

Module structure of the homology of right-angled Artin kernels

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In this paper, we study the module structure of the homology of Artin kernels, i.e., kernels of non-resonant characters from right-angled Artin groups onto the integer numbers, the module structure being with respect to the ring $\mathbb{K}[t^{\pm 1}]$, where \mathbb{K} is a field of characteristic zero. Papadima and Suciu determined some part of this structure by means of the flag complex of the graph of the Artin group. In this work, we provide more properties of the torsion part of this module, e.g., the dimension of each primary part and the maximal size of Jordan forms (if we interpret the torsion structure in terms of a linear map). These properties are stated in terms of homology properties of suitable filtrations of the flag complex and suitable double covers of an associated toric complex.

Introduction

The homological properties of Artin groups have been extensively studied in the recent years, see [7, 5, 1] and references therein. These groups are especially attractive because of the combinatorial nature of their definition that might influence the properties of the groups. In this work we deal with a particular type of Artin groups called right-angled Artin groups, or RAAG for short, and the goal is to describe the homology of the kernels of most integer characters of these groups in combinatorial terms.

The definition of a RAAG can be given starting with a simplicial graph $\Gamma = (V, E)$, where V is the set of vertices and $E \subset \binom{V}{2}$ is the set of edges (seen as subsets of two elements in V). The RAAG associated with Γ can be defined via a finite presentation as

$$(1) \quad A_{\Gamma} := \langle v \in V \mid [v, w] = 1 \text{ if } \{v, w\} \in E \rangle.$$

Given a surjective character $\chi : A_\Gamma \twoheadrightarrow \mathbb{Z}$, one is interested in the *Artin kernel* $A_\Gamma^\chi := \ker \chi$. As a first approach one can consider their homology groups with a module structure given as follows. Let $t \in A_\Gamma$ such that $\chi(t) = 1$; conjugation by t defines an automorphism of A_Γ^χ which induces a $\mathbb{K}[t^{\pm 1}]$ -module structure on $H_k(A_\Gamma^\chi; \mathbb{K})$ for any field \mathbb{K} ; this structure does not depend on the particular choice of t . The main goal of this paper is to describe the structure of such modules.

In particular, detailed properties on their structure will be given when \mathbb{K} has characteristic 0 and the character χ satisfies $\chi(v) \neq 0, \forall v \in V$, a *non-resonant* character. Moreover, a complete description will be given for $k \leq 2$ in terms of geometric properties of objects associated with Γ and χ .

This problem was considered in [5] and the following result summarizes the starting point of this paper.

Theorem 1 ([5])

$$H_{k+1}(A_\Gamma^\chi; \mathbb{K}) \cong \mathbb{K}[t^{\pm 1}]^{r_k} \bigoplus_{d \in \mathbb{Z}_{>0}} \bigoplus_{j > 0} \left(\mathbb{K}[t^{\pm 1}] / \langle \Phi_d^j \rangle \right)^{r_{k,j}(d)}$$

where Φ_d is the d -cyclotomic polynomial and the exponents $r_k, r_{k,j}(d)$ depend on Γ .

Our purpose is to provide more properties concerning the torsion part. A general strategy is proposed to study the Φ_d -primary part of this module $H_{k+1}(A_\Gamma^\chi; \mathbb{K})$ by reducing it to the study of the $(t+1)$ -primary part of the homology of $H_{k+1}(A_\Gamma^\rho; \mathbb{K})$ with respect to an even character ρ associated with χ . The study of the Φ_2 -primary part for each H_{k+1} is done in terms of a filtration of the flag complex depending on k and ρ and the two-fold cover defined by ρ . Complete formulas are given for $k = 0, 1$, and in a large family of cases, the results obtained determine the torsion of the homology.

Nevertheless, these formulas are quite complicated and it is hard to interpret them in terms of the properties of Γ . In the case the flag complex defined by Γ satisfies certain acyclicity conditions more precise formulas can be given for the exponents in [Theorem 1](#). These formulas completely determine the Artin exponents for $k = 0, 1$ as shown in [Theorem 5.1](#). For $k > 1$, we obtain the dimension of the Φ_d -primary part, the number of its factors and the exponent $r_{k,k+2}(d)$. Moreover, if this exponent vanishes we can detect the maximal j such that $r_{k,j}(d) > 0$.

Let us rephrase this strategy in another language. Let us denote by $T_{k+1}(A_\Gamma^\chi) := \text{Tor } H_{k+1}(A_\Gamma^\chi; \mathbb{K})$; it is a finitely dimensional \mathbb{K} -vector space which coincides with the whole homology when the flag complex of Γ is k -connected. The action of t induces an endomorphism $t : T_{k+1}(A_\Gamma^\chi) \rightarrow$

$T_{k+1}(A_\Gamma^\chi)$. The Jordan form of this endomorphism is equivalent to the decomposition of the torsion in [Theorem 1](#). As a combination of some results in [\[5\]](#) and part of the results of this work, we obtain a version of the Monodromy Theorem for complex singularities.

Monodromy Theorem *Let χ be a non-resonant character. Let $t : T_{k+1}(A_\Gamma^\chi) \rightarrow T_{k+1}(A_\Gamma^\chi)$ be the endomorphism defined by the monodromy action. Then*

- (1) *The characteristic polynomial of t is a product of cyclotomic polynomials. More precisely, the roots of the characteristic polynomial are roots of unity whose order is a divisor of $\chi(v)$ for some $v \in V$.*
- (2) *The Jordan blocks for the eigenvalue 1 have size 1.*
- (3) *The Jordan blocks of t acting on $T_{k+1}(A_\Gamma^\chi)$ have size at most $k + 2$.*

Items (1) and (2) have been proved in [\[5\]](#); (3) is a consequence of [Corollary 3.13](#). In addition, using [Proposition 3.14](#) the number of such blocks can be recovered. Moreover, [Proposition 3.15](#) explains how to recover the actual maximal size of Jordan blocks in each case. Also, formulas in [Theorem 3.9](#) allow to recover the dimension of each primary part. The dimension of the eigenspaces can be recovered from [Theorem 4.2](#). All these results are specially simple to state in the particular case when the flag complex is k -acyclic, see [Theorem 5.1](#).

The moment-angle variety \mathbb{T} of the flag complex of Γ with respect to $(\mathbb{S}^1, 1)$, see [\(1.3\)](#), is an Eilenberg-McLane space for A_Γ , which can be naturally realized as a subvariety of $(\mathbb{S}^1)^V$. Let us denote by f^χ the restriction of the monomial $\prod_{v \in V} t_v^{\chi(v)}$ to \mathbb{T} . The pull-back

$$(2) \quad \begin{array}{ccc} \mathbb{T}^\chi & \longrightarrow & \mathbb{R} \\ \tilde{\chi} \downarrow & & \downarrow \\ \mathbb{T} & \xrightarrow{f^\chi} & \mathbb{S}^1 \end{array}$$

yields the infinite cyclic cover defined by χ , whose fundamental group is A_Γ^χ . The moment-angle variety with respect to $(\mathbb{C}^*, 1)$ has the same homotopy type as \mathbb{T} ; the map f^χ can be defined in this setting and is a holomorphic map, which connects the classical Monodromy Theorem with the one above.

The paper is organized as follows. In [Section 1](#) we give the classical description of the toric complex \mathbb{T} associated with a RAAG, say A_Γ . This complex provides an Eilenberg-McLane space of A_Γ and hence, its homological properties and those of cyclic covers will be key for our purposes. In

particular, since $\mathbb{K}[t^{\pm 1}]$ is a principal ideal domain, the space $H_{k+1}(A_{\Gamma}^X; \mathbb{K})$, as a finitely generated $\mathbb{K}[t^{\pm 1}]$ -module, decomposes as a direct sum of its free and torsion parts, which will be the focus of this work. The required tools to attack this problem will be presented in [Section 2](#). In this expository section we recover results by Papadima-Suciu [5] on the $(t - 1)$ -torsion part of the homology $H_{k+1}(A_{\Gamma}^X; \mathbb{K})$ in terms of the graph Γ . The Fitting ideals of the module $H_{k+1}(A_{\Gamma}^X; \mathbb{K})$ are considered in [Section 3](#), the purpose is to recover its Φ_d -primary torsion part in terms of both the flag complex associated with Γ and a weight filtration coming from d . This will give a precise bound of $k + 2$ for the maximal order of Φ_d as a torsion element in $H_{k+1}(A_{\Gamma}^X; \mathbb{K})$ as well as a formula for the weighted sum of exponents $\sum_{j=1}^{k+2} r_{k,j}(d)$ in terms of topological invariants of the weight filtration.

In [Section 4](#) we introduce the concept of *even characters* and prove a reduction of the general problem of studying the Φ_d -primary part associated with a given character χ to the $(t + 1)$ -primary part associated with an even character $\rho_{\chi,d}$ that depends on χ and $d > 1$. This allows us to give explicit formulas for the sum of the exponents $\sum_{j=1}^{k+2} r_{k,j}(d)$ in terms of the k -homology of a double cover of the toric complex \mathbb{T} associated with Γ . This reduction is key in the effective computation of the torsion invariants, however explicit formulas are too intricate to be presented here. In the particular case when the flag complex \mathcal{F} has some acyclicity properties, explicit formulas for the torsion exponents are presented in [Section 5](#). In the final [Section 6](#) we present some characteristic examples that describe the reduction strategy to study the torsion part in terms of topological properties of the flag complex and its filtrations.

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1 Settings

1.1 The toric complex \mathbb{T}

Most of this section appears in [5]; we recall the parts we will need. As in the Introduction, we fix a finite simplicial graph Γ , whose set of vertices is denoted by V and set of edges $E \subset \binom{V}{2}$. We define the right-angled group A_Γ as in (1).

Let \mathcal{F} be the flag complex of Γ . This is a simplicial complex with set of vertices V and such that $X = \{v_0, v_1, \dots, v_r\}$ is an r -simplex of \mathcal{F} if and only if X is a clique of Γ . For the sake of simplicity we will work over a field \mathbb{K} of characteristic 0. We will denote by $(\tilde{C}_*(\mathcal{F}), \partial)$ the augmented chain complex of \mathcal{F} over \mathbb{K} , i.e., $\tilde{C}_{-1}(\mathcal{F}) = \mathbb{K}$ and $\partial : \tilde{C}_0(\mathcal{F}) \rightarrow \tilde{C}_{-1}(\mathcal{F})$ sends any vertex to 1.

Let us associate more objects to Γ . The moment-angle complex \mathbb{T} of \mathcal{F} associated to the pair $(\mathbb{S}^1, 1)$ is the CW-subcomplex of $(\mathbb{S}^1)^V$ defined as

$$(1.3) \quad \mathbb{T} := \bigcup_{\sigma \in \mathcal{F}} (\mathbb{S}^1)^\sigma$$

We consider the empty simplex as a (-1) -dimensional simplex and by convention $(\mathbb{S}^1)^\emptyset = \mathbf{1} := (1)_{v \in V}$. The circle \mathbb{S}^1 has a natural CW-cell decomposition with cells $\{1\}$ and $\mathbb{S}^1 \setminus \{1\}$. It induces a CW-complex structure on \mathbb{T} such that an r -simplex $\sigma \in \mathcal{F}$ determines an $(r+1)$ -cell

$$\sigma^\mathbb{T} := (\mathbb{S}^1 \setminus \{1\})^\sigma \times \{1\}^{V \setminus \sigma}.$$

In particular, $\emptyset^\mathbb{T} = \{1\}$ is the only 0-cell. For each $v \in V$, $\overline{v^\mathbb{T}}$ is a circle and for each edge $e = \{v, w\}$, $\overline{e^\mathbb{T}}$ is a two-dimensional torus where $\overline{v^\mathbb{T}}$, $\overline{w^\mathbb{T}}$ are two circles intersected at one point. We have the following classical result.

