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ORIGINAL PAPER

Spherical Bessel functions and critical lengths

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Abstract
The critical length of a space of functions can be described as the supremum of the length of the intervals where Hermite interpolation problems are unisolvent for any choice of nodes. We analyze the critical length for spaces containing products of algebraic polynomials and trigonometric functions. We show the relation of these spaces with spherical Bessel functions and bound above their critical length by the first positive zero of a Bessel function of the first kind.

Keywords Critical length · Bessel functions · Shape preserving representations

Mathematics Subject Classification 41A05 · 41A10 · 42A10 · 65D05 · 65D17

1 Introduction

Interpolation methods in spaces containing oscillating functions, such as the trigonometric functions, can fail in large intervals. An extended Chebyshev space on an interval is a finite dimensional space such that the Hermite interpolation problem has a unique solution for any choice of nodes. This property is equivalent to the fact that the number of zeros (counting multiplicities) of any nonzero function of the space is less than or equal to the dimension of the space. The *critical length* is the supremum of the lengths of the intervals where Hermite interpolation problems are unisolvent for any choice of nodes (see [6]). The critical length is also relevant in the construction of Bernstein-like operators [2, 3] and in Computer-Aided Geometric Design [6–8].

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Spaces containing oscillating functions can be used for design purposes since they represent exactly some curves and solutions of differential equations related with the physical nature of some problems. In particular, the spaces of solutions of differential equations with constant coefficients can be used to model versatile shapes of curves. These spaces are invariant under translations, which allows us to represent the same curve in different parameter intervals of the same length. In order to obtain shape preserving representations in spaces invariant under translations, the length of the parameter domain must be less than a given value, called the critical length for design purposes. In [6], is is shown that the critical length for design purposes is the critical length of the space of the derivatives.

The critical length of an $(n+1)$-dimensional extended Chebyshev space containing the trigonometric functions $\cos x$, $\sin x$ must be not greater than $(n+1)\pi$. The cycloidal spaces C_n, generated by algebraic polynomials of degree less than or equal to $n-2$ and the trigonometric functions $\cos x$, $\sin x$, have been analyzed by many authors [2, 3, 6, 10, 12, 14]. In [9], it was shown that the critical lengths of cycloidal spaces are related to zeros of Bessel functions of the first kind, given by

$$J_\nu(t) := \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j+\nu+1)} \left(\frac{t}{2}\right)^{2j+\nu}. \tag{1}$$

Some spaces invariant under translations can be described as the space generated by products of functions in other spaces. In particular, the space $P_n \odot C_1$ is the space generated by the functions $x^k \cos x$, $x^k \sin x$, $k = 0, \dots, n$. These spaces are the set of solutions of differential equations with constant coefficients. In this paper, we show that the fundamental solutions of these differential equations can be expressed in terms of Bessel functions. A description of a canonical basis in terms of Bessel functions is obtained. We prove that the critical length of the space $P_n \odot C_1$ is bounded above by $j_{n+1/2,1}$, the first positive zero of the Bessel function $J_{n+1/2}$.

The paper is organized as follows. Section 2 describes the space $P_n \odot C_1$, its fundamental solution and a canonical basis for this space. We focus on the relation of the basis with spherical Bessel functions. In Sect. 3 we discuss the critical lengths of $P_n \odot C_1$ and obtain an upper bound. In Sect. 4, we derive some formulae for simplifying the computation of some wronskians arising in the determination of the critical length. We use them to deduce that the critical length of the spaces $P_n \odot C_1$ is $j_{n+1/2,1}$ for $n = 0, 1$. We have also evaluated several wronskians to confirm numerically that $\ell(P_n \odot C_1) = j_{n+1/2,1}$ for $n = 2, 3$. In Sect. 5, some applications to Computer Aided Design are discussed.

