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Infinite families of shape invariant potentials with n parameters subject to translation

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Abstract

We find new families of shape invariant potentials depending on $n \geq 1$ parameters subject to translation by the inclusion of non-trivial invariants. New dependencies of the spectra are found, and it opens the door to the engineering of physical quantities in a novel way. A number of examples are explicitly constructed.

Keywords: Shape Invariance, $n \geq 1$ parameters, quantum engineering, invariants

1. Introduction

The concept of Shape Invariance in Quantum Mechanics has a long tradition beginning in some of the works of Schrödinger himself [39, 40, 41] and developed afterwards in a classic work by Infeld and Hull [22]. These works dealt with exactly solvable problems in Quantum Mechanics. Some years later, Gendenshteïn and Krive renamed these cases as shape invariant potentials [12, 13], although it has been shown later a complete equivalence of the two approaches [4]. The list of known shape invariant potentials, summarized for example in the by now classic work [6] remained unchanged until key developments by Gómez–Ullate, Kamran and Milson [15] fostered the finding of isospectral rational extensions of some of the known shape invariant potentials, first by Quesne [30, 31] and afterwards followed by the same or other researchers, also with related questions [1, 32, 33, 34, 24, 25, 27, 26, 28, 29, 14, 16, 17, 18, 19, 20, 21, 43, 42]. In the meanwhile, they have appeared some works showing a compatibility condition that the (rational) extensions should satisfy [36, 37] and its equivalence with a group theory condition that appeared in [43] is shown in the later work [38]. In this last paper the question was open so as to explore the existence of shape invariant potentials (and their extensions) depending on more than one parameter subject to translation, inspired by the article [5].

That research is the subject of the present paper. In fact, the classic work of Infeld and Hull [22] dealt with a series in the parameter subject to translation which was unable to detect the existence of the rational extensions known much later as we have indicated. Likewise, [4] dealt with series expansions in the $n > 1$ parameters subject to translation, and essentially only one solution was found, being unable to find more solutions. Both approaches revealed to be not the most appropriate, as the rational extensions appeared much later by other means. And in this work we find an infinite number of shape invariant potentials depending on $n \geq 1$ parameters subject to translation. Let us remark that

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in [24, 25, 27, 28, 29] infinite families of shape invariant potentials subject to translation have been found, but they are iso-spectral (rational) extensions of given ones, and in the cases that we present here, the spectrum can be varied to a great extent.

The rest of the paper is organized as follows. Section 2 summarizes the main idea in [5]. In Section 3 we propose the new solutions. In Section 4 we find the basic specific solutions obeying the general setting. In Section 5 we discuss explicitly the relation of our approach with the previously known shape invariant cases in two parameters. In Section 6 we briefly describe the use of the $n \geq 1$ parameters subject to translation and rational extensions. In Section 7 we discuss the generalization to $n \geq 1$ parameters of the cases of shape invariant potentials whose rational extensions are not known to date. Finally, in the last Section we offer some conclusions.

2. The original idea

In [5] it is proposed the following form of the superpotential depending on $m_1, m_2, \ldots, m_n$ parameters subject to simultaneous translation $m_i \to m_i - 1$, $i = 1, \ldots, n$, where $n > 1$ is a natural number:

$$W(x; m_1, \ldots, m_n) = g_0(x) + \sum_{i=1}^{n} m_i g_i(x)$$

where $g_0(x), g_1(x), \ldots, g_n(x)$ are functions to be determined to satisfy the shape invariance condition (in an obvious notation)

$$W^2(x; m_i + 1) - W^2(x; m_i) + W'(x; m_i + 1) + W'(x; m_i) = L(m_i) - L(m_i + 1) \tag{1}$$

