Appendix A

Fundamentals of general topology

This section aims to provide the reader with a brief but self-contained reminder of the definitions and results from general topology and that are needed in chapter 1.

A.1 Normed spaces and metric spaces

Definition A.1. Let E be a vector space over $\mathbb{R}$. A **norm** is a function $|\cdot|\colon E \to \mathbb{R}$ such that the following holds:

1. $|x+y| \leq |x| + |y| \qquad x, y \in E,$
2. $|kx| = |k||x| \qquad k \in \mathbb{R}, x \in E,$
3. $|x| \geq 0$ for all $x \in E$, and $|x| = 0$ if and only if $x = 0_E$.

The pair $(E, |\cdot|)$ is called a **normed space**.

Definition A.2. Let $f\colon E \to F$ be a map between normed vector spaces E and F. We say that it is continuous at $x \in E$ if for every $\varepsilon > 0$ there exists $\delta(x, \varepsilon) > 0$ such that for every $y \in E \setminus \{x\}$, $|y - x| < \delta \implies |f(y) - f(x)| < \varepsilon$. If the above condition holds for every $x \in E$ we say that f is **continuous**. Moreover, if f is continuous and δ does not depend on x, we say that f is **uniformly continuous**.

Theorem A.3. *Let $A\colon E \to F$ be a linear map between two normed vector spaces E and F. The following statements are equivalent:*

1. *A is continuous at 0_E.*
2. *A is uniformly continuous.*
3. *A is bounded (there exists a real number $K \geq 0$ such that $|Ax| \leq K|x|$ for all $x \in E$).*

Definition A.4. Let X be a set. A **metric** is a function $d\colon X \times X \to \mathbb{R}$ such that the following holds:

1. $d(x,y) \leq d(x,z) + d(z,y) \qquad x, y, z \in X,$
2. $d(x,y) = d(y,x) \qquad x, y \in X,$
3. $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0 \iff x = y$.

The pair (E, d) is called a **metric space**.

Lemma A.5. *Every normed space $(E, |\cdot|)$ is also a metric space $\left(E, d_{|\cdot|}\right)$, where the metric is defined as $d_{|\cdot|}\colon X \times X \to \mathbb{R}$, $d_{|\cdot|}(x,y) \coloneqq |y - x|$.*

A.2 Topological spaces

Definition A.6. Let X be a set. A **topology** over X is a subset $\tau \subseteq \mathscr{P}(X)$ satisfying the following axioms: (i) $\varnothing, X \in \tau$, (ii) any arbitrary (finite or infinite) union of elements of τ belongs to τ, (iii) finite intersections of elements of τ belong to τ. The pair (X, τ) is called a **topological space**.

Lemma A.7. *Every metric space (E, d) is also a topological space (E, τ_d), where the topology τ_d is defined as*

$$\tau_d := \{U \subseteq E \mid \textit{for every } x \in U \textit{ there exists } r > 0 \textit{ such that } B(x, r) \subseteq U\},$$

with $B(x, r) := \{y \in E \mid d(x, y) < r\}$, called an open ball with centre x and radius r. The topology τ_d is called the induced topology (by the metric d).

Remark. Applying lemmas A.5 and A.7, we can define a topology over a normed space $(E, |\cdot|)$, which is also called the induced topology (by the norm $|\cdot|$)).

Definition A.8. A subset $\mathcal{B} \subseteq \tau$ is called a **base** of the topology τ if for every $U \in \tau$ and $x \in U$ exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Definition A.9. If $x \in X$, a **neighbourhood** of x is a subset $N \subseteq X$ such that exists an $U \in \tau$ with $x \in U \subseteq N$ (i.e., a subset of X which contains x as an interior point). We shall denote the set of neighbourhoods of x by $\mathcal{N}(x)$.

Definition A.10. A **local base** (or **neighbourhood base**) of $x \in X$ is a subset $\mathcal{B}_x \subseteq \mathcal{N}(x) \cap \tau =: \varepsilon(x)$ such that for every $N \in \mathcal{N}(x)$ exists a $B \in \mathcal{B}_x$ satisfying $B \subseteq N$.

Lemma A.11. *If $\mathcal{B}$ is a base of τ, $\mathcal{B} \cap \varepsilon(x)$ is a local base of x. Conversely, $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ is a base of τ.*

Definition A.12. Every subset $\mathcal{S} \subseteq \mathscr{P}(X)$ such that $X = \bigcup S_j$ determines a topology over X. We call $\mathcal{S}$ a **subbase** of the topology $\tau_{\mathcal{S}}$, which has $\mathcal{B}_{\mathcal{S}} = \{\text{finite intersections of } S_j \in \mathcal{S}\}$ as a base.

Definition A.13. Let $f\colon X \to Y$ be a map from a topological space (X, τ) into a topological space (Y, ω). We say that it is continuous at $x \in X$ if for every $V \in \omega$ such that $f(x) \in V$ exists an $U \in \tau$ with $x \in U$ and $f(U) \subseteq V$. If the above condition holds for every $x \in X$ we say that f is **continuous**. A bijection whose inverse is also continuous is called an **homeomorphism**.

Proposition A.14. *A map $f\colon X \to Y$ is continuous if and only if $f^{-1}(V) \in \tau$ for every $V \in \omega$.*

Definition A.15. We define the **product topology** on the Cartesian product of two topological spaces $X \times Y$ as the smallest topology that makes the canonical projections continuous. That is, the topology generated by the subbase

$$\mathcal{S}_p = \{U \times Y \mid U \in \tau\} \cup \{X \times V \mid V \in \omega\}.$$

Definition A.16. We say that a topological space (X, τ) is **Hausdorff** if for every pair of different points $x, y \in X$, $x \neq y$, there exists a pair of open sets such that $U \in \varepsilon(x)$, $V \in \varepsilon(y)$ and $U \cap V = \varnothing$.