2 Fundamental solutions of the differential equation and Bessel functions

Let us recall that the wronskian matrix of a system of functions $(b_0, \dots, b_n)$ at x is

$$W(b_0, \dots, b_n) = (b_j^{(i)}(x))_{i,j=0,\dots,n}.$$

If a space of functions is the set of solutions of a linear differential equation with constant coefficients, then the wronskian matrix of any basis is nonsingular at any point. A *canonical basis* at the origin is any basis $(b_0, \dots, b_{2n+1})$ such that

$$b_i^{(j)}(0) = 0, \quad j < i, \qquad b_i^{(i)}(0) \neq 0. \tag{2}$$

The last element of a canonical basis is a multiple of the fundamental solution of the linear differential equation. The fundamental solution can be defined as the unique element ϕ_n in the space satisfying

$$\phi_n(0) = \phi_n'(0) = \cdots = \phi_n^{(2n)}(0) = 0, \quad \phi_n^{(2n+1)}(0) = 1.$$

Canonical bases play a relevant role in the field of total positivity [11] and can be used to compute the critical length of an extended Chebyshev space.

Let us denote by $Df = f'$, the derivative operator. The $2(n+1)$-dimensional space of solutions of the differential equation

$$(D^2 + I)^{n+1} y = 0, \tag{3}$$

is generated by the functions

$$\cos x, \sin x, x\cos x, x\sin x, \ldots, x^n \cos x, x^n \sin x \tag{4}$$

and can be described as the space $P_n \odot C_1$, generated by the set of products of a function of the space P_n of polynomials of degree less than or equal to n and a trigonometric function of the space $C_1 = \langle \cos x, \sin x \rangle$.

Spherical Bessel functions can be defined by the Rayleigh's formula (formula 10.1.25 of [1])

$$j_n(x) := (-1)^n x^n \left(\frac{1}{x} D\right)^n \left(\frac{\sin x}{x}\right). \tag{5}$$

In particular, we have

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x - x\cos x}{x^2}, \quad j_2(x) = \frac{(3 - x^2)\sin x - 3x\cos x}{x^3},$$

and the following recurrence relation holds

$$j_{n+1}(x) = -x^n \left(\frac{j_n(x)}{x^n}\right)'. \tag{6}$$

The spherical Bessel function j_n satisfies the second order self-adjoint differential equation

$$(x^2 j_n'(x))' + (x^2 - n(n+1)) j_n(x) = 0$$

and can be related with the Bessel function of the first kind (1) with index $\nu = n + 1/2$ by the following formula (see 10.1.1 of [1])

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x).$$

Let $j_{\nu,k}$ denote the k-th positive zero of the Bessel function J_ν. Then $j_{n+1/2,1}$ is the first positive zero of the spherical Bessel function j_n.

For any nonnegative integer n, we define

$$f_n(x) := x^{n+1} j_n(x), \quad x \in \mathbb{R}, \tag{7}$$

and we have that

$$f_0(x) = \sin x, \quad f_1(x) = \sin x - x\cos x, \quad f_2(x) = (3 - x^2)\sin x - 3x\cos x.$$

Now, we derive some properties of f_n to show that the fundamental solution of the differential equation (3) can be expressed in terms of spherical Bessel functions. We shall use the double factorial notation

$$k!! := \prod_{j=0}^{\lfloor (k-1)/2 \rfloor} (k-2j),$$

where $\lfloor (k-1)/2 \rfloor$ is the greatest integer less than or equal to $(k-1)/2$.

Proposition 1 *The following properties hold for the functions f_n defined in (7):*

(a) $f_{n+1}(x) = (2n+1) f_n(x) - x f_n'(x)$.
(b) $f_n(x) = c_n(x)\cos x + s_n(x)\sin x$, *where c_n is an odd polynomial in P_n and s_n is an even polynomial in P_n. Hence f_n is an odd function in $P_n \odot C_1$.*
(c) $f_n'(x) = x f_{n-1}(x)$ *and* $f_n(x) = \int_0^x t f_{n-1}(t)dt$.
(d) $f_{n+1}(x) = (2n+1) f_n(x) - x^2 f_{n-1}(x)$.
(e) *f_n is a solution of the second order differential equation*

$$x(f_n(x) + f_n''(x)) = 2n f_n'(x).$$

(f) $f_n(x) + f_n''(x) = 2n f_{n-1}(x)$ *and hence f_n satisfies* $(D^2 + I)^{n+1} f_n = 0$.
(g) *f_n has a zero of multiplicity $2n+1$ at the origin. Moreover,*

$$\lim_{x \to 0} \frac{f_n(x)}{x^{2n+1}} = \frac{1}{(2n+1)!!}, \quad f_n^{(2n+1)}(0) = (2n)!! \tag{8}$$

