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New recurrence relations for several classical families of polynomials

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Abstract

In this paper we derive new recurrence relations for the following families of polynomials: Nörlund polynomials, generalized Bernoulli polynomials, generalized Euler polynomials, Bernoulli polynomials of the second kind, Buchholz polynomials, generalized Bessel polynomials and generalized Apostol-Euler polynomials. The recurrence relations are derived from a differential equation of first order and a Cauchy integral representation obtained from the generating function of these polynomials.

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1 Introduction

Several families of polynomials arise in many different areas of mathematics with important applications in overall branches of science. The study of different properties and representations for these families, such as explicit representations, symmetries, limit relations, recurrence relations and derivatives, integral representations, generating functions, zeros, differential equations, asymptotic expansions, etc., have been subject of investigation during centuries. In this paper, we consider the following families of polynomials: Nörlund polynomials B_m^α, generalized Bernoulli polynomials $B_m^\alpha(x)$, generalized Euler polynomials

$E_m^\alpha(x)$, Bernoulli polynomials of the second kind $b_m^\alpha(x)$, Buchholz polynomials $P_m^a(x)$, generalized Bessel polynomials $Y_m^a(x)$ and generalized Apostol-Euler polynomials $E_m^\alpha(x;\lambda)$. Several (more or less involved) explicit formulas, recurrence relations and properties of the generalized Bernoulli polynomials may be found in [3,4,6,8,19,22,23,28,29,31]; of the generalized Euler polynomials in [6,8,12,29,31]; of the Bernoulli polynomials of the second kind in [9,25,30]; of the Buchholz polynomials in [1,2,7,18]; of the generalized Bessel polynomials in [10, 13, 16, 24]; and of the generalized Apostol-Euler polynomials in [5, 14, 20, 21, 27, 32]. See also references therein for further information.

In this paper, we investigate new recurrence relations for the above mentioned families of polynomials. The starting point is the generating function $f_\alpha(t,x,\lambda)$ of these polynomials $p_m^\alpha(x;\lambda)$, that is of the form

$$f_\alpha(t,x,\lambda) := (g(t,x,\lambda))^\alpha h(t,x) = \sum_{m=0}^{\infty} p_m^\alpha(x;\lambda)t^m, \tag{1}$$

where $g(t,x,\lambda)$ and $h(t,x)$ are given in Table 1 for the different families of polynomials considered in this paper. For the sake of simplicity in the analysis, the polynomials $p_m^\alpha(x;\lambda)$ that we consider here differ by a factor $m!$ from the standard polynomials $\tilde{p}_m^\alpha(x;\lambda)$ considered so far in the literature, that is: $p_m^\alpha(x;\lambda) := \tilde{p}_m^\alpha(x;\lambda)/m!$.

On the one hand, upon taking the logarithmic derivative of $f_\alpha(t,x,\lambda)$ in (1) we get

$$u_\alpha(t,x,\lambda)f'_\alpha(t,x,\lambda) = v_\alpha(t,x,\lambda)f_\alpha(t,x,\lambda), \tag{2}$$

where $u_\alpha(t,x,\lambda)$ and $v_\alpha(t,x,\lambda)$ admit a Taylor expansion at $t=0$. By substituting the Taylor expansion at $t=0$ of $u_\alpha(t,x,\lambda)$, $v_\alpha(t,x,\lambda)$, $f_\alpha(t,x,\lambda)$ and $f'_\alpha(t,x,\lambda)$ in (2), and equating the coefficients of equal powers of t, we obtain a recurrence relation for $p_m^\alpha(x;\lambda)$ in terms of $p_k^\alpha(x;\lambda)$, for $k=0,1,2,\ldots,m-1$ with $p_0^\alpha(x;\lambda)$ given (second column of Table 2).

On the other hand, we consider the Cauchy integral representation of $p_m^\alpha(x;\lambda)$,

$$p_m^\alpha(x;\lambda) = \frac{1}{2\pi i}\oint f_\alpha(t,x,\lambda)\frac{dt}{t^{m+1}}, \tag{3}$$