Upon substitution and selecting the coefficients of the different expressions in m_i, we have that the following system of equations have to be satisfied:

$$g'_j + g_j \sum_{i=1}^{n} g_i = c_j, \qquad j = 1, \ldots, n \tag{2}$$

$$g'_0 + g_0 \sum_{i=1}^{n} g_i = c_0 \tag{3}$$

where $c_0, c_1, \ldots, c_n$ are constants. This system is difficult to be solved directly, as [6] pointed out. A way to solve it was found in [5] and is to take barycentre coordinates of the g_i:

$$\begin{aligned} g_{\rm cm}(x) &= \frac{1}{n} \sum_{i=1}^{n} g_i(x) \\ v_i(x) &= g_i(x) - g_{\rm cm}(x), \quad i = 1, \ldots, n \\ c_{\rm cm} &= \frac{1}{n} \sum_{i=1}^{n} c_i \end{aligned}$$

Then, the system (2), (3) decouples as follows (note that $v_1(x) = -\sum_{j=2}^{n} v_j(x)$):

$$(ng_{\rm cm})' + (ng_{\rm cm})^2 = nc_{\rm cm} \tag{4}$$

$$v'_j + v_j n g_{\rm cm} = c_j - c_{\rm cm}, \quad j = 2, \ldots, n \tag{5}$$

$$g'_0 + g_0 n g_{\rm cm} = c_0 \tag{6}$$

out of which we can distinguish a Riccati equation with constant coefficients for $ng_{\rm cm}$ and once solved, linear equations for the rest of functions, and this becomes a standard problem, very well known in the literature [22, 4].

What interests more us is that in terms of the functions $ng_{\rm cm}$, v_j the initial superpotential can be written in the following way:

$$W(x; m_i) = g_0(x) + \sum_{j=2}^{n} (m_j - m_1) v_j + \left(\frac{1}{n} \sum_{i=1}^{n} m_i \right) n g_{\rm cm}(x) \tag{7}$$

We can observe that under the change $m_i \to m_i - 1$, $i = 1, \ldots, n$, the quantity

$$M = \frac{1}{n}\sum_{i=1}^{n} m_i$$

changes as $M \to M - 1$ and that $m_j - m_1$ are *invariant* for all $j = 2, \ldots, n$.

3. A new proposal

Inspired by the previous idea, we propose a new form for the superpotential:

$$W(x; m_i) = \sum_{j=1}^{r} I_j v_j(x) + M\,G(x) \tag{8}$$

where M has been defined in the preceding Section, but in what follows $n \geq 1$, $r \geq 1$ is a positive integer, and I_j are expressions in the m_i, $i = 1, \ldots, n$ *invariant* under translation, that is

$$I_j(m_1 - k, m_2 - k, \ldots, m_n - k) = I_j(m_1, m_2, \ldots, m_n)$$

for all $j = 1, \ldots, r$ and for all $k \in \mathbb{N}$. There is an infinite number of such invariants, and they can be linear or non-linear in m_i. This setting allows in particular *invariants in only one parameter*, that is, if $n = 1$, we can have

$$I_j(m_1 - k) = I_j(m_1), \quad j = 1, \ldots, r$$

for all $k \in \mathbb{N}$. That is, periodic functions in one parameter m_1 with period 1 will do the job. The $G(x)$ and $v_j(x)$ appearing in (8) are required to satisfy the following system of differential equations:

$$G' + G^2 = \alpha \tag{9}$$

$$v_j' + v_j G = \beta_j, \quad j = 1, \ldots, r \tag{10}$$

Then, let us check that the shape invariance condition is met (in order to avoid excessive notation we will assume in what follows summation in the repeated index j in an obvious way):

$$\begin{aligned} &W^2(x; m_i + 1) - W^2(x; m_i) + W'(x; m_i + 1) + W'(x; m_i) = \\ &(I_j v_j + (M+1)G)^2 - (I_j v_j + MG)^2 + (I_j v_j + (M+1)G)' + (I_j v_j + MG)' \\ &= (2M+1)(G^2 + G') + 2I_j(v_j G + v_j') = (2M+1)\alpha + 2\beta_j I_j = L(m_i) - L(m_i + 1) = R(m_i) \end{aligned} \tag{11}$$

Let us remark that we have that the difference $L(m_i) - L(m_i + 1) = (2M+1)\alpha + 2\beta_j I_j$ but the individual $L(m_i)$ has some indetermination, like in [4]: it can be written that $L(m_i) = -\alpha M^2 - 2\beta_j I_j M + H$, where H is another invariant of the m_i.