and so, the fundamental solution of the equation $(D^2 + I)^{n+1} y = 0$ is

$$\phi_n(x) = \frac{f_n(x)}{(2n)!!}.$$

Proof (a) From (6), it follows that

$$f_{n+1}(x) = -x^{2n+2}(f_n(x)/x^{2n+1})' = (2n+1) f_n(x) - x f_n'(x).$$

(b) It follows by induction. The result is clear for $n = 0$, since $f_0(x) = \sin x$. Let us assume that (b) holds for n. Then we have by (a) that

$$\begin{aligned} f_{n+1}(x) &= (2n+1)(c_n(x)\cos x + s_n(x)\sin x) - x((c_n'(x) + s_n(x))\cos x \\ &\quad + (s_n'(x) - c_n(x))\sin x) \\ &= ((2n+1)c_n(x) - x c_n'(x) - x s_n(x))\cos x + ((2n+1)s_n(x) \\ &\quad - x s_n'(x) + x c_n(x))\sin x. \end{aligned}$$

Since $c_n, s_n \in P_n$ are respectively odd and even polynomials, we deduce that

$$\begin{aligned} c_{n+1}(x) &:= (2n+1)c_n(x) - x c_n'(x) - x s_n(x), \qquad s_{n+1}(x) \\ &:= (2n+1)s_n(x) - x s_n'(x) + x c_n(x) \end{aligned}$$

are odd and even polynomials in P_{n+1}, respectively.

(c) It follows by induction. Clearly $f_1'(x) = x \sin x = x f_0(x)$. Assuming that (c) holds for $n \geq 1$, we have from (a) that

$$f_{n+1}(x) = (2n+1) f_n(x) - x^2 f_{n-1}(x).$$

Differentiating and applying the induction hypothesis, we use (a) to deduce that

$$f'_{n+1}(x) = (2n+1)f'_n(x) - 2xf_{n-1}(x) - x^2 f'_{n-1}(x)$$
$$= x((2n-1)f_{n-1}(x) - xf'_{n-1}(x)) = xf_n(x)$$

and, since $f_n(0) = 0$, we also deduce that $f_n(x) = \int_0^x tf_{n-1}(t)dt$.

(d) follows directly from (a) and (c).

(e) Differentiating (a) and using (c), we derive

$$xf_n(x) = f'_{n+1}(x) = 2nf'_n(x) - xf''_n(x).$$

So, (e) follows.

(f) The first statement readily follows from (e) and (c). The second statement follows by induction on n. Clearly $f_0(x) = \sin(x)$ satisfies $(D^2 + I)f_0 = 0$. Assuming that f_{n-1} satisfies $(D^2 + I)^n f_{n-1} = 0$, it follows that

$$(D^2 + I)^{n+1} f_n = (D^2 + I)^n (f''_n + f_n) = 2n(D^2 + I)^n f_{n-1} = 0.$$

(g) It can be derived easily from (c) by induction. Clearly, $f_0(x) = \sin x = x + O(x^3)$. Assuming that

$$f_{n-1}(x) = \frac{x^{2n-1}}{(2n-1)!!} + O(x^{2n+1}),$$

we deduce that

$$f_n(x) = \int_0^x tf_{n-1}(t)dt = \frac{x^{2n+1}}{(2n+1)(2n-1)!!} + O(x^{2n+3}) = \frac{x^{2n+1}}{(2n+1)!!} + O(x^{2n+3}).$$

□

A canonical basis of $P_n \odot C_1$ at the origin can be obtained from the fundamental solution $\phi_n = f_n/(2n)!!$ by succesive differentiation $(\phi_n^{(2n+1)}, \phi_n^{(2n)}, \ldots, \phi'_n, \phi_n)$. From Proposition 1 (g), the next result follows.

Proposition 2 *A canonical basis of $P_n \odot C_1$ at the origin is given by the functions $f'_0, f_0, f'_1, f_1, \ldots, f'_n, f_n$.*

3 An upper bound for critical lengths of $P_n \odot C_1$

Let us recall that an $(n+1)$-dimensional space of $C^n(I)$ functions defined on the interval I is an *extended Chebyshev* space if the number of zeros, counting multiplicities, of any nonzero function of the space is less than or equal to n.