Appendix B

Differentiability on topological vector spaces

In section 1.1 we introduced the concept of differentiable maps between Banach spaces. Nonetheless, a notion of derivative can still be defined on topological vector spaces (TVS). We first cover the definitions and some results that hold in TVS. Afterwards, we give a detailed example of some calculational rules which are particularly interesting for a usual phase space in geometric quantum mechanics, the Schwartz space. The content of the first section of this appendix can be found in [11] and [18].

B.1 Differential calculus

In a general TVS we lack the norm that was used to define the derivative in 1.8. That is why the concept of *tangent to 0* is introduced.

Definition B.1. A real valued function of a real variable, defined on some neighbourhood of 0 is said to be $o(t)$ if $\lim_{t\to 0} o(t)/t = 0$.

Definition B.2. Let E, F be (locally convex) topological vector spaces, and φ a mapping of a neighbourhood of 0 in E into F. We say that φ is **tangent to** 0 if, given a neighbourhood W of 0 in F, there exists a neighbourhood V of 0 in E such that

$$\varphi(tV) \subset o(t)W \tag{B.1}$$

for some function $o(t)$.

Definition B.3. Let E, F be locally convex topological vector spaces and U open in E. Let $f : U \subset E \to F$ be a continuous map. We shall say that f is **Fréchet differentiable** at a point $x_0 \in U$ if there exists a continuous linear map λ of E into F such that, if we let

$$\varphi(y) \coloneqq f(x_0 + y) - f(x_0) - \lambda y \tag{B.2}$$

then φ is tangent to 0. It then follows trivially that λ is uniquely determined, and we say that it is the **Fréchet derivative** of f at x_0. We denote the derivative by $f'(x_0)$, which is an element of $L(E, F)$, the space of continuous linear maps from E into F. If f is differentiable at every point of U, then we say that f is **Fréchet differentiable** and f' is a map

$$f' : U \to L(E, F).$$

Remark. Let N_0 be a neighbourhood of 0 in E. Note that if $\varphi\colon N_0 \subset E \to F$ is tangent to 0 and both E, F are normed (Banach spaces), then (B.1) amounts to the condition

$$|\varphi(y)| \leq |y|\psi(y) \quad y \in N_0$$

for some function $\psi\colon N_0 \subset E \to \mathbb{R}$ such that $\lim_{y\to 0}\psi(y) = 0$. Then, from equation (B.2) we obtain

$$|f(x_0 + y) - f(x_0) - \lambda y| \leq |y|\psi(y)$$

for some function $\psi\colon N_0 \subset E \to \mathbb{R}$ with $\lim_{y\to 0}\psi(y) = 0$. That is to say,

$$\lim_{y\to 0} \frac{|f(x_0 + y) - f(x_0) - \lambda y|}{|y|} = 0 \in \mathbb{R}.$$

Therefore, definitions B.3 and 1.8 are equivalent.

The chain rule also holds in TVS.

Proposition B.4 (Chain rule)**.** *Let U, V, W be open subsets of locally convex topological vector spaces E_1, E_2, E_3 (respectively). If $f\colon U \to V$ is Fréchet differentiable at x_0 and $g\colon V \to W$ is Fréchet differentiable at $f(x_0)$, then $g \circ f$ is Fréchet differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(f(x_0)) \circ f'(x_0).$$

Lemma B.5. *Let E, F be locally convex topological vector spaces and U open in E. Let f be a continuous linear map, $f\colon U \subset E \to F$. Then, for every $x \in U$, f is Fréchet differentiable at x and*

$$f'(x) = f.$$

Proof. Let $x \in U$, and let $\varphi(y) = f(x + y) - f(x) - \lambda y$ be a function defined on some neighbourhood of 0, with $\lambda \in L(E, F)$. If $\lambda = f$, by the linearity of f we have

$$\varphi(y) = f(x + y) - f(x) - f(y) = f(0) = 0,$$

and therefore φ is tangent to 0. The result follows by definition B.3 (the Fréchet derivative is uniquely determined). □

B.2 A relevant example: derivatives in the Schwartz space

Definition B.6. The **Schwartz space**, or the space of **rapidly decreasing functions** is defined as

$$\mathscr{S}(\mathbb{R}^d) := \{\psi\colon \mathbb{R}^d \to \mathbb{C} \mid \psi \in \mathcal{C}^\infty(\mathbb{R}^d) \text{ and } p_{\alpha,\beta}(\psi) := \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta \psi(x)| < \infty\}$$

where $d \geq 1$ and $\alpha, \beta \in \mathbb{N}_0^d$.

Remark. In this example we will use the following notation: x will denote a point of $\mathbb{R}^d$, ψ will denote a point of our TVS, $\mathscr{S}$, and $\mathscr{F}$ will denote a functional from $\mathscr{S}$ to another Banach space, usually $\mathbb{R}$.

We now introduce without proof a technical result that will later be needed.

Lemma B.7. *Let $\mathscr{S}$ be the Schwartz space. The space of continuous n-multilinear maps of $\mathscr{S}^{\otimes n}$ into $\mathscr{S}$, denoted by $L(\mathscr{S}^{\otimes n}, \mathscr{S})$, is a locally convex topological vector space for every $n > 1$. Furthermore, the algebraic canonical isomorphism*

$$L(\mathscr{S}^{\otimes n}, \mathscr{S}) \cong L\left(\mathscr{S}, L\left(\mathscr{S}, \ldots, L\left(\mathscr{S}, \mathscr{S}\right)\right)\right)$$

is toplinear.

