A Revisit to Stability of Schauder Bases: Fractalizing Multivariate Faber-Schauder System

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Abstract. Let X be a Banach space with a Schauder basis $(x_m)_{m=0}^{\infty}$, and I be the identity operator on X. It is known, at least in essence, that if $(T_m)_{m=0}^{\infty}$ is a sequence of bounded linear operators on X such that $\sum_{m=0}^{\infty} \|I - T_m\| < \infty$, then $\big(T_m(x_m)\big)_{m=0}^{\infty}$ is also a basis. The first part of this work acts as an expository note to formally record the aforementioned stability result. In the second part, we apply this stability result to construct a Schauder basis consisting of bivariate fractal functions for the space of continuous functions defined on a rectangle. To this end, fractal perturbations of the elements in the classical bivariate Faber-Schauder system are formulated using a sequence of bounded linear fractal operators close to the identity operator in accordance with the stability result mentioned above. This illustration although emphasized only for the bivariate case, can easily be extended to higher dimensions. Further, the perturbation technique used here acts as a companion for a few researches on fractal bases in the univariate setting.

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1. Introduction

This work, broadly, lies at the intersection of perturbation theory of Schauder basis and the theory of fractal (interpolation) function, a relatively recent field in interpolation and approximation. To be specific, we target to construct Schauder bases consisting of fractal functions (self-referential functions) for the space of bivariate continuous functions by using a suitable result on the stability of Schauder bases and the notion of fractal operators. The exposition

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herein is influenced by our attempt to make this note accessible to both analyst ignorant of fractal interpolation function and researchers in fractal approximation inexperienced in the perturbation theory of Schauder basis.

1.1. Schauder Basis

Let X be an infinite dimensional Banach space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A sequence $(x_m)_{m=0}^{\infty}$ in a Banach space X is a Schauder basis for X if for each $x \in X$ there is a unique sequence of scalars $(a_m(x))_{m=0}^{\infty}$ such that

$$x = \sum_{m=0}^{\infty} a_m(x) x_m,$$

where the convergence is taken with respect to the norm on X. There is a vast literature dedicated to the subject of Schauder basis; we shall just refer to the well-known books [6, 21] for background information and [8] for a systematic survey. The simplest and perhaps the most obvious way of constructing new basis from a known basis is through a suitable topological isomorphism (a bounded linear map with bounded inverse) of the underlying space. Let us recall that if $(x_m)_{m=0}^{\infty}$ is a Schauder basis for a Banach space X and T is a topological isomorphism on X that transforms $(x_m)_{m=0}^{\infty}$ to $(y_m)_{m=0}^{\infty}$, that is,

$$y_m = T(x_m), \quad m = 0, 1, 2, \dots,$$

then $(y_m)_{m=0}^{\infty}$ is also a basis for X. The question on "stability under small perturbations", which is obviously connected to the process of transforming a known basis is classical. That is, one may ask:

Question. Let $(x_m)_{m=0}^{\infty}$ be a fixed but arbitrary basis for a Banach space X. If a sequence $(y_m)_{m=0}^{\infty}$ in X is "close" to $(x_m)_{m=0}^{\infty}$, then must $(y_m)_{m=0}^{\infty}$ be a basis for X?

There are affirmative answers to the previous question with various interpretations attached to the notion of closeness; see, for instance, an albeit incomplete list of references [2, 5, 7, 17, 23]. The Paley-Wiener theorem [24] and most of the other approaches to the stability of Schauder bases seek to construct an operator T which, in some sense, is close to the identity operator.

For the sake of exposition and record, we shall explicitly note down the following result on the stability of Schauder bases by considering a perturbation through an appropriate sequence of linear operators $(T_m)_{m=0}^{\infty}$. Let us recall that a sequence $(x_m)_{m=0}^{\infty}$ in a Banach space is ω-independent if $\sum_{m=0}^{\infty} c_m x_m = 0$ implies $c_m = 0$ for all m.

Theorem 1.1 (A stability result). Let X be a Banach space and $(x_m)_{m=0}^{\infty}$ be a Schauder basis for X. If $(T_m)_{m=0}^{\infty}$ is a sequence of linear operators preserving the ω-independence of $(x_m)_{m=0}^{\infty}$ and $\sum_{m=0}^{\infty} \|I - T_m\| < \infty$, then $\big(T_m(x_m)\big)_{m=0}^{\infty}$ is a basis for X.

We do not claim complete originality to the aforementioned stability result, as it resembles some stability results proven by various authors; for instance, it may be seen, at least in essence, in [24]. However, we were unable to find a proper reference to this version in the literature, which befits the current purpose, and hence decided to include the details here. The novelty of the current note lies in the identification of suitable stability result on Schauder basis and its effective application in the construction of Schauder bases consisting of fractal functions for the space of bivariate continuous functions.

1.2. Fractal Interpolation

Fractal interpolation function and its connection with various branches of mathematics continue to receive a considerable research interest over the last three decades; see, for instance, [3, 9]. A special type of fractal interpolation function, referred to as the *α-fractal function* and the associated notion of *fractal operator* were introduced and popularized by the first author [10, 11, 12]. Using the notion of univariate α-fractal function and fractal operator associated with it, the existence of systems of fractal functions that constitute Schauder bases for some standard function spaces has been established; see, for example, [11, 12, 13]. The second part of the current note provides a more general approach to construct fractal bases in the bivariate setting, and acts as a supplement to the researches in [11, 13].

We could have started this part of our note on fractal interpolation with the univariate fractal interpolation by Barnsley [3] or its most general bivariate analogue by Ruan and Xu [19]. However, we decided to choose a special case of [19], namely, the bivariate α-fractal function, due to the independency of its treatment, and its relevance and adequacy for the current study.

Let $I_x = [x_0, x_m]$, $I_y = [y_0, y_n]$ be closed and bounded intervals in $\mathbb{R}$. Suppose that $D = I_x \times I_y$. We shall denote by $\mathcal{C}(D)$, the linear space of all real-valued continuous functions on D endowed with the uniform norm $\|.\|_\infty$.

Let $f \in \mathcal{C}(D)$, which is usually referred to as the *germ function* or *seed function*. Consider the following.