Constructing the following partner potentials and remainder R:

$$\begin{aligned} V(x; m_i) &= W(x; m_i)^2 - W'(x; m_i) \\ &= M(M+1)G^2 + (2M+1)I_j v_j G + (I_j v_j)^2 - M\alpha - \beta_j I_j \\ \tilde{V}(x; m_i) &= W(x; m_i)^2 + W'(x; m_i) \\ &= M(M-1)G^2 + (2M-1)I_j v_j G + (I_j v_j)^2 + M\alpha + \beta_j I_j \\ R(m_i) &= (2M+1)\alpha + 2\beta_j I_j \end{aligned}$$

Then, it is immediate to check that it holds

$$\tilde{V}(x; m_i) = V(x; m_i - 1) + R(m_i - 1)$$

so it is satisfied the shape invariance condition.

4. Specific solutions

The solutions of (9) are very well-known, that are summarized for example in [22, 8, 4]. Let us remark that the general solutions of a Riccati equation like (9) with $\alpha > 0$ have been found in [4], and therein they appear as logarithmic derivatives or rational functions. They appear to be rather long, although their knowledge is not a trivial matter. For example, the claimed new case of additive shape invariant potential in [3] and afterwards qualified in [23] can be easily related to the classical Morse potential [4, 35]. But we have chosen the specific situations below for the only sake of simplicity, and taking advantage of the fact that the general solutions of (9) with $\alpha > 0$ can be carried to an hyperbolic tangent or cotangent, or to a positive/negative constant (with a suitable rescaling equal to ± 1). Similar considerations hold for the cases of (9) with $\alpha = 0$ or $\alpha < 0$ as we will see below.

In fact, if $\alpha > 0$, by a convenient re-scaling of the variable can be carried into $\alpha = 1$, and the solutions can be reduced to any of the following four types:

$$G(x) = \tanh(x) \tag{12}$$
$$G(x) = \coth(x) \tag{13}$$
$$G(x) = 1 \tag{14}$$
$$G(x) = -1 \tag{15}$$

The corresponding solutions of (10) can be written as, respectively:

$$v_j(x) = \beta_j \tanh(x) + d_j \operatorname{sech}(x) \tag{16}$$
$$v_j(x) = \beta_j \coth(x) - d_j \operatorname{csch}(x) \tag{17}$$
$$v_j(x) = \beta_j - d_j \mathrm{e}^{-x} \tag{18}$$
$$v_j(x) = -\beta_j - d_j \mathrm{e}^{x} \tag{19}$$

where $\beta_j, d_j,\ j = 1, \ldots, r$ are real constants.

If $\alpha = 0$ the following two basic types can be obtained:

$$G(x) = 1/x \tag{20}$$
$$G(x) = 0 \tag{21}$$

and the corresponding solutions to (10) can be written, respectively, as:

$$v_j(x) = \frac{\beta_j}{2} x + \frac{d_j}{x} \tag{22}$$
$$v_j(x) = \beta_j x + d_j \tag{23}$$

where $\beta_j, d_j,\ j = 1, \ldots, r$ are real constants.

Likewise, if $\alpha < 0$, by a convenient re-scaling of the variable, can be carried into the case $\alpha = -1$; they can be found two types of basic *real* solutions[1]:

$$G(x) = -\tan(x) \tag{24}$$
$$G(x) = \cot(x) \tag{25}$$

and the corresponding solutions to (10) can be written, respectively, as:

$$v_j(x) = \beta_j \tan(x) - d_j \sec(x) \tag{26}$$
$$v_j(x) = -\beta_j \cot(x) + d_j \csc(x) \tag{27}$$

[1] There exist as well complex constant solutions $G(x) = \pm i$, with i being the imaginary unit, and they are not considered further in this paper, see also, e.g., [22].

where $\beta_j, d_j,\ j = 1, \ldots, r$ are real constants.