The space $P_n \odot C_1$ is invariant under translations because it is the set of solutions of a differential equation of order $2n + 2$ with constant coefficients. For this kind of spaces a critical length can be defined as follows.

Definition 1 Let U be a finite dimensional space of differentiable functions defined on $\mathbb{R}$ *invariant under translations*, that is, if $u \in U$, then $u_h(x) := u(x - h)$ also belongs to U for any $h \in \mathbb{R}$. Then *the critical length* of U is $\ell(U) \in (0, +\infty]$ such that U is an extended Chebyshev space on an interval I if and only if I does not contain a compact interval of length $\ell(U)$.

If the space is the set of solutions of a differential equation of order $2n + 2$ whose characteristic polynomial is an even polynomial, then it is *invariant under reflections* in the sense that if $u \in U$, then $x \mapsto u(\xi - x)$ also belongs to U (see Sect. 3 of [6]). Let us observe that $P_n \odot C_1$ is the space of solutions of the differential Eq. (3), whose characteristic polynomial is $(\lambda^2 + 1)^{n+1}$. So, the space $P_n \odot C_1$ is invariant under reflections.

The following result can be used to compute critical lengths of finite dimensional spaces of differentiable functions invariant under reflections.

Proposition 3 *Let U be a finite dimensional space of differentiable functions defined on $\mathbb{R}$ invariant under reflections. Let $(u_0, \ldots, u_n)$ be any canonical basis at the origin. If the critical length $\ell(U)$ is finite, then it coincides with the least positive zero of the functions*

$$w_{j,n}(x) := \det W(u_j, \ldots, u_n)(x), \quad j > n/2,$$

where $W(u_j, \ldots, u_n)(x)$ denotes the Wronskian matrix at x of the system of functions $(u_j, \ldots, u_n)$.

Proof Let us first assume that $u_i^{(i)}(0) > 0$, $i = 0, \ldots, n$. If the functions $w_{j,n}$, $j > n/2$, do not vanish on $(0, +\infty)$, we deduce from Proposition 3.2 of [6] that U is an extended Chebyshev space on any interval of arbitrary length, including the whole real line. So, the critical length is infinite. Otherwise, we can define $\alpha > 0$ as the least positive zero of the functions $w_{j,n}$, $j > n/2$. From the fact that $u_i^{(i)}(0) > 0$, $i = 0, \ldots, n$, it can be deduced that

$$w_{j,n}(x) > 0, \quad j > n/2, \quad x \in (0, \alpha).$$

By Proposition 3.2 (ii) of [6], U is an extended Chebyshev space on each interval of length less than α, that is, $\ell(U) \geq \alpha$. Since $w_{j,n}(\alpha) = 0$ for some $j > n/2$, we deduce from Proposition 3.2 (ii) of [6] that U is not an extended Chebyshev space on compact intervals of length α and $\ell(U) \leq \alpha$. So, if $u_i^{(i)}(0) > 0$, $i = 0, \ldots, n$, then we conclude that $\alpha = \ell(U)$.

For the general case, let $s_i \in \{-1, 1\}$ be the sign of $u_i^{(i)}(0)$. Then, we can apply the result to the basis $(s_0 u_0, \ldots, s_n u_n)$. Since

$$w_{j,n}(x) = \det W(u_j, u_{j+1}, \ldots, u_n)(x) = s_j s_{j+1} \cdots s_n \det W(s_j u_j, s_{j+1} u_{j+1}, \ldots, s_n u_n)(x),$$

we conclude that the wronskian associated to both bases $(u_0, \ldots, u_n)$ and $(s_0 u_0, \ldots, s_n u_n)$ coincide up to a sign and both have the same set of zeros. Then the result follows. □

In the following result, we show that the critical length of $P_n \odot C_1$ is not greater than the first positive zero of the spherical Bessel function j_n.