Proposition B.8. *Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz space, and let p be a polynomial in $\mathbb{C}$ such that $p(0) = 0$. The map*

$$\begin{aligned} \mathscr{F} \colon \mathscr{S}(\mathbb{R}^d) &\longrightarrow \mathbb{C} \\ \psi &\longmapsto \mathscr{F}(\psi) = \int_{\mathbb{R}^d} (p \circ \psi)\, \mathrm{d}m \end{aligned}$$

is Fréchet differentiable at every $\psi \in \mathscr{S}(\mathbb{R}^d)$.

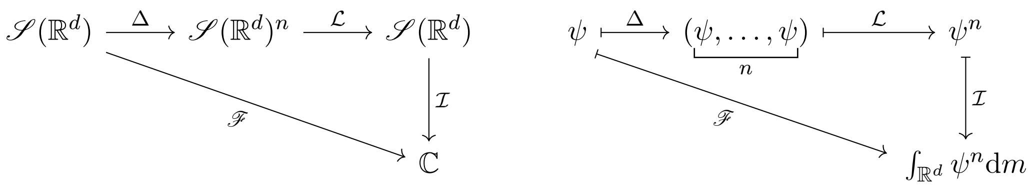
Proof. To prove that the map $\mathscr{F}$ is well-defined, let us note that $\mathscr{S}(\mathbb{R}^d)$ is a vector space over $\mathbb{C}$ and also a pointwise algebra. Hence, $p \circ \psi \in \mathscr{S}(\mathbb{R}^d)$, which is a (dense) subset of $L^1(\mathbb{R}^d)$.

Now, let us prove the Fréchet differentiability. Let $n > 1$ be a natural number, and let Δ, $\mathcal{L}$ and $\mathcal{I}$ be operators defined as

$$\begin{aligned} \Delta \colon \mathscr{S}(\mathbb{R}^d) &\longrightarrow \mathscr{S}(\mathbb{R}^d)^n \\ \psi &\longmapsto \underbrace{(\psi, \ldots, \psi)}_{n} \end{aligned} \qquad \begin{aligned} \mathcal{L} \colon \mathscr{S}(\mathbb{R}^d)^n &\longrightarrow \mathscr{S}(\mathbb{R}^d) \\ (\psi_1, \ldots, \psi_n) &\longmapsto \psi_1 \cdots \psi_n \end{aligned} \qquad \begin{aligned} \mathcal{I} \colon \mathscr{S}(\mathbb{R}^d) &\longrightarrow \mathbb{C} \\ \psi &\longmapsto \int_{\mathbb{R}^d} \psi\, \mathrm{d}m\,. \end{aligned}$$

Note that Δ and $\mathcal{I}$ are linear continuous maps, and therefore they are Fréchet differentiable by lemma B.5. As $\mathcal{L}$ is n-multilinear and continuous, we can apply the isomorphism given in lemma B.7 to write $\mathcal{L}$ as a composition of linear maps, which are all again Fréchet differentiable. Then, using the chain rule stated in B.4, we have the Fréchet differentiability of $\mathcal{L}$.

If the complex polynomial p is $p(z) = z^n$, $n \in \mathbb{N}$, the diagram



commutes. By the chain rule, $\mathscr{F}$ is Fréchet differentiable. For a general polynomial $p(z) = \sum_{k=1}^{n} a_k z^k$, the result follows from the linearity of the Fréchet derivative. □

To obtain the explicit form of the Fréchet derivative we are going to introduce a weaker, but more convenient to apply, definition of derivative. Then, we will use a theorem which relates the two definitions, allowing us to calculate functional derivatives on the Schwartz space.

Definition B.9. Let E be a topological vector space, U an open subset of E. The map $\mathscr{F} \colon U \subset E \to \mathbb{C}$ is called **Gâteaux differentiable** at $\psi \in E$ in direction $\eta \in E$, if and only if the function $s \longmapsto \mathscr{F}(\psi + s\eta)$ with $s \in \mathbb{R}$ is differentiable at $s = 0$, and we call

$$D_\eta \mathscr{F}(\psi) \coloneqq \frac{\mathrm{d}}{\mathrm{d}s} \mathscr{F}(\psi + s\eta)\Big|_{s=0} = \lim_{s \to 0} \frac{\mathscr{F}(\psi + s\eta) - \mathscr{F}(\psi)}{s} \in \mathbb{C}$$

the **Gâteaux derivative** of $\mathscr{F}$ at ψ in direction η.

Theorem B.10. *Let us assume that E is a countably Banach space, U is open in E, and the map $\mathscr{F}: U \subset E \to \mathbb{C}$ is Fréchet differentiable at $\psi \in U$. Then, $\mathscr{F}$ is Gâteaux differentiable at ψ in any direction, and for all $\eta \in E$, $D_\eta \mathscr{F}(\psi) = \mathscr{F}'(\psi)(\eta)$.*

Proposition B.11 (Calculational rule)**.** *Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz space, let p be a polynomial in $\mathbb{C}$ such that $p(0) = p'(0) = 0$, and let $\mathscr{F}$ be the map defined in proposition B.8. For every $\psi, \eta \in \mathscr{S}(\mathbb{R}^d)$, $\mathscr{F}$ is Gâteaux differentiable at ψ and the Gâteaux derivative is $D_\eta \mathscr{F}(\psi) = \int_{\mathbb{R}^d} (p' \circ \psi)\eta \, \mathrm{dm}$. Consequently, the Fréchet derivative is the continuous linear map defined as*

$$\mathscr{F}'(\psi)\colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathbb{C} \qquad \eta \longmapsto \mathscr{F}'(\psi)(\eta) = \int_{\mathbb{R}^d} (p' \circ \psi)\eta \, \mathrm{dm} := \int_{\mathbb{R}^d} \frac{\delta \mathscr{F}}{\delta \psi} \eta \, \mathrm{dm} \, .$$