Proposition 1.1 ([2, 4]) *The fundamental group of \mathbb{T} is isomorphic to A_Γ . Moreover, \mathbb{T} is an Eilenberg-McLane space for this group.*

The cellular complex $(C_*(\mathbb{T}), \partial_\mathbb{T})$ is minimal (i.e., $\partial_\mathbb{T} = 0$) and, if we drop the differential maps, $C_*(\mathbb{T}) \cong \tilde{C}_{*-1}(\mathcal{F})$. In particular, $H_*(\mathbb{T}; \mathbb{K}) = \tilde{C}_{*-1}(\mathcal{F})$ as the differential of $C_*(\mathbb{T})$ vanishes.

1.2 The associated \mathbb{T}^\times complex

In the Introduction we have considered a surjective character $\chi : A_\Gamma \rightarrow \mathbb{Z}$; for $v \in V$ we denote $n_v := \chi(g_v)$. Note that this character is completely determined by the tuple $(n_v)_{v \in V}$. For technical reasons we impose the restrictions $n_v \neq 0, \forall v \in V$ since for these characters we will be able to

express the cohomology of Artin kernels in an easier way in terms of filtrations of the flag complex; this could be done without the restriction but the statements become much more complicated and much less geometrical. Without loss of generality, we are going to assume that $n_v > 0$. The epimorphism condition is equivalent to $\gcd_{v \in V} \{n_v\} = 1$.

Let $\tilde{\chi} : \mathbb{T}^\chi \rightarrow \mathbb{T}$ be the infinite cyclic covering associated to χ as in (2). The fiber $\tilde{\chi}^{-1}(\mathbf{1})$ is identified with \mathbb{Z} . We pick up a generator $t : \mathbb{T}^\chi \rightarrow \mathbb{T}^\chi$ of the deck automorphism group of $\tilde{\chi}$, such that if $\tilde{\mathbf{1}} \in \tilde{\chi}^{-1}(\mathbf{1})$ is associated to 0, then the identification $\mathbb{Z} \equiv \tilde{\chi}^{-1}(\mathbf{1})$ is given by $n \equiv t^n(\tilde{\mathbf{1}})$. Recall that $\pi_1(\mathbb{T}^\chi) \cong \ker \chi = A_\Gamma^\chi$, the Artin kernel.

The CW-complex structure of \mathbb{T} induces another such structure in \mathbb{T}^χ and $(C_*(\mathbb{T}^\chi), \partial_\chi)$ is a complex of $\mathbb{K}[t^{\pm 1}]$ -modules (the structure is induced by the deck automorphisms of the covering). Note that \mathbb{T}^χ is an Eilenberg-McLane space for A_Γ and then $H_*(\mathbb{T}^\chi; \mathbb{K})$ is isomorphic (as $\mathbb{K}[t^{\pm 1}]$ -module) with $H_*(A_\Gamma^\chi; \mathbb{K})$.

Let $\sigma \in \mathcal{F}$ be a simplex; recall that $\sigma^\mathbb{T}$ is a cell of \mathbb{T} . Then, the decomposition of $\tilde{\chi}^{-1}(\sigma)$ in connected components is a disjoint union of cells. We fix one of them and denote it by σ^χ ; below we will describe the concrete choice for each σ^χ . Then

$$\tilde{\chi}^{-1}(\sigma) = \bigcup_{n \in \mathbb{Z}} t^n \cdot \sigma^\chi.$$

With its $\mathbb{K}[t^{\pm 1}]$ -module structure, $C_*(\mathbb{T}^\chi)$ is free and there is a basis $\{\sigma^\chi \mid \sigma \in \mathcal{F}\}$. Note that $\dim \sigma^\chi = \dim \sigma + 1$.

By fixing an order in the set of vertices V one can define an *incidence number* $\langle \sigma | \tau \rangle$ for any $\sigma, \tau \in \mathcal{F}$ as follows

$$\langle \sigma | \tau \rangle = \begin{cases} (-1)^s & \text{if } \tau = \sigma \setminus \{v_i\}, \text{ and } s = \#\{j \mid v_j \in \tau, i < j\} \\ 0 & \text{otherwise.} \end{cases}$$

This number fits in the boundary map formula as

$$\partial(\sigma) = \sum_{\tau \in \mathcal{F}} \langle \sigma | \tau \rangle \tau = \sum_{v \in \sigma} \langle \sigma | \sigma_v \rangle \sigma_v.$$

where, for any $v \in \sigma$, $\sigma_v := \sigma \setminus \{v\}$. Then, once \emptyset^χ has been fixed, there is a unique choice of oriented cells σ^χ , such that

$$(\chi 1) \quad \partial^\chi(v^\chi) = (t^{n_v} - 1) \cdot \emptyset^\chi;$$

$$(\chi 2) \quad \partial^\chi(\sigma^\chi) = \sum_{v \in \sigma} \langle \sigma | \sigma_v \rangle (t^v - 1) \sigma_v^\chi.$$

Figure 1 visualizes this choice in the case where $\sigma = \{v, w\}$ is an edge in \mathcal{F} . Summarizing, a simplex σ in \mathcal{F} determines elements $\sigma \in C_*(\mathcal{F})$, $\sigma^\mathbb{T} \in C_*(\mathbb{T})$ and $\sigma^\chi \in C_*(\mathbb{T}^\chi)$.

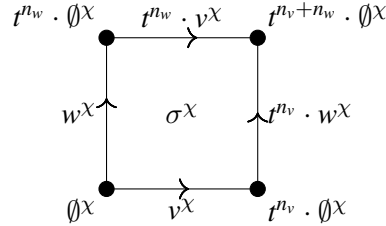


Figure 1: Special lift of the edge σ

Notation 1.1 For $v \in V$, we define $p_v := t^{n_v} - 1$, and for σ we define $p_\sigma := \prod_{v \in \sigma} p_v$. With this notation the formula in (χ2) is equivalent to:

$$\frac{1}{p_\sigma} \partial^\chi(\sigma^\chi) = \sum_{v \in \sigma} \langle \sigma | \sigma_v \rangle \frac{1}{p_{\sigma_v}} \sigma_v$$

Given $k \in \mathbb{Z}$, for the morphism $\partial_k : \tilde{C}_k(\mathcal{F}) \rightarrow \tilde{C}_{k-1}(\mathcal{F})$, its matrix in the bases $\{\sigma\}_{k\text{-simplices}}$ and $\{\tau\}_{(k-1)\text{-simplices}}$ is the incidence matrix $(\langle \sigma | \tau \rangle)_{\sigma, \tau}$.

Remark 1.2 The matrix of $\partial_{k+1}^\chi : C_{k+1}(\mathbb{T}^\chi) \rightarrow C_k(\mathbb{T}^\chi)$, as free $\mathbb{K}[t^{\pm 1}]$ -modules with bases $\{\sigma^\chi\}_{k\text{-simplices}}$ and $\{\tau^\chi\}_{(k-1)\text{-simplices}}$ is $(\langle \sigma | \tau \rangle \frac{p_\sigma}{p_\tau})_{\sigma, \tau}$. Note that the entries are actually in $\mathbb{K}[t^{\pm 1}]$, since $\langle \sigma | \tau \rangle = 0$ when τ is not a face of σ . If $\tau = \sigma_v$ the entry is $\langle \sigma | \sigma_v \rangle p_v$.

1.3 First results on the module structure of Artin kernels

Most of the results in this section come from [5]; they are included since they help the reading. Since $H_*(\mathbb{T}^\chi; \mathbb{K})$ is a finitely generated $\mathbb{K}[t^{\pm 1}]$ -module, eventually replacing \mathbb{K} by a finite extension, there is an isomorphism

$$H_{k+1}(\mathbb{T}^\chi; \mathbb{K}) \cong \mathbb{K}[t^{\pm 1}]^{r_k} \oplus \bigoplus_{\lambda \in \mathbb{K}^*} H_{k, \lambda},$$

where,

$$(1.4) \quad H_{k, \lambda} = \bigoplus_{j=1}^{\infty} \left(\mathbb{K}[t^{\pm 1}] / \langle (t - \lambda)^j \rangle \right)^{r_{k, j}^\lambda}.$$

In order to recover each term of the direct sum it is useful to compare this homology with the homology of some tensored complexes. Recall that $H_{k+1}(\mathbb{T}^X; \mathbb{K}) = H_{k+1}(C_*(\mathbb{T}^X))$.

Lemma 1.3 *Let $R \supset \mathbb{K}[t^{\pm 1}]$ be an integral domain with its natural structure as $\mathbb{K}[t^{\pm 1}]$ -module. Then, there is a natural isomorphism*

$$H_{k+1}(C_*(\mathbb{T}^X) \otimes_{\mathbb{K}[t^{\pm 1}]} R) \cong H_{k+1}(C_*(\mathbb{T}^X)) \otimes_{\mathbb{K}[t^{\pm 1}]} R.$$

Proof The Universal Coefficient Theorem yields the following exact sequence:

$$0 \rightarrow H_{k+1}(C_*(\mathbb{T}^X)) \otimes_{\mathbb{K}[t^{\pm 1}]} R \rightarrow H_{k+1}(C_*(\mathbb{T}^X) \otimes_{\mathbb{K}[t^{\pm 1}]} R) \rightarrow \text{Tor}(H_k(C_*(\mathbb{T}^X)), R) \rightarrow 0.$$

If A, B are $\mathbb{K}[t^{\pm 1}]$ -modules then $\text{Tor}(A \oplus B, R) = \text{Tor}(A, R) \oplus \text{Tor}(B, R)$. Since R is an integral domain, from its very definition $\text{Tor}(\mathbb{K}[t^{\pm 1}]/\langle p(t) \rangle, R) = 0$, and hence the last term of the sequence vanishes. \square

Lemma 1.4 *Let R be as in Lemma 1.3. Assume that $\forall v \in V$, the polynomial $p_v(t) = t^{n_v} - 1$ is invertible in R . Then,*

$$R^{n_k} \cong H_{k+1}(C_*(\mathbb{T}^X) \otimes_{\mathbb{K}[t^{\pm 1}]} R) \cong \tilde{H}_k(\mathcal{F}; R).$$

Proof From the hypotheses, $\left\{ \sigma^X \otimes \frac{1}{p_\sigma} \right\}_{k\text{-simplices}}$ and $\left\{ \tau^X \otimes \frac{1}{p_\tau} \right\}_{(k-1)\text{-simplices}}$ are bases of the spaces $C_{k+1}(\mathbb{T}^X) \otimes_{\mathbb{K}[t^{\pm 1}]} R$ and $C_k(\mathbb{T}^X) \otimes_{\mathbb{K}[t^{\pm 1}]} R$, respectively. The matrix of $\partial_{k+1}^X \otimes 1_R$ on those bases is $(\langle \sigma | \tau \rangle)_{\sigma, \tau}$. As a consequence, this complex is naturally isomorphic to $(C_{*-1}(\mathcal{F}) \otimes_{\mathbb{K}} R, \partial)$ which yields the second automorphism. The first one is a consequence of Lemma 1.3 and the hypotheses of the statement and the fact that $\tilde{H}_k(\mathcal{F}; R)$ is free. \square

We recover two results in [5].

(PS1) The number r_k coincides with $\dim_{\mathbb{K}} \tilde{H}_k(\mathcal{F}; \mathbb{K})$.

(PS2) If λ is not a root of unity of order a divisor of some n_v , $v \in V$, then $r_{k,j}^\lambda = 0$.