Theorem 1 *The critical length of the space $P_n \odot C_1$ is the first positive zero of the functions*

$$w_{2i,2n+1}(x) = \det W(f_i', f_i, \ldots, f_n', f_n)(x), \quad \frac{n+1}{2} \leq i \leq n,$$

$$w_{2i+1,2n+1}(x) = \det W(f_i, \ldots, f_n', f_n)(x), \quad \frac{n}{2} \leq i \leq n.$$

Moreover,

$$\ell(P_n \odot C_1) \leq j_{n+1/2,1}.$$

Proof Since $P_n \odot C_1$ is invariant under translations and reflections, the first part of the statement follows from Proposition 3. The wronskian function $w_{2n+1,2n+1}(x) = f_n(x) = x^{n+1} j_n(x)$ coincides with the last function of the canonical basis of $P_n \odot C_1$. Therefore, the critical length must be less than or equal to its first positive zero. □

In [9], it was shown that the critical lengths of the cycloidal spaces

$$C_n := \ker(D^{n-1}(D^2+I)) = \langle 1, x, \ldots, x^{n-2}, \cos x, \sin x\rangle$$

are

$$\ell(C_{2n}) = \ell(C_{2n+1}) = 2j_{n-1/2,1}.$$

Observe that C_{2n+2} and $P_n \odot C_1$ are both $2(n+1)$-dimensional spaces containing trigonometric functions and

$$\ell(C_{2n+2}) = 2j_{n+1/2,1} \geq 2\ell(P_n \odot C_1).$$

This implies that Hermite interpolation problems on cycloidal spaces can be posed on longer intervals than the same kind of problems on spaces $P_n \odot C_1$ of the same dimension.

4 Critical lengths of the spaces $P_n \odot C_1$, for $n \leq 1$

Let us first show that the function

$$w_{2n,2n+1}(x) = \det W(f_n', f_n)(x) = \det \begin{pmatrix} f_n'(x) & f_n(x) \\ f_n''(x) & f_n'(x) \end{pmatrix}$$

has no positive zeros.

Proposition 4 *For any $x > 0$, the following inequalities hold*

(a) $\det \begin{pmatrix} f_{n-1}(x) & f_n(x) \\ f_{n-1}'(x) & f_n'(x) \end{pmatrix} > 0,$

(b) $\det \begin{pmatrix} f_{n-1}(x) & f_n(x) \\ f_{n-2}(x) & f_{n-1}(x) \end{pmatrix} > 0,$

(c) $\det \begin{pmatrix} f_n'(x) & f_n(x) \\ f_n''(x) & f_n'(x) \end{pmatrix} > 0.$

Proof (a) Let us show that

$$w(x) := \det W(f_{n-1}, f_n)(x) = \det \begin{pmatrix} f_{n-1}(x) & f_n(x) \\ f_{n-1}'(x) & f_n'(x) \end{pmatrix} > 0, \quad x > 0.$$

Differentiating, we have that

$$w'(x) = \det \begin{pmatrix} f_{n-1}(x) & f_n(x) \\ f_{n-1}''(x) & f_n''(x) \end{pmatrix}.$$

Using Proposition 1 (e) and Proposition 1 (c) we deduce that

$$\begin{aligned} xw'(x) &= \det \begin{pmatrix} f_{n-1}(x) & f_n(x) \\ 2(n-1)f_{n-1}'(x) - xf_{n-1}(x) & 2nf_n'(x) - xf_n(x) \end{pmatrix} = 2(n-1)w(x) + 2f_{n-1}(x)f_n'(x) \\ &= 2(n-1)w(x) + 2xf_{n-1}(x)^2. \end{aligned}$$

Dividing by x^{2n-1} we have that

$$\begin{aligned} (x^{-2(n-1)}w(x))' &= x^{-2(n-1)}w'(x) - 2(n-1)x^{-(2n-1)}w(x) \\ &= 2x^{-2(n-1)}f_{n-1}(x)^2 \geq 0, \quad x > 0. \end{aligned}$$

Hence $x^{-2(n-1)}w(x)$ is a nondecreasing function. Now, we use Proposition 1 (c) and (g) to deduce that

$$\lim_{x\to 0} x^{-4n+1}w(x) = \lim_{x\to 0}\det\begin{pmatrix} f_{n-1}(x)/x^{2n-1} & f_n(x)/x^{2n+1} \\ f_{n-2}(x)/x^{2n-3} & f_{n-1}(x)/x^{2n-1}\end{pmatrix}$$
$$= \det\begin{pmatrix} 1/(2n-1)!! & 1/(2n+1)!! \\ 1/(2n-3)!! & 1/(2n-1)!!\end{pmatrix} = \frac{2}{(2n+1)!!(2n-1)!!} > 0.$$

Therefore $x^{-2(n-1)}w(x)$ is positive in $(0, +\infty)$.