where the integration contour is a simple closed loop, traversed in the positive sense, that encircles the point $t=0$ and does not encircle any singularity of $f_\alpha(t,x,\lambda)$. A simple algebra lets us write $p_m^\alpha(x;\lambda)$ in terms of $p_k^{\alpha+1}(x;\lambda)$ for $k=0,1,2,\ldots,m$ (except in the case of Buchholz polynomials, that is written in terms of $p_k^{\alpha-1}(x;\lambda)$). Integrating by parts in (3), we obtain a relation between some $p_k^{\alpha+1}(x;\lambda)$ for $k=0,1,2,\ldots,m$, and some $p_k^\alpha(x;\lambda)$ for $k=0,1,2,\ldots,m$. Solving the two equations for $\{p_m^\alpha(x;\lambda), p_m^{\alpha+1}(x;\lambda)\}$ (for $\{p_m^\alpha(x;\lambda), p_m^{\alpha-1}(x;\lambda)\}$ in the case of the Buchholz polynomials), we get a recurrence relation for $p_m^{\alpha+1}(x;\lambda)$ in terms of $p_k^{\alpha+1}(x;\lambda)$ for $k=0,1,2,\ldots,m-1$, and some $p_k^\alpha(x;\lambda)$ for $k=0,1,2,\ldots,m-1$ (third column of Table 2).

Except in the case of the generalized Apostol-Euler, these polynomials $p_m^\alpha(x;\lambda)$ are also polynomials in α of degree m (of degree $\lfloor m/2\rfloor$ for the Buchholz polynomials). Therefore,

Polynomials	$p_m^\alpha(x;\lambda)$	$g(t,x,\lambda)$	$h(t,x)$	t
Nörlund	B_m^α	$\dfrac{t}{e^t-1}$	1	$\|t\|<2\pi$
Generalized Bernoulli	$B_m^\alpha(x)$	$\dfrac{t}{e^t-1}$	e^{xt}	$\|t\|<2\pi$
Generalized Euler	$E_m^\alpha(x)$	$\dfrac{2}{e^t+1}$	e^{xt}	$\|t\|<\pi$
Bernoulli of the second kind	$b_m^\alpha(x)$	$\dfrac{t}{\log(1+t)}$	$(1+t)^x$	$\|t\|<1$
Buchholz	$P_m^a(x)$	$\dfrac{\sin t}{t}$	$e^{\frac{x}{2}(\cot t-\frac{1}{t})}$	$\|t\|<\pi$
Generalized Bessel	$Y_m^a(x)$	$\dfrac{2}{1+\sqrt{1-2xt}}$	$\dfrac{e^{2t/(1+\sqrt{1-2xt})}}{\sqrt{1-2xt}}$	$\|2xt\|<1$
Generalized Apostol-Euler	$E_m^\alpha(x;\lambda)$	$\dfrac{2}{\lambda e^t+1}$	e^{xt}	$\|t\|<\|\log(-\lambda)\|$ $\lambda\neq-1$

Table 1: Functions $g(t,x,\lambda)$ and $h(t,x)$ in equation (1) for the families of polynomials considered in this paper.

they can be written in the form

$$p_m^\alpha(x;\lambda)=\sum_{k=0}^{m}a_k^m(x,\lambda)\alpha^k, \tag{4}$$

as well as in the form

$$p_m^\alpha(x;\lambda)=\sum_{k=0}^{m}b_k^m(\alpha,\lambda)x^k. \tag{5}$$

Then, by replacing the expansions (4) and (5) into the relation obtained by integration by parts that we have mentioned above, we get recurrence relations for the coefficients $a_k^m(x,\lambda)$ (Table 3) and $b_k^m(\alpha,\lambda)$ (Table 4).

The paper is organized as follows. In Section 2, we summarize the recurrence relations derived for the different families of polynomials considered in the paper and based on the techniques explained above, that are based on the differential equation (2) and the integral representation (3). The proofs of the results given in Section 2 are given in Section 3.

2 Main results

In Table 2 we show the recurrence relations satisfied by the different families of polynomials considered in the paper and obtained by applying the method based on the differential equation (2) (second column) and the method based on the Cauchy integral representation (3) of $p_m^\alpha(x;\lambda)$ (third column). Table 3 includes the expansion of $p_m^\alpha(x;\lambda)$ in powers of α (4) and the corresponding recurrence relations for the coeficients $a_k^m(x,\lambda)$, and Table 4 the expansion of $p_m^\alpha(x;\lambda)$ in powers of x (5) and a recurrence relation for the coeficients $b_k^m(\alpha,\lambda)$.

For all these polynomials, $p_0^\alpha(x;\lambda) = 1$, except for the generalized Apostol-Euler polynomials, for which $p_0^\alpha(x;\lambda) = \left(\frac{2}{\lambda+1}\right)^\alpha$. From either of the two recursive formulas given in Table 2, it is obvious that, except in the case of the generalized Apostol-Euler polynomials, $p_m^\alpha(x;\lambda)$ is a polynomial in α of degree m (of degree $\lfloor m/2 \rfloor$ for the Buchholz polynomials).