The proof of (PS1) is a consequence of Lemma 1.4, for $R = \mathbb{K}(t)$. The proof of (PS2) is a consequence of Lemma 1.4 for the localized ring $R = \mathbb{K}[t^{\pm 1}]_{\langle t-\lambda \rangle}$. Using Lemma 1.3, we deduce that there is no torsion for these λ . A third result from [5] is immediately available

(PS3) If \mathcal{F} is n -acyclic for \mathbb{K} , then $\dim_{\mathbb{K}} H_m(\mathbb{T}^X; \mathbb{K}) < \infty$ for $0 \leq m \leq n + 1$. Moreover, if $\tilde{H}_{n+1}(\mathcal{F}; \mathbb{K}) \neq 0$, then $H_{n+2}(\mathbb{T}^X; \mathbb{K})$ has infinite dimension.

2 The torsion part of the homology

2.1 Primary parts via localization

Fix $\lambda \in \mathbb{K}^*$ a primitive d -root of unity for some d dividing $\text{lcm}\{n_v\}_{v \in V}$.

There is a natural monomorphism $\eta_\lambda : \mathbb{K}[t^{\pm 1}] \rightarrow \mathbb{K}[[s]]$ such that $t \mapsto \lambda + s$. Note that

$$\eta_\lambda(t^n - 1) = s^{\alpha_\lambda} q_n(s) \text{ where } q_n \in \mathbb{K}[[s]]^\times \text{ and } \alpha_\lambda = \begin{cases} 0 & \text{if } \lambda^n \neq 1 \\ 1 & \text{if } \lambda^n = 1. \end{cases}$$

The use of this map has been borrowed from [5]. As a consequence

$$\eta_\lambda(p_\sigma) = s^{\omega(\lambda, \sigma)} u_\sigma(s) \text{ where } u_\sigma \in \mathbb{K}[[s]]^\times \text{ and } \omega(\lambda, \sigma) = \#\{v \in \sigma \mid \lambda^{n_v} = 1\}.$$

Note that $\omega(\lambda, \sigma)$ depends on d rather than a particular choice of the primitive d -root of unity. Hence we denote $\omega_d(\sigma) := \omega(\lambda, \sigma)$.

The following lemma is useful for the study of the torsion part of the homology of \mathbb{T}^\times .

Lemma 2.1 *The torsion part of $H_{k+1}(C_*(\mathbb{T}^\times) \otimes_{\eta_\lambda} \mathbb{K}[[s]])$ coincides with the one of $\text{Coker}(\partial_{k+2}^\times \otimes 1_{\mathbb{K}[[s]])}$ and with*

$$H_{k,\lambda} \otimes_{\eta_\lambda} \mathbb{K}[[s]] = \bigoplus_{j=1}^{\infty} \left(\mathbb{K}[[s]] / \langle s^j \rangle \right)^{r_{k,j}^\lambda}.$$

Proof The following exact sequence holds:

$$\begin{array}{ccccccc} 0 & \rightarrow & H_{k+1}(C_*(\mathbb{T}^\times) \otimes_{\eta_\lambda} \mathbb{K}[[s]]) & \longrightarrow & \text{Coker } \partial_{k+2}^\times \otimes 1_{\mathbb{K}[[s]]} & \longrightarrow & \frac{C_{k+1}(\mathbb{T}^\times) \otimes_{\eta_\lambda} \mathbb{K}[[s]]}{\ker \partial_{k+1}^\times \otimes 1_{\mathbb{K}[[s]]}} \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & \frac{\ker \partial_{k+1}^\times \otimes 1_{\mathbb{K}[[s]]}}{\text{Im } \partial_{k+2}^\times \otimes 1_{\mathbb{K}[[s]]}} & & \frac{C_{k+1}(\mathbb{T}^\times) \otimes_{\eta_\lambda} \mathbb{K}[[s]]}{\text{Im } \partial_{k+2}^\times \otimes 1_{\mathbb{K}[[s]]}} & & \text{Im } \partial_{k+1}^\times \otimes 1_{\mathbb{K}[[s]]} \end{array}$$

Since the last module is free, the torsion part of $H_{k+1}(C_*(\mathbb{T}^\times) \otimes_{\eta_\lambda} \mathbb{K}[[s]])$ coincides with the one of $\text{Coker } \partial_{k+2}^\times \otimes 1_{\mathbb{K}[[s]]}$. The matrix of $\partial_{k+2}^\times \otimes 1_{\mathbb{K}[[s]]}$ is a presentation matrix for $\text{Coker } \partial_{k+2}^\times \otimes 1_{\mathbb{K}[[s]]}$. The last statement is a consequence of Lemma 1.3. \square

Lemma 2.2 *The matrix of the differential $\partial_{k+2}^\times \otimes 1_{\mathbb{K}[[s]]}$ is $(\langle \sigma | \tau \rangle s^{\omega_d(\sigma) - \omega_d(\tau)})_{\sigma, \tau}$, for the bases $\left\{ \frac{\sigma^\times \otimes 1}{u_\sigma} \right\}_{k\text{-simplices}}$ and $\left\{ \frac{\tau^\times \otimes 1}{u_\tau} \right\}_{(k-1)\text{-simplices}}$.*

Proof It is not hard to check that the matrix in the bases $\{\sigma^\chi \otimes 1\}_{k\text{-simplices}}$ and $\{\tau^\chi \otimes 1\}_{(k-1)\text{-simplices}}$ is equal to $\left(\langle \sigma | \tau \rangle s^{\omega_d(\sigma) - \omega_d(\tau)} \frac{u_\sigma}{u_\tau} \right)_{\sigma, \tau}$. The result follows. \square

It is clear that the values $\omega_d(\sigma)$ are interesting. The following result is straightforward.

Lemma 2.3 $\omega_d(\sigma) = \#\{v \in \sigma \mid d \text{ divides } n_v\}$.

2.2 The $(t - 1)$ -primary part

This corresponds with considering the case $d = 1$ in the construction above. We will recover another result in [5].

$$(PS4) \quad H_{k,1} = \left(\mathbb{K}[t^{\pm 1}] / \langle t - 1 \rangle \right)^{\text{Rank } \partial_{k+1}}.$$

We need to compute a matrix for the differential $\partial_{k+2}^\chi \otimes 1_{\mathbb{K}[[s]]}$ for the morphism $\eta_1 : \mathbb{K}[t^{\pm 1}] \rightarrow \mathbb{K}[[s]]$. Such a matrix is $(\langle \sigma | \tau \rangle)_{\sigma, \tau}$ and (PS4) follows. Note that, in particular, the $(t - 1)$ -primary part is semisimple.

2.3 The $(t - \lambda)$ -primary part

Consider λ be a d -primitive root of unity for $d > 1$. The $(t - \lambda)$ -primary part of $H_{k+1}(A_\Gamma^\chi; \mathbb{K})$ can be recovered using the following result.

Proposition 2.4 *Let λ be a d -primitive root of unity. The exponents $r_{k,j}^\lambda$ depend completely on the function ω_d (in fact, on $\omega_d|_V$). These exponents $r_{k,j}^\lambda$ coincide with the exponents $r_{k,j}(d)$ of Theorem 1*

We will call this part in Theorem 1 the Φ_d -torsion part of $H_{k+1}(A_\Gamma^\chi; \mathbb{K})$. By Lemma 2.1 this submodule is completely determined by the torsion part of the module $\text{Coker}(\partial_{k+2}^\chi \otimes 1_{\mathbb{K}[[s]]})$. A matrix of this map is $(\langle \sigma, \tau \rangle s^{\omega_d(\sigma) - \omega_d(\tau)})_{\sigma, \tau}$. This torsion part is determined by its Smith form since $\mathbb{K}[[s]]$ is a Euclidean ring.

Notation 2 Let us assume that this Smith form has its first r diagonal entries equal to 1 and then the other non-zero entries are $(s^{a_{k,1}}, \dots, s^{a_{k,\ell_k}})$ where $0 < a_{k,1} \leq \dots \leq a_{k,\ell_k}$, $\ell_k \geq 0$. In this case

$$H_{k,\lambda} \cong \bigoplus_{j=1}^{\ell_k} \mathbb{K}[t^{\pm 1}] / \langle (t - \lambda)^{a_{k,j}} \rangle$$

and $r_{k,j}(d)$ is the number of entries such $a_{k,i} = j$. In fact, if Φ_d is the cyclotomic polynomial whose roots are the d -primitive roots of unity, then

$$\bigoplus_{j=1}^{\ell_k} \mathbb{K}[t^{\pm 1}] / \langle \Phi_d(t)^{a_{k,j}} \rangle$$

is a factor of $H_{k+1}(\mathbb{T}^X; \mathbb{K})$.

The goal will be to compute some of these numbers relating them to the sequence of non-zero Fitting ideals of the above matrix. The first r Fitting ideals equal $\mathbb{K}[[s]]$, while the next ones will be of the form $\langle s^{b_{k,1}} \rangle, \dots, \langle s^{b_{k,\ell_k}} \rangle$ where

$$(2.1) \quad b_{k,1} \leq \dots \leq b_{k,\ell_k}, \quad b_{k,j} = a_{k,1} + \dots + a_{k,j}.$$

Our goal is to compute these numbers. Note that this means that we have $r + \ell_k$ non-zero ideals.

3 Fitting ideals

As in the previous section, we will fix $\lambda \in \mathbb{K}$ a primitive d -th root of unity for $d > 1$. Prior to the computation of the minors of the matrix $(\langle \sigma, \tau \rangle s^{\omega_d(\sigma) - \omega_d(\tau)})_{\sigma, \tau}$, let us study its square submatrices in order to characterize non-zero minors. For simplicity from now on the subindex in ω_d will be dropped. These submatrices are parametrized by two subsets of simplices: the subset K of $(k + 1)$ -simplices, corresponding to the rows of the submatrix, and the subset L of k -simplices, corresponding to the columns *not* in the submatrix. We denote these submatrices as $M_{K,L}^X$. We will compare this submatrix with $M_{K,L} = (\langle \sigma | \tau \rangle)_{\sigma \in K}^{\tau \in L}$. Since we deal with square matrices, $K \cup L$ is bijective with the set of k -simplices of \mathcal{F} .

Let us introduce some helpful notation:

- The k -skeleton of \mathcal{F} is denoted by \mathcal{F}^k .
- The set of simplices of dimension k of \mathcal{F} is denoted as $S^k(\mathcal{F})$.

- For a subset K of $S^k(\mathcal{F})$ we define its *weight* is defined as $\omega(K) := \sum_{\sigma \in K} \omega(\sigma)$.

Lemma 3.1 $\det M_{K,L}^\chi = \det M_{K,L} s^{\omega(K) + \omega(L) - \omega(S^k(\mathcal{F}))}$.

Proof If we multiply each column of the matrix of $\partial_{k+2}^\chi \otimes 1_{\mathbb{K}[[s]]}$ by $s^{\omega(\sigma)}$ and divide each row by $\omega(\tau)$, we obtain the matrix for ∂_{k+1} . Hence,

$$\det M_{K,L} = \frac{s^{\omega(S^k(\mathcal{F}))}}{s^{\omega(L)} s^{\omega(K)}} \det M_{K,L}^\chi \quad \square$$

The next goal is to characterize geometrically when these minors do not vanish. Let us denote:

$$M_K = \mathcal{F}^k \cup \bigcup_{\sigma \in K} \sigma, \quad M_L = \mathcal{F}^{k-1} \cup \bigcup_{\tau \in L} \tau.$$

Note that $\chi(M_K) - \chi(M_L) = (-1)^{k+1} \#K + (-1)^k (\#S^k(\mathcal{F}) - \#L) = 0$, i.e $\chi(M_K, M_L) = 0$.

