

ORIGINAL ARTICLE

Vanishing of higher order Alexander-type invariants of plane curves

José I. Cogolludo-Agustín¹ | Eva Elduque² 

¹Departamento de Matemáticas, IUMA, Universidad de Zaragoza, Zaragoza, Spain

²Departamento de Matemáticas, Universidad Autónoma de Madrid, and ICMAT, Madrid, Spain

Correspondence

Eva Elduque, Departamento de Matemáticas, Universidad Autónoma de Madrid, and ICMAT, 28049 Madrid, Spain.
Email: eva.elduque@uam.es

Funding information

Gobierno de Aragón, Grant/Award Number: E22_20R; Ministerio de Ciencia e Innovación, Grant/Award Number: PID2020-114750GB-C31; Simons Foundation, Grant/Award Number: AMS-Simons Travel Grant

Abstract

The higher order degrees are Alexander-type invariants of complements to an affine plane curve. In this paper, we characterize the vanishing of such invariants for a curve C given as a transversal union of plane curves C' and C'' in terms of the finiteness and the vanishing properties of the invariants of C' and C'' , and whether or not they are irreducible. As a consequence, we prove that the multi-variable Alexander polynomial Δ_C^{multi} is a power of $(t - 1)$, and we characterize when $\Delta_C^{\text{multi}} = 1$ in terms of the defining equations of C' and C'' . Our results impose obstructions on the class of groups that can be realized as fundamental groups of complements of a transversal union of curves.

KEYWORDS

Alexander invariants, Alexander polynomials, derived series, line arrangements, plane curves

MSC (2020)

32S25, 32S55, 32S05, 32S20, 57K31

1 | INTRODUCTION

The study of curve complements goes back to the work of Zariski ([21]), who observed that the position of the singularities of a plane curve can affect the embedded topology of the curve, and that the fundamental group of the complement of the curve can detect this phenomenon. Alexander-type invariants, which appeared originally in classical knot theory and were first adapted to the study of singularities of plane curves by Libgober ([12, 14]), are easier to handle than the fundamental group, and they can also be sensitive to the type and position of singularities.

In knot theory, a possible strategy to address problems for which the Alexander polynomial is not fine enough to distinguish embedded topologies is to consider non-abelian Alexander-type invariants, such as the *higher order degrees* (e.g., see Cochran, [2]), which have been shown to give better bounds for the knot genus than the Alexander polynomial (Horn, [8]). These invariants also have striking applications in the world of 3-manifolds (Harvey, [7]). For example, in *loc. cit.*, lower bounds for the Thurston norm are given and new algebraic obstructions to a 4-manifold of the form $M^3 \times S^1$ admitting a symplectic structure are provided.

Leidy and Maxim initiated in [10, 11] the study of higher order Alexander-type invariants for complex affine plane curve complements, and Maxim and the second author continued this work in [6]. To any affine plane curve $C \subset \mathbb{C}^2$

This is an open access article under the terms of the [Creative Commons Attribution-NonCommercial-NoDerivs](https://creativecommons.org/licenses/by-nc-nd/4.0/) License, which permits use and distribution in any medium, provided the original work is properly cited, the use is non-commercial and no modifications or adaptations are made.

© 2023 The Authors. *Mathematische Nachrichten* published by Wiley-VCH GmbH.

(given by the zeros of a reduced nonconstant polynomial $f \in \mathbb{C}[x, y]$), in [10] one associates a sequence $\{\delta_n(C)\}_n$ of (possibly infinite) integers, called the higher order degrees of C . Roughly speaking, these integers measure the “sizes” of quotients of successive terms in the rational derived series $\{G_r^{(n)}\}_{n \geq 0}$ of the fundamental group $G = \pi_1(\mathbb{C}^2 \setminus C)$ of the curve complement.

It was also noted in [10] that the higher order degrees of plane curves (at any level n) are sensitive to the position of singular points. These integers can also be interpreted as L^2 -Betti numbers associated to the tower of coverings of $\mathbb{C}^2 \setminus C$ corresponding to the subgroups $G_r^{(i)}$ (the first of which is the universal abelian cover), so in principle, there is no reason to expect that such invariants have any good vanishing or finiteness properties. Some finiteness results obtained in [6, 10] are summarized in this theorem.

Theorem 1.1. *Let $C \subset \mathbb{C}^2$ be a plane curve of degree m . If one of the following conditions hold, then $\delta_n(C)$ is finite:*

1. C is irreducible [10, Remark 3.3].
2. C is in general position at infinity [10, Corollary 4.8].
3. C is an essential line arrangement [6, Theorem 1.2].
4. C has only nodes or simple tangents at infinity [6, Theorem 1.4].

Moreover, in the cases (2), (3), and (4), we have that $\delta_n(C) \leq m(m-2)$ for all $n \geq 0$. That is, there is a uniform bound for all the higher order degrees that depends only on the degree of the curve.

In relation to an old question of Serre [1, 18], finiteness results impose restrictions on which groups can be realized as fundamental groups of curve complements, but vanishing results, on top of being stronger, shed more light on what type of problems these invariants are well suited for.

An example of a “vanishing” (or “triviality”) result is the following theorem of Oka. In general terms, it tells us that the univariable Alexander polynomial (see Definition 2.1) of a transversal union of curves $C = C' \cup C''$ does not remember information about the topology of C' or C'' , even though the fundamental group does (See Theorem 3.4).

Theorem 1.2 [16, Theorem 34]. *Let C be a plane curve of the form $C = C' \cup C''$, where C' and C'' are curves in \mathbb{C}^2 of degrees m' and m'' , respectively. Assume that C is in general position at infinity, and assume that $C' \cap C''$ consists on $m'm''$ distinct points. Then,*

$$\Delta_C^{\text{uni}}(t) = (t-1)^{s-1},$$

where s is the number of irreducible components of C and $\Delta_C^{\text{uni}}(t)$ is the (univariable) Alexander polynomial of C .

It is natural to ask whether more involved Alexander invariants also exhibit this behavior. In this paper, we completely characterize the vanishing of the higher order degrees of a union C of two curves C' and C'' that intersect transversally (even when C is not in general position at infinity). This characterization is done in terms of the finiteness and vanishing properties of the higher order degrees of C' and C'' , obtaining vanishing results in most cases (and finiteness in all cases). More concretely, we obtain the following.

Theorem 1.3. *Let $C = C' \cup C'' \subset \mathbb{C}^2$ be the union of two affine plane curves, with $\deg C' = m'$ and $\deg C'' = m''$. Suppose that $C' \cap C''$ consists on $m'm''$ distinct points in \mathbb{C}^2 . Then,*

1. if C' and C'' are either both irreducible or both not irreducible, then $\delta_n(C) = 0$ for all $n \geq 0$;
2. if C' is irreducible and C'' is not irreducible, then $\delta_n(C) \leq m'' - 1$ for all $n \geq 0$.

Moreover, in this case,

- (a) if $\delta_0(C') \neq 0$, then

$$\delta_n(C) = 0 \text{ for all } n \geq 1, \text{ and}$$

$$\delta_0(C) = 0 \Leftrightarrow \delta_0(C'') < \infty,$$

(b) if $\delta_0(C') = 0$, then

$$\delta_n(C) = 0 \Leftrightarrow \delta_n(C'') < \infty \text{ for all } n \geq 0.$$

In particular, $\delta_n(C)$ is finite for all $n \geq 0$.

This provides a broad generalization of the vanishing results of [6], where the fundamental group of one of the curve complements was assumed to be isomorphic to \mathbb{Z} , and δ_n of the other curve was assumed to be finite.

The paper is structured as follows. In Section 2, we recall the relevant definitions of the Alexander-type invariants that are used throughout the paper and the relationships between them. In Section 3, we prove the main result (Theorem 1.3). In Section 4, we characterize which curves have $\delta_0(C) = \infty$ in terms of their defining equations (curves of affine pencil type, as defined in Lemma 4.5) and arrive at Corollary 1.4 below about the triviality of the multivariable Alexander polynomial of a transversal union of curves (see Definition 2.2). This corollary provides concrete restrictions as to which groups can be realized as fundamental groups of a complement of a transversal union of curves (see Remark 4.6).

Corollary 1.4. *Under the same hypotheses as in Theorem 1.3,*

1. if C' and C'' are either both irreducible or both not irreducible, then $\Delta_C^{\text{multi}} = 1$;
2. if C' is irreducible and C'' is not irreducible, then

$$\Delta_C^{\text{multi}} = (t - 1)^{\delta_0(C)},$$

where t is the variable corresponding to a positively oriented meridian around C' .

Moreover, $0 \leq \delta_0(C) \leq m'' - 1$, and $\delta_0(C) \neq 0$ if and only if C' is irreducible and $\delta_0(C'') = \infty$.

In particular, $\Delta_C^{\text{multi}} \neq 1$ if and only if C' is irreducible and C'' is of affine pencil type.

2 | DEFINITIONS OF CLASSICAL AND HIGHER ORDER ALEXANDER INVARIANTS

In this section, we recall the basic definitions of the notions that will be used throughout this note. For a more detailed explanation of the different Alexander invariants used in this paper, we refer the reader to [12] (for univariable Alexander polynomials), [20] (for multivariable Alexander polynomials), and [10] (for higher order degrees of plane curve complements), for example.

