Final Degree Project
Complementary material

EMERGENCE OF POLARIZATION IN COMPLEX NETWORKS

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Contents

A Example network construction:

- **Erdös and Rényi (ER) model**

  Erdös and Rényi [1] started the systematic study of random graphs. Starting with N unconnected nodes, ER random graphs are constructed by linking pairs of nodes chosen with a random probability p. Obviously, given the construction procedure of an ER graph $P(k)$ follows a binomial distribution that, in the continuous limit and for large N, can be described as a Poissonian function.

- **Watts-Strogatz (WS) model**

  The starting point is a N nodes ring, in which each node is symmetrically connected to its $2m$ nearest neighbors for a total of $K = mN$ edges. Then, for every node, each link connected to a clockwise neighbor is rewired to a randomly chosen node with a probability p, and preserved with a probability $1 - p$.

  Notice that for $p = 0$ we have a regular lattice, while for $p = 1$ the model produces a random graph with the constraint that each node has a minimum connectivity $k_{min} = m$. For intermediate values of p the procedure generates graphs with the small-world property [2].

- **Barabàsi-Albert (BA) model**

  This model proposes a simple way of generating scale-free networks by means of a growing (time-dependant) graph in which the number of nodes increases and new links are created at each time step. In particular, starting with a m_0 fully connected graph, at each time step $t = 1, 2, ..., N - m_0$ a new node j with $m \leq m_0$ links is added to the network. The probability that one of this m links brought by j will connect it to an existing node i is linearly proportional to the actual degree of i:

$$\prod_{j \longrightarrow i}(t) \approx \frac{k_i(t)}{\sum_l k_l} \tag{1}$$

  As at every time step is being added one node and m links, at time t the network will have $N = m0 + t$ nodes and $K = m \cdot t + m_0(m_0 - 1)/2$ links, corresponding to an average degree $\langle k \rangle = 2m$ for large times [3]. Since the probability is proportional to nodal degree, initially more connected nodes will have a higher likelihood of receiving a new connection. Thus, the more nodes we introduce in the network, the bigger the degree of the initial nodes will be as the likelihood of new connections increases along with them. The outcome will be a network where the vast majority of nodes are only moderately connected and a small number of nodes are extremely connected (hubs).

B Simulation Parameters

B.1 Modeling echo chambers

Parameters defined in [4] and [5]:

Number of agents	$N = 1000$
Activity Driven parameters	$(m, \epsilon, \gamma) = (10, 10^{-2}, 2.1)$
Reciprocity parameter	$r = 0.5$

B.2 Kuramoto Model

Number of agents	$N = 100$
Cauchy distribution parameters	$(x_0, \gamma) = (0, 1)$

B.3 Kuramoto Model with memory

Parameters defined by [6]:

Memory time	$T = 100$
Cauchy distribution parameters	$(x_0, \gamma) = (0, 1)$

B.4 Modelling echo chambers with adaptive networks and memory

Number of agents	$N = 300$
Uniform ω distribution	$\omega \in [-1/2, 1/2]$
Coupling	$K = 10$

C Kuramoto

In these systems, we consider an ensemble of N weakly interacting limit-cycle oscillators with phase θ_i with $i = 1, .., N$. Each oscillator rapidly relax to their limit cycles and can therefore be solely characterized by their natural frequency ω_i. On a longer time scale, each unit exerts a phase-dependant influence on the other units changing their rhythm, $\dot{\theta}_i$. Oscillators can display the temporal analogue of a phase transition [7, 8]. As long as the spread of the natural frequencies is large compared to the coupling, each oscillator moves at its own frequency and the system behaves incoherently. However, when a certain coupling threshold is crossed, collective synchronization appears. The oscillators freeze into a common mode despite their different individual frequencies.

The hypothesis is that individual phases θ_i are pulled towards the mean-field phase Φ and not to the phase of any other oscillator. Furthermore, the magnitude of this effect is proportional to the extent of coherence, i.e., to the fraction of oscillators frozen in synchrony (positive-feedback loop).

Kuramoto explored stable solutions where the parameter $r(t)$ is constant r and $\Phi(t)$ rotates uniformly at frequency Ω. The governing equations of the Kuramoto model can be rewritten in terms of the order parameter r (See Appendix C.2) as:

$$\dot{\theta}_n(t) = \omega_n + \lambda N \cdot r \cdot \sin(\Phi - \theta_n(t)) \qquad n = 1, ...N \tag{2}$$

Changing the origin for the rotating frame with frequency Ω one can set in Eq.[4] $\Phi = 0$ without loss of generality. This equation states that each oscillator interacts with all the others only through the mean field quantities r and ϕ. The second term provides a positive feedback loop to the system's collective rhythm, θ_i is drawn toward the mean phase Φ rather than toward any individual oscillator.