1. The set $\Delta := \big\{(x_i, y_j) : i = 0, 1, \dots, m; j = 0, 1, \dots, n\big\}$, where $x_0 < x_1 < \cdots < x_m$ and $y_0 < y_1 < \cdots < y_n$. Note that $\Delta_x = \{x_i \in \mathbb{R} : i = 0, 1, \dots, m$ such that $x_0 < x_1 < \cdots < x_m\}$ is a partition of I_x and $\Delta_y = \{y_j \in \mathbb{R} : j = 0, 1, \dots, n$ such that $y_0 < y_1 < \cdots < y_n\}$ is a partition of I_y. Also, $\Delta = \Delta_x \times \Delta_y$ provides a *partition* of D into sub-rectangles.
2. Let $\alpha : D \to \mathbb{R}$ be a fixed continuous function with $\|\alpha\|_\infty = \sup\{|\alpha(x, y)| : (x, y) \in D\} < 1$, called a *scaling function*.
3. Let $L : \mathcal{C}(D) \to \mathcal{C}(D)$ be a bounded linear operator such that $L \neq I$, and $L(f)(x_i, y_j) = f(x_i, y_j)$ for $i \in \{0, m\}$ and $j \in \{0, n\}$. That is, $L(f)$ interpolates to f at four vertices of D.

For $i \in \mathbb{N}_m = \{1, 2, \dots, m\}$, let $u_i : I_x \to [x_{i-1}, x_i]$ be linear maps $u_i(x) = a_i x + b_i$, where constants are determined such that

$$\begin{cases} u_i(x_0) = x_{i-1}, \quad u_i(x_m) = x_i, \text{ if } i \text{ is odd,} \\ u_i(x_0) = x_i, \quad u_i(x_m) = x_{i-1}, \text{ if } i \text{ is even.} \end{cases}$$

Similarly, for $j \in \mathbb{N}_n = \{1, 2, \ldots, n\}$, let $v_j : I_y \to [y_{j-1}, y_j]$ be linear maps $v_j(y) = c_j y + d_j$ satisfying

$$\begin{cases} v_j(y_0) = y_{j-1}, \quad v_j(y_n) = y_j, \text{ if } j \text{ is odd}, \\ v_j(y_0) = y_j, \quad v_j(y_n) = y_{j-1}, \text{ if } j \text{ is even}. \end{cases}$$

Note that

$$u_{i+1}^{-1}(x_i) = u_i^{-1}(x_i) \ \forall\, i \in \mathbb{N}_{m-1}, \text{ and } v_{j+1}^{-1}(y_j) = v_j^{-1}(y_j) \ \forall\, j \in \mathbb{N}_{n-1}.$$

Consider

$$\begin{aligned} \mathcal{C}_f(D) = \Big\{ & g : D \to \mathbb{R}; g \in \mathcal{C}(D) \text{ and } g(x_i, y_j) = f(x_i, y_j), \\ & \forall\, i \in \{0, m\},\ j \in \{0, n\} \Big\}, \end{aligned}$$

which is a complete metric space under the sup-metric. Now define a map $\Phi^\alpha_{\Delta,L}$, which is a form of the Read-Bajraktarević operator (see [3, 9]), as follows. For $(x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, where $(i, j) \in \mathbb{N}_m \times \mathbb{N}_n$

$$\Phi^\alpha_{\Delta,L}(g)(x, y) = f(x, y) + \alpha(x, y)\big(g - L(f)\big)\big(u_i^{-1}(x), v_j^{-1}(y)\big),$$

for all $g \in \mathcal{C}_f(D)$. It is easy to see that the mapping $\Phi^\alpha_{\Delta,L}$ is well-defined and it satisfies

$$d_\infty\big(\Phi^\alpha_{\Delta,L}(g_1), \Phi^\alpha_{\Delta,L}(g_2)\big) \le \|\alpha\|_\infty \ d_\infty(g_1, g_2) \ \forall\ g_1, g_2 \in \mathcal{C}_f(D).$$

Consequently, $\Phi^\alpha_{\Delta,L}$ possesses a unique fixed point $f^\alpha_{\Delta,L}$, which satisfies the self-referential equation

$$f^\alpha_{\Delta,L}(x, y) = f(x, y) + \alpha(x, y)\ \big(f^\alpha_{\Delta,L} - L(f)\big)\big(u_i^{-1}(x), v_j^{-1}(y)\big), \tag{1.1}$$

for $(x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j],\ (i, j) \in \mathbb{N}_m \times \mathbb{N}_n$. Note that

$$\begin{aligned} f^\alpha_{\Delta,L}(x_i, y_j) =& f(x_i, y_j) + \alpha(x_i, y_j)\ \big(f^\alpha_{\Delta,L} - L(f)\big)\big(u_i^{-1}(x_i), v_j^{-1}(y_j)\big) \\ =& f(x_i, y_j), \text{ for all } i \in \mathbb{N}_m, \text{ and } j \in \mathbb{N}_n. \end{aligned} \tag{1.2}$$

Analogous to the univariate setting [10], we call the function $f^\alpha_{\Delta,L}$, a *bivariate α-fractal function* corresponding to f with respect to the parameters Δ, α and L. Note that $f^\alpha_{\Delta,L}$ is a special type of fractal interpolation function corresponding to f. In the construction mentioned above, the fractal functions $f^\alpha_{\Delta,L}$ interpolate the germ function f at points in Δ, but not on the entire coordinate lines (x_i, y), (x, y_j) for $i \in \mathbb{N}_m$, and $j \in \mathbb{N}_n$; see also the construction in [14].

For a fixed choice of scaling function α, partition Δ, and operator L, we associate with each $f \in \mathcal{C}(D)$ its fractal perturbation $f^\alpha_{\Delta,L}$ to provide a linear operator referred to as *fractal operator* denoted by $\mathcal{F}^\alpha_{\Delta,L}$ or $\mathcal{F}^\alpha$.

$$\mathcal{F}^\alpha_{\Delta,L} : \mathcal{C}(D) \to \mathcal{C}(D), \quad \mathcal{F}^\alpha_{\Delta,L}(f) = f^\alpha_{\Delta,L}.$$

In loose terms, one may interpret $f^\alpha_{\Delta,L}$ as a "fractal perturbation" of the original function f and $\mathcal{F}^\alpha_{\Delta,L}$ as a "fractal perturbation operator". Depending on the approximation problem at hand, the parameters Δ, α and L involved in the perturbation process may be selected so that perturbations $f^\alpha_{\Delta,L}$ may preserve or modify the properties inherent in the original function f. Self-referentiality of the function $f^{\alpha}_{\Delta,L}$ may be an added advantage.