2.1 | Alexander polynomials

Let $C = \{f(x, y) = 0\} \subset \mathbb{C}^2$ be a plane curve given by the zeros of a reduced polynomial f , with complement $U := \mathbb{C}^2 \setminus C$, and denote by $G := \pi_1(U)$ the fundamental group of its complement. Note that all the tools described in this paper for the fundamental group are independent of the choice of the base point, so it will be omitted. If C has s irreducible components, then

$$H_1(G; \mathbb{Z}) = H_1(U; \mathbb{Z}) = G/G' = \mathbb{Z}^s \quad (2.1)$$

is generated by meridian loops about the smooth parts of the irreducible components of C .

Consider a homotopy class α in G . The inclusion $i : U \hookrightarrow \mathbb{C}^2$ implies that $i_*\alpha$ can be described by a loop that bounds a disk. Moreover, this disk can be assumed to intersect C transversally in a finite number of points. The natural orientation of both α and C in \mathbb{C}^2 naturally defines a linking number homomorphism $\psi : G \xrightarrow{\psi} \mathbb{Z}$, where $\alpha \mapsto \text{lk}(\alpha, C)$. Since f is a reduced polynomial, ψ is the map induced in fundamental groups by the polynomial map $f : U \rightarrow \mathbb{C}^*$. Let $\text{Ab} : G \rightarrow \mathbb{Z}^s$ be the abelianization homomorphism, which sends a positively oriented meridian about the i -th component of C to the i -th element of the canonical basis of \mathbb{Z}^s . Let \mathcal{L}^ψ and \mathcal{L}^{Ab} be the local systems of $\mathbb{Q}[t^{\pm 1}]$ -modules and $\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ -modules

induced by ψ and Ab , respectively. More explicitly, \mathcal{L}^ψ and \mathcal{L}^{Ab} are given by

$$\begin{aligned} G &\rightarrow \text{Aut}(\mathbb{Q}[t^{\pm 1}]) \\ \gamma &\mapsto (1 \mapsto t^{\psi(\gamma)}) \end{aligned}$$

and

$$\begin{aligned} G &\rightarrow \text{Aut}(\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]) \\ \gamma &\mapsto (1 \mapsto t^{\text{Ab}(\gamma)}), \end{aligned}$$

where $t^{(a_1, \dots, a_s)} := t_1^{a_1} \cdot \dots \cdot t_s^{a_s}$ for all $(a_1, \dots, a_s) \in \mathbb{Z}^s$.

For the following two definitions, let $F_i(M)$ be the i -th fitting ideal of a module M over a commutative ring.

Definition 2.1. The *univariable Alexander polynomial* of U , denoted by $\Delta_C^{\text{uni}}(t)$ is defined as

$$\Delta_C^{\text{uni}}(t) := \text{a generator of } F_0(H_1(U; \mathcal{L}^\psi)) \in \mathbb{Q}[t^{\pm 1}],$$

which is well defined up to multiplication by a unit of $\mathbb{Q}[t^{\pm 1}]$.

Definition 2.2. The *multivariable Alexander polynomial* of U , denoted by $\Delta_C^{\text{multi}}(t_1, \dots, t_s)$ is defined as

$$\Delta_C^{\text{multi}}(t_1, \dots, t_s) := \gcd(F_1(H_1(U, u_0; \mathcal{L}^{\text{Ab}}))) \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}],$$

where u_0 is a base point. It is well defined up to multiplication by a unit of $\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$.

Remark 2.3. Note that $H_1(U; \mathcal{L}^\psi) \cong H_1(U^\psi; \mathbb{Q})$ ([9, Theorem 2.1]) as modules over $\mathbb{Q}[t^{\pm 1}]$, where U^ψ is the infinite cyclic cover of U induced by $\ker \psi$, whose deck group is isomorphic to \mathbb{Z} . Also note that the definition and computations are easier in the univariable case because $\mathbb{Q}[t^{\pm 1}]$ is a principal ideal domain (PID). In the definition of the higher order degrees, a (noncommutative) PID is constructed to help generalize this construction of the univariable Alexander polynomial to other covers of U that lie above U^ψ .

Remark 2.4. Note that $\Delta_C^{\text{uni}}(t)$ is defined using the homology of U whereas $\Delta_C^{\text{multi}}(t_1, \dots, t_s)$ uses the homology of U relative to a base point. In general, they are far from being the same, even after specializing $\Delta_C^{\text{multi}}(t_1, \dots, t_s)$ to a single variable polynomial. However, if C is irreducible, both definitions coincide. Indeed, from [6, Remark 5.10], we know that $\Delta_C^{\text{multi}}(t)$ divides $\Delta_C^{\text{uni}}(t)$ in $\mathbb{Q}[t^{\pm 1}]$, and the same argument of the proof of [6, Theorem 5.18] (for $s = 1$) shows that both polynomials are the same up to multiplication by a unit in $\mathbb{Q}[t^{\pm 1}]$.

2.2 | Higher order degrees

Definition 2.5. The *rational derived series* of the group G is defined inductively by: $G_r^{(0)} = G$, and for $n \geq 1$,

$$G_r^{(n)} = \{g \in G_r^{(n-1)} \mid g^k \in [G_r^{(n-1)}, G_r^{(n-1)}], \text{ for some } k \in \mathbb{Z} \setminus \{0\}\}.$$

It is easy to see that $G_r^{(i)} \triangleleft G_r^{(j)} \triangleleft G$, if $i \geq j \geq 0$. The successive quotients of the rational derived series are torsion-free abelian groups. In fact (cf. [7, Lemma 3.5]),

$$G_r^{(n)} / G_r^{(n+1)} \cong \left(G_r^{(n)} / [G_r^{(n)}, G_r^{(n)}] \right) / \{\mathbb{Z} - \text{torsion}\}.$$

Therefore, for $G = \pi_1(\mathbb{C}^2 \setminus C)$, one has from (2.1) that $G' = G_r^{(1)}$.

The use of the rational derived series instead of the usual derived series is needed in order to avoid zero divisors in the group ring $\mathbb{Z}\Gamma_n$, where

$$\Gamma_n := G/G_r^{(n+1)}.$$

Γ_n is a poly-torsion-free-abelian (PTFA) group, that is, it admits a normal series of subgroups such that each of the successive quotients of the series is torsion-free abelian ([7, Corollary 3.6]). Thus, $\mathbb{Z}\Gamma_n$ is a right and left Ore domain, so it embeds in its classical right ring of quotients \mathcal{K}_n , which is a skew field. Every module over \mathcal{K}_n is a free module, and such modules have a well-defined rank $\text{rk}_{\mathcal{K}_n}$ which is additive on short exact sequences.

In [10], one associates to any plane curve C a sequence of nonnegative integers $\delta_n(C)$ as follows. Since G' is in the kernel of ψ (the linking number homomorphism), we have a well-defined induced epimorphism $\bar{\psi} : \Gamma_n \rightarrow \mathbb{Z}$. Let $\bar{\Gamma}_n = \ker \bar{\psi}$. Then, $\bar{\Gamma}_n$ is a PTFA group, so $\mathbb{Z}\bar{\Gamma}_n$ has a right ring of quotients $\mathbb{K}_n = (\mathbb{Z}\bar{\Gamma}_n)S_n^{-1}$, where $S_n = \mathbb{Z}\bar{\Gamma}_n \setminus \{0\}$. Let $R_n := (\mathbb{Z}\Gamma_n)S_n^{-1}$. R_n and \mathcal{K}_n are flat left $\mathbb{Z}\Gamma_n$ -modules.

A very important role in what follows is played by the fact that R_n is a PID; in fact, R_n is isomorphic to the ring of skew-Laurent polynomials $\mathbb{K}_n[t^{\pm 1}]$. This can be seen as follows: By choosing a $t \in \Gamma_n$ such that $\bar{\psi}(t) = 1$, one obtains a splitting ϕ of $\bar{\psi}$, and the embedding $\mathbb{Z}\bar{\Gamma}_n \subset \mathbb{K}_n$ extends to an isomorphism $R_n \cong \mathbb{K}_n[t^{\pm 1}]$. However, this isomorphism depends in general on the choice of splitting of $\bar{\psi}$.

Definition 2.6.

1. The n -th order localized Alexander module of the plane curve C is defined to be

$$\mathcal{A}_n(C) = H_1(U; R_n) := H_1(U; \mathbb{Z}\Gamma_n) \otimes_{\mathbb{Z}\Gamma_n} R_n,$$

viewed as a right R_n -module. The coefficients in the rightmost expression are the rank 1 local system induced by the projection $G \twoheadrightarrow \Gamma_n$ [7, section 5]. If we choose a splitting ϕ to identify R_n with $\mathbb{K}_n[t^{\pm 1}]$, we define $\mathcal{A}_n^\phi(C) = H_1(U; \mathbb{K}_n[t^{\pm 1}])$.