The effective strength of the coupling is proportional to the coherence r, which means that as long as the coupling between the oscillators is strengthened in each turn, more of them tend to be recruited into the synchronized set. The solutions of Eq.[4] show two types of long-term behaviour depending on the relation on how large is $|\omega_i|$ in comparison to λN. When $|\omega_i| \leq \lambda N \cdot r^*$, a group of oscillators are phase-locked at frequency Ω in the original frame approaching the stable fixed point defined as Eq.[5] (See Appendix C.2):

$$\omega_i = \lambda N \cdot r^* \cdot sin(\theta_i^*) \quad \text{where } |\theta_i| \leq \frac{1}{2} \cdot \pi \tag{3}$$

The rest of oscillators for which $|\omega_i| > \lambda N \cdot r^*$ is verified, are drifting in a non-uniform manner around the circle, sometimes accelerating and sometimes rotating at lower frequencies.

C.1 Mean-field analysis

Kuramoto explored stable solutions where the parameter $r(t)$ is constant, r, and the mean phase $\Phi(t)$ rotates uniformly at frequency Ω. The governing equations of the Kuramoto model can be rewritten in terms of the order parameter r (See Appendix C.2) as:

$$\dot{\theta}_n(t) = \omega_n + \lambda N \cdot r \cdot \sin(\Phi - \theta_n(t)) \qquad n = 1, ...N \tag{4}$$

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The rest of oscillators for which $|\omega_i| > \lambda N \cdot r^*$ is verified, are drifting in a non-uniform manner around the circle, sometimes accelerating and sometimes rotating at lower frequencies.

C.2 Kuramoto's governing equation

The governing equation in Kuramoto model introduced in Section 4.1 (Eq.[7]):

$$\dot{\theta}_n(t) = \omega_n + \lambda \sum_{m=1}^{N} sin(\theta_m(t) - \theta_n(t)) = \omega_n + \frac{K}{N} \sum_{m=1}^{N} sin(\theta_m(t) - \theta_n(t)), \qquad n = 1, ..., N$$

can be rewritten in terms of the order parameter Eq.[**??**]:

$$r(t) \cdot e^{i \cdot \phi(t)} = \frac{1}{N} \sum_{m=1}^{N} e^{i \cdot \theta_m(t)}$$

Now, multiplying both parts of Eq.[7] by $e^{-i\theta_n(t)}$:

$$r \cdot e^{i \cdot (\cdot \phi(t) - \theta_n(t))} = \frac{1}{N} \sum_{m=1}^{N} e^{i \cdot (\cdot \theta_m(t) - \theta_n(t))}$$

Equating both imaginary parts yields

$$r(t) \cdot \sin(i \cdot (\phi(t) - \theta_m(t))) = \frac{1}{N} \sum_{m=1}^{N} \sin(i \cdot (\theta_m(t) - \theta_n(t)))$$

Substituting in Eq.[7] we obtain the following expression:

$$\dot{\theta}_n(t) = \omega_i + K \cdot r(t) \cdot \sin(\phi(t) - \theta_n(t)) \qquad n = 1, ..., N$$

Using the mean field parameters r and Φ we obtain Eq.[4]:

$$\dot{\theta}_<(t) = \omega_i + K \cdot r \cdot \sin(\phi - \theta_n(t)) \qquad n = 1, ..., N$$

Linear stability analysis:

The steady-state condition in Eq.[4] reads:

$$\omega_n - Kr\sin\theta_n = 0 \tag{6}$$

Steady states for $|\omega_n| \leq Kr$ since $|\sin\theta_n| \leq 1$ Steady states are stable if:

$$\frac{d}{d\theta}(-\sin\theta) < 0 \quad for \quad |\theta_n| < \frac{\pi}{2} \tag{7}$$

For such oscillators θ_i^*=constant.

C.3 Kuramoto's analysis

The aim is now to solve the resultant oscillator motions (which will depend on r as a parameter). These movements must be self-consistent, which means that values for the order parameter must be compatible with the initial assumptions in Eq.[14].

