,2]·Norm[Inverse[A],2]. Since these Wronskian matrices are all STP, by Theorem 6.2 of [1], all their eigenvalues are positive and distinct. Let us observe that the eigenvalues of these Wronskian matrices are $0!, \ldots, n!$, so in this case the relative errors are 0 with both methods. On the other hand, the approximations of the singular values obtained by means of `TNSingularValues` are very accurate for all considered n, whereas the approximations of the singular values obtained with the Matlab command `svd` are not very accurate when

n increases. Table 5 shows the relative errors of the approximations to the lowest singular value obtained with both methods.

Table 5 Condition number of Wronskian matrices of monomial bases at $x_0 = 0.3$ (left) and relative errors when computing the lowest singular value of these matrices (middle and right).

n+1	$\kappa_2(\mathbf{W_n})$	**svd**	`TNSingularValues`
10	4.5×10^{5}	1.5898×10^{-12}	3.9691×10^{-16}
15	1.1×10^{11}	7.2111×10^{-8}	2.6461×10^{-16}
20	1.5×10^{17}	2.4313×10^{-1}	6.6151×10^{-16}
25	7.7×10^{23}	7.4909×10^{-1}	2.6461×10^{-16}

Let us also consider Wronskian matrices of the exponential polynomial bases at $x = 1/2$ with $\lambda_i = i/(n+2)$, $i = 1,\dots,n+1$. The approximations of the eigenvalues and singular values obtained by means of the proposed factorization are very accurate for all considered n, whereas the approximations of the eigenvalues and singular values obtained with the Matlab commands `eig` and `svd` are not very accurate when n increases. Table 6 shows the relative errors of the approximations to the lowest eigenvalue and singular value obtained with both methods.

Table 6 Relative errors when computing the lowest eigenvalue (left) and the lowest singular value (right) of Wronskian matrices of exponential bases at $x_0 = 1/2$ and $\lambda_i = i/(n+2)$, $i = 1,\dots,n+1$.

n+1	**eig**	`TNEigenValues`	**svd**	`TNSingularValues`
10	1.8449×10^{-11}	3.1595×10^{-16}	1.7818×10^{-10}	1.5487×10^{-16}
15	1.8701×10^{-6}	7.9152×10^{-16}	3.0235×10^{-6}	1.1653×10^{-15}
20	1.1279×10^{-2}	1.1208×10^{-15}	7.0058×10^{-1}	8.6431×10^{-16}
25	1.4512×10^{3}	1.6727×10^{-15}	1.0646×10^{2}	2.4382×10^{-15}

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