The distance between the seed function and its fractal counterpart satisfies the following inequality (see [11, 22] for details)

$$\left\|f^{\alpha}_{\Delta,L} - f\right\|_{\infty} \leq \frac{\|\alpha\|_{\infty}}{1-\|\alpha\|_{\infty}}\|f - L(f)\|_{\infty}. \tag{1.3}$$

Consequently, the corresponding operator norm satisfies

$$\left\|\mathcal{F}^{\alpha}_{\Delta,L} - I\right\| \leq \frac{\|\alpha\|_{\infty}}{1-\|\alpha\|_{\infty}}\|I - L\|. \tag{1.4}$$

It is quite natural that the aforementioned bounded linear fractal operator assists the field of fractal interpolation to interact fruitfully with functional analysis and operator theory - a fact which is well-explored in the univariate setting [10, 11, 12]. Now, its development in multivariate setting is still in its infancy, yet as inroads are made [14, 19, 22], interest is gathering steam.

In this note, we target to construct Schauder bases consisting of fractal functions for $\mathcal{C}(D)$ by using bivariate fractal operators and the stability result that we alluded to. In fact, the efficacy of the univariate fractal operator in the construction of particular examples of Schauder bases (consisting of fractal functions) by perturbations of classical bases (see, for instance, [10, 11, 12]) stimulated the search for a suitable stability result for Schauder bases and its application discussed herein. On the one hand, this note provides a fractal Schauder basis in the bivariate setting, and on the other, a more general approach herein supplements the study of fractal bases undertaken in the univariate setting. Let us close this section with a remark that our approach in this note can be easily adapted to higher dimensions via the stability result for Schauder basis given here and the notion of multivariate fractal operator introduced recently in [15].

2. Basics of Bases

We recall here a few basic facts on the notion of Schauder basis and Riesz basis needed in the sequel; for details, the reader may refer [20, 21].
Let X be a Banach space and $(x_m)_{m=0}^{\infty}$ be a Schauder basis for X.

Definition 2.1. A basis $(x_m)_{m=0}^{\infty}$ is a bounded basis if $(x_m)_{m=0}^{\infty}$ is norm-bounded both above and below, i.e., if $0 < \inf\|x_m\| \leq \sup\|x_m\| < \infty$. A basis $(x_m)_{m=0}^{\infty}$ is a normalized basis if $\|x_m\| = 1$ for every m.

Definition 2.2. The maps $\sum_{m=0}^{\infty} a_m(x)x_m \mapsto a_m(x)$ and $P_N : X \to X$ defined by $P_N\left(\sum_{m=0}^{\infty} a_m(x)x_m\right) = \sum_{m=0}^{N} a_m(x)x_m$ are called the m-th coefficient functional and the N-th natural projection associated with $(x_m)_{m=0}^{\infty}$, respectively.

Remark 2.1. In fact, for each m, a_m is a continuous linear functional and the partial sum operator P_N is a bounded linear map. To refer explicitly to both the basis and the associated coefficient functionals, we shall write $\{(x_m)_{m=0}^{\infty}, (a_m)_{m=0}^{\infty}\}$ to say that $(x_m)_{m=0}^{\infty}$ is a basis with the associated coefficient functionals $(a_m)_{m=0}^{\infty}$.

Definition 2.3. Let X be a Banach space. Two bases $(x_m)_{m=0}^{\infty}$ and $(y_m)_{m=0}^{\infty}$ for X are said to be equivalent if

$$\sum_{m=0}^{\infty} c_m x_m \text{ is convergent} \iff \sum_{m=0}^{\infty} c_m y_m \text{ is convergent.}$$

Furthermore, two bases for X are equivalent if and only if there is a topological isomorphism on X transforming one basis into the other.

Definition 2.4. If $\{(x_m)_{m=0}^{\infty}, (a_m)_{m=0}^{\infty}\}$ is a basis for a Banach space X, then its basis constant is the finite number $C = \sup_N \|P_N\|$ satisfying $C \geq 1$.

Theorem 2.1. *The coefficient functionals a_m are continuous linear functionals on X which satisfy $1 \leq \|a_m\|\|x_m\| \leq 2C$, where C is the basis constant for $\{(x_m)_{m=0}^{\infty}, (a_m)_{m=0}^{\infty}\}$.*

Definition 2.5. A sequence $(x_m)_{m=0}^{\infty}$ in a Hilbert space X is called a Bessel sequence if there exists $B > 0$ such that $\sum_{m=0}^{\infty} |\langle x, x_m\rangle|^2 \leq B\|x\|^2$ for all $x \in X$. A Bessel sequence is a frame if there exists $A > 0$ such that $A\|x\|^2 \leq \sum_{m=0}^{\infty} |\langle x, x_m\rangle|^2 \leq B\|x\|^2$ for all $x \in X$.

Definition 2.6. A sequence $(x_m)_{m=0}^{\infty} \subseteq X$, where X is a Hilbert space, is a Riesz sequence if there exist $k_1, k_2 > 0$ such that for any $(c_m)_{m=0}^{\infty} \in l^2$ $k_1 \sum_{m=0}^{\infty} |c_m|^2 \leq \left\| \sum_{m=0}^{\infty} c_m x_m \right\|^2 \leq k_2 \sum_{m=0}^{\infty} |c_m|^2$. A Riesz sequence is a Riesz basis for its closed linear span $[x_m] := \overline{\text{span}}(x_m)_{m=0}^{\infty}$. If $[x_m] = X$, then $(x_m)_{m=0}^{\infty}$ is a Riesz basis.

Definition 2.7. A sequence $(x_m)_{m=0}^{\infty} \subseteq X$, where X is a Hilbert space, is a Riesz basis if there is an orthonormal basis $(y_m)_{m=0}^{\infty}$ for X and a topological isomorphism $S : X \to X$ such that $x_m = S(y_m)$.

Definition 2.8. Let X and Y be Banach spaces. We shall denote by $X \otimes_\lambda Y$ the completion of the algebraic tensor product $X \otimes Y$ in the norm

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\| = \sup\left\{ \|\phi(x_i) y_i\| : \phi \in X^*, \|\phi\| = 1 \right\}$$

and we denote by $X \otimes_\pi Y$ the completion of $X \otimes Y$ in the norm

$$\|z\| = \inf\left\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i \right\},$$

where the infimum is taken over all possible representations of $z \in X \otimes Y$. Both the norms on $X \otimes Y$ defined above satisfy $\|x \otimes y\| = \|x\|\|y\|$ for a typical element $x \otimes y$ in $X \otimes Y$.

Theorem 2.2. *We have*

$$\mathcal{C}\big([a,b]\big) \otimes_\lambda \mathcal{C}\big([c,d]\big) = \mathcal{C}\big([a,b]\times[c,d]\big),$$

where the equality sign is interpreted as isometric isomorphism under the association $\sum_{i=1}^{n} x_i \otimes y_i \mapsto \sum_{i=1}^{n} x_i(s)y_i(t)$.