Proposition 3.2 *The minor $\det M_{K,L}$ (and hence $\det M_{K,L}^\chi$) does not vanish if and only if the pair (M_K, M_L) is \mathbb{K} -acyclic.*

Proof Let us denote $\mathbb{K}\langle K \rangle, \mathbb{K}\langle L \rangle$, the subspaces of $C_{k+1}(\mathcal{F})$ and $C_k(\mathcal{F})$ generated by K, L , respectively. The relative complex is as follows:

$$C_j(M_K, M_L) = \begin{cases} 0 & \text{if } j < k \text{ or } j > k + 1, \\ \mathbb{K}\langle K \rangle & \text{if } j = k + 1, \\ \tilde{C}_k(\mathcal{F}) / \mathbb{K}\langle L \rangle & \text{if } j = k. \end{cases}$$

In the natural bases the matrix of the only relevant differential is $M_{K,L}$, and the statement follows. \square

Let us express this acyclicity condition using the long exact sequence of pairs:

$$\begin{array}{ccccccc} 0 & \tilde{C}_{k+1}(\mathcal{F}) & 0 & \tilde{C}_k(\mathcal{F}) & & & \\ \parallel & \cup & \parallel & \cup & & & \\ \tilde{H}_{k+1}(M_L) & \rightarrow \tilde{H}_{k+1}(M_K) & \rightarrow \tilde{H}_{k+1}(M_K, M_L) & \rightarrow \tilde{H}_k(M_L) & \longrightarrow \tilde{H}_k(M_K) & \longrightarrow & \\ & & & & & & \\ & \longrightarrow \tilde{H}_k(M_K, M_L) & \longrightarrow \tilde{H}_{k-1}(M_L) & \longrightarrow \tilde{H}_{k-1}(M_K) & \rightarrow \tilde{H}_{k-1}(M_K, M_L) & & \\ & \parallel & & \parallel & \parallel & & \\ & 0 & & \tilde{H}_{k-1}(\mathcal{F}) & 0 & & \end{array}$$

Proposition 3.3 *The minor $M_{K,L}$ does not vanish if and only if*

(Ac1) $\tilde{H}_{k+1}(M_K) = 0$, i.e., $\ker \partial_{k+1} \cap \mathbb{K}\langle K \rangle = 0$.

(Ac2) *The map $\tilde{H}_k(M_L) \rightarrow \tilde{H}_k(M_K)$, induced by the inclusion, is an isomorphism.*

Proof From the previous diagram (\Rightarrow) is evident. For (\Leftarrow), we need to prove that $\tilde{H}_{k+1}(M_K, M_L) = 0 = \tilde{H}_k(M_K, M_L)$ since the other equalities are obvious. The two conditions of the statement imply $\tilde{H}_{k+1}(M_K, M_L) = 0$. The Euler characteristic of the pair implies $\tilde{H}_k(M_K, M_L) = 0$. \square

Definition 3.4 We say that a pair (K, L) is *acyclic* if (M_K, M_L) is acyclic. An acyclic pair (K, L) is *minimal* if the sum of its multiplicities is minimal among the acyclic pairs of the same size. The *size* of an acyclic pair (K, L) is $\#K$.

The next step in our strategy is to determine the maximal size of the non-zero minors.

Proposition 3.5 *The maximal size of an acyclic pair is $\text{Rank } \partial_{k+1}$.*

A pair (K, L) is acyclic of size $\text{Rank } \partial_{k+1}$ if and only if the following conditions hold:

- (1) $\#K = \text{Rank } \partial_{k+1}$,
- (2) $\tilde{H}_k(M_K) = \tilde{H}_k(\mathcal{F})$,
- (3) $\#L = \#S^k(\mathcal{F}) - \text{Rank } \partial_{k+1}$,
- (4) *The map $\tilde{H}_k(M_L) \rightarrow \tilde{H}_k(\mathcal{F})$, induced by inclusion, is an isomorphism.*

In particular, the conditions for K and L are independent.

Proof Let (K, L) be an acyclic pair. Note that the condition (Ac1) implies that

$$\#K \leq \#S^{k+1}(\mathcal{F}) - \dim \ker \partial_{k+1} = \dim \tilde{C}_{k+1}(\mathcal{F}) - \dim \ker \partial_{k+1} = \text{Rank } \partial_{k+1}.$$

It is clear that there exists $K \subset S^{k+1}(\mathcal{F})$ such that $\#K = \text{Rank } \partial_{k+1}$ and $\tilde{C}_{k+1}(\mathcal{F}) = \ker \partial_{k+1} \oplus \mathbb{K}\langle K \rangle$. Note that $\tilde{H}_k(M_K) = \tilde{H}_k(\mathcal{F})$.

In order to satisfy (Ac2) we need to find L such that $\tilde{H}_k(M_L) \rightarrow \tilde{H}_k(\mathcal{F})$ is an isomorphism. As the map

$$\ker \partial_k = \tilde{H}_k(\mathcal{F}^k) \longrightarrow \tilde{H}_k(\mathcal{F}) = \ker \partial_k / \text{Im } \partial_{k+1}$$

is clearly surjective, any $L \subset S^k(\mathcal{F})$ such that $\#L = \#S^k(\mathcal{F}) - \text{Rank } \partial_{k+1}$ and $\tilde{C}_k(\mathcal{F}) = \text{Im } \partial_{k+1} \oplus \mathbb{K}\langle L \rangle$. \square

One can define a natural filtration given by

$$(3.1) \quad \mathcal{S}_j^m(\mathcal{F}) := \{\sigma \in \mathcal{S}^m(\mathcal{F}) \mid \omega(\sigma) \leq j\}, \quad \mathcal{F}_j^m := \mathcal{F}^{m-1} \cup \bigcup \mathcal{S}_j^m(\mathcal{F}), \quad 0 \leq j \leq m+1.$$

Note that

$$(3.2) \quad \begin{aligned} \mathcal{F}^{k-1} &\subset \mathcal{F}_0^k \subset \mathcal{F}_1^k \dots \subset \mathcal{F}_k^k \subset \mathcal{F}_{k+1}^k = \\ \mathcal{F}^k &\subset \mathcal{F}_0^{k+1} \subset \mathcal{F}_1^{k+1} \dots \subset \mathcal{F}_{k+1}^{k+1} \subset \mathcal{F}_{k+2}^{k+1} = \mathcal{F}^{k+1}. \end{aligned}$$

In what follows we will describe certain properties of minimal acyclic pairs in terms of the homology of blocks of this filtration.

Lemma 3.6 *Let $0 \leq j \leq k+2$. There exists $K_j \subset \mathcal{S}_j^{k+1}(\mathcal{F})$ such that $\tilde{H}_{k+1}(M_{K_j}) = 0$ and $\tilde{H}_k(M_{K_j}) \rightarrow \tilde{H}_k(\mathcal{F}_j^{k+1})$ is an isomorphism. The cardinality of any such K_j depends only on j . Its cardinality is*

$$u_{k+1,j} := \dim \tilde{C}_{k+1}(\mathcal{F}_j^{k+1}) - \dim \tilde{H}_{k+1}(\mathcal{F}_j^{k+1})$$

and

$$(3.3) \quad u_{k+1,j} - u_{k+1,j-1} = \dim \tilde{H}_k(\mathcal{F}_{j-1}^{k+1}) - \dim \tilde{H}_k(\mathcal{F}_j^{k+1}).$$

Proof Let $K_j \subset \mathcal{S}_j^{k+1}(\mathcal{F})$. The condition $\tilde{H}_{k+1}(M_{K_j}) = 0$ is equivalent to $\mathbb{K}\langle K_j \rangle \cap \ker \partial_{k+1} = 0$. Maximality is obtained when $\mathbb{K}\langle K_j \rangle \oplus \ker \partial_{k+1}|_{\tilde{C}_{k+1}(\mathcal{F}_j^{k+1})} = \tilde{C}_{k+1}(\mathcal{F}_j^{k+1})$. Hence, the cardinality of K_j equals $\dim \partial_{k+1}(\tilde{C}_{k+1}(\mathcal{F}_j^{k+1}))$. The formula for $u_{k+1,j}$ follows since $\tilde{H}_{k+1}(\mathcal{F}_j^{k+1}) = \ker \partial_{k+1}|_{\tilde{C}_{k+1}(\mathcal{F}_j^{k+1})}$. Let us consider the long exact sequence for the pair $(\mathcal{F}_j^{k+1}, \mathcal{F}_{j-1}^{k+1})$ and recall that the relative complex vanishes outside index $k+1$, i.e., $H_{k+1}(\mathcal{F}_j^{k+1}, \mathcal{F}_{j-1}^{k+1}) = \frac{\tilde{C}_{k+1}(\mathcal{F}_j^{k+1})}{\tilde{C}_{k+1}(\mathcal{F}_{j-1}^{k+1})}$ and the other homology groups vanish:

$$0 \rightarrow H_{k+1}(\mathcal{F}_{j-1}^{k+1}) \rightarrow H_{k+1}(\mathcal{F}_j^{k+1}) \rightarrow \frac{\tilde{C}_{k+1}(\mathcal{F}_j^{k+1})}{\tilde{C}_{k+1}(\mathcal{F}_{j-1}^{k+1})} \rightarrow H_k(\mathcal{F}_{j-1}^{k+1}) \rightarrow H_k(\mathcal{F}_j^{k+1}) \rightarrow 0$$

Since the difference $u_{k+1,j} - u_{k+1,j-1}$ is the alternating sum of the first three terms, the result follows. \square

Lemma 3.7 *Let $0 \leq j \leq k+1$. There exists $L_j \subset \mathcal{S}_j^k(\mathcal{F})$ such that $\tilde{H}_k(M_{L_j}) \rightarrow \tilde{H}_k(\mathcal{F})$ is injective, it has the same image as $\tilde{H}_k(\mathcal{F}_j^k) \rightarrow \tilde{H}_k(\mathcal{F})$, and $\tilde{H}_{k-1}(M_{L_j}) \rightarrow \tilde{H}_{k-1}(\mathcal{F}_j^k)$ is an isomorphism. The cardinality of any such L_j depends only on j . Its cardinality is*

$$u_{k,j} := \dim \tilde{C}_k(\mathcal{F}_j^k) - \dim \ker(\tilde{H}_k(\mathcal{F}_j^k) \rightarrow \tilde{H}_k(\mathcal{F})).$$

and

$$(3.4) \quad u_{k,j} - u_{k,j-1} = \dim \tilde{H}_k(\mathcal{F}_{j-1}^{k+1}, \mathcal{F}_{j-1}^k) - \dim \tilde{H}_k(\mathcal{F}_j^{k+1}, \mathcal{F}_j^k).$$

Proof Let $L_j \subset S_j^{k+1}(\mathcal{F})$. The condition $\tilde{H}_k(M_{L_j}) = 0$ is equivalent to $\mathbb{K}\langle L_j \rangle \cap \text{Im } \partial_{k+1} = 0$ and maximality is obtained when $\mathbb{K}\langle L_j \rangle \oplus (\text{Im } \partial_{k+1} \cap \tilde{C}_k(\mathcal{F}_j^k)) = \tilde{C}_k(\mathcal{F}_j^k)$ and the computation of $u_{k,j}$ follows.