2. The n -th order degree of C is defined to be:

$$\delta_n(C) = \text{rk}_{\mathbb{K}_n} \mathcal{A}_n(C) = \text{rk}_{\mathbb{K}_n} \mathcal{A}_n^\phi(C).$$

The higher order degrees $\delta_n(C)$ are integral invariants of the fundamental group G of the complement (endowed with the linking number homomorphism). Indeed, by [7], one has:

$$\delta_n(C) = \text{rk}_{\mathbb{K}_n} \left(G_r^{(n+1)} / [G_r^{(n+1)}, G_r^{(n+1)}] \otimes_{\mathbb{Z}\bar{\Gamma}_n} \mathbb{K}_n \right). \quad (2.2)$$

Note that since the isomorphism between R_n and $\mathbb{K}_n[t^{\pm 1}]$ depends on the choice of splitting, one cannot define a higher order version of the (univariable) Alexander polynomial in a canonical way. However, for any choice of splitting, the degree of the associated higher order Alexander polynomial is the same, hence this yields a well-defined invariant of G , which is exactly the higher order degree δ_n defined above.

2.3 | An effective method to compute $\delta_n(C)$

The higher order degrees of C may be computed by means of Fox free calculus from a presentation of $G = \pi_1(U)$, where $U := \mathbb{C}^2 \setminus C$, see [7, Section 6] for details, although the computations can be quite tedious in practice. Such techniques will be used freely in this paper, as summarized in this section.

Consider the matrix of Fox derivatives for a presentation of $\pi_1(U)$ given by

$$G = \pi_1(U) = \langle a_1, \dots, a_m \mid r_j, \quad j = 1, \dots, l \rangle,$$

that is, the matrix

$$\left(\frac{\partial r_j}{\partial a_i} \right)_{i,j}, \quad 1 \leq i \leq m, 1 \leq j \leq l,$$

which has entries in $\mathbb{Z}G$. Let $\bar{\cdot} : \mathbb{Z}G \rightarrow \mathbb{Z}G$ be the involution of $\mathbb{Z}G$ defined as the \mathbb{Z} -linear map that takes elements of G to their inverses. Let A be the matrix with entries in $\mathbb{Z}G$ defined by

$$A = \left(\frac{\partial r_j}{\partial a_i} \right)_{i,j}. \quad (2.3)$$

Let $q_n : G \rightarrow \Gamma_n$ be the projection, and let $q'_n : \mathbb{Z}G \rightarrow \mathbb{Z}\Gamma_n$ be the induced map on group rings. Let

$$B(n) = A^{q'_n},$$

that is, the matrix formed by the images of the entries of A by q'_n .

With this notation, $B(n)$ is a presentation matrix for the right $\mathbb{Z}\Gamma_n$ -module $H_1(U, u_0; \mathbb{Z}\Gamma_n)$, where u_0 is some base point. As noted in [7, section 6], the involution $\bar{\cdot}$, used for the definition of A in (2.3), is necessary for $B(n)$ to be a presentation matrix for $H_1(U, u_0; \mathbb{Z}\Gamma_n)$ as a $\mathbb{Z}\Gamma_n$ -module, since we work with right rather than left $\mathbb{Z}G$ -modules.

Moreover, since R_n and \mathcal{K}_n are flat over $\mathbb{Z}\Gamma_n$, we have that $B(n)$ is a presentation matrix for the right R_n -module (resp. \mathcal{K}_n -module) $H_1(U, u_0; R_n)$ (resp. $H_1(U, u_0; \mathcal{K}_n)$). By [7, Proposition 5.6], one obtains the following property for $B(n)$.

Lemma 2.7. *The rank of the left \mathcal{K}_n -module generated by the rows of $B(n)$ is $\leq m - 1$, and the rank of the left \mathcal{K}_n -module generated by the rows of $B(n)$ is $m - 1$ if and only if $\delta_n(C)$ is finite.*

By performing only allowed row and column operations to $B(n)$ in $R_n \cong \mathbb{K}_n[t^{\pm 1}]$ ([7, Lemma 9.2]), we can turn $B(n)$ into a different presentation matrix of $H_1(U, u_0; R_n)$ of the form

$$\left(\begin{array}{ccc|ccc} D & & & 0 & & \\ 0 & \dots & 0 & 0 & \dots & 0 \end{array} \right), \quad (2.4)$$

where D is a diagonal matrix with entries in $\mathbb{K}_n[t^{\pm 1}]$ and the last row is a row of zeros.

We define the degree of a nonzero polynomial of $\mathbb{K}_n[t^{\pm 1}]$ to be the difference between the highest and lowest exponents of t appearing in the polynomial. The following result is immediate and allows one to obtain the invariant $\delta_n(C)$.

Proposition 2.8. *The higher order degree $\delta_n(C)$ is the degree of the product of the diagonal elements of D in (2.4) if all of those elements are nonzero, and $\delta_n(C) = \infty$ otherwise.*

3 | VANISHING OF HIGHER ORDER DEGREES OF TRANSVERSAL INTERSECTIONS

The goal of this section is to prove Theorem 1.3, which characterizes the vanishing of the higher order degrees of a curve that is the union of two curves that intersect transversally and do not intersect at infinity.

Remark 3.1. The right-hand side of the “ \Leftrightarrow ” equivalences in Theorem 1.3 is always satisfied in the cases described in Theorem 1.1.

Theorem 1.3 is a consequence of Lemmas 3.6, 3.8, and 3.9. Before we prove them, let us write down some facts that will be used throughout the section.

Proposition 3.2 [10, Remark 3.3, Remark 3.9, Proposition 5.1]. *If C is an irreducible curve, then*

$$\delta_0(C) = 0 \iff G_r^{(1)} = G_r^{(2)} \iff \delta_n(C) = 0 \text{ for all } n \geq 0.$$

Remark 3.3. There are three curves in the statement of Theorem 1.3, namely, C , C' , and C'' . We will use $'$ or $''$ to refer to the objects corresponding to C' and C'' , respectively. For example, $U' := \mathbb{C}^2 \setminus C'$, $G'' := \pi_1(U'')$, and so forth.

Theorem 3.4 The Oka–Sakamoto theorem, [17]. Let $C = C' \cup C'' \subset \mathbb{C}^2$ be the union of two affine plane curves, with $\deg C' = m'$ and $\deg C'' = m''$. Suppose that $C' \cap C''$ consists on $m'm''$ distinct points in \mathbb{C}^2 . Then, $G \cong G' \times G''$.

Remark 3.5. In the conditions of the Oka–Sakamoto theorem, we can consider a presentation for G with generators $a_1, \dots, a_{m'}, b_1, \dots, b_{m''}$, where the a_i s are a choice of positively oriented meridians around irreducible components of C' generating G' , and the b_j s are a choice of positively oriented meridians around irreducible components of C'' generating G'' [13]. Recall that a_i and a_k are conjugate in $\pi_1(\mathbb{C}^2 \setminus C')$ if and only if they both correspond to positively oriented meridians about the same irreducible component of C' , and the analogous statement holds for the b s in $\pi_1(\mathbb{C}^2 \setminus C'')$ (see, for instance, [3, Proposition 1.34]). Let R' and R'' be a set of relations of a presentation of G' and G'' where the generators are the a s and b s, respectively. Then, we have the following presentation for G :

$$G = \langle a_1, \dots, a_{m'}, b_1, \dots, b_{m''} \mid [a_i, b_j] = 1 \text{ for all } i = 1, \dots, m' \text{ and } j = 1, \dots, m''; R'; R'' \rangle.$$

The first of our three key lemmas deals with the 0-th order degree of a union of two transversal irreducible curves.

Lemma 3.6. Let $C = C' \cup C'' \subset \mathbb{C}^2$ be as in Theorem 3.4. Moreover, suppose that both C' and C'' are irreducible. Then, $\delta_0(C) = 0$.

Proof. By Theorem 3.4, $G \cong G' \times G''$. We have that

$$G_r^{(1)}/G_r^{(2)} \cong (G'_r)^{(1)}/(G'_r)^{(2)} \times (G''_r)^{(1)}/(G''_r)^{(2)},$$

By Equation (2.2), one has:

$$\delta_n(C) = \text{rk}_{\mathbb{K}_n} \left(G_r^{(n+1)}/[G_r^{(n+1)}, G_r^{(n+1)}] \otimes_{\mathbb{Z}\bar{\Gamma}_n} \mathbb{K}_n \right).$$

Notice that the tensor product kills the \mathbb{Z} -torsion, so this is equivalent to

$$\delta_n(C) = \text{rk}_{\mathbb{K}_n} \left(G_r^{(n+1)}/G_r^{(n+2)} \otimes_{\mathbb{Z}\bar{\Gamma}_n} \mathbb{K}_n \right). \quad (3.1)$$

Note that $\mathbb{Z}\bar{\Gamma}_0 \cong \mathbb{Z}[t^{\pm 1}]$ in this case, and hence $\mathbb{K}_0 \cong Q(\mathbb{Z}[t^{\pm 1}])$, where $Q(R)$ denotes the field of fractions of R . Since both C' and C'' are irreducible, we have that $\Gamma'_0 \cong \Gamma''_0 \cong \mathbb{Z}$, $\bar{\Gamma}'_0 \cong \bar{\Gamma}''_0$ are the trivial group, and $\mathbb{K}'_0 \cong \mathbb{K}''_0 \cong \mathbb{Q}$. By Proposition 3.2, δ_n of any irreducible curve is finite for all $n \geq 0$, thus

$$\delta_0(C') = \text{rk}_{\mathbb{Q}} \left((G'_r)^{(1)}/(G'_r)^{(2)} \otimes_{\mathbb{Z}} \mathbb{Q} \right) < \infty$$

and the same statement holds for C'' , which means that $(G'_r)^{(1)}/(G'_r)^{(2)}$ and $(G''_r)^{(1)}/(G''_r)^{(2)}$ are both finite rank free abelian groups. Let us call them A and B for simplicity.