Theorem 2.3. *Let* $\{(x_m)_{m=0}^{\infty}, (a_m)_{m=0}^{\infty}\}$ *be a Schauder basis for the Banach space* X *and* $\{(y_m)_{m=0}^{\infty}, (b_m)_{m=0}^{\infty}\}$ *be a Schauder basis for the Banach space* Y*.Then the sequence* $\{x_m \otimes y_n\}_{m,n=0}^{\infty}$ *ordered in the following way*

$$x_0 \otimes y_0, x_0 \otimes y_1, x_1 \otimes y_1, x_1 \otimes y_0, x_0 \otimes y_2, x_1 \otimes y_2, x_2 \otimes y_2, x_2 \otimes y_1, x_2 \otimes y_0 \dots$$

is a Schauder basis for both $X \otimes_\lambda Y$ *and* $X \otimes_\pi Y$ *with associated sequence of coefficient functionals* $\{a_m \otimes b_n\}_{m,n=0}^{\infty}$.

3. A Revisit to Stability of Schauder Basis and Allied Results

3.1. Banach Space Setting

Let X be a Banach space and $\{(x_m)_{m=0}^{\infty}, (a_m)_{m=0}^{\infty}\}$ be a normalized basis. Then for each $x \in X$

$$x = \sum_{m=0}^{\infty} a_m(x) x_m.$$

Let us commence with the following straightforward observations.

Remark 3.1. Let P_n be the n-th partial sum operator corresponding to the normalized basis $\{(x_m)_{m=0}^{\infty}, (a_m)_{m=0}^{\infty}\}$. We have

$$a_n(x)x_n = P_n x - P_{n-1}x. \tag{3.1}$$

Taking C as the basis constant of $\{(x_m)_{m=0}^{\infty}, (a_m)_{m=0}^{\infty}\}$, by (3.1) we have

$$\begin{aligned} |a_n(x)| &= \|a_n(x)x_n\| \\ &\le \|P_n x\| + \|P_{n-1}x\| \\ &\le 2C\|x\|. \end{aligned} \tag{3.2}$$

Consequently, $\|a_n\| \le 2C$.

Remark 3.2. Let X be a Banach space. A closed subspace $Y \subset X$ is said to be complemented in X if there exists a closed subspace $Z \subset X$ such that $X = Y \oplus Z$, the direct sum. The codimension of a subspace Y of X is defined as the dimension of the quotient space X/Y. Every closed subspace $Y \subset X$ of finite codimension is complemented in X (see, for example, [18]). To be explicit, let us mention the following. Let $q : X \to X/Y$ be the quotient map, and $\{\eta_1, \eta_2, \dots, \eta_k\}$ be a basis for X/Y. Choose $z_i \in X$ such that $q(z_i) = \eta_i$ for $i = 1, 2, \dots, k$, and define $Z = \operatorname{span}\{z_1, z_2, \dots, z_k\}$. Then $X = Y \oplus Z$.

The following result, popularly known as the principle of small perturbations, is fundamental in the theory of Schauder basis; see, for instance, [1].

Theorem 3.1. *Let $(x_m)_{m=0}^{\infty}$ be a Schauder basis for a Banach space X with the basis constant C. If $(y_m)_{m=0}^{\infty}$ is a sequence in X such that*

$$2C\sum_{m=0}^{\infty}\frac{\|x_m-y_m\|}{\|x_m\|}=\theta<1,$$

then $(y_m)_{m=0}^{\infty}$ is a basis for X equivalent to $(x_m)_{m=0}^{\infty}$.

Lemma 3.1. *Let $(x_m)_{m=0}^{\infty}$ be a Schauder basis for a Banach space X. Suppose that $(T_m)_{m=0}^{\infty}$ is a sequence of bounded linear operators on X such that*

$$\sum_{m=0}^{\infty}\|I-T_m\|<\infty.$$

Then there exists $n\in\mathbb{N}$ such that the sequence $(y_m)_{m=0}^{\infty}$ defined by

$$y_m=\begin{cases}x_m \text{ for } m\le n,\\ T_m(x_m) \text{ for } m>n.\end{cases}$$

is a Schauder basis for X equivalent to $(x_m)_{m=0}^{\infty}$.

Proof. Let C be the basis constant for $(x_m)_{m=0}^{\infty}$. Since $\sum_{m=0}^{\infty}\|I-T_m\|<\infty$, we can find $n\in\mathbb{N}\cup\{0\}$ large enough such that

$$\sum_{m=n+1}^{\infty}\|I-T_m\|<\frac{1}{2C}.$$

Fix such an $n\in\mathbb{N}\cup\{0\}$ and construct the sequence $(y_m)_{m=0}^{\infty}$ as prescribed in the statement. We have

$$\sum_{m=0}^{\infty}\frac{\|x_m-y_m\|}{\|x_m\|}=\sum_{m=n+1}^{\infty}\frac{\|x_m-T_m(x_m)\|}{\|x_m\|}\le\sum_{m=n+1}^{\infty}\|I-T_m\|<\frac{1}{2C}.$$

Hence, from the principle of small perturbations of Schauder basis (Theorem 3.1), it follows that $(y_m)_{m=0}^{\infty}$ is a Schauder basis equivalent to $(x_m)_{m=0}^{\infty}$. □

Lemma 3.2. *Let $(x_m)_{m=0}^{\infty}$ be a Schauder basis for a Banach space X, and let $(y_m)_{m=0}^{\infty}$ be a sequence in X with $x_m=y_m$ except for finitely many indices $m\in\mathbb{N}\cup\{0\}$. If, in addition, $(y_m)_{m=0}^{\infty}$ is ω-independent, then it is a Schauder basis equivalent to $(x_m)_{m=0}^{\infty}$.*

Proof. Choose $n\in\mathbb{N}\cup\{0\}$ such that $x_m=y_m$ for all $m>n$ and define $Y=\overline{\operatorname{span}\{x_m:m>n\}}$, the closure of the linear span of $\{x_m:m>n\}$. Note that X/Y is isomorphic to $[x_0,x_1,x_2,\dots,x_n]$ and hence, the codimension of Y, that is $\dim(X/Y)$, is $n+1$. Further, both the families $(x_m+Y)_{m=0}^{n}$ and $(y_m+Y)_{m=0}^{n}$ are linearly independent. Hence, there exists an isomorphism $\tilde{T}:X/Y\to X/Y$ with $\tilde{T}(x_m+Y)=y_m+Y$ for all $0\le m\le n$.