Using angular brackets to denote population averages we distinguish between locked and drifting phases so that we consider to different populations of oscillators as follows:

$$\langle e^{i\theta}\rangle = \langle e^{i\theta}\rangle_{lock} + \langle e^{i\theta}\rangle_{drift} \tag{8}$$

Choosing the precise origin for the rotating frame with frequency Ω one can set in the main field equation $\Phi = 0$ without loss of generality. Since $\Phi = 0 \Longrightarrow \langle e^{i\theta}\rangle = r \cdot e^{i\Phi} = r$.

We evaluate the locked and the drifted contribution separately using Euler's formula: [1]:

- In the locked state:

$$\sin\theta^* = \frac{\omega}{K \cdot r} \quad \forall |\omega| \leq Kr.$$

  As $N \longrightarrow \infty$, the distribution of oscillators' locked phases is symmetric about $\theta = 0$ since $g(\omega) = g(-\omega)$. Hence:

$$\langle\sin\theta\rangle_{lock} = 0 \implies \langle e^{i\theta}\rangle_{lock} = \langle\cos\theta\rangle_{lock} = \int_{-Kr}^{Kr} \cos\theta(\omega) g(\omega) d\omega$$

  where $\theta(\omega)$ is defined by the stable solution (Eq.5) so that we use the change of variable $\omega \longrightarrow Kr\sin\theta$ and $d\omega \longrightarrow Kr\cos\theta \cdot d\theta$:

$$\langle e^{i\theta}\rangle_{lock} = \int_{-Kr}^{Kr} \cos\theta \cdot g(Kr\sin\theta) \cdot Kr\cos\theta \cdot d\theta = Kr\int_{-Kr}^{Kr} \cos^2\theta \cdot g(Kr\sin\theta) \cdot d\theta$$

- For the contribution of the drifting oscillators we use the probability distribution of them:

$$\langle e^{i\theta}\rangle_{drift} = \int_{-\pi}^{\pi}\int_{|\omega|>Kr} e^{i\theta}\rho(\theta,\omega) \cdot g(\omega) d\omega d\theta$$

[1] Euler's formula: $e^{i\theta} = cos(\theta) + i \cdot sin(\theta)$

Using the properties of $g(\omega) = g(-\omega)$ and the symmetry $\rho(\theta+\pi, -\omega) = \rho(\theta, \omega)$, the integral goes to zero so $\langle e^{i\theta} \rangle_{drift} = 0$.

Therefore the self-consistency condition Eq.[8] results in:

$$r = Kr \int_{-\pi/2}^{\pi/2} \cos^2\theta \cdot g(Kr\sin\theta) \mathrm{d}\theta \tag{9}$$

The solutions for Eq.[9] are:

- Always have a trivial zero solution $r = 0$ for any value of K. This associated with a completely incoherent state with $\rho(\theta, \omega) = \frac{1}{2\pi}\ \forall \theta, \omega$.
- Eq.[9] admits a non-trivial branch of solutions corresponding to the existence of partially synchronized oscillators. These soutions fulfil:

$$1 = K \int_{-\pi/2}^{\pi/2} \cos^2\theta \cdot g(Kr\sin\theta) \mathrm{d}\theta \tag{10}$$

This solutions bifurcates continuously from the zero solution $r = 0$ at a critical coupling $K = K_c$ obtained by letting $r \longrightarrow 0^+$ in Eq.[10]:

$$K_c = \frac{2}{\pi \cdot g(0)} \tag{11}$$

By expanding in powers of r the integrand in Eq.[10] it understand the bifurcation behaviour:

  - Supercritical bifurcation if $g''(0) < 0$
  - Subcritical bifurcation if $g''(0) > 0$

The value of K_c value indicates the onset of collective synchronization. Moreover, near these onset, the order parameter, r, obeys the usual square-root scaling law for mean-field models:

$$r \approx \sqrt{\frac{16}{\pi \cdot K_c^3}} \cdot \sqrt{\frac{\mu}{-g''(0)}} \text{ with } \mu = \frac{K - K_c}{K_c} \tag{12}$$

Where μ is the normalized distance above the threshold. For the specific case of Lorentzian or Cauchy density:

$$r = \left(1 - \frac{K_c}{K}\right)^{\beta} \text{ with } \beta = \frac{1}{2}\ \forall\ K \geq K_c \tag{13}$$

C.4 Kuramoto Model in Complex Networks

A local order parameter is defined as:

$$r_n \cdot e^{i \cdot \phi_n} = \sum_{m=1}^{N} A_{nm} \cdot \langle e^{i \cdot \theta_m} \rangle_t \tag{14}$$

where $\langle ... \rangle_t$ stands for time average.