Now,

$$\delta_0(C) = \text{rk}_{Q(\mathbb{Z}[t^{\pm 1}])} \left((A \oplus B) \otimes_{\mathbb{Z}[t^{\pm 1}]} Q(\mathbb{Z}[t^{\pm 1}]) \right).$$

Let $a \in A$, and let k be an integer bigger than the rank of A . We have that a, at, \dots, at^k are linearly dependent, so a is annihilated by some polynomial in $\mathbb{Z}[t^{\pm 1}]$. The same holds for all $b \in B$. Hence, $\delta_0(C) = 0$. \square

The following result will be used in the proofs of Lemmas 3.8 and 3.9.

Lemma 3.7. Let $C = C' \cup C''$ be as in Theorem 3.4. Suppose moreover that C' is irreducible. Under the notation in Remark 3.5, let $y_j = b_j a_1^{-1}$ for all $j = 1, \dots, m''$. Let $f_n : \Gamma_n'' \rightarrow \bar{\Gamma}_n$ be the group homomorphism defined on generators by $f_n(b_j) = y_j$ for all $j = 1, \dots, m''$.

Then, f_n is an isomorphism if $n = 0$. Moreover, if $\delta_0(C') = 0$, f_n is an isomorphism for all $n \geq 0$.

Proof. First of all, notice that $\psi(b_j a_1^{-1}) = 0$ for all j , that is, $y_j \in \bar{\Gamma}_n$. f_n is induced by the group homomorphism $f : G'' \rightarrow G$ that takes b_j to $b_j a_1^{-1}$, which can be seen to be well defined using the presentations of the groups G'' and G described in Remark 3.5. Hence, f_n is a well-defined homomorphism for all n .

Theorem 3.4 induces an isomorphism $F_n : \Gamma'_n \times \Gamma''_n \rightarrow \Gamma_n$. Hence, one obtains a monomorphism $\Gamma''_n \hookrightarrow \Gamma_n$, which takes b_j to b_j for all j . Since a_1 and b_j commute in Γ_n for all j , f_n is injective for all n .

Since C' is irreducible, $\Gamma'_0 \cong \mathbb{Z}$ is generated by a_1 . Moreover, if $\delta_0(C') = 0$, $\Gamma'_0 = \Gamma'_n$ for all $n \geq 0$ by Proposition 3.2. Hence, to conclude the proof of the lemma, it suffices to show that f_n is surjective if $\Gamma'_n \cong \mathbb{Z}$ is generated by a_1 .

Suppose that $\Gamma'_n \cong \mathbb{Z}$ is generated by a_1 . Using the isomorphism F_n , we can write any element in Γ_n as wa_1^k , where $k \in \mathbb{Z}$ and w is a word on the y_j s. Note that $\psi(wa_1^k) = k$, so $wa_1^k \in \bar{\Gamma}_n$ if and only if $k = 0$. Hence, f_n is an epimorphism. \square

The proofs of Lemmas 3.8 and 3.9 consist on applying the techniques of Section 2.3 to conveniently chosen presentations of the fundamental group.

Lemma 3.8. *Let n be a fixed integer, with $n \geq 0$. Let $C = C' \cup C'' \subset \mathbb{C}^2$ be as in Theorem 3.4. Moreover, suppose that C' is irreducible, with $\delta_0(C') = 0$. Then, $\delta_n(C) \leq m'' - 1$, and $\delta_n(C) = 0$ if and only if $\delta_n(C'') < \infty$.*

Proof. By Theorem 3.4, $G \cong G' \times G''$. We first consider the case where C'' is also an irreducible curve such that $\delta_0(C'') = 0$. In this situation, we know that $\delta_n(C'') = 0$ for all $n \geq 0$, and, in fact, the stronger statement $(G'')_r^{(1)} = (G'')_r^{(2)}$ holds (Proposition 3.2). Since G is the direct product of G' and G'' , we have that

$$G_r^{(n+1)} / G_r^{(n+2)} \cong (G')_r^{(n+1)} / (G')_r^{(n+2)} \times (G'')_r^{(n+1)} / (G'')_r^{(n+2)},$$

which is the trivial group for all $n \geq 0$. By Equation (3.1), one obtains that $\delta_n(C) = 0$ for all $n \geq 0$.

From now on, we assume that C'' is either not irreducible, or if it is irreducible, then $\delta_0(C'') \neq 0$.

We consider the presentation of G described in Remark 3.5. Since $(G')_r^{(1)} = (G')_r^{(2)}$ (Proposition 3.2), $a_i a_k^{-1} = 1$ in $\mathbb{Z}\Gamma_n$ for all $i, k \in \{1, \dots, m'\}$, $n \geq 0$.

From now on, n is some integer such that $n \geq 1$ if C'' is an irreducible curve with $\delta_0(C'') \neq 0$, and $n \geq 0$ if C'' is not irreducible. Note that, if C'' is irreducible, the result for $n = 0$ is already proved in Lemma 3.6.

Let $x_1 = a_1$, and $x_i = a_i a_1^{-1}$ for all $i = 2, \dots, m'$. Let $y_j = b_j a_1^{-1}$ for all $j = 1, \dots, m''$. We obtain the presentation

$$G = \langle x_1, \dots, x_{m'}, y_1, \dots, y_{m''} \mid r_j, s_{i,j} \text{ for all } i = 2, \dots, m', j = 1, \dots, m''; \tilde{R}'; \tilde{R}'' \rangle,$$

where $r_j = [x_1, y_j]$ and $s_{i,j} = x_i x_1 y_j x_i^{-1} x_1^{-1} y_j^{-1}$ for all $i = 2, \dots, m'$, $j = 1, \dots, m''$, \tilde{R}' are some relations in $x_1, \dots, x_{m'}$, and \tilde{R}'' are the same relations as R'' if we switch the letter b_j for y_j , for all $j = 1, \dots, m''$. Indeed, if we plug in $y_j x_1$ for b_j in the relations R'' , the x_1 s cancel out because they commute with all the y_j s and because the linking number homomorphism takes any word in the b letters to the sum of the exponents appearing on that word, so the sum of the exponents of words in R'' must be zero.

We may assume by reordering that $y_1 \neq y_2$ in Γ_n , where $n \geq 1$ if C'' is irreducible, and $n \geq 0$ otherwise. Let us see this. Indeed, if C'' is not irreducible, this amounts to b_1 and b_2 being positively oriented meridians around different irreducible components of C'' , which we can assume after reordering. If C'' is irreducible but $\delta_0(C'') \neq 0$, Proposition 3.2 says that $(G'')_r^{(2)} \subsetneq (G'')_r^{(1)}$, which implies that there exist $j \neq l$ in $\{1, \dots, m''\}$ such that $b_j b_l^{-1} \neq 1$ in Γ''_1 . Reordering, we may assume that $j = 1$ and $l = 2$, and hence $y_1 \neq y_2$ in Γ_n .

Consider the involution of the matrix of Fox derivatives for this presentation of G with coefficients in $\mathbb{Z}\Gamma_n$ ($B(n)$ in the notation of Section 2.3),

$$\left(\begin{array}{c|c|c|c|c|c} -(1 - y_j^{-1}) - & -(x_2^{-1} - y_j^{-1}) - & \cdots & -(x_{m'}^{-1} - y_j^{-1}) - & & \\ -0 - & -(1 - x_1^{-1} y_j^{-1}) - & \cdots & -0 - & & \\ -0 - & -0 - & \cdots & -0 - & A' & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ -0 - & -0 - & \cdots & -(1 - x_1^{-1} y_j^{-1}) - & & \\ \hline (x_1^{-1} - 1)I_{m''} & (x_1^{-1} x_2^{-1} - 1)I_{m''} & \cdots & (x_1^{-1} x_{m'}^{-1} - 1)I_{m''} & 0 & A'' \end{array} \right),$$

where “ $-z_j-$ ” denotes a row of m'' elements whose j -th entry is z_j for $j = 1, \dots, m''$, and $I_{m''}$ is the identity matrix of dimension $m'' \times m''$. The columns of this matrix correspond to the relations appearing in the presentation of G , the first $m' \cdot m''$ of them being $r_1, \dots, r_{m''}, s_{2,1}, \dots, s_{2,m''}, \dots, s_{m',1}, \dots, s_{m',m''}$, in order. A' is the matrix corresponding to the relations \bar{R}' , and A'' is the matrix that computes $\delta_n(C'')$ with coefficients in $\mathbb{Z}\Gamma_n''$, which is identified with $\mathbb{Z}\bar{\Gamma}_n$ by the isomorphism of groups $f_n : \Gamma_n'' \rightarrow \bar{\Gamma}_n$ of Lemma 3.7. For example, the entry in the first row and column is $\frac{\partial[x_1, y_1]}{\partial x_1} = 1 - y_1 = 1 - y_1^{-1}$. Some entries in the matrix above have been simplified using the relations in the presentation of G . For example, the entry in the first row and $(m'' + 1)$ -th column corresponds to the involution of $\frac{\partial s_{2,1}}{\partial x_1} = x_2 - x_2 x_1 y_1 x_2^{-1} x_1^{-1} = x_2 - s_{2,1} y_2 = x_2 - y_2$ (as an element in $\mathbb{Z}\Gamma_n$).