Let us deal with the difference. As a first step, let us consider the long exact sequence for the pair $(\mathcal{F}^{k+1}, \mathcal{F}_j^k)$:

$$0 \longrightarrow \tilde{H}_{k+1}(\mathcal{F}^{k+1}) \longrightarrow \tilde{H}_{k+1}(\mathcal{F}^{k+1}, \mathcal{F}_j^k) \longrightarrow \tilde{H}_k(\mathcal{F}_j^k) \longrightarrow \tilde{H}_k(\mathcal{F}^{k+1}) \longrightarrow 0.$$

Replacing the two terms by the kernel of this map, we obtain that

$$u_{k,j} := \dim \tilde{C}_k(\mathcal{F}_j^k) - \dim \tilde{H}_{k+1}(\mathcal{F}^{k+1}, \mathcal{F}_j^k) + \tilde{H}_{k+1}(\mathcal{F}^{k+1}).$$

Let us consider the long exact sequence for the triple $(\mathcal{F}^{k+1}, \mathcal{F}_j^k, \mathcal{F}_{j-1}^k)$:

$$\begin{aligned} 0 \longrightarrow \tilde{H}_{k+1}(\mathcal{F}^{k+1}, \mathcal{F}_{j-1}^{k+1}) \longrightarrow \tilde{H}_{k+1}(\mathcal{F}^{k+1}, \mathcal{F}_j^{k+1}) \longrightarrow \frac{\tilde{C}_k(\mathcal{F}_j^k)}{\tilde{C}_k(\mathcal{F}_{j-1}^k)} \longrightarrow \\ \longrightarrow \tilde{H}_k(\mathcal{F}^{k+1}, \mathcal{F}_{j-1}^{k+1}) \longrightarrow \tilde{H}_k(\mathcal{F}^{k+1}, \mathcal{F}_j^{k+1}) \longrightarrow 0. \end{aligned}$$

The alternating sum yields the result. \square

Proposition 3.8 *Let (K, L) be a minimal acyclic pair of size $\text{Rank } \partial_{k+1}$. The simplices of K of weight $\leq j$ form a subset K_j as in Lemma 3.6 and the number of simplices of weight equal to j is the right-hand side of (3.3).*

The simplices of L of weight $\leq j$ form a subset L_j as in Lemma 3.7 and the number of simplices of weight equal to j is the right-hand side of (3.4). Any (K, L) with the above properties is minimal.

Proof By Proposition 3.5, such an acyclic pair can be constructed. Let (K, L) be an acyclic pair of size $\text{Rank } \partial_{k+1}$. Let us denote by $v_{k+1,j}$ (resp. $v_{k,j}$) the number of simplices in K (resp. L) of weight $\leq j$. Since $u_{k+1,k+2} = v_{k+1,k+2}$, let j be the first value for which $u_{k+1,j} = v_{k+1,j}$ but $u_{k+1,j-1} > v_{k+1,j-1}$. If such a j exists it is easy to construct an acyclic pair (K', L) where $\omega(K') < \omega(K)$.

A similar argument works for L . Then, if (K, L) is minimal, no such j exists, i.e., K, L satisfy the conditions of the statement. \square

Minimal acyclic pairs are important since they determine their corresponding Fitting ideals. The following result gives a formula for the maximal multiplicity of $s = (t - \lambda)$ in the Fitting ideals (see (2.1) for the definition of $b_{k,j}$).

Theorem 3.9 *There are Rank $\partial_{k+1} = r + \ell_k$ non-zero Fitting ideals. Then*

$$\sum_j j \cdot r_{k,j}(d) = b_{k,\ell_k} = \sum_{j=0}^{k+1} \dim \tilde{H}_k(\mathcal{F}_j^{k+1}) - (k+2) \dim \tilde{H}_k(\mathcal{F}) + \sum_{j=0}^k \dim \tilde{H}_{k+1}(\mathcal{F}^{k+1}, \mathcal{F}_j^k) - (k+1) \dim \tilde{H}_{k+1}(\mathcal{F}^{k+1}, \mathcal{F}^k).$$

Proof From Lemma 3.1 we need to compute $\omega(K) + \omega(L) - \omega(\mathcal{F}^k)$ for a minimal acyclic pair (K, L) of size Rank ∂_{k+1} .

From Proposition 3.8 and Lemma 3.6, we have the following value for $\omega(K)$:

$$\sum_{j=1}^{k+2} j \dim \tilde{H}_k(\mathcal{F}_{j-1}^{k+1}) - \sum_{j=1}^{k+2} j \dim \tilde{H}_k(\mathcal{F}_j^{k+1}) = \sum_{j=0}^{k+1} \dim \tilde{H}_k(\mathcal{F}_j^{k+1}) - (k+2) \dim \tilde{H}_k(\mathcal{F}^{k+1}).$$

From Proposition 3.8 and Lemma 3.7, we have the following value for $\omega(L)$:

$$\sum_{j=1}^{k+1} j \dim \tilde{H}_k(\mathcal{F}^{k+1}, \mathcal{F}_{j-1}^k) - \sum_{j=1}^{k+1} j \dim \tilde{H}_k(\mathcal{F}^{k+1}, \mathcal{F}_j^k) = \sum_{j=0}^k \dim \tilde{H}_k(\mathcal{F}^{k+1}, \mathcal{F}_j^k) - (k+1) \dim \tilde{H}_k(\mathcal{F}^{k+1}, \mathcal{F}^k).$$

Finally, note that $\dim \tilde{C}_k(\mathcal{F}^k, \mathcal{F}_j^k)$ is the number of k -cells of weight $> j$. Hence

$$\omega(\mathcal{F}^k) = \sum_{j=0}^k \dim \tilde{C}_k(\mathcal{F}^k, \mathcal{F}_j^k) = \sum_{j=0}^k \dim \tilde{H}_k(\mathcal{F}^k, \mathcal{F}_j^k).$$

Let us consider the triple $(\mathcal{F}^{k+1}, \mathcal{F}^k, \mathcal{F}_j^k)$ and its homology long exact sequence:

$$0 \rightarrow \tilde{H}_{k+1}(\mathcal{F}^{k+1}, \mathcal{F}_j^k) \rightarrow \tilde{H}_{k+1}(\mathcal{F}^{k+1}, \mathcal{F}^k) \rightarrow \tilde{H}_k(\mathcal{F}^k, \mathcal{F}_j^k) \rightarrow \tilde{H}_k(\mathcal{F}^{k+1}, \mathcal{F}_j^k) \rightarrow 0$$

We obtain that

$$\omega(L) - \omega(\mathcal{F}^k) = \sum_{j=0}^k \dim \tilde{H}_{k+1}(\mathcal{F}^{k+1}, \mathcal{F}_j^k) - (k+1) \dim \tilde{H}_{k+1}(\mathcal{F}^{k+1}, \mathcal{F}^k). \square$$

3.1 Bounding the Jordan blocks

The purpose of this section is to give a bound for the size of the Jordan blocks in the $(t - \lambda)$ -primary part of $H_{k+1}(\mathbb{T}^x; \mathbb{K})$. This will be done by computing the drop in multiplicities in the Fitting ideals. This is measured by the acyclic pairs (K, L) as considered above. Their construction is studied next.

Proposition 3.10 *Let (K, L) be an acyclic pair with size $< \text{Rank } \partial_{k+1}$. Then there exists $\sigma \in S^{k+1}(\mathcal{F}) \setminus K$ such that $\mathbb{K}\langle K \cup \{\sigma\} \rangle \cap \ker \partial_{k+1} = 0$. Moreover, for each such σ there exists $\tau \in L$ such that $(K \cup \{\sigma\}, L \setminus \{\tau\})$ is an acyclic pair.*

Proof Since $\mathbb{K}\langle K \rangle \cap \ker \partial_{k+1} = 0$ and $\#K < \text{Rank } \partial_{k+1}$, then we can find a simplex σ as in the statement. Fix one such σ . Note that $M_{K \cup \{\sigma\}} = M_K \cup |\sigma|$ and $\partial|\sigma| = M_K \cap |\sigma|$. Using Mayer-Viétoris long exact sequence we have

$$\begin{array}{ccccccccc} \tilde{H}_{k+1}(M_{K \cup \{\sigma\}}) & \rightarrow & \tilde{H}_k(\partial|\sigma|) & \rightarrow & \tilde{H}_k(M_K) \oplus \tilde{H}_k(|\sigma|) & \rightarrow & \tilde{H}_k(M_{K \cup \{\sigma\}}) & \rightarrow & \tilde{H}_{k-1}(\partial|\sigma|) \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ 0 & & \mathbb{Z} & & 0 & & 0 & & 0 \end{array}$$

i.e.,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{H}_k(M_K) \longrightarrow \tilde{H}_k(M_{K \cup \{\sigma\}}) \longrightarrow 0.$$

In particular, the composition $\phi : \tilde{H}_k(M_L) \rightarrow \tilde{H}_k(M_{K \cup \{\sigma\}})$ has a one-dimensional kernel. Recall that $\tilde{H}_k(M_{K \cup \{\sigma\}}) = \ker \alpha_{k|\mathbb{K}\langle L \rangle}$. Let $\tau \in L$ in the support of the non-zero elements of the kernel of ϕ . We have:

$$\begin{array}{ccccccc} 0 & & 0 & & \mathbb{Z} \\ \parallel & & \parallel & & \parallel \\ \tilde{H}_k(\partial|\tau|) & \longrightarrow & \tilde{H}_k(M_{L \setminus \{\tau\}}) \oplus \tilde{H}_k(|\tau|) & \longrightarrow & \tilde{H}_k(M_L) & \longrightarrow & \tilde{H}_{k-1}(\partial|\tau|) \\ \\ \tilde{H}_{k-1}(\partial|\tau|) & \longrightarrow & \tilde{H}_{k-1}(M_{L \setminus \{\tau\}}) \oplus \tilde{H}_{k-1}(|\tau|) & \longrightarrow & \tilde{H}_{k-1}(M_L) & \longrightarrow & \tilde{H}_{k-2}(\partial|\tau|) \\ \parallel & & \parallel & & \parallel \\ \mathbb{Z} & & 0 & & 0 \end{array}$$

i.e.,

$$0 \longrightarrow \tilde{H}_k(M_{L \setminus \{\tau\}}) \longrightarrow \tilde{H}_k(M_L) \longrightarrow \mathbb{Z} \longrightarrow \tilde{H}_{k-1}(M_{L \setminus \{\tau\}}) \longrightarrow \tilde{H}_{k-1}(M_L) \longrightarrow 0.$$

For the choice of τ the map $\mathbb{Z} \rightarrow \tilde{H}_{k-1}(M_{L \setminus \{\tau\}})$ is zero and the first terms yield a short exact sequence. Moreover, by construction, the map $\tilde{H}_k(M_{L \setminus \{\tau\}}) \rightarrow \tilde{H}_k(M_{K \cup \{\sigma\}})$ is injective; as the dimensions are equal, it is bijective and $(K \cup \{\sigma\}, L \setminus \{\tau\})$ is an acyclic pair. \square

Corollary 3.11 *Any acyclic pair (K', L') is obtained from an acyclic pair (K, L) of size $\text{Rank } \partial_{k+1}$ by removing simplices from K and adding simplices to L .*

The following result provides information about acyclic pairs after adding simplices, rather than removing them.

Proposition 3.12 *Let (K, L) be an acyclic pair of positive size. Let $\sigma \in K$ and let $\tau \in S^k(\mathcal{F}) \setminus L$. Then $(K \setminus \{\sigma\}, L \cup \{\tau\})$ is an acyclic pair if and only if $\partial_{k+1}(\sigma)$ and τ are both non trivial $\text{mod } \partial_{k+1}(\mathbb{K}\langle K \setminus \{\sigma\} \rangle) \oplus \mathbb{K}\langle L \rangle$.*

Proof Since (K, L) is an acyclic pair, we know that $\partial_{k+1}(\mathbb{K}\langle K \rangle) \oplus \mathbb{K}\langle L \rangle = \tilde{C}_k(\mathcal{F})$ and $\partial_{k+1}|_{\mathbb{K}\langle K \rangle}$ is injective. Given σ, τ as in the statement, we clearly have that the pair $(K \setminus \{\sigma\}, L \cup \{\tau\})$ is acyclic if and only if $\partial_{k+1}(\mathbb{K}\langle K \setminus \{\sigma\} \rangle) \oplus \mathbb{K}\langle L \cup \{\tau\} \rangle = \tilde{C}_k(\mathcal{F})$.