We call $\frac{\delta \mathscr{F}}{\delta \psi}$ the ***functional Fréchet derivative*** *of $\mathscr{F}$ at ψ.*

Proof. (Sketch) As the Schwartz space is countably Banach and we know that $\mathscr{F}$ is Fréchet differentiable at every $\psi \in \mathscr{S}(\mathbb{R}^d)$ from B.8, we are in conditions to apply theorem B.10. By the dominated convergence theorem we can calculate the Gâteaux derivative, whose existence is now guaranteed:

$$D_\eta \mathscr{F}(\psi) = \lim_{s \to 0} \frac{\mathscr{F}(\psi + s\eta) - \mathscr{F}(\psi)}{s} = \int_{\mathbb{R}^d} (p' \circ \psi)\eta \, \mathrm{dm} \, .$$

Finally, the explicit form of the Fréchet derivative follows directly from theorem B.10. □

Complete proof of proposition B.11

We give now the detailed proof of the preceding proposition. Firstly, we need some auxiliary results.

Lemma B.12. *Let $\psi \in \mathscr{S}(\mathbb{R}^d)$. Let $p\colon \mathbb{C} \to \mathbb{C}$ be a polynomial in $\mathbb{C}$ such that $p(0) = 0$. Then, $p \circ \psi \in \mathscr{S}(\mathbb{R}^d)$.*

Proof. It follows trivially from the fact that $\mathscr{S}(\mathbb{R}^d)$ is a $\mathbb{C}$-vector space and an algebra with the pointwise product. □

Lemma B.13. *Let p be a polynomial in $\mathbb{C}$ such that $p(0) = p'(0) = 0$, and let $\psi, \eta \in \mathscr{S}(\mathbb{R}^d)$. Then, the following holds:*

(i) Let $\varepsilon > 0$. The sequence $(\varphi_s) := \left(\frac{p\circ(\psi + s\eta) - p\circ\psi}{s}\right)_s$ with $s \in (-\varepsilon, \varepsilon)$ is a subset of the Schwartz space.

(ii) It converges pointwise to $\varphi := \eta(p' \circ \psi) \in \mathscr{S}(\mathbb{R}^d)$ when $s \to 0$.

(iii) $p_{\alpha,0}(\varphi_s - \varphi) \underset{s \to 0}{\longrightarrow} 0$ for all $\alpha \in \mathbb{N}_0^d$, where the seminorm $p_{\alpha,0}$ is defined as

$$p_{\alpha,0}(\psi) := \sup_{x \in \mathbb{R}^d} |x^\alpha \psi(x)| \qquad \psi \in \mathscr{S}(\mathbb{R}^d).$$

Proof.

(i) Trivial from Lemma B.12.

(ii) Let $x \in \mathbb{R}^d$. We have that

$$\lim_{s\to 0} \left(\varphi_s(x) - \varphi(x)\right) = \lim_{s\to 0} \frac{p(\psi(x) + s\eta(x)) - p\psi(x)}{s} - \eta(x)p'(\psi(x)) \,,$$

which is a limit of a mapping from $\mathbb{R}$ to $\mathbb{C}$. If we let $h \in \mathbb{C}$, it holds that

$$\lim_{h\to 0} \eta(x)\frac{p(\psi(x) + h) - p(\psi(x))}{h} - \eta(x)p'(\psi(x)) = 0.$$

And then we have that the previous limit is 0 as well, since it must hold for any trajectory in $\mathbb{C}$ such that $h \to 0$, in particular for $h = s\eta(x)$ with $s \to 0$.

(iii) We want to prove that, for $\alpha \in \mathbb{N}_0^d$,

$$p_{\alpha,0}(\varphi_s - \varphi) = \sup_{x\in\mathbb{R}^d} \left|x^\alpha \left(\varphi_s(x) - \varphi(x)\right)\right| \underset{s\to 0}{\longrightarrow} 0.$$

As p is a polynomial in $\mathbb{C}$ it holds that, for some $N \in \mathbb{N}$, $\{a_k\}_{k=2}^N \subset \mathbb{C}$,

$$p(z) = \sum_{k=2}^{N} a_k z^k \quad \text{and} \quad p'(z) = \sum_{k=1}^{N-1} a_{k+1}(k+1)z^k.$$

Let $x \in \mathbb{R}^d$. By substituting the expressions for φ_s and φ given in (i) and (ii) respectively,

$$x^\alpha \left(\varphi_s(x) - \varphi(x)\right) = x^\alpha \left[\frac{p\left(\psi(x) + s\eta(x)\right) - p\left(\psi(x)\right)}{s} - \eta(x)p'\left(\psi(x)\right)\right]. \tag{B.3}$$

In the following we will denote $\psi^k(x) := (\psi(x))^k$. Expanding p we obtain

$$p\left(\psi(x) + s\eta(x)\right) - p\left(\psi(x)\right) = \sum_{k=2}^{N} a_k \left(\psi(x) + s\eta(x)\right)^k - \sum_{k=2}^{N} a_k\psi^k(x) =$$
$$= \sum_{k=2}^{N} a_k \left(\sum_{i=0}^{k} \binom{k}{i} \psi^{k-i}(x)s^i\eta^i(x) - \psi^k(x)\right) = \sum_{k=2}^{N} a_k \sum_{i=1}^{k} \binom{k}{i} \psi^{k-i}(x)s^i\eta^i(x).$$