As far as we know, the recurrence relations given in Tables 2, 3 and 4 are new, with the exception of the one given for the Nörlund polynomials in the second column of Table 2, that is proved in [11, Theorem 2.1]. An alternative recursive formula for the Nörlund polynomials to the one given in the third column of Table 2 may be found in [17], although it is more intricate. Other more or less involved recurrence relations may be found in the literature cited in the Introduction and references therein.

3 Proofs of the results in Section 2

In this section we give a complete and detailed proof of the different recurrence relations obtained for the generalized Euler polynomials in Tables 2, 3 and 4. For the other families of polynomials we provide the more important details of the demonstration and refer to the proof for the generalized Euler polynomials.

3.1 Generalized Euler polynomials

The generalized Euler polynomials, $E_m^\alpha(x)$, are generated by the function [23, chap. 6]

$$G_\alpha(x,t) := \left(\frac{2}{e^t+1}\right)^\alpha e^{xt} = \sum_{m=0}^\infty E_m^\alpha(x) t^m, \quad |t| < \pi. \tag{6}$$

It is clear that $E_0^\alpha(x) = 1$. The recurrence relation given in the fourth line and second column of Table 2 can be proved by introducing the expansion (6) into the partial differential equation

$$(e^t+1)\frac{\partial G_\alpha(x,t)}{\partial t} = \left[e^t(x-\alpha)+x\right] G_\alpha(x,t) \tag{7}$$

and equating the coefficients of equal powers of t.

In order to prove the recurrence relation given in the fourth line and third column of Table 2, consider $\alpha > 1$. From (6) we have that

$$E_m^\alpha(x) = \frac{1}{2\pi i}\oint \left(\frac{2}{e^t+1}\right)^\alpha e^{xt}\frac{dt}{t^{m+1}}, \tag{8}$$

where the integration contour is a closed loop around the point $t = 0$, contained inside the disk $D_0(\pi)$ and traversed once in the positive sense. Then, on the one hand, a simple algebra shows that

$$\begin{aligned} E_m^\alpha(x) &= \frac{1}{2\pi i}\oint\left(\frac{2}{e^t+1}\right)^{\alpha+1} e^{xt}\frac{e^t+1}{2}\frac{dt}{t^{m+1}} \\ &= \frac{1}{2\pi i}\oint\left(\frac{2}{e^t+1}\right)^{\alpha+1} e^{xt}\frac{dt}{t^{m+1}} + \frac{1}{2}\sum_{k=1}^{m}\frac{1}{k!}\frac{1}{2\pi i}\oint\left(\frac{2}{e^t+1}\right)^{\alpha+1} e^{xt}\frac{dt}{t^{m-k+1}} \\ &= E_m^{\alpha+1}(x) + \frac{1}{2}\sum_{k=1}^{m}\frac{E_{m-k}^{\alpha+1}(x)}{k!}. \end{aligned} \tag{9}$$

On the other hand, integrating by parts in (8) we find that, for $m = 1, 2, 3, \dots$,

$$mE_m^\alpha(x) = (x-\alpha)E_{m-1}^\alpha(x) + \frac{\alpha}{2}E_{m-1}^{\alpha+1}(x). \tag{10}$$

Solving the two equations (9) and (10) for $\{E_m^\alpha(x), E_m^{\alpha+1}(x)\}$ we find the recurrence relation given in the fourth line and third column of Table 2.

In order to prove the recurrence relation given in Table 3, replace the right hand side of $E_m(\alpha) = \sum_{k=0}^{m} a_k^m(x)\alpha^k$ into formula (10), by writting

$$E_{m-1}^{\alpha+1}(x) = \sum_{k=0}^{m-1} a_k^{m-1}(x)(\alpha+1)^k = \sum_{k=0}^{m-1} a_k^{m-1}(x)\sum_{n=0}^{k}\binom{n}{k}\alpha^n = \sum_{k=0}^{m-1}\alpha^k\sum_{n=k}^{m-1}\binom{n}{k}a_n^{m-1}(x).$$

We obtain, for $m = 0, 1, 2, \dots$,

$$\begin{aligned} m\,a_0^m(x) - xa_0^{m-1}(x) + \sum_{k=1}^{m}\left[m\,a_k^m(x)\alpha^k + a_{k-1}^{m-1}(x)\right]\alpha^k - x\sum_{k=1}^{m-1} a_k^{m-1}(x)\alpha^k \\ = \frac{1}{2}\sum_{k=1}^{m}\alpha^k\sum_{n=k-1}^{m-1}\binom{n}{k-1}a_n^{m-1}(x). \end{aligned} \tag{11}$$

When we identify the coefficients of every power α^k, $k = 0, 1, 2, \dots, m$, we obtain the recurrence relation given in Table 3.