Let $H := \partial_{k+1}(\mathbb{K}\langle K \setminus \{\sigma\} \rangle) \oplus \mathbb{K}\langle L \rangle$. Since $\partial_{k+1}(\mathbb{K}\langle K \rangle) \oplus \mathbb{K}\langle L \rangle = \tilde{C}_k(\mathcal{F})$ we deduce that both $\partial_{k+1}(\sigma)$ and τ are non trivial $\text{mod } H$. \square

This provides an upper bound for the Jordan blocks of the $(t - \lambda)$ -primary part of the homology $H_{k+1}(\mathbb{T}^x; \mathbb{K})$.

Corollary 3.13 *The value a_{k, ℓ_k} is at most $k + 2$, i.e., any exponent in the torsion of $H_{k+1}(\mathbb{T}^x)$ is at most $k + 2$.*

Proof We have to find minimal acyclic pairs of size $\text{Rank } \partial_{k+1} - 1$. It must come from a minimal acyclic pair of size $\text{Rank } \partial_{k+1}$. The highest gap is to take out a $(k + 1)$ -simplex σ of weight $k + 2$ and introduce a k -simplex τ of weight 0. \square

The number of blocks of size $(k + 2)$ is given in terms of the flag complex.

Proposition 3.14 *The number $\#\{j \geq 1 \mid a_{k, j} = k + 2\}$ equals the rank of the natural map*

$$\ker(\tilde{H}_k(\mathcal{F}_0^k) \rightarrow \tilde{H}_k(\mathcal{F})) \longrightarrow \ker(\tilde{H}_k(\mathcal{F}_{k+1}^{k+1}) \rightarrow \tilde{H}_k(\mathcal{F})).$$

Proof Let us fix an arbitrary minimal acyclic pair (K, L) of size $\text{Rank } \partial_{k+1}$. The set of k -simplices of weight 0 not in L is a natural basis of $H_k(\mathcal{F}_0^k, M_{L_0})$. In the same way, the set of $(k + 1)$ -simplices

of weight $k + 2$ in K is a natural basis of $H_k(M_K, M_{K_{k+1}})$. We are going to use long exact sequences of pairs for $(\mathcal{F}_0^k, M_{L_0})$. For the first one:

$$\begin{array}{ccccccccc} H_{k+1}(\mathcal{F}_0^k, M_{L_0}) & \rightarrow & H_k(M_{L_0}) & \rightarrow & H_k(\mathcal{F}_0^k) & \rightarrow & H_k(\mathcal{F}_0^k, M_{L_0}) & \rightarrow & H_{k-1}(M_{L_0}) & \xrightarrow{\cong} & H_{k-1}(\mathcal{F}_0^k). \\ & & \parallel & & & & & & & & \\ & & 0 & & & & & & & & \end{array}$$

We obtain a short exact sequence. Recall that $H_k(M_{L_0}) \rightarrow H_k(\mathcal{F})$ is injective and its image coincides with the one of $H_k(\mathcal{F}_0^k) \rightarrow H_k(\mathcal{F})$. We deduce a natural isomorphism

$$\text{Coker}(H_k(M_{L_0}) \rightarrow H_k(\mathcal{F}_0^k)) \cong \ker(H_k(\mathcal{F}_0^k) \rightarrow H_k(\mathcal{F})).$$

Hence $\ker(H_k(\mathcal{F}_0^k) \rightarrow H_k(\mathcal{F})) \rightarrow H_k(\mathcal{F}_0^k, M_{L_0})$ is an isomorphism.

A similar argument shows that $H_{k+1}(K, K_{k+1}) \rightarrow \ker(H_k(\mathcal{F}_{k+1}^{k+1}) \rightarrow H_k(\mathcal{F}))$ is an isomorphism. The result follows by [Proposition 3.12](#). \square

With the same ideas we can compute the maximal size of the exponents and give a bound for its number. Let us denote by

$$c_{k,i,j} := \text{Rank}(\ker(\tilde{H}_k(\mathcal{F}_j^k) \rightarrow \tilde{H}_k(\mathcal{F})) \rightarrow \ker(\tilde{H}_k(\mathcal{F}_{i-1}^{k+1}) \rightarrow \tilde{H}_k(\mathcal{F}))), \quad i > j.$$

Proposition 3.15 *Assume that $\text{Rank } \partial_{k+1} > 1$ and define $a_{k,0} = 0$ if $\ell_k = 1$. Then, a_{k,ℓ_k} is equal to the maximum value a such that $\exists i, j, j - i = a$, and $c_{k,i,j} \neq 0$. Moreover the number of terms $a_{k,h}$ with the same value is at least $\max\{a_{k,i,j} \mid i - j = a\}$.*

In particular, the semisimple case can be characterized when this value equals 1.

4 A reduction to even characters

The purpose of this section is twofold. First we present a special type of characters called *even characters* and then we show that the Φ_d -torsion part of $H_*(A_\Gamma^X; \mathbb{K})$ can be described as the $(t + 1)$ -torsion part of an even character.

A character $\rho : A_\Gamma \rightarrow \mathbb{Z}$ such that $\{\rho(v) \mid v \in V\} = \{1, 2\}$ will be called *even character*.

4.1 Even characters associated with a pair (χ, d)

Let $\chi : A_\Gamma \rightarrow \mathbb{Z}$ be a surjective character such that for at least a vertex v one has $\chi(v) > 1$. Given any $d > 1$ one can associate an even character to χ and d as follows $\rho_{\chi,d} : A_\Gamma \rightarrow \mathbb{Z}$, $\rho_{\chi,d}(v) := 2^{\omega_d(v)}$. Since χ is surjective and $d > 1$, one has the following equality for the set of indices $\{\rho_{\chi,d}(v)\}_{v \in V} = \{1, 2\}$. In particular, $\rho_{\chi,d}$ is surjective and one can consider the exponents $r_{k,j}(2)$ for $\rho_{\chi,d}$. The following result is straightforward.

Corollary 4.1 *The exponents $r_{k,j}(d)$ for χ coincide with the exponents $r_{k,j}(2)$ for $\rho_{\chi,d}$.*

4.2 The $(t + 1)$ -primary part of an even character

In this section we fix a surjective *weight map* $\omega : V \rightarrow \{0, 1\}$, defining an even character $\rho : A_\Gamma \rightarrow \mathbb{Z}$, defined as $\rho(v) := 2^{\omega(v)}$. We denote also by ω its extension to \mathcal{F} , where $\omega(\sigma) := \sum_{v \in \sigma} \omega(v)$. The goal is to compute the $(t + 1)$ -primary part of $H_{k+1}(\mathbb{T}^{\rho_2}; \mathbb{K})$. The double cover \mathbb{T}^{ρ_2} of \mathbb{T} defined by the even character ρ detects ℓ_k , the number of summands in the $(t + 1)$ -primary part as defined in (2.1).

Theorem 4.2 *The number ℓ_k coincides with*

$$\dim H_{k+1}(\mathbb{T}^{\rho_2}; \mathbb{K}) - \dim \tilde{H}_k(\mathcal{F}) - \dim \tilde{C}_k(\mathcal{F}) - \ell_{k-1} = \dim H_{k+1}^-(\mathbb{T}^{\rho_2}; \mathbb{K}) - \dim \tilde{H}_k(\mathcal{F}) - \ell_{k-1},$$

where $H_{k+1}^-(\mathbb{T}^{\rho_2}; \mathbb{K})$ is the anti-invariant part of the homology of \mathbb{T}^{ρ_2} .

Proof We recover ideas from Sakuma's formula [6] for the homology of unramified coverings. The character $\rho_2 : A_\Gamma \rightarrow \mathbb{Z}/2$, where $\rho_2(g) := \rho(g) \bmod 2$ determines an unramified double covering of \mathbb{T} denoted $\tilde{\rho}_2 : \mathbb{T}^{\rho_2} \rightarrow \mathbb{T}$. For the sake of brevity we denote $\Lambda = \mathbb{K}[t^{\pm 1}]$, $\Lambda_+ := \Lambda / \langle t - 1 \rangle \cong \mathbb{K}$ and $\Lambda_- := \Lambda / \langle t + 1 \rangle \cong \mathbb{K}$. Recall that $\Lambda \otimes \Lambda_- = \Lambda_-$ and that if $\lambda \in \mathbb{K}$ and $n > 0$ then $\Lambda_- \otimes \Lambda / \langle (t - \lambda)^n \rangle = 0$ if $\lambda \neq -1$ and $\Lambda_- \otimes \Lambda / \langle (t + 1)^n \rangle = \Lambda_-$:

$$\ell_k = \dim H_{k+1}(\mathbb{T}^{\rho_2}; \mathbb{K}) \otimes_{\Lambda} \Lambda_- - r_k.$$

From (PS1), $r_k = \dim \tilde{H}_k(\mathcal{F})$ and by definition $H_{k+1}(\mathbb{T}^{\rho^2}; \mathbb{K}) = H_{k+1}(C_*(\mathbb{T}^{\rho^2}))$. By the Universal Coefficient Theorem

$$\dim H_{k+1}(C_*(\mathbb{T}^{\rho^2})) \otimes_{\Lambda} \Lambda_- = \dim H_{k+1}(C_*(\mathbb{T}^{\rho^2}) \otimes_{\Lambda} \Lambda_-) - \dim \operatorname{Tor}_1(H_k(C_*(\mathbb{T}^{\rho^2})), \Lambda_-).$$

By the properties of the Tor functor $\dim \operatorname{Tor}_1(H_k(C_*(\mathbb{T}^{\rho^2})), \Lambda_-) = \ell_{k-1}$. The homology of \mathbb{T}^{ρ^2} decomposes in the invariant and anti-invariant part for the monodromy of the covering:

$$\dim H_{k+1}(C_*(\mathbb{T}^{\rho^2}) \otimes_{\Lambda} \Lambda_-) = \dim H_{k+1}(\mathbb{T}^{\rho^2}; \mathbb{K}) - \dim H_{k+1}(C_*(\mathbb{T}^{\rho^2}) \otimes_{\Lambda} \Lambda_+),$$

where $H_{k+1}(C_*(\mathbb{T}^{\rho^2}) \otimes_{\Lambda} \Lambda_+) = \tilde{C}_k(\mathcal{F})$. Summarizing,

$$\ell_k = \dim H_{k+1}(\mathbb{T}^{\rho^2}; \mathbb{K}) - \dim \tilde{H}_k(\mathcal{F}) - \dim \tilde{C}_k(\mathcal{F}) - \ell_{k-1}. \square$$

Remark 4.3 From the computational point of view $H_{*+1}^-(\mathbb{T}^{\rho^2}; \mathbb{K})$, is the homology of the complex $C_*(\mathcal{F}^{\rho}, \mathbb{K})$ after the evaluation $t = -1$, i.e., tensoring by Λ_- .

4.3 An application to even characters

The above results for $H_1(\mathbb{T}^{\rho}, \mathbb{K})$ for an even character ρ give the following. They depend only on Γ . Let us denote by $V_0 = \mathcal{F}_0^0$ the set of vertices of weight 0. We denote also by $\Gamma_0 = \mathcal{F}_0^1$ (resp. $\Gamma_1 = \mathcal{F}_1^1$) the graph whose set of vertices is V and the set of edges consist of the edges of Γ with weight 0 (resp. ≤ 1).