Now we have that,

$$\frac{1}{s}\left[p\left(\psi(x) + s\eta(x)\right) - p\left(\psi(x)\right)\right] = \frac{1}{s}\sum_{k=2}^{N} a_k \sum_{i=1}^{k} \binom{k}{i} \psi^{k-i}(x)s^i\eta^i(x) =$$
$$= \sum_{k=2}^{N} a_k \sum_{i=1}^{k} s^{i-1}\binom{k}{i} \psi^{k-i}(x)\eta^i(x) =$$
$$= \sum_{k=2}^{N} a_k \sum_{i=2}^{k} s^{i-1}\binom{k}{i} \psi^{k-i}(x)\eta^i(x) + \sum_{k=2}^{N} a_k \binom{k}{1} \psi^{k-1}(x)\eta(x).$$

Therefore, substituting in (B.3),

$$x^\alpha \left(\varphi_s(x) - \varphi(x)\right) = x^\alpha\Big[\sum_{k=2}^{N} a_k \sum_{i=2}^{k} s^{i-1}\binom{k}{i} \psi^{k-i}(x)\eta^i(x) +$$
$$+ \sum_{k=2}^{N} a_k\, k\, \psi^{k-1}(x)\eta(x) - \sum_{k=1}^{N-1} a_{k+1}(k+1)\psi^k(x)\eta(x)\Big] =$$
$$= \sum_{k=2}^{N} a_k \sum_{i=2}^{k} s^{i-1}\binom{k}{i} x^\alpha\psi^{k-i}(x)\eta^i(x).$$

Finally, we have the seminorm convergence,

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d} |x^\alpha (\varphi_s(x) - \varphi(x))| &= \sup_{x\in\mathbb{R}^d} \Big| \sum_{k=2}^{N} a_k \sum_{i=2}^{k} s^{i-1} \binom{k}{i} x^\alpha \psi^{k-i}(x)\eta^i(x) \Big| \leq \\
&\leq \sup_{x\in\mathbb{R}^d} \sum_{k=2}^{N} |a_k| \sum_{i=2}^{k} |s|^{i-1} \binom{k}{i} |x^\alpha \psi^{k-i}(x)\eta^i(x)| = \\
&= \sum_{k=2}^{N} |a_k| \sum_{i=2}^{k} |s|^{i-1} \binom{k}{i} \sup_{x\in\mathbb{R}^d} |x^\alpha \psi^{k-i}(x)\eta^i(x)| = \\
&= \sum_{k=2}^{N} |a_k| \sum_{i=2}^{k} |s|^{i-1} \binom{k}{i} p_{\alpha,0}\left(\psi^{k-i}\eta^i\right) \underset{s\to 0}{\longrightarrow} 0.
\end{aligned}$$

Note that $p_{\alpha,0}\left(\psi^{k-i}\eta^i\right)$ is well defined again, as the pointwise-product is closed in the Schwartz space. The limit is clearly 0 since each term of the sum is multiplied by some power $|s|^k$ with $k \geq 1$.

□

In the following lines we prove some technical results about domination on the Schwartz space.

Lemma B.14. *Let $(\varphi_k)_{k=1}^\infty \subset \mathscr{S}(\mathbb{R}^d)$ be a sequence of functions of the Schwartz space which satisfies that*

$$p_{\alpha,0}(\varphi_k) \coloneqq \sup_{x\in\mathbb{R}^d} |x^\alpha \varphi_k(x)| \to 0, \quad k \to \infty,$$

for all $\alpha \in \mathbb{N}_0^d$. Then, there exists $g \in L^1(\mathbb{R}^d)$ such that $|\varphi_k(x)| \leq g(x)$ for all $x \in \mathbb{R}^d$.

Proof. If $x \in \mathbb{R}^d$,

$$\left(1 + |x|^2\right)^d |\varphi_k(x)| = \left(1 + x_1^2 + \cdots + x_d^2\right)^d |\varphi_k(x)|\,,$$

and expanding the d-th power it can be written as a sum of terms $Cx^\alpha|\varphi_k(x)|$, with the constant $C > 0$ and every index of α even. Note that, by hypothesis, each $Cx^\alpha|\varphi_k(x)|$ is bounded by $Cp_{\alpha,0}(\varphi_k)$. Therefore,

$$M_k \coloneqq \sup_{x\in\mathbb{R}^d} \left(1 + |x|^2\right) |\varphi_k(x)|$$

is bounded by some linear combination of $p_{\alpha,0}(\varphi_k)$, which all vanish when $k \to \infty$. Consequently $M_k \to 0$ and there exists $M > 0$ such that $M_k \leq M$ for all $k \in \mathbb{N}$. This implies that, for every $k \in \mathbb{N}$,

$$|\varphi_k(x)| = \frac{(1 + |x|^2)^d |\varphi_k(x)|}{(1 + |x|^2)^d} \leq \frac{M_k}{(1 + |x|^2)^d} \leq \frac{M}{(1 + |x|^2)^d} =: g(x)$$

for all $x \in \mathbb{R}^d$. It can be verified that $g \in L^1(\mathbb{R}^d)$. □

Proposition B.15. *Let $(\varphi_k)_{k=1}^\infty \subset \mathscr{S}(\mathbb{R}^d)$ be a sequence of functions of the Schwartz space which satisfies that*

$$p_{\alpha,0}(\varphi_k - \varphi) \coloneqq \sup_{x\in\mathbb{R}^d} |x^\alpha (\varphi_k(x) - \varphi(x))| \to 0, \quad k \to \infty,$$

for all $\alpha \in \mathbb{N}_0^d$, for some $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Then, there exists $g \in L^1(\mathbb{R}^d)$ such that $|\varphi_k(x)| \leq g(x)$ for all $x \in \mathbb{R}^d$.