First, note that $x_i = 1$ in Γ_n , for all $i = 2, \dots, m'$. In addition, the left part of this matrix consists on m' blocks of dimensions $(m' + m'') \times m''$. We subtract the i -th column to the i -th column of the j -th block, for all $i = 1, \dots, m'', j = 2, \dots, m'$, to get

$$\left(\begin{array}{c|c|c|c|c|c} \begin{array}{c} -(1 - y_j^{-1}) - \\ -0 - \\ -0 - \\ \vdots \\ -0 - \end{array} & \begin{array}{c} -0 - \\ -(1 - x_1^{-1} y_j^{-1}) - \\ -0 - \\ \vdots \\ -0 - \end{array} & \begin{array}{c} \cdots \\ \cdots \\ \cdots \\ \vdots \\ \cdots \end{array} & \begin{array}{c} -0 - \\ -0 - \\ -0 - \\ \vdots \\ -(1 - x_1^{-1} y_j^{-1}) - \end{array} & \begin{array}{c} \\ \\ A' \\ \\ \end{array} & \begin{array}{c} \\ \\ 0 \\ \\ \end{array} \\ \hline \begin{array}{c} (x_1^{-1} - 1)I_{m''} \\ \\ \\ \end{array} & \begin{array}{c} 0 \\ \\ \\ \end{array} & \begin{array}{c} \cdots \\ \cdots \\ \cdots \\ \vdots \\ \cdots \end{array} & \begin{array}{c} 0 \\ \\ \\ \end{array} & \begin{array}{c} \\ \\ 0 \\ \\ \end{array} & \begin{array}{c} \\ \\ A'' \\ \\ \end{array} \end{array} \right). \quad (3.2)$$

Note that $1 - y_j^{-1} \neq 0$ in $\mathbb{Z}\bar{\Gamma}_n$ for any $j = 1, \dots, m''$, since $1 - y_j^{-1} \neq 0$ in $\mathbb{Z}\bar{\Gamma}_0$. We multiply row $m' + 1$ by $1 - y_1^{-1}$ on the left, and add to it the first row times $1 - x_1^{-1}$, and the $(m' + j)$ -th row times $1 - y_j^{-1}$ for all $j = 2, \dots, m''$, to get

$$\left(\begin{array}{c|c|c|c|c|c} \begin{array}{c} -(1 - y_j^{-1}) - \\ -0 - \\ -0 - \\ \vdots \\ -0 - \\ -0 - \end{array} & \begin{array}{c} -0 - \\ -(1 - x_1^{-1} y_j^{-1}) - \\ -0 - \\ \vdots \\ -0 - \\ -0 - \end{array} & \begin{array}{c} \cdots \\ \cdots \\ \cdots \\ \vdots \\ \cdots \\ \cdots \end{array} & \begin{array}{c} -0 - \\ -0 - \\ -0 - \\ \vdots \\ -(1 - x_1^{-1} y_j^{-1}) - \\ -0 - \end{array} & \begin{array}{c} \\ \\ A' \\ \\ (1 - x_1^{-1})a'_{1,*} \\ \end{array} & \begin{array}{c} \\ \\ 0 \\ \\ \\ \end{array} \\ \hline \begin{array}{c} | \\ 0 \\ | \end{array} \begin{array}{c} (x_1^{-1} - 1)I_{m''} \\ \\ \end{array} & \begin{array}{c} 0 \\ \\ \end{array} & \begin{array}{c} \cdots \\ \cdots \end{array} & \begin{array}{c} 0 \\ \\ \end{array} & \begin{array}{c} 0 \\ \\ \end{array} & \begin{array}{c} \\ \\ A'' \\ \end{array} \end{array} \right),$$

where $a'_{1,*}$ is the first row of A' , so its entries are polynomials in $\mathbb{Z}[x_1]$, which commute with elements of \mathbb{K}_n , which is identified by f_n with \mathcal{K}_n'' .

We now focus on the second to m' -th blocks of size $m' \times m''$ at the top of the matrix. We can multiply the j -th column (on the right) of each of these blocks by y_j for all $j = 1, \dots, m''$, and subtract the second from the first column of each of these blocks to get $y_1 - y_2$ as the first entry and $y_j - x_1^{-1}$ as the j -th entry of the k -th row of the k -th block, where $k = 2, \dots, m', j = 2, \dots, m''$. Note that $y_1 \neq y_2$ in $\mathbb{Z}\bar{\Gamma}_n$, so $y_1 - y_2$ has an inverse in R_n . Now, we multiply the first column (on the right) by the inverse of $1 - y_1^{-1}$, and the first column of the j -th block of size $m' \times m''$ by the inverse of $y_1 - y_2$ for all $j = 2, \dots, m''$. Reordering the columns, putting the ones corresponding to the first column of every $m' \times m''$ block first, we get

$$\left(\begin{array}{c|c|c|c|c} I_{m'} & 0 & * & A' & * \\ \hline & & & (1 - x_1^{-1})a'_{1,*} & \\ \hline 0 & A'' & B & 0 & 0 \end{array} \right). \quad (3.3)$$

where B is the matrix

$$\begin{pmatrix} -0- \\ (x_1^{-1} - 1)I_{m''-1} \end{pmatrix}.$$

Hence, performing column operations we can turn matrix (3.3) into

$$\left(\begin{array}{c|c|c|c} I_{m'} & 0 & 0 & 0 \\ \hline 0 & A'' & B & 0 \end{array} \middle| \begin{array}{c} 0 \\ (1 - x_1^{-1})a'_{1,*} \\ 0 \end{array} \right). \quad (3.4)$$

Let k be the rank of the left \mathcal{K}_n'' -module spanned by the rows of A'' . By Proposition 2.8, k is at most $m'' - 1$, and k is equal to $m'' - 1$ if and only if $\delta_n(C'') < \infty$. Identifying \mathcal{K}_n'' with \mathbb{K}_n by f_n , we get that the rank of the left \mathbb{K}_n -module spanned by the rows of A'' is k as well. Hence, doing row and column operations in \mathcal{K}_n'' , and noting that x_1 commutes with \mathcal{K}_n'' in R_n , we can turn the matrix (3.4) into

$$\left(\begin{array}{c|c|c|c} I_{m'} & 0 & 0 & 0 \\ \hline 0 & I_k & & \\ \hline 0 & 0 & (x_1^{-1} - 1)\tilde{B} & (x_1^{-1} - 1)E \end{array} \middle| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right),$$

where \tilde{B} is an $m'' \times m''$ matrix in \mathcal{K}_n'' such that the rank of the left \mathcal{K}_n'' -module spanned by its rows is $m'' - 1$, and E is a matrix with entries in R_n . In particular, the rank of the left \mathcal{K}_n'' -module spanned by the last $m'' - k$ rows of \tilde{B} is greater than or equal to $m'' - k - 1$, and at most $m'' - k$. Let us denote by D and F the matrices formed by the last $m'' - k$ rows of \tilde{B} and E , respectively. Performing column operations, one obtains

$$\left(\begin{array}{c|c|c|c} I_{m'} & 0 & 0 & 0 \\ \hline 0 & I_k & 0 & 0 \\ \hline 0 & 0 & (x_1^{-1} - 1)D & (x_1^{-1} - 1)F \end{array} \middle| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right). \quad (3.5)$$

By Lemma 2.7, the rank of the left \mathcal{K}_n -module generated by the rows of this matrix should be less than or equal to $m' + m'' - 1$, which rules out the possibility of the rank of the left \mathcal{K}_n'' -module spanned by the rows of D being $m'' - k$. Hence, the rank of the left \mathcal{K}_n'' -module spanned by the rows of D is $m'' - k - 1$. If we keep doing row and column operations to D in \mathcal{K}_n'' , and perhaps permuting some of the last $m'' - k$ rows of matrix (3.5) at the end, one obtains

$$\left(\begin{array}{c|c|c|c} I_{m'} & 0 & 0 & 0 \\ \hline 0 & I_k & 0 & 0 \\ \hline 0 & 0 & (x_1^{-1} - 1)I_{m''-k-1} & \\ \hline -0- & -0- & -0- & (1 - x_1^{-1}) * \end{array} \right),$$

where $*$ is a matrix in R_n . But again, by rank considerations, the last row of this matrix must be identically 0. Performing column operations, we can turn $(1 - x_1^{-1}) *$ into the zero matrix. Hence, $\delta_n(C) = m'' - k - 1 \leq m'' - 1$. This means that $\delta_n(C) = 0$ if and only if $\delta_n(C'')$ is finite. \square

Lemma 3.9. *Let $C = C' \cup C'' \subset \mathbb{C}^2$ be as in Theorem 3.4. Moreover, assume that neither C' nor C'' are irreducible with $\delta_0 = 0$. Then,*

$$\delta_n(C) = 0 \text{ for all } n \geq 1.$$

Furthermore, if either both C' and C'' are not irreducible, or both irreducible, the equality holds for all $n \geq 0$. However, if one of the curves, say C' , is irreducible and the other one (C'') is not, then $\delta_0(C) \leq m'' - 1$, and

$$\delta_0(C) = 0 \Leftrightarrow \delta_0(C'') < \infty.$$

Proof. If both C' and C'' are irreducible, the result for $n = 0$ follows from Lemma 3.6.