Proposition 4.4 *The dimension of the $(t+1)$ -primary part of $H_1(\mathbb{T}^{\rho}, \mathbb{K})$ equals*

$$\dim \tilde{H}_0(\Gamma_0) + \dim \tilde{H}_0(\Gamma_0) - 2 \dim \tilde{H}_0(\Gamma) + \dim \tilde{H}_0(\Gamma, V_0) - \dim \tilde{H}_0(V, V_0).$$

The number of non-semisimple factors (exponent 2) is the rank of the map

$$\ker(\tilde{H}_0(V_0) \rightarrow \tilde{H}_0(\Gamma)) \longrightarrow \tilde{H}_0(\Gamma_1).$$

Corollary 4.5 *Assume Γ is connected. Then the dimension of the $(t+1)$ -primary part of the space $H_1(\mathbb{T}^{\rho}, \mathbb{K})$ equals*

$$\dim H_0(\Gamma_0) + \dim H_0(\Gamma_1) - 2 - \omega(V).$$

and the number of non-semisimple blocks is $\operatorname{Rank}(\tilde{H}_0(V_0) \rightarrow \tilde{H}_0(\Gamma_1))$.

5 Main result

Let

$$H_{k+1}(A_\Gamma^\chi; \mathbb{K}) \cong \mathbb{K}[t^{\pm 1}]^{r_k} \bigoplus_{d \in \mathbb{Z}_{>0}} \bigoplus_{j > 0} \left(\mathbb{K}[t^{\pm 1}] / \langle \Phi_d^j \rangle \right)^{r_{k,j}(d)}$$

be the decomposition of this finitely generated $\mathbb{K}[t^{\pm 1}]$ -module. If one assumes certain acyclicity conditions on the flag complex \mathcal{F} , then a more concise description of $H_{k+1}(A_\Gamma^\chi; \mathbb{K})$ can be given. Consider $d > 1$ and the weighted graph obtained from Γ by attaching the weight

$$\omega_d(v) = \begin{cases} 1 & \text{if } d|n_v \\ 0 & \text{otherwise,} \end{cases}$$

as defined in [Section 2](#). This weight defines a filtration in the flag complex \mathcal{F}_j^k as defined in [\(3.1\)](#). Note that the filtration \mathcal{F}_j^k depends on d , we drop d to avoid cumbersome notation.

We denote by $\tilde{h}_i(\mathcal{F}_j^m) := \dim_{\mathbb{K}} \tilde{H}_i(\mathcal{F}_j^m; \mathbb{K})$ and by $\omega(S^k(\mathcal{F}))$ the weight of the set of k -simplices of \mathcal{F} as defined in [Section 3](#).

Theorem 5.1 *Assume \mathcal{F} is k -acyclic, then*

$$H_{k+1}(A_\Gamma^\chi; \mathbb{K}) \cong \left(\frac{\mathbb{K}[t^{\pm 1}]}{\langle (t-1) \rangle} \right)^{\text{Rank } \partial_{k+1}} \bigoplus_{d \in \mathbb{Z}_{>0}} \left(\bigoplus_{j=1}^{k+2} \left(\frac{\mathbb{K}[t^{\pm 1}]}{\langle \Phi_d^j \rangle} \right)^{r_{k,j}(d)} \right),$$

where

$$\begin{aligned} \sum_{j=1}^{k+2} j \cdot r_{k,j}(d) &= \sum_{i=k}^{k+1} \sum_{j=0}^i \tilde{h}_k(\mathcal{F}_j^i) - (k+1)\tilde{h}_k(\mathcal{F}^k), \\ \ell_k(d) &= \sum_{j=1}^{k+2} r_{k,j}(d) = \sum_{i=0}^k (-1)^{k-i} \dim H_{i+1}^-(\mathbb{T}^{\rho_2}; \mathbb{K}) \\ r_{k,k+2}(d) &= \text{Rank} \left(\tilde{H}_k(\mathcal{F}_0^k) \rightarrow \tilde{H}_k(\mathcal{F}_{k+1}^{k+1}) \right). \end{aligned}$$

The maximal h such that $r_{k,h} > 0$ is

$$\max \{ i - j + 1 \mid \text{Rank} \left(\tilde{H}_k(\mathcal{F}_j^k) \rightarrow \tilde{H}_k(\mathcal{F}_{i-1}^{k+1}) \right) > 0 \}.$$

This determines inductively the $\mathbb{K}[t^{\pm 1}]$ -module structure of $H_1(A_\Gamma^\chi; \mathbb{K})$ and $H_2(A_\Gamma^\chi; \mathbb{K})$ completely.

Proof From the acyclicity condition there is no Λ -free part. The first formula follows from [Theorem 3.9](#) since $\tilde{h}_k(\mathcal{F}) = 0$ by hypothesis, $\tilde{h}_k(\mathcal{F}^{k+1}, \mathcal{F}_j^k) = \tilde{h}_{k-1}(\mathcal{F}_j^k)$ using the exact sequence of pairs and the acyclicity of \mathcal{F} , and $\sum_{j=0}^k \dim \tilde{C}_k(\mathcal{F}^k, \mathcal{F}_j^k) = \omega(\mathcal{S}^k(\mathcal{F}))$. The second formula follows [Theorem 4.2](#) and the third equality can be obtained from [Proposition 3.14](#) and the acyclicity of \mathcal{F} . The maximality statement follows [Proposition 3.15](#). \square

6 Examples

In this section we present computations for a variety of examples. An interested reader may apply the results to other examples and check the computations below using the notebook in <https://github.com/enriqueartal/ArtinKernels>, and using SageMath [8] or Binder [3].

Example 6.1 Let Γ be the graph in [Figure 2](#). The labels on the vertices correspond to the values $\rho_{\chi,d}(v)$. The structure of $H_1(A_\Gamma^\chi; \mathbb{C})$ is:

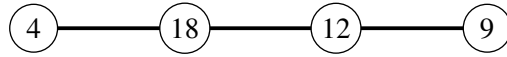


Figure 2: A linear tree

$$H_1(A_\Gamma^\chi; \mathbb{K}) = \left(\frac{\mathbb{K}[t^{\pm 1}]}{\Phi_1} \right)^3 \oplus \left(\frac{\mathbb{K}[t^{\pm 1}]}{\Phi_2 \Phi_3} \right)^2 \oplus \left(\frac{\mathbb{K}[t^{\pm 1}]}{\Phi_6^2 \Phi_4 \Phi_9 \Phi_{12} \Phi_{18}} \right).$$

Let us study each one of the Φ_d -primary factors, for $d = 1, 2, 3, 4, 6, 9, 12, 18$.

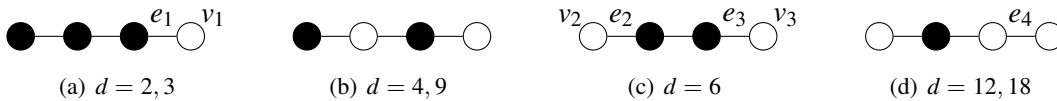


Figure 3: Weighted graphs. Weight-one vertices in black.

- $d = 1$: Recall that $r_{0,1}(1) = \text{Rank } \partial_1$. The map ∂_1 is injective as Γ is a tree; hence $r_{0,1}(1) = 3$.
- $d = 2, 3$: The primary parts for Φ_2 and Φ_3 are obtained considering the associated even character ρ as in [Figure 3\(a\)](#), up to automorphism of the graph. We have:

$$\Gamma_0 = V, \quad \Gamma_1 = V \cup e_1, \quad V_0 = \{v_1\}.$$

The dimension of the $(t+1)$ -primary part for this ρ is $\tilde{h}_0(\Gamma_1) + \tilde{h}_0(\Gamma_0) - \omega(V) = 2 + 3 - 3 = 2$, according to [Corollary 4.5](#). Since V_0 is connected this part is semisimple and hence the primary parts for Φ_2 and Φ_3 are $(\mathbb{K}[t^{\pm 1}]/\Phi_2)^2$ and $(\mathbb{K}[t^{\pm 1}]/\Phi_3)^2$ respectively.

- $d = 4, 9$: The associated even character ρ is described in [Figure 3\(b\)](#). We have:

$$\Gamma_0 = V, \quad \Gamma_1 = \Gamma.$$

The dimension of the $(t+1)$ -primary part for ρ is $4 + 1 - 2 - 2 = 1$. This gives the result for Φ_4 and Φ_9 .

- $d = 6$: The associated even character ρ is described in [Figure 3\(c\)](#). We have:

$$\Gamma_0 = V, \quad \Gamma_1 = e_2 \cup e_3, \quad V_0 = \{v_2, v_3\}.$$

The dimension of the corresponding primary part is $4 + 2 - 2 - 2 = 2$. In this case V_0 consists of the end vertices and the map $\tilde{H}_0(V_0) \rightarrow \tilde{H}_0(\Gamma_1)$ is bijective. Hence the primary part for Φ_6 is $\mathbb{K}[t^{\pm 1}]/\Phi_6^2$.

- $d = 12, 18$: The associated even character ρ is described in [Figure 3\(d\)](#). We have:

$$\Gamma_0 = V \cup e_4, \quad \Gamma_1 = \Gamma.$$

The dimension of the corresponding primary part is $3 + 1 - 2 - 1 = 1$. The computation ends.

Example 6.2 This example shows that the non-vanishing restrictions $\chi(v) = n_v \neq 0$ on the character χ are necessary. Let us consider the same graph in [Figure 2](#) where the values of n_v 's are $(1, 0, 2, 2)$. The matrices for ∂_j^X are

$$j = 1 : \begin{pmatrix} t-1 & 0 & t^2-1 & t^2-1 \end{pmatrix} \quad j = 2 : \begin{pmatrix} 0 & 0 & 0 \\ t-1 & 1-t^2 & 0 \\ 0 & 0 & 1-t^2 \\ 0 & 0 & t^2-1 \end{pmatrix}$$

Hence $H_0(\mathbb{T}^X; \mathbb{K}) = \mathbb{K}[t^{\pm 1}]/\langle t-1 \rangle \cong \mathbb{K}$. If we denote by v_j the vertices, then $\ker \partial_1^X$ is the free module generated by $v_2, v_4 - v_3, (t+1)v_1 - v_3$. Since the image of ∂_2^X is the free module generated by $(t-1)v_2$ and $(t^2-1)(v_4 - v_3)$ we have that $H_1(\mathbb{T}^X; \mathbb{K}) = \mathbb{K}[t^{\pm 1}] \oplus (\mathbb{K}[t^{\pm 1}]/\langle t-1 \rangle)^2 \oplus \mathbb{K}[t^{\pm 1}]/\langle t+1 \rangle$. In particular, it does not satisfy [Theorem 5.1](#) since $\tilde{h}_0(\mathcal{F}) = 0$, but $H_1(\mathbb{T}^X; \mathbb{K})$ is not a torsion module.

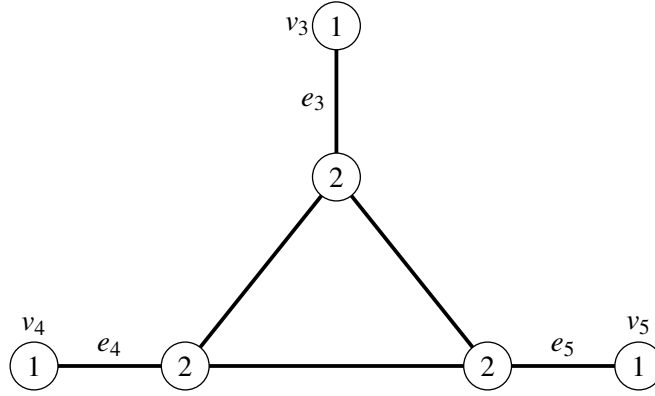


Figure 4: Kite graph

Example 6.3 We consider the graph Γ in [Figure 4](#) and the even character ρ defined by its labels. The flag complex \mathcal{F} is obtained by adding the triangle.