Proof. We can apply Lemma B.14, since $(\varphi_k - \varphi)_{k=1}^{\infty} \subset \mathscr{S}(\mathbb{R}^d)$ and, by hypothesis, the sequence $p_{\alpha,0}(\varphi_k - \varphi) \to 0$ when $k \to \infty$ for all $\alpha \in \mathbb{N}_0^d$. It follows that there is $f \in L^1(\mathbb{R}^d)$ such that $|\varphi_k(x) - \varphi(x)| \leq f(x)$ for all $x \in \mathbb{R}^d$. Now,

$$|\varphi_k(x)| \leq |\varphi_k(x) - \varphi(x)| + |\varphi(x)| \leq f(x) + |\varphi(x)| =: g(x).$$

As $|\varphi| \in \mathscr{S}(\mathbb{R}^d)$, $g \in L^1(\mathbb{R}^d)$. □

Remark. Note that we can easily adapt the two previous results to continuous sequences in the Schwartz space, $(\varphi_s)_s \subset \mathscr{S}(\mathbb{R}^d)$, $s \in (-\varepsilon, \varepsilon)$ and therefore this holds for the sequence of B.13.

Proof of proposition B.11.

Proof. Applying the dominated convergence theorem (DCT), whose hypotheses are guaranteed by Lemma *B*.13 and Proposition *B*.15, we have that

$$\begin{aligned} D_\eta \mathscr{F}(\psi) &= \lim_{s\to 0} \frac{\mathscr{F}(\psi + s\eta) - \mathscr{F}(\psi)}{s} = \lim_{s\to 0} \int_{\mathbb{R}^d} \frac{p\left(\psi(x) + s\eta(x)\right) - p\left(\psi(x)\right)}{s} \,\mathrm{d}m(x) = \\ &= \lim_{s\to 0} \int_{\mathbb{R}^d} \varphi_s \,\mathrm{d}m \underset{(DCT)}{=} \int_{\mathbb{R}^d} \lim_{s\to 0} \varphi_s \,\mathrm{d}m = \int_{\mathbb{R}^d} \varphi \,\mathrm{d}m = \int_{\mathbb{R}^d} \eta(p' \circ \psi) \,\mathrm{d}m \,. \end{aligned}$$

□

Appendix C

Operations with vector bundles

The aim of this appendix is to provide a thorough presentation of the constructions given at the end of section 1.2, namely the cotangent bundle and general tensor bundles. For that purpose, some basic concepts from category theory are required. As in previous sections, see [11] for a more detailed discussion.

C.1 Fundamentals of category theory

Definition C.1. A **category** is a collection of objects $\{X, Y, \dots\}$ such that for every two objects X, Y we have a set $\mathrm{Mor}(X, Y)$ and for every three objects X, Y, Z, a mapping (composition law)

$$\circ\colon \mathrm{Mor}(Y, Z) \times \mathrm{Mor}(X, Y) \longrightarrow \mathrm{Mor}(X, Z)$$

satisfying the following axioms:

1. Two sets $\mathrm{Mor}(X, Y)$ and $\mathrm{Mor}(X', Y')$ are disjoint unless $X = X'$ and $Y = Y'$, in which case they are equal.
2. Each set $\mathrm{Mor}(X, X)$ has an element id_X which acts as a left and right identity under the composition law.
3. The composition law is associative.

The elements of $\mathrm{Mor}(X, Y)$ are called **morphisms**, and we write frequently $f\colon X \to Y$ for such a morphism. The composition of two morphisms g, f is written $g \circ f$ or gf.

Definition C.2. A (covariant) **functor** $\lambda\colon \mathfrak{A} \to \mathfrak{A}'$ from a category $\mathfrak{A}$ into a category $\mathfrak{A}'$ is a map which associates with each object X in $\mathfrak{A}$ an object $\lambda(X)$ in $\mathfrak{A}'$, and with each morphism $f\colon X \to Y$ a morphism $\lambda(f)\colon \lambda(X) \to \lambda(Y)$ in $\mathfrak{A}'$ such that whenever f and g are morphisms in $\mathfrak{A}$ that can be composed, then $\lambda(gf) = \lambda(g)\lambda(f)$ and $\lambda(\mathrm{id}_X) = \mathrm{id}_{\lambda(X)}$.

A contravariant functor is defined by reversing the arrows (so that for a morphism f we have $\lambda(f)\colon \lambda(Y) \to \lambda(X)$ and consequently $\lambda(gf) = \lambda(f)\lambda(g)$). A multivariable functor, which can be covariant in some variables and contravariant in others, can be defined in a similar way.

Definition C.3. The functors of the same variance from one category $\mathfrak{A}$ to another $\mathfrak{A}'$ form themselves the objects of a category $\mathrm{Fun}(\mathfrak{A}, \mathfrak{A}')$. Its morphisms are called **natural transformations** and they are defined as follows. If λ, μ are two functors from $\mathfrak{A}$ to $\mathfrak{A}'$ (say covariant), the natural transformation $t\colon \lambda \to \mu$ consists of a collection of morphisms

$$t_X\colon \lambda(X) \longrightarrow \mu(X)$$

as X ranges over $\mathfrak{A}$, which makes the following diagram commutative for any $f\colon X \to Y$, morphism in $\mathfrak{A}$:

$$\begin{array}{ccc} \lambda(X) & \xrightarrow{t_X} & \mu(X) \\ \downarrow{\scriptstyle \lambda(f)} & & \downarrow{\scriptstyle \mu(f)} \\ \lambda(Y) & \xrightarrow[t_Y]{} & \mu(Y) \end{array}$$

Definition C.4. In any category $\mathfrak{A}$, we say that a morphism $f\colon X \to Y$ is an **isomorphism** if there exists a morphism $g\colon Y \to X$ such that gf and fg are the identities.

Definition C.5. If $f\colon X \to Y$ is a morphism, then a **section** of f is defined to be a morphism $g\colon Y \to X$ such that $fg = \mathrm{id}_Y$.