Since Y is a closed subspace of finite codimension, Y can be complemented in X, that is, there exists a closed subspace Z of X such that $X=Y\oplus Z$; in fact, $Z=\operatorname{span}\{x_0,x_1,\dots,x_n\}$ (see Remark 3.2). Consequently, we infer that there is an isomorphism $T:X\to X$ such that $T(x_m)=y_m$ for all $m=0,1,\dots,$ which completes the proof. □

The following stability result follows at once from the above mentioned pair of lemmas.

Theorem 1.1. Let X be a Banach space and $(x_m)_{m=0}^{\infty}$ be a normalized Schauder basis for X. If $(T_m)_{m=0}^{\infty}$ is a sequence of linear operators such that $\big(T_m(x_m)\big)$ is ω-independent and

$$\sum_{m=0}^{\infty} \|I - T_m\| < \infty,$$

then $\big(T_m(x_m)\big)_{m=0}^{\infty}$ is a Schauder basis for X equivalent to $(x_m)_{m=0}^{\infty}$.

Let $\{(x_m)_{m=0}^{\infty}, (a_m)_{m=0}^{\infty}\}$ be a bounded Schauder basis for X such that

$$0 < k_1 \leq \|x_m\| \leq k_2 \quad \forall \quad m = 0, 1, \dots$$

with the basis constant C. Consider a sequence of bounded linear operators $(T_m)_{m=0}^{\infty}$ such that $\big(T_m(x_m)\big)_{m=0}^{\infty}$ is ω-independent and

$$k := \sum_{m=0}^{\infty} \|I - T_m\| < \infty.$$

Let us define $y_m = T_m(x_m)$, for $m = 0, 1, \dots$ With a series of simple propositions, we shall estimate bounds of the perturbed basis $\big(T_m(x_m)\big)_{m=0}^{\infty}$, its basis constant and the norm of the associated coefficient functionals.

First define the operator S on X by setting

$$S(x) = \sum_{m=0}^{\infty} a_m(x) y_m. \tag{3.3}$$

Using the assumption $\sum_{m=0}^{\infty} \|I - T_m\| < \infty$, one can prove that $S_N(x) = \sum_{m=0}^{N} a_m(x) y_m$ is a Cauchy sequence, and hence converges in X. Thus, S is a well-defined linear operator. Consider $F = I - S$ on X, that is,

$$F(x) = \sum_{m=0}^{\infty} a_m(x)(x_m - y_m).$$

Proposition 3.1. *The map F is a bounded linear map with*

$$\|F\| \leq 2Ck_1^{-1}k_2k.$$

Consequently, S is a bounded linear map, and

$$\|S\| \leq 1 + 2Ck_1^{-1}k_2k.$$

Proof. We have

$$\begin{aligned} |a_m(x)| = \|x_m\|^{-1}\|a_m(x)x_m\| &\leq k_1^{-1}\|a_m(x)x_m\| \\ &= k_1^{-1}\|S_m x - S_{m-1}x\| \\ &\leq k_1^{-1}2C\|x\|, \end{aligned}$$

from which it follows that $\|a_m\| \leq 2Ck_1^{-1}$. Therefore,

$$\|F(x)\| \leq \sum_{m=0}^{\infty} |a_m(x)| \|x_m - y_m\| \leq \sum_{m=0}^{\infty} |a_m(x)| \|I - T_m\| \|x_m\| \leq k_1^{-1} 2Ckk_2 \|x\|,$$

from which one can read $\|F\| \leq 2Ck_1^{-1}k_2k$. It follows at once that

$$\|S\| = \|I - F\| \leq 1 + 2Ck_1^{-1}k_2k,$$

establishing the claim. □

Remark 3.3. Following the second part of the proof of Theorem 12 in Chapter 1 of the book [24], we can prove that (i) F is a compact operator, (ii) the kernel of $S = I - F$ is $\{0\}$, and (iii) consequently, by the Fredholm alternative $S = I - F$ is invertible. Since S is a topological isomorphism and $S(x_m) = y_m$, we obtain an alternative approach to show that (y_m) is a Schauder basis.

Proposition 3.2. *If $2Ck_1^{-1}k_2k < 1$, then the basis constant $\tilde{C}$ corresponding to $\big(T_m(x_m)\big)_{m=0}^{\infty}$ is such that*

$$1 \leq \tilde{C} \leq C\Big(\frac{1 + 2Ck_1^{-1}k_2k}{1 - 2Ck_1^{-1}k_2k}\Big),$$

where C is the basis constant of $(x_m)_{m=0}^{\infty}$.

Proof. The basis constant $\tilde{C} = \sup_M \|\tilde{S_M}\|$, where $\tilde{S_M}$ is the M-th partial sum operator corresponding to the basis $\big(T_m(x_m)\big)_{m=0}^{\infty}$. Then

$$x = S \circ S^{-1}(x) = \sum_{m=0}^{\infty} a_m\big(S^{-1}(x)\big) y_m,$$

and

$$\begin{aligned} \|\tilde{S_M} x\| &= \Big\| \sum_{m=0}^{M} a_m\big(S^{-1}(x)\big) y_m \Big\| \\ &\leq \|S\| \Big\| \sum_{m=0}^{M} a_m\big(S^{-1}(x)\big) x_m \Big\| \\ &\leq \|S\| \|S_M\| \|S^{-1}\| \|x\|. \end{aligned}$$

By Proposition 3.1

$$\|S^{-1}\| = \|(I - F)^{-1}\| \leq \frac{1}{1 - \|F\|} \leq \frac{1}{1 - 2Ck_1^{-1}k_2k}. \tag{3.4}$$

Consequently in view of Proposition 3.1 and Equation (3.4)

$$\|\tilde{S_M}\| \leq C\|S\|\|S^{-1}\| \leq C\Big(\frac{1 + 2Ck_1^{-1}k_2k}{1 - 2Ck_1^{-1}k_2k}\Big),$$

and the result follows. □

Proposition 3.3. *If $2Ck_1^{-1}k_2k < 1$, then the coefficient functionals $\tilde{a_m}$ corresponding to the basis $\big(T_m(x_m)\big)_{m=0}^{\infty}$ satisfy*

$$\frac{1}{k_2(1+k)} \leq \|\tilde{a_m}\| \leq \frac{2Ck_1^{-1}}{1-2Ck_1^{-1}k_2k}.$$