In order to prove the recurrence relation given in Table 4, replace the right hand side of $E_m(\alpha) = \sum_{k=0}^{m} b_k^m(\alpha)x^k$ into formula (10). We obtain, for $m = 0, 1, 2, \dots$,

$$\begin{aligned} m\,b_0^m(\alpha) + m\sum_{k=1}^{m-1} b_k^m(\alpha)x^k + mb_m^m(\alpha)x^m &= \sum_{k=1}^{m-1} b_{k-1}^{m-1}(\alpha)x^k + b_{m-1}^{m-1}(\alpha)x^m \\ &- \alpha\,b_0^{m-1}(\alpha) - \alpha\sum_{k=1}^{m-1} b_k^{m-1}(\alpha)x^k + \frac{\alpha}{2}\,b_0^{m-1}(\alpha+1) + \frac{\alpha}{2}\sum_{k=1}^{m-1} b_k^{m-1}(\alpha+1)x^k. \end{aligned} \tag{12}$$

When we identify the coefficients of every power x^k, $k = 0, 1, 2, \dots, m$, we obtain the recurrence relation given in Table 4.

$p_m^\alpha(x;\lambda)$	Recurrence relation derived from (2)	Recurrence relation derived from (3)
B_m^α	$B_m^\alpha = \frac{1}{m}\sum_{k=0}^{m-1}\frac{\alpha(k-m)-k}{(m+1-k)!}B_k^\alpha$	$B_m^{\alpha+1} = \frac{\alpha-m}{m}\sum_{k=1}^{m}\frac{B_{m-k}^{\alpha+1}}{(k+1)!} - \frac{\alpha}{m}B_{m-1}^\alpha$
$B_m^\alpha(x)$	$B_m^\alpha(x) = \frac{1}{m}\sum_{k=0}^{m-1}\frac{\alpha(k-m)+x(m+1-k)-k}{(m+1-k)!}B_k^\alpha(x)$	$B_m^{\alpha+1}(x) = \frac{\alpha-m}{m}\sum_{k=1}^{m}\frac{B_{m-k}^{\alpha+1}(x)}{(k+1)!} + \frac{x-\alpha}{m}B_{m-1}^\alpha(x)$
$E_m^\alpha(x)$	$E_m^\alpha(x) = \frac{1}{2m}\Big[\sum_{k=0}^{m-2}\frac{(x-\alpha)(m-k)-k}{(m-k)!}E_k^\alpha(x)$ $+(2x-\alpha-m+1)E_{m-1}^\alpha(x)\Big]$	$E_m^{\alpha+1}(x) = -\frac{1}{2}\sum_{k=2}^{m}\frac{E_{m-k}^{\alpha+1}(x)}{k!} + \frac{\alpha-m}{2m}E_{m-1}^{\alpha+1}(x)$ $+\frac{x-\alpha}{m}E_{m-1}^\alpha(x)$
$b_m^\alpha(x)$	$b_m^\alpha(x) = \frac{1}{m}\sum_{k=0}^{m-1}(-1)^{m+1-k}\frac{\alpha-k+x(m+1-k)}{(m+1-k)(m-k)}b_k^\alpha(x)$	$b_m^{\alpha+1}(x) = -\frac{1}{m}\sum_{k=1}^{m}\frac{\alpha k+m}{k+1}b_{m-k}^{\alpha+1}(x) + \frac{x}{m}b_{m-1}^\alpha(x-1)$
$P_m^a(x)$	$P_{2m-1}^a(x) = \frac{1}{2m-1}\Big(\frac{x}{2}\sum_{k=0}^{m-1}(-1)^{m-k}\frac{4^{m-k}(2(m-k)-1)\tilde{B}_{2(m-k)}}{(2(m-k))!}P_{2k}^a(x)$ $+ a\sum_{k=0}^{m-2}\frac{(-1)^{m-k-1}2^{2(m-k-1)-1}\left(\tilde{B}_{2(m-k-1)}+\tilde{B}_{2(m-k-1)}(1)\right)}{(2(m-k-1))!}P_{2k+1}^a(x)\Big)$, $P_{2m}^a(x) = \frac{1}{2m}\Big(\frac{x}{2}\sum_{k=0}^{m-1}(-1)^{m-k}\frac{4^{m-k}(2(m-k)-1)\tilde{B}_{2(m-k)}}{(2(m-k))!}P_{2k+1}^a(x)$ $+ a\sum_{k=0}^{m-1}\frac{(-1)^{m-k}2^{2(m-k)-1}\left(\tilde{B}_{2(m-k)}+\tilde{B}_{2(m-k)}(1)\right)}{(2(m-k))!}P_{2k}^a(x)\Big)$	$P_m^a(x) = \frac{1}{m}\Bigg(\frac{x}{2}\sum_{k=1}^{\lfloor\frac{m}{2}\rfloor+\frac{1-(-1)^m}{2}}(-1)^k\frac{4^k(2k-1)\tilde{B}_{2k}}{(2k)!}P_{m-2k+1}^a(x)$ $+ 2a\sum_{k=1}^{\lfloor\frac{m}{2}\rfloor}\frac{(-1)^k k}{(2k+1)!}P_{m-2k}^{a-1}(x)\Bigg)$
$Y_m^a(x)$	$Y_m^a(x) = \frac{1}{m}\sum_{k=0}^{m-1}(2x)^{m-k-1}\Big(x+(-1)^{m-k-1}\binom{-\frac{1}{2}}{m-k-1}$ $+(-1)^{m-k}ax\binom{-\frac{1}{2}}{m-k}\Big)Y_k^a(x)$	$Y_m^{a+1}(x) = \sum_{k=0}^{m-1}\frac{(2x)^k}{m}\left(x+(-1)^k\binom{-\frac{1}{2}}{k}\right)Y_{m-k-1}^a(x)$ $+\sum_{k=0}^{m-1}(-1)^k(2x)^k\left(\frac{ax}{2m}\binom{-\frac{1}{2}}{k}+2x\binom{\frac{1}{2}}{k+1}\right)Y_{m-k-1}^{a+1}(x)$
$E_m^\alpha(x;\lambda)$	$E_m^\alpha(x;\lambda) = \frac{1}{(\lambda+1)m}\Big[\lambda\sum_{k=0}^{m-2}\frac{(x-\alpha)(m-k)-k}{(m-k)!}E_k^\alpha(x;\lambda)$ $+(\lambda(x-\alpha-m+1)+x)E_{m-1}^\alpha(x;\lambda)\Big]$	$E_m^{\alpha+1}(x;\lambda) = \frac{2}{\lambda+1}\Big[-\frac{\lambda}{2}\sum_{k=2}^{m}\frac{E_{m-k}^{\alpha+1}(x;\lambda)}{k!}$ $+\frac{\alpha-\lambda m}{2m}E_{m-1}^{\alpha+1}(x;\lambda) + \frac{x-\alpha}{m}E_{m-1}^\alpha(x;\lambda)\Big]$