We consider the presentation of G described in Remark 3.5. If C' is not irreducible, we can assume that $a_1 \neq a_2$ in Γ_0 by reordering, so $a_2 a_1^{-1} \neq 1$ in $\mathbb{Z}\bar{\Gamma}_n$ for any $n \geq 0$. Similarly, if C'' is not irreducible, we can assume that $b_2 b_1^{-1} \neq 1$ in $\mathbb{Z}\bar{\Gamma}_n$ for any $n \geq 0$. After reordering, we may assume the same condition if C'' is irreducible with $\delta_0(C'') \neq 0$ (resp. C'), but this time for $n \geq 1$, as justified in the proof of Lemma 3.8.

We deal with the case when n is some integer greater than or equal to 1 if either C' or C'' are irreducible, and ≥ 0 otherwise.

Let $x_1 = a_1$, and $x_i = a_i a_1^{-1}$ for all $i = 2, \dots, m'$. Let $y_1 = b_1$, and $y_j = b_j b_1^{-1}$ for all $j = 2, \dots, m''$. We obtain the presentation

$$G = \langle x_1, \dots, x_{m'}, y_1, \dots, y_{m''} \mid [x_i, y_j] = 1 \text{ for all } i = 1, \dots, m' \text{ and } j = 1, \dots, m''; \tilde{R}'; \tilde{R}'' \rangle,$$

where \tilde{R}' (resp. \tilde{R}'') are the defining relations for $\pi_1(\mathbb{C}^2 \setminus C')$ (resp. $\pi_1(\mathbb{C}^2 \setminus C'')$) in $x_1, \dots, x_{m'}$ (resp. $y_1, \dots, y_{m''}$).

Consider the matrix $B(n)$ described in Section 2.3, that is,

$$\left(\begin{array}{c|c|c|c|c} -(1 - y_j^{-1}) - & -0 - & \cdots & -0 - & \\ -0 - & -(1 - y_j^{-1}) - & \cdots & -0 - & \\ \vdots & \vdots & \cdots & \vdots & * \\ -0 - & -0 - & \cdots & -(1 - y_j^{-1}) - & \\ \hline (x_1^{-1} - 1)I_{m''} & (x_2^{-1} - 1)I_{m''} & \cdots & (x_{m'}^{-1} - 1)I_{m''} & * \end{array} \right), \quad (3.6)$$

where the rightmost columns correspond to \tilde{R}' and \tilde{R}'' .

Note that $x_2^{-1} - 1$ is nonzero in $\mathbb{Z}\bar{\Gamma}_n$ if $n \geq 1$, and, if C' is not irreducible, also for $n = 0$. We begin by multiplying the last m'' rows by the inverse of $x_2^{-1} - 1$ (on the left), and then, by performing column operations, one obtains

$$\left(\begin{array}{c|c|c|c|c} -(1 - y_j^{-1}) - & -0 - & \cdots & -0 - & \\ - * - & -(1 - y_j^{-1}) - & \cdots & - * - & \\ \vdots & \vdots & \cdots & \vdots & * \\ -0 - & -0 - & \cdots & -(1 - y_j^{-1}) - & \\ \hline 0 & I_{m''} & \cdots & 0 & 0 \end{array} \right).$$

Hence, we may compute $\delta_n(C)$ using the matrix formed by the first m' rows of the matrix above without the columns of the second block. This new matrix consists on $m' - 1$ blocks of $m' \times m''$ matrices, plus another matrix at the end, represented by the rightmost submatrix after the last vertical line. One can permute the first and second rows in this new matrix of m' rows, and then permute columns so that the first m' columns of the resulting matrix are the second columns of each of the first $(m' - 1)$ blocks of size $m' \times m''$. This way one obtains a matrix of the form,

$$\left(\begin{array}{c|c} * & * \\ \hline (1 - y_2^{-1})I_{m'-1} & * \end{array} \right).$$

Note that $1 - y_2^{-1}$ is nonzero in $\mathbb{Z}\bar{\Gamma}_n$ for $n \geq 1$, and, if C'' is not irreducible, also in $\mathbb{Z}\bar{\Gamma}_0$. Finally, multiplying each row on the left by the inverse of $1 - y_2^{-1}$, and performing column and row operations, we see that $\delta_n(C) = 0$.

Finally, we consider the case when $n = 0$, C' is irreducible, and C'' is not irreducible. The proof of this is done by considering the same presentation for G as the one explained in the proof of Lemma 3.8, and following the same computations done there, the only difference being that $n = 0$ in this case and that, since C'' is not irreducible, we know that $y_1 \neq y_2$ in $\mathbb{Z}\bar{\Gamma}_0$. In particular, we can apply Lemma 3.7. Note that $x_i = 1$ in $\mathbb{Z}\bar{\Gamma}_0$ for $i = 2, \dots, m'$ because C' is irreducible. Using the same notation as in the proof of Lemma 3.8, it follows that $\delta_0(C) = m'' - k - 1 \leq m'' - 1$, where k is the rank of the left \mathcal{K}_0'' -module spanned by the rows of A'' , and $\delta_0(C'')$ is finite if and only if $k = m'' - 1$. This means that $\delta_0(C) = 0$ if and only if $\delta_0(C'')$ is finite. \square

Proof of Theorem 1.3. Part (1) is divided into two cases, depending on whether or not both curves are irreducible. If both curves are irreducible, Lemma 3.6 ends the proof for $n = 0$. If $n \geq 1$ and $\delta_0(C') = 0$, then Lemma 3.8 applies since $\delta_n(C'') < \infty$ by Theorem 1.1(1) (and analogously if $\delta_0(C'') = 0$). If $n \geq 1$, $\delta_0(C') \neq 0$, and $\delta_0(C'') \neq 0$, then it follows from Lemma 3.9. Otherwise, if both curves are not irreducible, it is a consequence of Lemma 3.9.

Part (a) follows from Lemma 3.9 and part (b) follows from Lemma 3.8. Finally, the bound $\delta_0(C) \leq m'' - 1$ is also stated in these lemmas. \square

Example 3.10. Let C' be an irreducible curve such that $\delta_0(C') \neq 0$. For example, we can take C' to be the cuspidal cubic, which has $\delta_0(C') = 2$, and $\delta_n(C') = 1$ for $n \geq 1$ [19, Example 9.8]. Let C'' be a collection of m'' parallel lines, each of which intersects C' in three distinct points. Let $C = C' \cup C''$. Then, following the proof and notations of Lemma 3.9, it follows that

$$\begin{cases} \delta_0(C) = m'' - 1 \\ \delta_n(C) = 0 \end{cases} \quad \text{for all } n \geq 1.$$

Indeed, the last paragraph of the proof of Lemma 3.9 shows that $\delta_0(C) = m'' - k - 1$, where k is the rank of the left \mathcal{K}_0'' -module spanned by the rows of A'' . In this case, the fundamental group of the complement to m'' parallel lines is the free group on m'' generators, and thus it has a presentation with no relations. Hence, A'' is the empty matrix, which implies $k = 0$ and $\delta_0(C) = m'' - 1$.

This example shows that δ_0 and δ_n can differ by arbitrarily large numbers, for $n \geq 1$. This cannot happen in the case of knots ([2]), where $\delta_0 \leq \delta_1 + 1 \leq \delta_2 + 1 \leq \dots$

4 | RESTATEMENT OF THE MAIN THEOREM IN TERMS OF MULTIVARIABLE ALEXANDER POLYNOMIALS

We start by recalling the relationship between the Alexander polynomials of a plane curve C and $\delta_0(C)$. If C is irreducible, the result below appears in [10, Remark 3.9], and the nonirreducible case was done in [6, Theorem 5.18].

Theorem 4.1. *Let $C \subset \mathbb{C}^2$ be a plane curve with s irreducible components. Then,*

$$\delta_0(C) = \deg \Delta_C^{\text{multi}}(t_1, \dots, t_s).$$

Remark 4.2. Conceptually, $\delta_0(C)$ is defined using the Alexander invariant, which is the homology of U as a module, whereas Δ_C^{multi} is an invariant of the Alexander module, which is the relative homology of U at a base point viewed as a module. However, the difference in the invariants of both modules can only be detected in prime ideals of length greater than one. This explains why this difference has no effect in the degree of Δ_C^{multi} .