Proof. From

$$x = S \circ S^{-1}(x) = \sum_{m=0}^{\infty} a_m\big(S^{-1}(x)\big)y_m,$$

we have

$$\tilde{a_m} = a_m \circ S^{-1}.$$

Proposition 3.1 and Equation (3.4) in conjunction with the above yield

$$\|\tilde{a_m}\| \leq \|a_m\|\|S^{-1}\| \leq \frac{2Ck_1^{-1}}{1-2Ck_1^{-1}k_2k}.$$

In view of Theorem 2.1, the basis $\big(T_m(x_m)\big)_{m=0}^{\infty} = (y_m)_{m=0}^{\infty}$ satisfies

$$1 \leq \|\tilde{a_m}\|\|y_m\| \leq 2\tilde{C}.$$

Further,

$$\|y_m\| - \|x_m\| \leq \|y_m - x_m\| = \|T_m(x_m) - x_m\| \leq \|I - T_m\|\|x_m\| \leq kk_2,$$

from which

$$\|T_m(x_m)\| = \|y_m\| \leq kk_2 + k_2 = k_2(1+k).$$

Therefore

$$\|\tilde{a_m}\| \geq \|y_m\|^{-1} \geq \frac{1}{k_2(1+k)},$$

completing the proof. □

3.2. Hilbert Space Setting

The special properties possessed by the inner product norm enable us to relax the previous hypothesis for the stability of the Schauder bases in Hilbert spaces. Let $(x_m)_{m=0}^{\infty}$ be an orthonormal basis for the Hilbert space X. Let $(T_m)_{m=0}^{\infty}$ be a sequence of bounded linear operators on X such that

$$k^* := \sum_{m=0}^{\infty} \|I - T_m\|^2 < \infty. \tag{3.5}$$

Proposition 3.4. *Let $(x_m)_{m=0}^{\infty}$ be an orthonormal basis for a Hilbert space X and $(T_m)_{m=0}^{\infty}$ be a sequence of bounded linear operators on X satisfying (3.5). Then the system $\big(x_m - T_m(x_m)\big)_{m=0}^{\infty}$ is a Bessel sequence for X.*

Proof. Let $x \in X$. By applying the Cauchy-Schwartz inequality

$$\begin{aligned}\sum_{m=0}^{\infty} \big|\langle x, x_m - T_m(x_m)\rangle\big|^2 &\leq \|x\|^2 \sum_{m=0}^{\infty} \|x_m - T_m(x_m)\|^2 \\ &\leq \|x\|^2 \sum_{m=0}^{\infty} \|I - T_m\|^2 \|x_m\|^2 \\ &= k^* \|x\|^2.\end{aligned}$$

Thus $\big(x_m - T_m(x_m)\big)_{m=0}^{\infty}$ is a Bessel sequence with the Bessel constant k^*. □

Theorem 3.2. *Let $(x_m)_{m=0}^{\infty}$ be an orthonormal basis for a Hilbert space X. Suppose that $(T_m)_{m=0}^{\infty}$ is a sequence of bounded linear operators on X preserving ω-independence of $(x_m)_{m=0}^{\infty}$ and satisfying (3.5). Then $\big(T_m(x_m)\big)_{m=0}^{\infty}$ is a Riesz basis, and hence an exact frame for X. In particular, there exist positive constants A, B such that*

$$A\|x\|^2 \leq \sum_{m=0}^{\infty} |\langle x, T_m(x_m)\rangle|^2 \leq B\|x\|^2.$$

Proof. As before let us consider now the operator on X,

$$F(x) = \sum_{m=0}^{\infty} a_m(x)(x_m - y_m),$$

where $y_m = T_m(x_m)$. Let F_M be the M-th partial sum operator corresponding to F. Then

$$\begin{aligned}\|F_M(x)\|^2 &= \sup\Big\{\big|\langle F_M(x), y\rangle\big|^2 : \|y\| = 1\Big\} \\ &= \sup\Big\{\Big|\langle \sum_{m=0}^{M} a_m(x)(x_m - y_m), y\rangle\Big|^2 : \|y\| = 1\Big\} \\ &= \sup\Big\{\Big|\sum_{m=0}^{M} a_m(x)\langle x_m - y_m, y\rangle\Big|^2 : \|y\| = 1\Big\}.\end{aligned}$$

Using the Cauchy-Schwartz inequality we obtain

$$\|F_M(x)\|^2 \leq \sup\Big\{\Big(\sum_{m=0}^{M} |a_m(x)|^2\Big)\Big(\sum_{m=0}^{M} |\langle x_m - y_m, y\rangle|^2\Big) : \|y\| = 1\Big\}.$$

By the Parseval identity and the previous proposition

$$\|F_M(x)\|^2 \leq k^* \|x\|^2,$$

from which we deduce that F_M is bounded. Since F_M is of finite rank, it is compact. One can easily see that F_M converges to F, which ensures that F is compact. Now as previously, we define $S = I - F$. Since F is compact, $\dim(\ker(I - F)) < \infty$. Moreover,

$$X = \ker(S) \oplus \overline{\mathrm{rg}(S^*)}.$$

Using ω-independence of $\big(T_m(x_m)\big)_{m=0}^{\infty}$, we can show that $\ker(S) = \{0\}$. Consequently, S is an isomorphism and $\big(S(x_m)\big)_{m=0}^{\infty} = \big(T_m(x_m)\big)_{m=0}^{\infty}$ is a Riesz basis for X. $\square$

Proposition 3.5. *Let $(x_m)_{m=0}^{\infty}$ be an orthonormal basis for a Hilbert space X. Suppose that $(T_m)_{m=0}^{\infty}$ is a sequence of bounded linear operators on X preserving ω-independence of $(x_m)_{m=0}^{\infty}$ and satisfying (3.5). Then for all $(c_m)_{m=0}^{\infty} \subset l^2$, there exist positive constants K_1 and K_2 such that*

$$K_1 \sum_{m=0}^{\infty} |c_m|^2 \leq \Big\| \sum_{m=0}^{\infty} c_m T_m(x_m) \Big\|^2 \leq K_2 \sum_{m=0}^{\infty} |c_m|^2.$$