Table 2: Recurrence relations satisfied by $p_m^\alpha(x;\lambda)$ for $m = 1, 2, 3, \ldots$. In the formula given in the third column for Buccholz polynomials, $P_m^a(x)$, $\tilde{B}_n$ is the standard Bernoulli number and $\tilde{B}_n(x)$ is the standard Bernoulli polynomial of degree n.

$p_m^\alpha(x;\lambda)$	Recurrence relation $p_m^\alpha(x;\lambda)=\sum_{k=0}^m a_k^m(x,\lambda)\alpha^k$
B_m^α	$a_0^0:=1;\quad a_0^m:=0,\quad m=1,2,3,\ldots;\quad a_m^m:=\frac{(-1)^m}{2^m m!}$, and for $k=m-1,m-2,m-3,\ldots,1$, and $m=2,3,4,\ldots$ $a_k^m=-\dfrac{1}{m+k}\left[a_{k-1}^{m-1}+\sum_{n=k+1}^{m}\binom{n}{k-1}a_n^m\right].$
$B_m^\alpha(x)$	$a_0^0(x):=1;\quad a_0^m(x):=\frac{x^m}{m!},\quad m=1,2,3,\ldots;\quad a_m^m(x):=\frac{(-1)^m}{2^m m!}$, and for $k=m-1,m-2,m-3,\ldots,1$, and $m=2,3,4,\ldots$ $a_k^m(x)=-\dfrac{1}{m+k}\left[a_{k-1}^{m-1}(x)-xa_k^{m-1}(x)+\sum_{n=k+1}^{m}\binom{n}{k-1}a_n^m(x)\right]$
$E_m^\alpha(x)$	$a_0^0(x):=1;\quad a_0^m(x):=\frac{x^m}{m!},\quad m=1,2,3,\ldots;\quad a_m^m(x):=\frac{(-1)^m}{2^m m!}$, and for $k=m-1,m-2,m-3,\ldots,1$, and $m=2,3,4,\ldots$ $a_k^m(x)=\dfrac{1}{m}\left[-\dfrac{1}{2}a_{k-1}^{m-1}(x)+\left(x+\dfrac{k}{2}\right)a_k^{m-1}(x)+\dfrac{1}{2}\sum_{n=k+1}^{m-1}\binom{n}{k-1}a_n^{m-1}(x)\right]$

Table 3: Expansion of $p_m^\alpha(x;\lambda)$ in powers of α (4) and the corresponding recurrence relations for the coefficients $a_k^m(x,\lambda)$.