(b) It follows follows from Proposition 1 (c)

$$\det\begin{pmatrix} f_{n-1}(x) & f_n(x) \\ f_{n-2}(x) & f_{n-1}(x)\end{pmatrix} = x^{-1}\det\begin{pmatrix} f_{n-1}(x) & f_n(x) \\ xf_{n-2}(x) & xf_{n-1}(x)\end{pmatrix} = x^{-1}w(x) > 0, \quad x > 0.$$

(c) Let us define

$$v(x) := \det\begin{pmatrix} f_n'(x) & f_n(x) \\ f_n''(x) & f_n'(x)\end{pmatrix}.$$

Differentiating and using Proposition 1 (f), we get

$$v'(x) = \det\begin{pmatrix} f_n'(x) & f_n(x) \\ f_n'''(x) & f_n''(x)\end{pmatrix} = \det\begin{pmatrix} f_n'(x) & f_n(x) \\ 2nf_{n-1}'(x) & 2nf_{n-1}(x)\end{pmatrix} = 2nw(x).$$

So $v(x)$ is a strictly increasing function on $(0, +\infty)$. By Proposition 1 (g), we have

$$\lim_{x\to 0} x^{-4n}v(x) = \lim_{x\to 0}\det\begin{pmatrix} f_n'(x)/x^{2n} & f_n(x)/x^{2n+1} \\ f_n''(x)/x^{2n-1} & f_n'(x)/x^{2n}\end{pmatrix}$$
$$= \frac{1}{(2n+1)!!^2}\det\begin{pmatrix} 2n+1 & 1 \\ (2n+1)2n & 2n+1\end{pmatrix} = \frac{1}{(2n+1)!!(2n-1)!!} > 0$$

and then $v(x)$ is positive in $(0, +\infty)$.

□

Let us observe that the proof of Proposition 4(c) is based on Proposition 4(b), which corresponds to the following Turán type inequality for Bessel functions

$$J_\nu(x)^2 - J_{\nu-1}(x)J_{\nu+1}(x) > 0,$$

with $\nu = n - 1/2$ (see [4, 13]). We have included a proof based on the properties shown in Proposition 1 for the sake of completeness.

By Proposition 3, it follows that $\ell(P_0 \odot C_1)$ is the first positive zero of the function

$$w_{1,1}(x) = f_0(x) = \sin x,$$

that is,

$$\ell(P_0 \odot C_1) = j_{1/2,1} = \pi.$$

If $n = 1$, we can use Proposition 3 to show that $\ell(P_0 \odot C_1)$ is the first positive zero of the functions

$$w_{3,3}(x) = f_1(x) = x^2 j_1(x), \quad w_{2,3}(x) = \det W(f_n', f_n)(x).$$

By Proposition 4 (c), $\det W(f_n', f_n)(x)$ does not vanish on $(0, +\infty)$. So