We have

$$\Gamma_0 = V, \quad V_0 = \{v_3, v_4, v_5\}, \quad \Gamma_1 = e_3 \cup e_4 \cup e_5, \quad \mathcal{F}_j^2 = \Gamma, \quad j = 0, 1, 2.$$

We deduce that the free part of the homology vanishes.

Since $\text{Rank } \partial_1 = 5$, the dimension of the $(t-1)$ -primary part for $H_1(A_\Gamma^\rho; \mathbb{K})$ is 5 and it is semisimple. Using [Corollary 4.5](#), the dimension of the $(t+1)$ -primary part for $H_1(A_\Gamma^\rho; \mathbb{K})$ is $6 + 3 - 2 - 3 = 4$. Also, since the map $\tilde{H}_0(V_0) \rightarrow \tilde{H}_0(\Gamma_1)$ is an isomorphism of vector spaces of dimension 2. Hence, the $(t+1)$ -primary part for $H_1(A_\Gamma^\rho; \mathbb{K})$ consists of two blocks with exponent 2.

As for $H_2(A_\Gamma^\rho; \mathbb{K})$, let us consider its $(t-1)$ -primary part whose exponent is given as $\text{Rank } \partial_2 = 1$. Using [Theorem 5.1](#), the dimension of the $(t+1)$ -primary part for $H_2(A_\Gamma^\rho; \mathbb{K})$ can be obtained by

$$r_{1,1} + 2r_{1,2} + 3r_{1,3} = \sum_{j=0}^2 \tilde{h}_1(\mathcal{F}_j^2) + \sum_{j=0}^1 \tilde{h}_1(\Gamma_j) - 2\tilde{h}_1(\Gamma) = 3 - 2 = 1,$$

i.e. $r_{1,1} = 1, r_{1,2} = r_{1,3} = 0$, hence it is semisimple with one block:

$$H_2(\mathbb{T}^\rho; \mathbb{K}) = \mathbb{K}[t^{\pm 1}] \oplus \frac{\mathbb{K}[t^{\pm 1}]}{\langle t-1 \rangle} \oplus \frac{\mathbb{K}[t^{\pm 1}]}{\langle t+1 \rangle}.$$

It is easy to check that $H_3(\mathbb{T}^\rho; \mathbb{K}) = 0$.

Example 6.4 Let us consider the graph Γ of [Figure 5](#) whose labels define a character ρ . The flag complex \mathcal{F} has dimension 2 (its realization is the big triangle in [Figure 5](#)). Note that

$$\Gamma_0 = V, \quad \Gamma_1 = \partial\mathcal{F}, \quad V_0 = \{v_1, v_2, v_3\}, \quad \mathcal{F}_0^2 = \mathcal{F}_1^2 = \Gamma, \quad \mathcal{F}_2^2 = \overline{\mathcal{F} \setminus \Delta}.$$

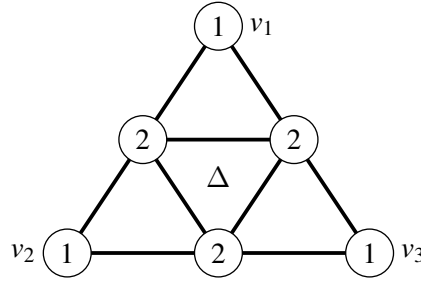


Figure 5

The dimension of the $(t - 1)$ -primary part for $H_1(A_\Gamma^\rho; \mathbb{K})$ is $\text{Rank } \partial_1 = 5$ and it is semisimple. The dimension of the $(t + 1)$ -primary part for $H_1(A_\Gamma^\rho; \mathbb{K})$ is $6 + 1 - 2 - 3 = 2$. The target of $\tilde{H}_0(V_0) \rightarrow \tilde{H}_0(\Gamma_1)$ is zero; hence, the $(t + 1)$ -primary part for $H_1(A_\Gamma^\rho; \mathbb{K})$ consists of two blocks with exponent 1.

The $(t - 1)$ -primary part of $H_2(A_\Gamma^\rho; \mathbb{K})$ is $\text{Rank } \partial_2 = 4$ (and semisimple). The dimension of the $(t + 1)$ -primary part for $H_2(A_\Gamma^\rho; \mathbb{K})$ is given by [Theorem 5.1](#) since

$$\tilde{h}_1(\mathcal{F}_0^2) = \tilde{h}_1(\mathcal{F}_1^2) = 4, \quad \tilde{h}_1(\mathcal{F}_2^2) = 1, \quad \tilde{h}_1(\Gamma_0) = 0, \quad \tilde{h}_1(\Gamma_1) = 1, \quad \tilde{h}_1(\Gamma) = 4.$$

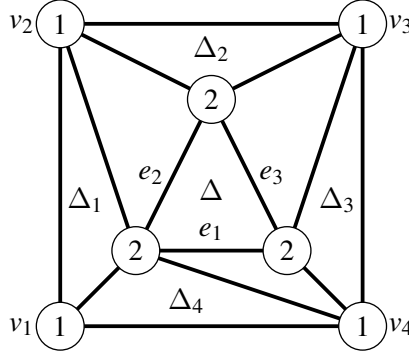
One has $r_{1,1} + 2r_{1,2} + 3r_{1,3} = 2$. Since $H_1(\Gamma_0) = 0$ there is no block with exponent 3 as expected. Moreover, the map $H_1(\Gamma_1) \rightarrow H_1(\mathcal{F}^2)$ is an isomorphism of vector spaces of dimension 1 and we deduce that the $(t + 1)$ -primary part for $H_2(A_\Gamma^\rho; \mathbb{K})$ consists of 1 block of exponent 2. It is easily checked that $H_3(A_\Gamma^\rho; \mathbb{K}) = 0$.

Example 6.5 The graph Γ together with the character ρ defined by the labels as shown in [Figure 6](#) provides an example of a maximal exponent 3 block in $H_2(A_\Gamma^\rho; \mathbb{K})$.

Let us consider the filtered subcomplexes:

$$V_0 = \{v_1, v_2, v_3, v_4\}, \quad \Gamma_0 = V \cup \partial\mathcal{F}, \quad \Gamma_1 = \overline{\Gamma \setminus (e_1 \cup e_2 \cup e_3)}, \quad \mathcal{F}_0^2 = \Gamma, \\ \mathcal{F}_1^2 = \Gamma \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4, \quad \mathcal{F}_2^2 = \overline{\mathcal{F} \setminus \Delta}.$$

Since Γ_1 is connected, $(t + 1)$ -primary part of $H_1(A_\Gamma^\rho; \mathbb{K})$ is semisimple. Using [Corollary 4.5](#), its dimension is $4 + 1 - 2 - 3 = 0$, i.e., this primary part vanishes. Hence $H_1(A_\Gamma^\rho; \mathbb{K}) \cong (\mathbb{K}[t^{\pm 1}]/\langle t - 1 \rangle)^6$.


 Figure 6: The labeled graph Γ

The map $\tilde{H}_1(\Gamma_0) \rightarrow \tilde{H}_1(\mathcal{F}_2^2)$ is an isomorphism between 1-dimensional spaces; by [Theorem 5.1](#) $r_{1,3}(2) = 1$, hence there is one copy $\mathbb{K}[t^{\pm 1}]/\langle (t+1)^3 \rangle$ in the $(t+1)$ -primary part of $H_2(A_\Gamma^\rho; \mathbb{K})$. Since

$$r_{1,1}(2) + 2r_{1,2}(2) + 3r_{1,3}(2) = \sum_{j=0}^2 \tilde{h}_1(\mathcal{F}_j^2) + \sum_{j=0}^1 \tilde{h}_1(\Gamma_j) - 2\tilde{h}_1(\Gamma) = (8 + 4 + 1) + (5 + 1) - 2 \cdot 8 = 3,$$

one obtains

$$H_2(A_\Gamma^\rho; \mathbb{K}) \cong (\mathbb{K}[t^{\pm 1}]/\langle (t-1) \rangle)^8 \oplus \mathbb{K}[t^{\pm 1}]/\langle (t+1)^3 \rangle.$$

Finally, one can check that $H_j^-(\mathbb{T}^{\rho_2}; \mathbb{K})$ has dimension 0, 0, 1, 1 for $j = 0, 1, 2, 3$; as a consequence, $H_3(A_\Gamma^\rho; \mathbb{K}) = 0$.

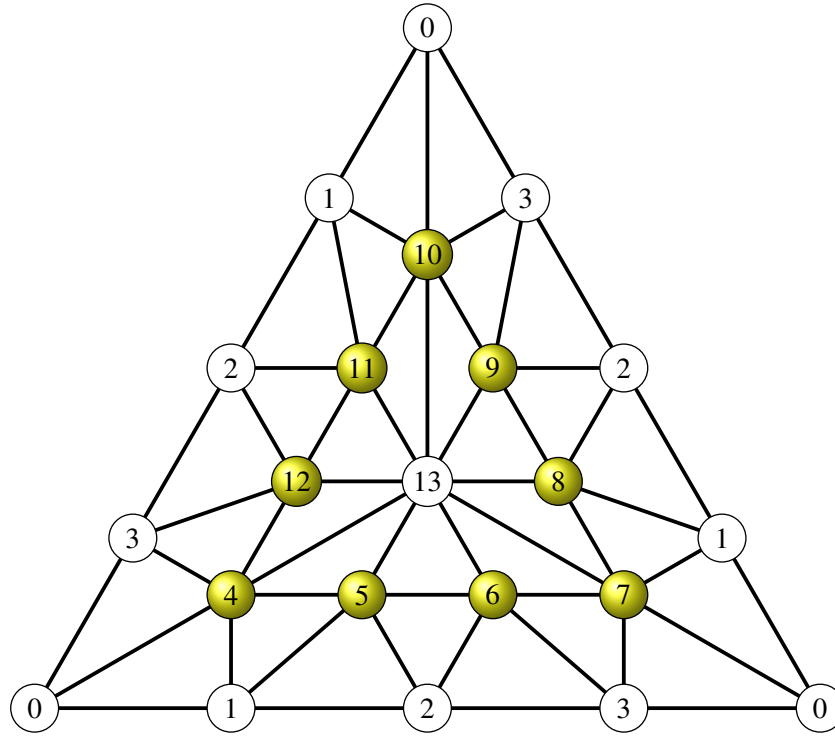
Example 6.6 In this final example we would like to point out how the acyclicity condition in [Theorem 5.1](#) depends on the base field. Note that the flag complex \mathcal{F} of the graph in [Figure 7](#) has the homotopy type of the CW-complex associated with the standard presentation of the cyclic group of 3 elements. Hence it is \mathbb{Q} -acyclic but $H_1(\mathcal{F}; \mathbb{F}_3)$ is of dimension 1.

Let χ be the even character which sends to 2 the vertices from 4 to 12. For this character the homology over \mathbb{C} is as expected

$$H_2(A_\Gamma^\chi; \mathbb{C}) \cong \left(\frac{\Lambda}{\langle t-1 \rangle} \right)^{30} \oplus \left(\frac{\Lambda}{\langle t+1 \rangle} \right)^8 \oplus \left(\frac{\Lambda}{\langle (t+1)^2 \rangle} \right),$$

where $\Lambda = \mathbb{C}[t^{\pm 1}]$. However, the formula changes if the field is of characteristic 3:

$$H_2(A_\Gamma^\chi; \mathbb{F}_3) \cong \Lambda \oplus \left(\frac{\Lambda}{\langle t-1 \rangle} \right)^{29} \oplus \left(\frac{\Lambda}{\langle t+1 \rangle} \right)^8,$$

Figure 7: A graph not 1-acyclic for \mathbb{F}_3 .

where $\Lambda = \mathbb{F}_3[t^{\pm 1}]$.

As mentioned above, computations can be checked using the notebook in <https://github.com/enriqueartal/ArtinKernels>, and using SageMath [8] or Binder [3] (different choices for the character can be selected).

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