C.2 The category of vector bundles

We now make the set of vector bundles (as defined in 1.19) into a category. The vector bundles will be the objects, and therefore we have to define their morphisms.

Definition C.6. Let $\pi\colon A \to X$ and $\pi'\colon A' \to X'$ be two vector bundles with fibres Banach spaces E and E' respectively. A **VB-morphism** $\pi \to \pi'$ consists of a pair of morphisms

$$f_0\colon X \longrightarrow X' \qquad \text{and} \qquad f\colon A \longrightarrow A'$$

satisfying the following conditions.

1. The diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow[f_0]{} & X' \end{array}$$

   is commutative, and the induced map for each $x \in X$

$$f_x\colon A_x \longrightarrow A'_{f_0(x)}$$

   is a continuous linear map (note the isomorphisms $A_x \cong E$, $A'_{f_0(x)} \cong E'$).

2. For each $x_0 \in X$, there exist trivialising maps

$$\tau : \pi^{-1}(U) \longrightarrow U \times E \qquad \text{and} \qquad \tau' : \pi'^{-1}(U') \longrightarrow U' \times E'$$

   at x_0 and $f_0(x_0)$ respectively, such that $f_0(U)$ is contained in U', and such that the map of U into $L(E, E')$ given by

$$x \longmapsto \tau'_{f_0(x)} \circ f_x \circ \tau_x^{-1}$$

   is a morphism.

We denote by $\mathbf{VB}(X, \mathfrak{A})$ the **category of vector bundles** with base space X and fibre E, an element of the subcategory $\mathfrak{A}$ of Banach spaces.

C.3 The cotangent bundle and general tensor bundles

The purpose of introducing category theory is to handle vector bundle constructions in an abstract way via functors. We first provide the necessary technical results and then we give some relevant examples.

Definition C.7. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be subcategories of Banach spaces. Let us consider a functor

$$\lambda\colon \mathfrak{A} \times \mathfrak{B} \longrightarrow \mathfrak{C}$$

which is contravariant in the first variable and covariant in the second. That is, this functor assigns Banach spaces as $(E, F) \mapsto \lambda(E, F)$, and for two continuous linear maps $f\colon E' \to E$, $g\colon F \to F'$ (f morphism of $\mathfrak{A}$, g morphism of $\mathfrak{B}$), we have a map

$$\begin{aligned} L(E', E) \times L(F, F') &\longrightarrow L\left(\lambda(E, F), \lambda(E', F')\right) \\ (f, g) &\longmapsto \lambda(f, g) \end{aligned} \quad .$$

We shall say that λ is of **class $\mathcal{C}^\infty$** if the following condition is satisfied: given a manifold U and two $\mathcal{C}^\infty$ maps

$$\varphi\colon U \longrightarrow L(E', E) \qquad \text{and} \qquad \psi\colon U \longrightarrow L(F, F')\,,$$

then the composite

$$U \longrightarrow L(E', E) \times L(F, F') \longrightarrow L\left(\lambda(E, F), \lambda(E', F')\right)$$

is also infinitely differentiable.

Theorem C.8. *Let λ be a functor as above, of class $\mathcal{C}^\infty$. Then, for each manifold X, there exists a functor λ_X, on vector bundles*

$$\lambda_X\colon \mathbf{VB}(X, \mathfrak{A}) \times \mathbf{VB}(X, \mathfrak{B}) \longrightarrow \mathbf{VB}(X, \mathfrak{C})$$

satisfying the following properties. For any bundles $\alpha := \{\pi\colon A \to X\}$, $\beta := \{\pi\colon A' \to X\}$ in $\mathbf{VB}(X, \mathfrak{A}), \mathbf{VB}(X, \mathfrak{B})$ *respectively, and VB-morphisms*

$$f\colon \alpha' \to \alpha \qquad \textit{and} \qquad g\colon \beta \to \beta'$$

in the respective categories, and for each $x \in X$, we have:

1. $\lambda_X(\alpha, \beta)_x = \lambda(\alpha_x, \beta_x)$.
2. $\lambda_X(f, g)_x = \lambda(f_x, g_x)$.
3. *If α is the trivial bundle $X \times E$ and β the trivial bundle $X \times F$, then $\lambda_X(\alpha, \beta)$ is the trivial bundle $X \times \lambda(E, F)$.*

Remark. In this theorem we have denoted the vector bundles by α and β, but usually we denote a bundle $\pi\colon A \to X$ by A (with a slight abuse of language).

Definition C.9. Let $\pi\colon A \to X$ be a vector bundle. We take λ to be the dual functor, that is $E \mapsto E^* := L(E, \mathbb{R})$. Then, $\lambda_X(A)$ is denoted by A^* and is called the **dual bundle**. Applying theorem C.8 1., the fibre at each point $x \in X$ is the dual space ${E_x}^*$. The dual bundle of the tangent bundle is called the **cotangent bundle**, $T^*(X)$. Similarly, taking λ to be the functor $E \mapsto L^r_a(E)$, the set of alternating r-multilinear forms on E, we denote $\lambda_X(A)$ by $L^r_a(A)$, the **bundle of alternating multilinear forms**. General tensor bundles can be defined in an analogous way.

Appendix D

The Lie derivative

We introduce in this appendix a concept that is needed for the proof of theorem 2.23: the Lie derivative. We shall define it on Banach manifolds, but note that the same construction can be replicated on Fréchet manifolds, as shown in [5], for vector fields such that a local flow exists.

We have already seen how a vector field induces a local diffeomorphism on a manifold by means of its integral curves. The purpose of the Lie derivative is to apply that transformation to all the geometric objects, namely to general tensor fields. Let us begin by introducing the definition on functions and vector fields.