$$\ell(P_1 \odot C_1) = j_{3/2,1} \approx 4.493409455874135.$$

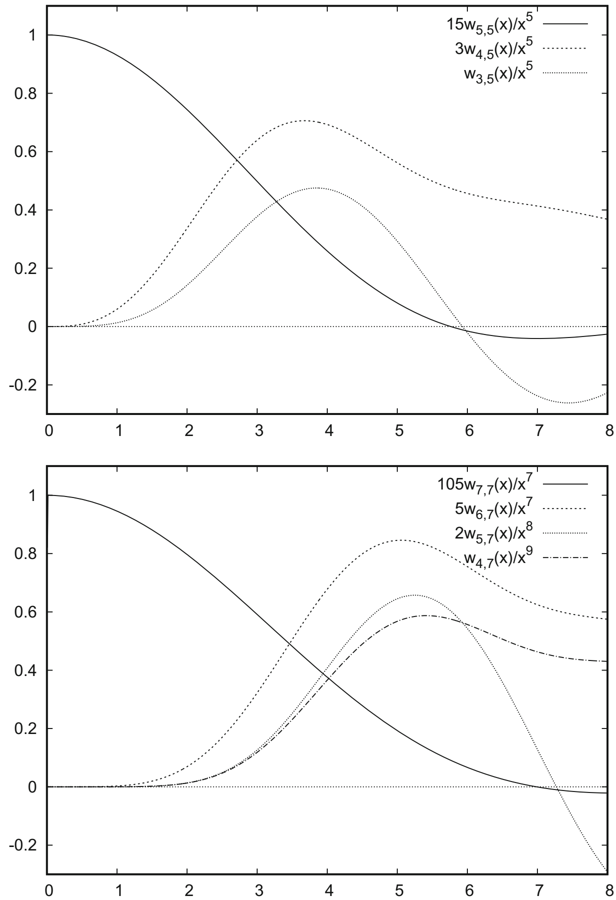

Fig. 1 The first positive zero of the wronskians $w_{i,2n+1}$, $i \geq n+1$, for $n = 2$ (up) and for $n = 3$ (down)

The difficulty of the analysis of the sign of $w_{j,2n+1}$ increases when $j < 2n$. However, we conjecture that the least positive zero of the functions $w_{j,2n+1}$ is attained for the function $w_{2n+1,2n+1} = f_n$, or equivalently, that $\ell(P_n \odot C_1) = j_{n+1/2,1}$ for $n > 1$.

We have computed several wronskians for low degree n and they confirm our conjecture at least for $n = 2, 3$, as shown in Figure 1. We have depicted the graphs of the wronskians, conveniently normalized and divided by a factor of the form x^k. The graphs show that the first positive zero of the wronskians is attained for $f_n = w_{2n+1,2n+1}$, giving rise to

$$\ell(P_2 \odot C_1) = j_{3/2,1} \approx 5.763459196842433, \quad \ell(P_3 \odot C_1) = j_{3/2,1} \approx 6.987924414992913.$$

5 Applications to computer-aided design

Most design tools describe curves in the parametric form

$$\gamma(t) = \sum_{i=0}^{n} P_i u_i(t), \quad t \in [a, b],$$

where $u_i : [a, b] \to \mathbb{R}, i = 0, \dots, n$, are nonnegative functions such that $\sum_{i=0}^{n} u_i(t) = 1$ for all $t \in [a, b]$. In order to avoid redundancy, $u_0, \dots, u_n$ are required to be linearly independent and they form a basis of a space $U = \langle u_0, \dots, u_n \rangle$. The polygon $P_0 \cdots P_n$ is called the *control polygon* of γ. Usually, such parametric representation of curves is *shape preserving*, in the sense that the shape of the curve imitates the shape of its control polygon. Shape preserving representations of curves are associated to the fact that the system of functions $(u_0, \dots, u_n)$ is *normalized and totally positive* [5].

A *totally positive matrix* is any matrix such that all its minors are nonnegative. A system $(u_0, \dots, u_n)$ of functions defined on an interval I, $u_i : I \to \mathbb{R}$, $i = 0, \dots, n$, is *totally positive* if any *collocation matrix* $(u_j(t_i))_{i,j=0,\dots,n}$ is a totally positive matrix for any $t_0 < \cdots < t_n$ in I. The system is *normalized* if all functions add up to one, that is $\sum_{i=0}^{n} u_i(t) = 1$, for all $t \in I$.

The spaces U where curves are designed might be invariant under translations. If they contain trigonometric functions, they do not possess normalized totally positive bases on intervals of arbitrary length (see Section 6 of [6]). This motivates the following definition.

Definition 2 Let U be a finite dimensional space of differentiable functions invariant under translations containing the constant functions. The *critical length for design purposes* is $\ell'(U) \in (0, +\infty]$ such that U is an extended Chebyshev space and has a normalized totally positive basis on an interval $[a, b]$ if and only $b - a < \ell'(U)$.

In Corollary 4.1 of [6] it was shown that the critical length for design purposes can be expressed in terms of the critical length.

Proposition 5 *The critical length for design purposes of a finite dimensional space of differentiable functions U invariant under translations and containing the constant functions coincides with the critical length of the space of its derivatives $DU = \{u' | u \in U\}$, that is*

$$\ell'(U) = \ell(DU).$$

For a given a space $U \subset C[a, b]$, let $D^{-1}U := \{u \in C^1[a, b] | u' \in U\}$. Then the results in the previous sections can be interpreted in terms of the existence of shape preserving representations of curves in the space $D^{-1}(P_n \odot C_1)$.