3.2 Generalized Bernoulli polynomials

The generalized Bernoulli polynomials, $B_m^\alpha(x)$, are generated by the function [23, chap. 6]

$$F_\alpha(x,t):=\left(\frac{t}{e^t-1}\right)^\alpha e^{xt}=\sum_{m=0}^\infty B_m^\alpha(x)t^m,\quad |t|<2\pi. \tag{13}$$

It is clear that $B_0^\alpha(x)=1$. The recurrence relations given in Tables 2 and 3 may be proved as in subsection 3.1 replacing, respectively, formulas (7), (9), (10) and (11) by

$$t(e^t-1)\frac{\partial F_\alpha(x,t)}{\partial t}=\big(\alpha[(1-t)e^t-1]+xt(e^t-1)\big)\,F_\alpha(x,t),$$

$$\begin{aligned} B_m^\alpha(x)&=\frac{1}{2\pi i}\oint\left(\frac{t}{e^t-1}\right)^{\alpha+1}e^{xt}(e^t-1)\frac{dt}{t^{m+2}}\\ &=\sum_{k=1}^{m+1}\frac{1}{k!}\frac{1}{2\pi i}\oint\left(\frac{t}{e^t-1}\right)^{\alpha+1}e^{xt}\frac{dt}{t^{m+2-k}}=\sum_{k=0}^{m}\frac{B_{m-k}^{\alpha+1}(x)}{(k+1)!}, \end{aligned}$$

$p_m^\alpha(x;\lambda)$	Recurrence relation $p_m^\alpha(x;\lambda)=\sum_{k=0}^m b_k^m(\alpha,\lambda)x^k$
$E_m^\alpha(x)$	$b_0^0(\alpha):=1;\ b_0^m(\alpha):=\frac{\alpha}{2m}[b_0^{m-1}(\alpha+1)-2b_0^{m-1}(\alpha)],\ m=1,2,3,\ldots;\ b_m^m(\alpha):=\frac{1}{m!}$, and for $k=m-1,m-2,m-3,\ldots,1$, and $m=2,3,4,\ldots$ $b_k^m(\alpha)=\dfrac{1}{m}\left[b_{k-1}^{m-1}(\alpha)-\alpha b_k^{m-1}(\alpha)+\dfrac{\alpha}{2}b_k^{m-1}(\alpha+1)\right]$
$E_m^\alpha(x;\lambda)$	$b_0^0(\alpha,\lambda):=\left(\frac{2}{\lambda+1}\right)^\alpha;\ b_0^m(\alpha,\lambda):=\frac{\alpha}{2m}[b_0^{m-1}(\alpha+1,\lambda)-2b_0^{m-1}(\alpha,\lambda)],\ m=1,2,3,\ldots$ $b_m^m(\alpha,\lambda):=\frac{1}{m!}\left(\frac{2}{\lambda+1}\right)^\alpha$, and for $k=m-1,m-2,m-3,\ldots,1,0$, and $m=2,3,4,\ldots$ $b_k^m(\alpha,\lambda)=\dfrac{1}{m}\left[b_{k-1}^{m-1}(\alpha,\lambda)-\alpha b_k^{m-1}(\alpha,\lambda)+\dfrac{\alpha}{2}b_k^{m-1}(\alpha+1,\lambda)\right]$

Table 4: Expansion of $p_m^\alpha(x;\lambda)$ in powers of x (5) and the corresponding recurrence relations for the coefficients $b_k^m(\alpha,\lambda)$.

$$mB_m^\alpha(x)=\alpha\left[B_m^\alpha(x)-B_{m-1}^\alpha(x)-B_m^{\alpha+1}(x)\right]+xB_{m-1}^\alpha(x),$$

and

$$\begin{aligned}m\,a_0^m(x)-xa_0^{m-1}(x)+\sum_{k=1}^{m}\left[m\,a_k^m(x)\alpha^k+a_{k-1}^{m-1}(x)\right]\alpha^k-x\sum_{k=1}^{m-1}a_k^{m-1}(x)\alpha^k\\=\sum_{k=1}^{m+1}\left[a_{k-1}^m(x)-\sum_{n=k-1}^{m}\binom{n}{k-1}a_n^m(x)\right]\alpha^k.\end{aligned}$$

3.3 Nörlund polynomials

Nörlund polynomials in the variable α, B_m^α, are a particular case of the generalized Bernoulli polynomials $B_m^\alpha(x)$ considered in subsection 3.2 with $x=0$.