Definition D.1. Let X be a manifold, U open in X, $\varphi \in \mathcal{C}^\infty(X)$, $\xi \in \mathfrak{X}(X)$. We define the **Lie derivative** of φ along ξ as $\mathscr{L}_\xi\, \varphi \coloneqq \xi\varphi$.

Definition D.2. Let X be a manifold, U open in X, $\xi, \eta \in \mathfrak{X}(X)$ vector fields on X. Given a point $x \in U$, let α be a local flow for ξ at x. The map

$$\mathscr{L}_\xi \colon \mathfrak{X}(U) \longrightarrow \mathfrak{X}(U) \qquad \eta \longmapsto \mathscr{L}_\xi\, \eta \coloneqq \left.\frac{d}{dt}\right|_{t=0} (\alpha_{-t})_* \circ \eta \circ \alpha_t$$

is called the **Lie derivative** of the vector field η along ξ.

Now, the Lie derivative is extended to general tensor fields algebraically.

Definition D.3. Let X be a manifold and $\xi \in \mathfrak{X}(X)$. Let λ be a differentiable functor on Banach spaces defined as in C.7, with r contravariant variables and s covariant ones. Consider the vector bundle $\lambda_X(T(X))$, whose sections $\Gamma_\lambda(X)$ will be r-contravariant and s-covariant tensor fields. The **Lie derivative** is the map $\mathscr{L}_\xi \colon \Gamma_\lambda(X) \longrightarrow \Gamma_\lambda(X)$ satisfying the following axioms:

1. The Lie derivative of a function is equal to the action of the vector field as derivation, $\mathscr{L}_\xi\, \varphi \coloneqq \xi\varphi$ for all $\varphi \in \mathcal{C}^\infty(X)$.

2. It commutes with the exterior derivative, $\mathscr{L}_\xi\, d\varphi = d\, \mathscr{L}_\xi\, \varphi$ for all $\varphi \in \mathcal{C}^\infty(X)$.

3. It satisfies the Leibniz rule with the tensor product, $\mathscr{L}_\xi\, (\eta_1 \otimes \eta_2) = \mathscr{L}_\xi \eta_1 \otimes \eta_2 + \eta_1 \otimes \mathscr{L}_\xi \eta_2$.

4. The Lie derivative is compatible with contraction, i.e. given $\eta \in \Gamma_\lambda(X) = \Gamma\left(T^r_s(X)\right)$

$$\begin{aligned}(\mathscr{L}_\xi \eta)\,(\varphi_1, \ldots, \varphi_r, \xi_1, \ldots, \xi_s) &= \mathscr{L}_\xi\left(\eta(\varphi_1, \ldots, \varphi_r, \xi_1, \ldots, \xi_s)\right) - \\ &- \eta\left(\mathscr{L}_\xi \varphi_1, \ldots, \varphi_r, \xi_1, \ldots, \xi_s\right) - \eta\left(\varphi_1, \ldots, \varphi_r, \xi_1, \ldots, \mathscr{L}_\xi \xi_s\right)\,.\end{aligned}$$

Appendix E

Proofs of results from geometric quantum mechanics

We provide in this appendix a detailed proof that was omitted in section 2.2 for the sake of brevity. Notice that because of isomorphisms (2.3), in the following we will assume the scalar product to be defined only on the fibre and the vector fields to be represented as pairs $(\psi, \xi(\psi))$.

Proof of proposition 2.18.

Proof. Let $\omega \in \Lambda^2(\mathscr{S}(\mathbb{R}^d))$ be the two-form defined as $\langle \omega, \xi \times \eta \rangle = 2\mathrm{Im}\langle \xi, \eta \rangle$, for $\xi, \eta \in \mathfrak{X}(\mathscr{S}(\mathbb{R}^d))$. To alleviate notation we shall write $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^d)$. Note that by $2\mathrm{Im}\langle \xi, \eta \rangle$ we mean the function of $\mathcal{C}^\infty(\mathscr{S})$ obtained by the inner product $\langle \xi(\psi), \eta(\psi) \rangle$ at every point $\psi \in \mathscr{S}$. We have to prove that the two-form is closed and its induced map is injective:

1. It can be proved that ω is locally exact, thus closed ($\omega = -d\theta$, see [5]).

2. The induced continuous linear map is, pointwise,

$$\begin{aligned} \cdot^\flat \colon T(\mathscr{S}) \cong \mathscr{S} \times \mathscr{S} &\longrightarrow T^*(\mathscr{S}) \cong \mathscr{S} \times \mathscr{S}' \\ (\psi, \xi(\psi)) &\longmapsto (\psi, \xi^\flat(\psi)) := (\psi, i_{\xi(\psi)}\, \omega) = (\psi, 2\mathrm{Im}\langle \xi(\psi), \cdot \rangle) \, . \end{aligned}$$

   In order to see that the map $\cdot^\flat$ is injective we rely on the Gel'fand triple structure. That is to say, we know that $\mathscr{S}$ is a dense topological vector subspace of $\mathscr{H} := L^2(\mathbb{R}^d)$ endowed with a topology that makes the inclusion j continuous. Hence, considering the canonical isomorphism determined by the inner product (Riesz representation theorem) and the dual of the inclusion, j^*, we have a continuous injection

$$\begin{aligned} \mathscr{S} &\xrightarrow{\ j\ } \mathscr{H} \cong \mathscr{H}^* \xrightarrow{\ j^*\ } \mathscr{S}' \\ \xi &\longmapsto |\xi\rangle \mapsto \langle \xi| \longmapsto j^*(\langle \xi|) = \langle \xi| \circ j = \langle \xi, \cdot \rangle \, . \end{aligned}$$

   Since the map $\xi \mapsto \langle \xi, \cdot \rangle$ is injective, so is the map $\xi \mapsto 2\mathrm{Im}\langle \xi, \cdot \rangle$.

□