To characterize the macroscopic coherence for the entire system, a global order parameter is defined as:

$$\tilde{r} \cdot e^{i \cdot \Phi} = \frac{\sum_{m=1}^{N} r_m}{\sum_{m=1}^{N} k_m} \tag{15}$$

Where k_m is defined as the degree of each node already introduced: $k_m = \sum_{m=1}^{N} A_{nm}$.

In terms of r_n we can express:

$$\dot{\theta}_n = \omega_n - K \cdot r_n \sin(\theta_i \phi_n) - K h_n(t) \tag{16}$$

where the term $h_i(t)$ takes into consideration the fluctuations with:

$$h_n(t) = Im \left[e^{-i\theta_n} \sum_m A_{nm} \left(\langle e^{i\theta_m} \rangle_t - e^{i\theta_m} \right) \right] \tag{17}$$

The terms in the sum are assumed to be statistically independent and assuming the number of connections of each node n to large enough so that we can neglet the fluctuations, $h_n \longrightarrow 0$.

So we the expression Eq.[] in terms of r_n reads:

$$\dot{\theta}_n(t) = \omega_n - k \cdot r_n \sin(\theta_i(t) - \phi_n) \tag{18}$$

From this equation we can obtain the locked condition $|\omega_n| \leq k r_n$ for the oscillators. The phase of the locked oscillators, θ_n settles at a value for which $\sin(\theta_n - \Phi_n) = \omega_n/(k r_n)$.

We can rewrite the local order parameter (Eq.[14] distinguishing between locked and drifting oscillators:

$$r_n \cdot e^{i \cdot \phi_n} = \sum_{m=1}^{N} A_{nm} \cdot \langle e^{i \cdot \theta_m(t)} \rangle_t \Longrightarrow r_n \cdot e^{i \cdot \phi_n - \theta_n(t)} = \sum_{m=1}^{N} A_{nm} \cdot e^{-\theta_n(t)} \langle e^{i \cdot \theta_m(t)} \rangle_t \Longrightarrow$$

$$\Longrightarrow r_n = \sum_{m=1}^{N} A_{nm} \cdot e^{-\Phi_n} \langle e^{i \cdot \theta_m(t)} \rangle_t = \sum_{m=1}^{N} A_{nm} \langle e^{i(\theta_m(t) - \Phi_n)} \rangle_t \Longrightarrow$$

$$\Longrightarrow r_n = \underbrace{\sum_{|\omega_n| \leq k r_n}^{N} A_{nm} \cdot e^{i(\theta_m(t) - \Phi_n)}}_{\text{locked oscillators}} + \underbrace{\sum_{|\omega_n| > k r_n}^{N} A_{nm} \langle e^{i(\theta_m(t) - \Phi_n)} \rangle_t}_{\text{Non-locked oscillators}} \longrightarrow 0$$

The sum over the drifting oscillator can be neglected. So r_n is defined only with the locked oscillators analysing the real and the imaginary part of the oscillators:

$$r_n = \sum_{m=1}^{N} A_{nm} \cdot e^{-\Phi_n(t)} \langle e^{i \cdot \theta_m(t)} \rangle_t = \sum_{|\omega_n| \leq k r_n}^{N} A_{nm} \cdot e^{i(\theta_m(t) - \Phi_n)} = Re \left[\sum_{|\omega_n| \leq k r_n}^{N} A_{nm} \cdot e^{i(\theta_m(t) - \Phi_m)} e^{i(\Phi_m - \Phi_n)} \right]$$

$$= \sum_{|\omega_n| \leq k r_n}^{N} A_{nm} \cdot [\cos(\theta_m - \Phi_m) \cdot \cos(\Phi_m - \Phi_n) - \sin(\theta_m - \Phi_m) \cdot \sin(\Phi_m - \Phi_n)] =$$

$$= \sum_{|\omega_n| \leq k r_n}^{N} A_{nm} \cdot \left[\sqrt{1 - \left(\frac{\omega_m}{\lambda r_m}\right)^2} \cdot \cos(\Phi_m - \Phi_n) - \cancelto{0}{\frac{\omega_m}{\lambda r_m} \cdot \sin(\Phi_m - \Phi_n)} \right] \tag{19}$$

and the second term goes to zero due to the symmetry of $g(\omega)$.