3.4 Bernoulli polynomials of the second kind of order α

The Bernoulli polynomials of the second kind of order α, $b_m^\alpha(x)$, are generated by the function [9]

$$H_\alpha(x,t):=\left(\frac{t}{\log(1+t)}\right)^\alpha(1+t)^x=\sum_{m=0}^\infty b_m^\alpha(x)t^m,\quad |t|<1. \tag{14}$$

It is clear that $b_0^\alpha(x)=1$. The recurrence relations given in Table 2 may be proved as in subsection 3.1 replacing, respectively, formulas (7), (9) and (10) by

$$t(t+1)\log(1+t)\frac{\partial H_\alpha(x,t)}{\partial t} = [\alpha(1+t)\log(1+t) - \alpha t + xt\log(1+t)]\, H_\alpha(x,t),$$

$$\begin{aligned} b_m^\alpha(x) &= \frac{1}{2\pi i}\oint \left(\frac{t}{\log(1+t)}\right)^{\alpha+1} \frac{(1+t)^x}{\log(1+t)}\frac{dt}{t^{m+2}} \\ &= \sum_{k=1}^{m+1}\frac{(-1)^{k+1}}{k}\frac{1}{2\pi i}\oint \left(\frac{t}{\log(1+t)}\right)^{\alpha+1}\frac{(1+t)^x}{t^{m+2-k}}dt = \sum_{k=0}^{m}\frac{(-1)^k}{k+1}b_{m-k}^{\alpha+1}(x), \end{aligned}$$

and

$$mb_m^\alpha(x) = \alpha b_m^\alpha(x) - \alpha\sum_{k=0}^{m}(-1)^k b_{m-k}^{\alpha+1}(x) + xb_{m-1}^\alpha(x-1). \tag{15}$$

3.5 Buchholz polynomials

Buchholz polynomials, $P_m^a(x)$, are generated by the function [7, Sec. 3]

$$H(x,a,t) := e^{\frac{x}{2}(\cot t - \frac{1}{t})}\left(\frac{\sin t}{t}\right)^a = \sum_{m=0}^{\infty} P_m^a(x)t^m, \quad |t| < \pi. \tag{16}$$

It is clear that $P_0^a(x) = 1$. The recurrence relation given in the second column of Table 2 may be proved as in subsection 3.1 replacing formula (7) by

$$t^2\frac{\partial H(x,a,t)}{\partial t} = \left[\frac{x}{2}(1 - t^2\csc^2 t) + at(t\cot t - 1)\right] H(x,a,t).$$

In order to prove the recurrence relation given in the third column of Table 2, consider $a > 1$, and integrate by parts in

$$P_m^a(x)(x) = \frac{1}{2\pi i}\oint e^{\frac{x}{2}(\cot t - \frac{1}{t})}\left(\frac{\sin t}{t}\right)^a \frac{dt}{t^{m+1}},$$

where the integration contour is a closed loop around the point $t = 0$, contained inside the disk $D_0(\pi)$ and traversed once in the positive sense.

3.6 Generalized Bessel polynomials

The generalized Bessel polynomials, $Y_m^a(x)$, were first introduced in [16] and are generated by the function [13]

$$H(x,a,t) := \left(\frac{2}{1+\sqrt{1-2xt}}\right)^a \frac{e^{2t/(1+\sqrt{1-2xt})}}{\sqrt{1-2xt}} = \sum_{m=0}^{\infty} Y_m^a(x)t^m. \tag{17}$$

It is clear that $Y_0^a(x)=1$. The recurrence relations given in Table 2 may be proved as in subsection 3.1 replacing, respectively, formulas (7), (9) and (10) by

$$\frac{\partial H(x,a,t)}{\partial t}=\left(\frac{x}{1-2xt}+\frac{ax}{\sqrt{1-2xt}(1+\sqrt{1-2xt})}+\frac{1}{\sqrt{1-2xt}}\right)H(x,a,t),$$

$$\begin{aligned}Y_m^a(x)&=\frac{1}{2\pi i}\oint\left(\frac{2}{1+\sqrt{1-2xt}}\right)^{a+1}\frac{e^{2t/(1+\sqrt{1-2xt})}}{\sqrt{1-2xt}}\frac{1+\sqrt{1-2xt}}{2}\frac{1}{t^{m+1}}dt\\&=Y_m^{a+1}(x)+\frac{1}{2}\sum_{k=1}^{m}(-1)^k(2x)^k\binom{\frac{1}{2}}{k}Y_{m-k}^{a+1}(x),\end{aligned}$$