Remark 4.3. We are using the convention $\deg 0 = \infty$. The proof of [6, Theorem 5.18] assumes $\delta_0(C)$ is finite, but the result is also true for $\delta_0(C) = \infty$ because

$$\delta_0(C) = \infty \Leftrightarrow F_1(H_1(U, u_0; R_0)) = 0 \Leftrightarrow F_1(H_1(U, u_0; \mathbb{Z}\Gamma_0)) = 0 \Leftrightarrow \Delta_C^{\text{multi}} = 0.$$

In this list of equivalences, we have used that the projection $G \rightarrow \Gamma_0$ is the abelianization morphism and that R_0 is flat as a $\mathbb{Z}\Gamma_0$ -module. Note that the leftmost equivalence is a consequence of Proposition 2.8.

Remark 4.4. Recall that Δ_C^{multi} is well defined up to a unit in $\mathbb{Z}\Gamma_0$. Theorem 4.1 tells us that $\delta_0(C) = 0$ if and only if Δ_C^{multi} has a representative, which is a nonzero homogeneous polynomial. Furthermore, under the hypotheses of Theorem 1.3, we claim that $\delta_0(C) = 0$ if and only if $\Delta_C^{\text{multi}} = 1$. Indeed, with the notation of Section 2.3, $B(0)$ is a presentation matrix for $H_1(U, u_0; \mathbb{Z}\Gamma_0) = H_1(U, u_0; \mathcal{L}^{\text{Ab}})$. Hence, Δ_C^{multi} is the gcd of the codimension 1 minors of $B(0)$, and the matrix $B(0)$ in Equation (3.6) has a codimension 1 minor whose only homogeneous factor is 1 (up to a unit). Namely, the minor that we are referring to is the one formed by the columns 1 through m'' , the columns $km'' + 1$, for $k = 1, \dots, m' - 1$, and the last

$m' + m'' - 1$ rows. The submatrix formed by those rows and columns is

$$\left(\begin{array}{c|cccc} 0 & & & & (1 - y_1^{-1})I_{m'-1} \\ \hline & x_2^{-1} - 1 & x_3^{-1} - 1 & \dots & x_{m'}^{-1} - 1 \\ \hline (x_1^{-1} - 1)I_{m''} & & & & 0 \end{array} \right),$$

where x_1 (seen in Γ_0) is the image of a meridian around an irreducible component of C' , and y_1 (as an element in Γ_0) is the image of a meridian around an irreducible component of C'' . Hence, reordering the variables, we can assume that $x_1 = t_1$ and $y_1 = t_2$. Thus, the relevant minor is, up to a sign, a product of powers of $(1 - t_1^{-1})$ and $(1 - t_2^{-1})$. Therefore, for curves C satisfying the hypotheses of Theorem 1.3, one has that

$$\delta_0(C) = 0 \Leftrightarrow \Delta_C^{\text{multi}} = 1.$$

Note that one does not need any hypotheses on the irreducibility of C' or C'' , or on the value of $\delta_0(C')$, to be able to use this minor for the purposes of this remark. Those hypotheses are used in the proof of Lemma 3.9 after the appearance of Equation (3.6).

Now, we characterize the plane curves with infinite $\delta_0(C)$.

Lemma 4.5. *Let C be a plane curve, with s irreducible components C_1, \dots, C_s . Then, the following are equivalent:*

1. $\delta_0(C) = \infty$.
2. $s \geq 2$ and C_i is the zero set of a polynomial of the form $f(x, y) + \lambda_i$ for all $1 \leq i \leq s$, where $f(x, y) \in \mathbb{C}[x, y]$ is a polynomial of degree $d \geq 1$, and $\lambda_i \in \mathbb{C}$ for all $1 \leq i \leq s$.
3. There exists an epimorphism $G \twoheadrightarrow \mathbb{F}_s$ onto the free group of rank $s \geq 2$.

We will refer to condition (2) as C being of *affine pencil type*.

Proof. Note that by Theorem 1.1, $\delta_0(C) = \infty \Rightarrow s \geq 2$.

With the notation of Section 2.3, $B(0)$ is a presentation matrix for $H_1(U, u_0; \mathbb{Z}\Gamma_0)$. Let $\mathcal{V}_1(U)$ be the first homology jump loci of U , namely,

$$\mathcal{V}_1(U) = \{\rho \in \text{Hom}(G, \mathbb{C}^*) \mid H_1(U, \mathbb{C}_\rho) \neq 0\},$$

where \mathbb{C}_ρ is the rank one \mathbb{C} -local system on U induced by ρ . By [6, Remark 5.14], one has the following:

$$\mathcal{V}_1(U) = (\mathbb{C}^*)^s \Leftrightarrow \text{all the codimension 1 minors of } B(0) \text{ are 0.}$$

This last condition is equivalent to the rank of the left \mathcal{K}_0 -module generated by the rows of $B(0)$ being strictly smaller than $m - 1$, which by Remark 2.3 is equivalent to $\delta_0(C) = \infty$.

Let \bar{C} be the projective completion of C , and let D be the curve in \mathbb{P}^2 defined by $D = \bar{C} \cup L_\infty$, where L_∞ is the line at infinity. By [5, Theorem 4.1], the condition $\mathcal{V}_1(U) = (\mathbb{C}^*)^s$ (which can be reformulated in terms of cohomology jump loci by [4, p. 50, (2.1)]) is equivalent to the existence of a primitive pencil $C_{[\alpha_1 : \alpha_2]} = \alpha_1 P_1(x, y, z) + \alpha_2 P_2(x, y, z)$ of plane curves on \mathbb{P}^2 having $s + 1$ fibers (corresponding to $s + 1$ different $[\alpha_1 : \alpha_2] \in \mathbb{P}^1$) whose reduced support forms a partition of the set of $s + 1$ irreducible components of D . Hence, the reduced support of those $s + 1$ fibers must be in one-to-one correspondence with the irreducible components of D , so we may write the pencil in the form $\beta_1 F(x, y, z) + \beta_2 z^d$, where $F(x, y, z)$ is a degree d irreducible polynomial in $\mathbb{C}[x, y, z]$ and $[\beta_1 : \beta_2] \in \mathbb{P}^1$. Restricting to the affine part (making $z = 1$) yields (1) \Leftrightarrow (2).

For (2) \Rightarrow (3), we see that the polynomial f induces a map $U \rightarrow \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_s\}$, which in turn induces the desired epimorphism $G \rightarrow \mathbb{F}_s$ in fundamental groups. By [15, Lemma 1.2.1] part (3) implies $\mathcal{V}_1(U) = (\mathbb{C}^*)^s$ and hence the argument above implies part (2). \square

As a corollary of Theorem 1.3, Theorem 4.1, Remark 4.4, and Lemma 4.5, one obtains Corollary 1.4, whose proof is below.

Proof of Corollary 1.4. Part (1) follows from Theorem 1.3(1) and Remark 4.4. Part (2) follows from Remark 4.4 if $\delta_0(C) = 0$. Otherwise, if $\delta_0(C) \neq 0$, then Theorem 1.3(2) and Lemma 4.5 imply that C' is irreducible and C'' is of affine pencil type.

In that case, note that Δ_C^{multi} can be computed using the matrix from Equation (3.2), since the operations performed to $B(0)$ in order to obtain (3.2) are all allowed in $\mathbb{Z}\Gamma_0 \cong \mathbb{Z}[t^{\pm 1}, t_1^{\pm 1}, \dots, t_s^{\pm 1}]$, where $s \geq 2$ is the number of components of C'' . The abelianization morphism identifies x_1 with t and y_j with $t_j t^{-1}$ in Equation (3.2) for $j = 1, \dots, s$ (possibly after reordering). Note that $(t_j - 1)^{m'-1}(1 - t)^{m''}$ are $(m' + m'' - 1)$ -minors of (3.2) for all $j = 1, \dots, s$, up to a unit in $\mathbb{Z}[t^{\pm 1}, t_1^{\pm 1}, \dots, t_s^{\pm 1}]$. Indeed, the j -th of these minors is the one formed by the columns 1 through m'' , columns $km'' + j$ for $k = 1, \dots, m' - 1$, and the last $m' + m'' - 1$ rows. Hence, Δ_C^{multi} divides the greatest common divisor of all these minors, which is $(t - 1)^{m''}$. In particular, Δ_C^{multi} is a power of $t - 1$ and, by Theorem 4.1, it has degree $\delta_0(C)$. Thus, $\Delta_C^{\text{multi}} = (t - 1)^{\delta_0(C)}$, as desired. \square

Remark 4.6. The higher order degrees depend on the linking number homomorphism, which, if the curve is not irreducible, is not an invariant of the fundamental group of the curve complement. However, the multivariable Alexander polynomial only depends on the fundamental group (up to a change of basis in the variables). Thus, Corollary 1.4 gives us direct restrictions for which groups can be realized as fundamental groups of the complement of a union of transversal plane curves, and those restrictions can be computed from a presentation of the group.

Example 4.7. As an example of a curve of affine pencil type, one can consider the cuspidal cubic $f(x, y) = y^2 - x^3$ and the curve $C'' = \{(x, y) \in \mathbb{C}^2 \mid f(f - 1) = 0\}$ of degree $m'' = 6$. One can check that

$$\pi_1(\mathbb{C}^2 \setminus C'') = \langle \alpha_1, \alpha_2, \gamma : [\gamma, \alpha_2] = [\gamma^{\alpha_1}, \alpha_2] = 1, \gamma = \gamma^{\alpha_1} \gamma^{\alpha_1^{-1}} \rangle.$$

Here, $a^b := b^{-1}ab$, α_1 and $\gamma\alpha_1$ are positively oriented meridians about $\{f = 0\}$, and α_2 is a positively oriented meridian about $\{f = 1\}$.