Proof. If $(c_m)_{m=0}^{\infty} \subset l^2$, then by the Riesz-Fischer theorem [18], the series $\sum_{m=0}^{\infty} c_m x_m$ converges in X, say to x. Therefore

$$S(x) = \sum_{m=0}^{\infty} c_m S(x_m),$$

and hence

$$\Big\| \sum_{m=0}^{\infty} c_m T_m(x_m) \Big\|^2 = \Big\| \sum_{m=0}^{\infty} c_m S(x_m) \Big\|^2 = \|Sx\|^2 \leq \|S\|^2 \|x\|^2 = \|S\|^2 \sum_{m=0}^{\infty} |c_m|^2,$$

due to the orthonormality of $(x_m)_{m=0}^{\infty}$. Since S is a topological isomorphism, there exists $K_1 > 0$ such that

$$K_1 \|x\|^2 \leq \|S(x)\|^2 \quad \forall\, x \in X.$$

In particular, for $x = \sum_{m=0}^{\infty} c_m x_m$,

$$K_1 \sum_{m=0}^{\infty} |c_m|^2 \leq \Big\| \sum_{m=0}^{\infty} c_m S(x_m) \Big\|^2 = \Big\| \sum_{m=0}^{\infty} c_m T_m(x_m) \Big\|^2,$$

providing the claim. $\square$

Remark 3.4. Recall that the invertibility of a bounded linear operator is a stable property and the set of topological isomorphisms is open. Since the operator S is a perturbation of the identity operator, by restricting the value of the constant k (Section 3.1) or k^* (Section 3.2) suitably, the system $\big(T_m(x_m)\big)_{m=0}^{\infty}$ turns out to be a basis with no additional conditions on the sequence $(T_m)_{m=0}^{\infty}$. For instance, bearing $S = I - F$ and $\|F\| \leq 2Ck$ in mind, it follows that for $k < \frac{1}{2C}$, the sequence $\big(T_m(x_m)\big)_{m=0}^{\infty}$ is a Schauder basis.

4. Bivariate Fractal Faber-Schauder System

Here we apply the stability results in the previous section to construct Schauder bases consisting of fractal functions for $\mathcal{C}(D)$ by perturbing known Schauder basis of this space. The new system presents important features, namely, (1) self-referentiality of its elements, and (2) lack of (piecewise) differentiability of its elements.

The construction of the (classical) Faber-Schauder system and its multivariate analogue using the idea of tensor product are well-known; see, for instance, [20, 21]. In what follows, we shall outline the construction of the bivariate Faber-Schauder system for the sake of completeness of the exposition.

For convenience, let us take $D = [0,1] \times [0,1]$. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of the interval $[0,1]$ and $T = (t_j)_{j=0}^n$. Denote by $\mathcal{B}_T$ the subspace of $\mathcal{C}([0,1])$ consisting of polygonal functions (piecewise linear functions) with nodes at $t_0, t_1, \dots, t_n$. The *hat functions* for partition T are the functions $h_{t_j}^T$ in $\mathcal{B}_T$ determined by $h_{t_j}^T(t_k) = \delta_{jk}$ for $j,k = 0,1,\dots,n$. Consider a dense sequence in $[0,1]$, say, $\mathcal{J} := \{0 = t_0, 1 = t_1, t_2, \dots\}$. Denote $T_n = \{t_0, t_1, \dots, t_n\}$ and define

$$\Phi_0^{\mathcal{J}} = h_{t_0}^{T_1},\ \Phi_1^{\mathcal{J}} = h_{t_0}^{T_2},\ \Phi_n^{\mathcal{J}} = h_{t_n}^{T_n},\ \text{for } n \geq 2. \tag{4.1}$$

The functions $\{\Phi_0^{\mathcal{J}}, \Phi_1^{\mathcal{J}}, \dots\}$ are the *Schauder hat functions* for the sequence (t_n).

Proposition 4.1. [20, Proposition 2.3.5] *The sequence* $\{\Phi_m^{\mathcal{J}}\}_{m=0}^\infty$ *is a Schauder basis for* $\mathcal{C}\big([0,1]\big)$*, known as the Faber-Schauder system.*

For instance, one can take the specified countable sequence of dyadic numbers and consider $\mathcal{J} := \Big\{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \dots\Big\}$. For notational convenience, we shall denote $\Phi_m^{\mathcal{J}}$ by Φ_m for $m \geq 0$. Then we have

$$\Phi_0(x) = 1 - x,\ \Phi_1(x) = x,$$

$$\Phi_{2^n+k}(x) = \max\big\{0, 1 - 2^{n+1}\big|x - \frac{2k-1}{2^{n+1}}\big|\big\},\ n = 0,1,\dots; k = 1,\dots,2^n.$$

$$\|\Phi_m\|_\infty = 1,\ m = 0,1,\dots$$

The tensor product of two copies of the Schauder system $\{\Phi_m\}_{m=0}^\infty$ in $\mathcal{C}([0,1])$ gives a basis for $\mathcal{C}(D)$ consisting of piecewise biaffine functions (see Section 2).

At this juncture, the reader, if needed, may revisit the notion of bivariate α-fractal function and associated fractal operator given in the introductory section. Here we shall give two examples for the operator L used in the construction of the fractal operator $\mathcal{F}_{\Delta,,L}^\alpha$.

1. One may consider $L(f) = f \circ c$ where $c : \mathcal{C}(D) \to \mathcal{C}(D)$ is a fixed continuous mapping such that $c(x_i, y_j) = (x_i, y_j)$ for $i \in \{0, m\}$ and $j \in \{0, n\}$, and $c \neq I$, the identity map on D. In this case, $\|L(f)\|_\infty = \|f \circ c\|_\infty = \|f\|_\infty$, and hence $\|L\| = 1$.
2. Another choice is $L(f) = \nu f$, where $\nu \in \mathcal{C}(D)$ is a fixed non-constant function such that $\nu(x_i, y_j) = 1$ for $i \in \{0, m\}$ and $j \in \{0, n\}$. Here $\|L\| = \|\nu\|_\infty$.

Using the stability result in the previous section, let us perturb the classical bivariate Faber-Schauder system via suitable fractal operators and obtain a basis for $\mathcal{C}(D)$ consisting of self-referential functions.

Let $X = \mathcal{C}(D)$ and assume that the tensor product basis $(\Phi_m \otimes \Phi_n)_{m,n=0}^{\infty}$ mentioned above takes the role of the normalized Schauder basis (x_m) in Section 3. We construct a sequence of fractal operators $(\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L})_{m,n=0}^{\infty}$ with choice of parameters as follows.