Proposition 6 *A canonical basis at the origin of $D^{-1}(P_n \odot C_1) = P_0 \oplus (P_n \odot C_1)$ is given by*

$$f_0'(x), f_0(x), \dots, f_n'(x), f_n(x), \int_0^x f_n(t)dt.$$

The critical length for design purposes satisfies

$$\ell'(D^{-1}(P_n \odot C_1)) \leq j_{n+1/2,1}.$$

Proof Since

$$D(P_0 \oplus (P_n \odot C_1)) = D(P_n \odot C_1) = P_n \odot C_1$$

and

$$\dim(P_0 \oplus (P_n \odot C_1)) = 2n + 3 = \dim(P_n \odot C_1) + 1,$$

we deduce that $D^{-1}(P_n \odot C_1) = P_0 \oplus (P_n \odot C_1)$. By Proposition 1 (g), we deduce that $\int_0^x f_n(t)dt$ has a zero of multiplicity $2n + 2$ at $x = 0$ and deduce that the given basis is canonical at the origin. Finally, we conclude from Theorem 1 and Proposition 5 that the critical length for design purposes of $D^{-1}(P_n \odot C_1)$ satisfies

$$\ell'(D^{-1}(P_n \odot C_1)) = \ell(P_n \odot C_1) \leq j_{n+1/2,1}.$$

□

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References

1. Abramowitz, M., Stegun, I.: Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards Applied Mathematics Series, 55, Washington D.C., NY (1972)
2. Aldaz, J.M., Kounchev, O., Render, H.: Bernstein operators for exponential polynomials. Constr. Approx. **29**(3), 345–367 (2009). https://doi.org/10.48550/arXiv.0805.1618
3. Aldaz, J.M., Kounchev, O., Render, H.: Bernstein operators for extended Chebyshev systems. Appl. Math. Comput. **217**(2), 790–800 (2010). https://doi.org/10.1016/j.amc.2010.06.018
4. Baricz, Á., Pogány, T.K.: Turán determinants of Bessel functions. Forum Math. **26**(1), 295–322 (2014). https://doi.org/10.1515/form.2011.160
5. Carnicer, J.M., Peña, J.M.: Totally positive bases for shape preserving curve design and optimality of B-splines. Comput. Aided Geom. Design **11**(6), 633–654 (1994). https://doi.org/10.1016/0167-8396(94)90056-6
6. Carnicer, J.M., Mainar, E., Peña, J.M.: Critical length for design purposes and extended Chebyshev spaces. Constr. Approx. **20**, 55–71 (2004). https://doi.org/10.1007/s00365-002-0530-1
7. Carnicer, J.M., Mainar, E., Peña, J.M.: On the critical length of cycloidal spaces. Constr. Approx. **39**, 573–583 (2014). https://doi.org/10.1007/s00365-013-9223-1
8. Carnicer, J.M., Mainar, E., Peña, J.M.: Greville abscissae of totally positive bases. Comput. Aided Geom. Design **48**, 60–74 (2016). https://doi.org/10.1016/j.cagd.2016.09.001
9. Carnicer, J.M., Mainar, E., Peña, J.M.: Critical lengths of cycloidal spaces are zeros of Bessel functions. Calcolo **54**, 1521–1531 (2017). https://doi.org/10.1007/s10092-017-0239-y
10. Chen, Q., Wang, G.: A class of Bézier-like curves. Comput. Aided Geom. Design **20**, 29–39 (2003). https://doi.org/10.1016/S0167-8396(03)00003-7
11. Karlin, S.: Total positivity, vol. I. Stanford University Press, Stanford (1968)
12. Manni, C., Pelosi, F., Sampoli, M.L.: Generalized B-splines as a tool in isogeometric analysis. Comput. Methods Appl. Mech. Eng. **200**, 867–881 (2011). https://doi.org/10.1016/j.cma.2010.10.010
13. Skovgaard, H.: On inequalities of the Turán type. Math. Scand. **2**, 65–73 (1954)
14. Zhang, J.: C-curves: an extension of cubic curves. Comput. Aided Geom. Design **13**, 199–217 (1996). https://doi.org/10.1016/0167-8396(95)00022-4