Let r_n be the rank of the left \mathcal{K}_n'' -module generated by the rows of the 3×3 matrix $B(n)$ of Section 2.3 computed from the presentation of $\pi_1(\mathbb{C}^2 \setminus C'')$ above. One has $\gamma^{-1} - 1 \in \mathbb{Z}\bar{\Gamma}_n''$ is 0 if $n = 0$ and a unit otherwise. Using this, it is straightforward to check that $r_0 = 1$, and $r_n = 2$ for all $n \geq 1$. By Lemma 2.7, $\delta_0(C'') = \infty$ and $\delta_n(C'') < \infty$ for all $n \geq 1$. In fact, using the methods of Section 2.3, one can check that $\delta_n(C'') = 0$ for all $n \geq 1$. Indeed,

$$\begin{aligned} B(n) &= \begin{bmatrix} 0 & -u\alpha_1 v & u\alpha_1 + \alpha_1^{-1}u\alpha_1 - u \\ -u & \alpha_1^{-1}u\alpha_1 & 0 \\ v & -\alpha_1 v & \alpha_1^{-2}\gamma^{-1}\alpha_1 - 1 + \alpha_1 \end{bmatrix} \cong \begin{bmatrix} 0 & \alpha_1 v & 1 - w - \alpha_1 \\ -u & uw & 0 \\ 0 & vw & \alpha_1^{-2}\gamma^{-1}\alpha_1 - w \end{bmatrix} \\ &\cong \begin{bmatrix} \alpha_1 v & 1 - w - \alpha_1 \\ 0 & w\alpha_1^{-1}w - w\alpha_1^{-1} + \alpha_1^{-2}\gamma^{-1}\alpha_1 \end{bmatrix} \cong \begin{bmatrix} \alpha_1 v & 1 - w - \alpha_1 \alpha_2^{-1} \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where $u = (\gamma^{-1} - 1)$, $v = (\alpha_2^{-1} - 1)$, and $w = u^{-1}\alpha_1^{-1}u\alpha_1 = [u^{-1}, \alpha_1^{-1}]$. The first transformation is a result of multiplying the first row by $-u^{-1}$ (on the left), and then adding row 1 and vu^{-1} times row 2 to row 3. The second transformation results from eliminating column 1 and row 2 (since u is a unit) and subtracting $w\alpha_1^{-1}$ times row 1 from row 2, using that v and w commute. The resulting last row is identically 0 because $\alpha_1 v \neq 0$ and r_n is at most 2 by Lemma 2.7. One obtains the last matrix after subtracting the first column from the second column. Finally, $1 - w - \alpha_1 \alpha_2^{-1} \in \mathbb{Z}\bar{\Gamma}_n''$ is a unit, as it is nonzero in $\mathbb{Z}\bar{\Gamma}_0''$, so $\delta_n(C'') = 0$ for all $n \geq 1$.

Let C' be an irreducible curve of degree m' such that C' and C'' intersect in $6m'$ distinct points, and let $C = C' \cup C''$. Using the statement above Lemma 2.7 and the proofs of Lemmas 3.8 and 3.9, we get that $\delta_0(C) =$

(row corank of a presentation. matrix of $H_1(U'', u''_0; \mathcal{K}_0'')$) $- 1 = (3 - r_0) - 1 = 1$, and, by Theorem 1.3, $\delta_n(C) = 0$ for all $n \geq 1$. By Corollary 1.4, $\Delta_C^{\text{multi}}(t, t_1, t_2) = t - 1$.

ACKNOWLEDGMENT

The authors would like to thank Moisés Herradón Cueto and Laurențiu Maxim for useful discussions.

The authors are partially supported by PID2020-114750GB-C31, funded by MCIN/AEI/10.13039/501100011033. The first author is partially supported by the Departamento de Ciencia, Universidad y Sociedad del Conocimiento del Gobierno de Aragón (Grupo de referencia E22_20R “Álgebra y Geometría”). The second author was partially supported by an AMS-Simons Travel Grant during the preparation of this paper.

ORCID

Eva Elduque  <https://orcid.org/0000-0003-1339-6852>

REFERENCES

- [1] D. Arapura, *Fundamental groups of smooth projective varieties, Current topics in complex algebraic geometry* (Berkeley, CA, 1992/93), Math. Sci. Res. Inst. Publ., Vol. 28, Cambridge Univ. Press, Cambridge, 1995, pp. 1–16. <https://doi.org/10.2977/prims/1195168852>.
- [2] T. D. Cochran, *Noncommutative knot theory*, Algebr. Geom. Topol. **4** (2004), 347–398.
- [3] J. Cogolludo-Agustín, *Braid monodromy of algebraic curves*, Ann. Math. Blaise Pascal **18** (2011), no. 1, 141–209. <http://doi.org/10.5802/ambp.295>.
- [4] A. Dimca, *Sheaves in topology*, Universitext, pp. xvi+236, Springer, Berlin, 2004. <https://doi.org/10.1007/978-3-642-18868-8>.
- [5] A. Dimca, *Pencils of plane curves and characteristic varieties, Arrangements, local systems and singularities*, Progr. Math., vol. 283, Birkhäuser, Basel, 2010, pp. 59–82.
- [6] E. Elduque and L. Maxim, *Higher-order degrees of affine plane curve complements*, Indiana Univ. Math. J. **70** (2021), no. 1, 179–211. <https://doi.org/10.1512/iumj.2021.70.8297>.
- [7] S. L. Harvey, *Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm*, Topology **44** (2005), no. 5, 895–945. <https://doi.org/10.1016/j.top.2005.03.001>.
- [8] P. D. Horn, *On computing the first higher-order Alexander modules of knots*, Exp. Math. **23** (2014), no. 2, 153–169. <https://doi.org/10.1080/10586458.2014.882806>.
- [9] P. Kirk and C. Livingston, *Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants*, Topology **38** (1999), no. 3, 635–661. [https://doi.org/10.1016/S0040-9383\(98\)00039-1](https://doi.org/10.1016/S0040-9383(98)00039-1).
- [10] C. Leidy and L. Maxim, *Higher-order Alexander invariants of plane algebraic curves*, Int. Math. Res. Not. (2006), Art. ID 12976, 23. <https://doi.org/10.1155/IMRN/2006/12976>.
- [11] C. Leidy and L. Maxim, *Obstructions on fundamental groups of plane curve complements, Real and complex singularities*, Contemp. Math., vol. 459, Amer. Math. Soc., Providence, RI, 2008, pp. 117–130. <https://doi.org/10.1090/conm/459/08966>.
- [12] A. Libgober, *Alexander polynomial of plane algebraic curves and cyclic multiple planes*, Duke Math. J. **49** (1982), no. 4, 833–851.
- [13] A. Libgober, *On the homotopy type of the complement to plane algebraic curves*, J. Reine Angew. Math. **367** (1986), 103–114.
- [14] A. Libgober, *The topology of complements to hypersurfaces and nonvanishing of a twisted de Rham cohomology*, Singularities and complex geometry (Beijing, 1994), AMS/IP Stud. Adv. Math., vol. 5, Amer. Math. Soc., Providence, RI, 1997, pp. 116–130.
- [15] A. Libgober, *Characteristic varieties of algebraic curves, Applications of algebraic geometry to coding theory, physics and computation* (Eilat, 2001), Kluwer Acad. Publ. Dordrecht, 2001, pp. 215–254.
- [16] M. Oka, *A survey on Alexander polynomials of plane curves*, Singularités Franco-Japonaises, Sémin. Congr., vol. 10, Soc. Math. France, Paris, 2005, pp. 209–232.
- [17] M. Oka and K. Sakamoto, *Product theorem of the fundamental group of a reducible curve*, J. Math. Soc. Japan **30** (1978), no. 4, 599–602. <https://doi.org/10.2969/jmsj/03040599>.
- [18] J.-P. Serre, *Sur la topologie des variétés algébriques en caractéristique p*, Symposium internacional de topología algebraica International symposium on algebraic topology, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 24–53.
- [19] Y. Su, *Higher-order Alexander invariants of hypersurface complements*. Preprint available at arXiv:1510.03467.
- [20] A. I. Suciu, *Fundamental groups, Alexander invariants, and cohomology jumping loci*, Topology of algebraic varieties and singularities, Contemp. Math., vol. 538, Amer. Math. Soc., Providence, RI, 2011, pp. 179–223. <https://doi.org/10.1090/conm/538/10600>.
- [21] O. Zariski, *On the irregularity of cyclic multiple planes*, Ann. of Math. (2) **32** (1931), no. 3, 485–511. <https://doi.org/10.2307/1968247>.

How to cite this article: J. I. Cogolludo-Agustín and E. Elduque, *Vanishing of higher order Alexander-type invariants of plane curves*, Math. Nachr. (2023), 1–15. <https://doi.org/10.1002/mana.202100610>