1. The parameter map L occurring in fractal operators $\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L}$ in the above sequence is fixed, and it is defined by
$$L(f) = f \circ c,$$
where $c : \mathcal{C}(D) \to \mathcal{C}(D)$ is a fixed continuous mapping such that $c(x_i, y_j) = (x_i, y_j)$ for $i \in \{0, m\}$ and $j \in \{0, n\}$, and $c \neq I$, the identity map on $\mathcal{C}(D)$.
2. For all m and n, in the construction of $\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L}$ we shall take $\Delta_{m,n}$ as the Cartesian product partition $T_r \times T_s$ of $D = [0,1] \times [0,1]$, where T_r is the segment of $\mathcal{J}$ used for the construction of Schauder hat functions Φ_m, $m \geq 0$ and T_s is the segment of $\mathcal{J}$ used for the construction of Φ_n, $n \geq 0$ (see (4.1)).
3. Assume that the sequence of scaling functions $(\alpha^{m,n})_{m,n=0}^{\infty}$ corresponding to the fractal operators $\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L}$ satisfies $\sum_{m,n=0}^{\infty} \|\alpha^{m,n}\|_{\infty} < \infty$.

Theorem 4.1. *For parameters $\alpha_{m,n}$, $\Delta_{m,n}$ and L as mentioned above, the sequence $\big(\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L}(\Phi_m \times \Phi_n)\big)_{m,n=0}^{\infty}$ constitutes a Schauder basis consisting of fractal functions (self-referential functions) for $\mathcal{C}(D)$.*

Proof. As $\|L\| = 1$, by (1.4) we have
$$\sum_{m,n=0}^{\infty} \big\|\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L} - I\big\| \leq \|I - L\| \sum_{m,n=0}^{\infty} \frac{\|\alpha^{m,n}\|_{\infty}}{1 - \|\alpha^{m,n}\|_{\infty}} \leq 2 \sum_{m,n=0}^{\infty} \frac{\|\alpha^{m,n}\|_{\infty}}{1 - \|\alpha^{m,n}\|_{\infty}}.$$
Since $\sum_{m,n=0}^{\infty} \|\alpha^{m,n}\|_{\infty} < \infty$, it follows that
$$\sum_{m,n=0}^{\infty} \big\|\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L} - I\big\| < \infty.$$
Next assume that $\sum_{m,n=0}^{\infty} c_{m,n} \mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L}\big(\Phi_m \otimes \Phi_n\big) = 0$. Then we have
$$\sum_{m,n=0}^{\infty} c_{m,n} \mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L}\big(\Phi_m \otimes \Phi_n\big)(x_r, x_s) = 0, \ \forall\, r, s \in \{0, 1, 2, \dots\}. \tag{4.2}$$
Due to the interpolatory properties of the bivariate α-fractal functions given in (1.2) we have
$$\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L}\big(\Phi_m \otimes \Phi_n\big)(x_i, x_j) = \big(\Phi_m \otimes \Phi_n\big)(x_i, x_j),$$
at the points $(x_i, x_j) \in \Delta_{m,n}$. Consequently, taking (x_i, y_j) sequentially for $i, j \in \{0, 1, 2, \dots\}$ in (4.2) one obtains
$$c_{m,n} = 0 \text{ for all } m, n \in \{0, 1, 2, \dots\}.$$

Thus, $(\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L})_{m,n=0}^{\infty}$ preserves the ω-independence of $(\Phi_m \times \Phi_n)_{m,n=0}^{\infty}$. That the system $\left(\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L}(\Phi_m \otimes \Phi_n)\right)_{m,n=0}^{\infty}$ is a Schauder basis for $\mathcal{C}(D)$ now follows at once from Theorem 1.1. □

An alternative procedure to perturb the bivariate Faber-Schauder system so as to obtain a fractal Faber-Schauder system is as follows. Now let us choose the parameters in the construction of fractal operators $\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L_{m,n}}$ as follows.

1. The partition $\Delta_{m,n}$ of D is chosen as in the previous case.
2. Scaling function $\alpha^{m,n} : D \to \mathbb{R}$ is so chosen that $\|\alpha^{m,n}\|_\infty \le \kappa < 1$ for all m, n in $\{0, 1, \dots\}$.
3. Further, let $L_{m,n}$ be such that $L_{m,n}(f) = \nu_{m,n} f$, where $\nu_{m,n} = 1$ at four vertices of D and $\sum_{m,n=0}^{\infty} \|\mathbb{1} - \nu_{m,n}\|_\infty < \infty$. Here $\mathbb{1}$ denotes the constant function $\mathbb{1} \in \mathcal{C}(D)$ defined by $\mathbb{1}(x) = 1$ for all $x \in D$.

Note that

$$\|(I - L_{m,n})(f)\|_\infty = \|f - \nu_{m,n} f\|_\infty \le \|\mathbb{1} - \nu_{m,n}\|_\infty \|f\|_\infty.$$

For the constant function $\mathbb{1}$, we have

$$\|(I - L_{m,n})(\mathbb{1})\|_\infty = \|\mathbb{1} - \nu_{m,n}\|_\infty \|\mathbb{1}\|_\infty.$$

Consequently, $\|I - L_{m,n}\| = \|\mathbb{1} - \nu_{m,n}\|_\infty$, and

$$\begin{aligned} \sum_{m,n=0}^{\infty} \left\|\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L_{m,n}} - I\right\| &\le \sum_{m,n=0}^{\infty} \frac{\|\alpha_{m,n}\|_\infty}{1 - \|\alpha_{m,n}\|_\infty} \|I - L_{m,n}\| \\ &\le \frac{k}{1-k} \sum_{m,n=0}^{\infty} \|\mathbb{1} - \nu_{m,n}\|_\infty \\ &< \infty. \end{aligned}$$

As in the proof of the previous theorem, using the stability result in Theorem 1.1 we can deduce that $(\mathcal{F}^{\alpha^{m,n}}_{\Delta_{m,n},L_{m,n}}(\Phi_m \otimes \Phi_n))_{m,n=0}^{\infty}$ is a Schauder basis for $\mathcal{C}(D)$ consisting of self-referential functions.

Remark 4.1. Alternatively, as the univariate fractal operator is well-studied, one could have used our stability result to fractalize univariate Faber-Schauder system $\{\Phi_m\}_{m=0}^{\infty}$ in $\mathcal{C}([0,1])$. In this case, with appropriate choices of parameters used in the definition of the (univariate) fractal operator, the fractal Faber-Schauder system $\left(\mathcal{F}^{\alpha^m}_{\Delta_m,L}(\Phi_m)\right)_{m=0}^{\infty}$ is a Schauder basis for $\mathcal{C}([0,1])$. Consequently, the tensor product $\left(\mathcal{F}^{\alpha^m}_{\Delta_m,L}(\Phi_m) \otimes \mathcal{F}^{\alpha^n}_{\Delta_n,L}(\Phi_n)\right)_{m,n=0}^{\infty}$ yields a Schauder basis for $\mathcal{C}(D)$.

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