and

$$\begin{aligned}mY_m^a(x)=&\frac{x}{2\pi i}\oint\left(\frac{2}{1+\sqrt{1-2xt}}\right)^{a}\frac{e^{2t/(1+\sqrt{1-2xt})}}{\sqrt{1-2xt}}\frac{1}{1-2xt}\frac{1}{t^m}dt\\&+\frac{1}{2\pi i}\oint\left(\frac{2}{1+\sqrt{1-2xt}}\right)^{a}\frac{e^{2t/(1+\sqrt{1-2xt})}}{\sqrt{1-2xt}}\frac{1}{\sqrt{1-2xt}}\frac{1}{t^m}dt\\&+\frac{ax}{2}\frac{1}{2\pi i}\oint\left(\frac{2}{1+\sqrt{1-2xt}}\right)^{a+1}\frac{e^{2t/(1+\sqrt{1-2xt})}}{\sqrt{1-2xt}}\frac{1}{\sqrt{1-2xt}}\frac{1}{t^m}dt\\=&x\sum_{k=0}^{m-1}(2x)^kY_{m-k-1}^a(x)+\sum_{k=0}^{m-1}(-1)^k(2x)^k\binom{-\frac{1}{2}}{k}Y_{m-k-1}^a(x)\\&+\frac{ax}{2}\sum_{k=0}^{m-1}(-1)^k(2x)^k\binom{-\frac{1}{2}}{k}Y_{m-k-1}^{a+1}(x).\end{aligned}$$

3.7 Generalized Apostol-Euler polynomials

The generalized Apostol-Euler polynomials, $E_m^\alpha(x;\lambda)$, are generated by the function [21, eq. 11]

$$G_\alpha(x,t,\lambda):=\left(\frac{2}{\lambda e^t+1}\right)^\alpha e^{xt}=\sum_{m=0}^{\infty}E_m^\alpha(x;\lambda)t^m,\quad |t|<|\log(-\lambda)|,\ \lambda\neq-1. \tag{18}$$

It is clear that $E_0^\alpha(x;\lambda)=\left(\frac{2}{\lambda+1}\right)^\alpha$. The recurrence relations given in Tables 2 and 4 may be proved as in subsection 3.1 replacing, respectively, formulas (7), (9), (10) and (12) by

$$(\lambda e^t+1)\frac{\partial G_\alpha(x,t,\lambda)}{\partial t}=\left(\lambda e^t(x-\alpha)+x\right)G_\alpha(x,t,\lambda),$$

$$E_m^\alpha(x;\lambda) = \frac{1}{2\pi i}\oint \left(\frac{2}{\lambda e^t+1}\right)^{\alpha+1} e^{xt}\frac{\lambda e^t+1}{2}\frac{dt}{t^{m+1}}$$
$$= \frac{\lambda+1}{2}\frac{1}{2\pi i}\oint \left(\frac{2}{\lambda e^t+1}\right)^{\alpha+1} e^{xt}\frac{dt}{t^{m+1}} + \frac{\lambda}{2}\sum_{k=1}^{m}\frac{1}{k!}\frac{1}{2\pi i}\oint \left(\frac{2}{\lambda e^t+1}\right)^{\alpha+1} e^{xt}\frac{dt}{t^{m-k+1}}$$
$$= \frac{\lambda+1}{2}E_m^{\alpha+1}(x;\lambda) + \frac{\lambda}{2}\sum_{k=1}^{m}\frac{E_{m-k}^{\alpha+1}(x;\lambda)}{k!},$$
$$mE_m^\alpha(x;\lambda) = (x-\alpha)E_{m-1}^\alpha(x;\lambda) + \frac{\alpha}{2}E_{m-1}^{\alpha+1}(x;\lambda),$$

and

$$m\, b_0^m(\alpha,\lambda) + m\sum_{k=1}^{m-1} b_k^m(\alpha,\lambda)x^k + mb_m^m(\alpha,\lambda)x^m = \sum_{k=1}^{m-1} b_{k-1}^{m-1}(\alpha,\lambda)x^k + b_{m-1}^{m-1}(\alpha,\lambda)x^m$$
$$-\alpha\, b_0^{m-1}(\alpha,\lambda) - \alpha\sum_{k=1}^{m-1} b_k^{m-1}(\alpha,\lambda)x^k + \frac{\alpha}{2}\, b_0^{m-1}(\alpha+1,\lambda) + \frac{\alpha}{2}\sum_{k=1}^{m-1} b_k^{m-1}(\alpha+1,\lambda)x^k.$$

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