In addition, the imaginary part of the oscillators reads:

$$0 = \sum_{|\omega_n| \leq k r_n}^{N} A_{nm} \cdot \left[\frac{\omega_m}{\lambda r_m} \cdot \cos(\Phi_m - \Phi_n) + \sqrt{1 - \left(\frac{\omega_m}{\lambda r_m}\right)^2} \cdot \sin(\Phi_m - \Phi_n) \right] \tag{20}$$

Now, the relation Eq.[19] reads as:

$$r_n = \sum_{|\omega_n| \leq k r_n}^{N} A_{nm} \cdot \sqrt{1 - \left(\frac{\omega_m}{\lambda r_m}\right)^2} \cdot \cos(\Phi_m - \Phi_n) \longrightarrow \sum_{|\omega_n| \leq k r_n}^{N} A_{nm} \cdot \sqrt{1 - \left(\frac{\omega_m}{\lambda r_m}\right)^2} \tag{21}$$

As we choose λ_c smallest possible λ. Thus, by using the Frequency distribution approximation (FDA):

$$r_n = \sum_{|\omega_n| \leq k r_n}^{N} A_{nm} \cdot \sqrt{1 - \left(\frac{\omega_m}{\lambda r_m}\right)^2} \cdot \sqrt{1 - \left(\frac{\omega_m}{\lambda r_m}\right)^2} \approx \sum_m A_{nm} \int_{-\lambda rm}^{\lambda r_m} g(\omega) \sqrt{1 - \left(\frac{\omega_m}{\lambda r_m}\right)^2} \cdot d\omega \tag{22}$$

Now we introduce the change of variable $x = \omega_m / \lambda r_m$ in Eq.[22].

$$r_n \approx \lambda \sum_m A_{nm} r_m \int_{-1}^{1} g(x \lambda r_m) \sqrt{1 - x^2} dx \tag{23}$$

At λ_c we have $r_n \longrightarrow 0^+$:

$$r_n \approx \frac{\lambda_c g(0) \pi}{2} \sum_m A_{nm} r_m \longrightarrow r_n \frac{2}{\lambda_c g(0) \pi} \approx \sum_m A_{nm} r_m$$

So that the critical coupling λ_c can be expressed as:

$$\lambda_c = \frac{2}{g(0) \cdot \pi} \frac{1}{\Lambda_{max}(\mathcal{A})} \tag{24}$$

This critical value is the corresponding all-to-all topology λ_c^{KM} (Eq.[25]) divided by $\Lambda_{max}(\mathcal{A})$, which is the largest eigenvalue of the adjacency matrix $\mathcal{A}$.

$$\lambda_c^{KM} = \frac{2}{N \cdot g(0) \pi} \longrightarrow \lambda_c = \frac{2}{g(0) \cdot \pi} \frac{1}{\Lambda_{max}(\mathcal{A})} = \lambda_c^{KM} \cdot N \frac{1}{\Lambda_{max}(\mathcal{A})} \tag{25}$$

Now, if we apply the Mean-field approximation $r_n = \tilde{r} k_n$ to Eq.[23]:

$$k_n \approx \lambda \sum_m A_{nm} r_m \int_{-1}^{1} g(x \lambda \tilde{r} k_m) \sqrt{1 - x^2} dx \longrightarrow$$

$$\sum_n k_n \approx \lambda \sum_{n,m} A_{nm} r_m \int_{-1}^{1} g(x\lambda\tilde{r}k_m)\sqrt{1-x^2}dx = \lambda \sum_m k_m^2 \int_{-1}^{1} g(x\lambda\tilde{r}k_m)\sqrt{1-x^2}dx$$

So that introducing the probability distribution of the degree $P(k)$:

$$\sum_k kP(k) \approx \lambda \sum_k P(k)k^2 \int_{-1}^{1} g(x\lambda\tilde{r}k)\sqrt{1-x^2}dx$$

At λ_c we have $r_i = \tilde{r} \cdot k_i \longrightarrow 0^+$:

$$\sum_k kP(k) \approx \lambda_c \sum_k P(k)k^2 g(0) \int_{-1}^{1} \sqrt{1-x^2}dx \longrightarrow \langle k \rangle \approx \lambda_c \sum_k P(k)k^2 g(0)\frac{\pi}{2}$$

And we can therefore express λ_c in this other way:

$$\lambda_c = \frac{2}{g(0)\pi} \frac{1}{\Lambda_{max}(\mathcal{A})} \tag{26}$$

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