The superpotentials so found, according to (8) can, respectively, be written as:

$$W(x;m_i) = \left(M + I_j\beta_j\right)\tanh(x) + I_j d_j \,\mathrm{sech}\,(x) \tag{28}$$
$$W(x;m_i) = \left(M + I_j\beta_j\right)\coth(x) - I_j d_j \,\mathrm{csch}\,(x) \tag{29}$$
$$W(x;m_i) = \left(M + I_j\beta_j\right) - I_j d_j \mathrm{e}^{-x} \tag{30}$$
$$W(x;m_i) = -\left(M + I_j\beta_j\right) - I_j d_j \mathrm{e}^{x} \tag{31}$$
$$W(x;m_i) = \frac{1}{2}\beta_j I_j x + \left(M + d_j I_j\right)\frac{1}{x} \tag{32}$$
$$W(x;m_i) = I_j\beta_j x + d_j I_j \tag{33}$$
$$W(x;m_i) = -\left(M - I_j\beta_j\right)\tan(x) - I_j d_j \sec(x) \tag{34}$$
$$W(x;m_i) = \left(M - I_j\beta_j\right)\cot(x) + I_j d_j \csc(x) \tag{35}$$

We will write these superpotentials in terms of three new quantities ϵ, ρ, β. In the cases (28), (29), (30), (31) we will set $\epsilon = M + \beta_j I_j$, $\rho = d_j I_j$. In the case (32) we will set $\epsilon = M + d_j I_j$ and $\rho = \frac{1}{2}\beta_j I_j$. In the case (33) we will set $\beta = \beta_j I_j$ and $\rho = d_j I_j$. And in the cases (34) and (35) we will set $\epsilon = M - \beta_j I_j$ and $\rho = d_j I_j$.

Let us describe in what follows the corresponding superpotential, partner potentials, remainder of the shape invariance condition, and eigenenergies and normalized eigenstates of the potential $V(x;\epsilon,\rho)$ or $V(x;\beta,\rho)$ for each case, and the following notations will be taken: W will denote superpotential, R will denote remainder of the shape invariance relation, E_k will denote the k^{th} eigenenergy, and ζ_k a k^{th} normalized eigenfunction.

4.1. Case (28) (Scarf 2 type)

We have set $\epsilon = M + \beta_j I_j$, $\rho = d_j I_j$. We have, for $\epsilon > 0$ and $\epsilon - k > 0$:

$$\begin{aligned}
W(x;\epsilon,\rho) &= \epsilon\tanh(x) + \rho\,\mathrm{sech}\,(x), \quad x \in (-\infty,\infty)\\
V(x;\epsilon,\rho) &= \epsilon^2\tanh^2(x) + \rho(2\epsilon+1)\tanh(x)\,\mathrm{sech}\,(x) + (\rho^2-\epsilon)\,\mathrm{sech}^2(x)\\
\tilde{V}(x;\epsilon,\rho) &= \epsilon^2\tanh^2(x) + \rho(2\epsilon-1)\tanh(x)\,\mathrm{sech}\,(x) + (\rho^2+\epsilon)\,\mathrm{sech}^2(x)\\
R(\epsilon,\rho) &= 2\epsilon+1\\
E_k &= (2\epsilon-k)k\\
\zeta_k(x;\epsilon,\rho) &= 2^{\epsilon-1/2}\frac{|\Gamma(1/2+\epsilon-k-i\rho)|}{\sqrt{\pi}\sqrt{\Gamma(2(\epsilon-k))}}k!\,i^k a_k(\epsilon)\mathrm{e}^{-\rho\arctan(\sinh(x))}(\cosh(x))^{-\epsilon}\\
&\quad P_k^{(-1/2-\epsilon+i\rho,-1/2-\epsilon-i\rho)}(-i\sinh(x))
\end{aligned}$$

where

$$a_k(\epsilon) = \begin{cases} 1, & k=0\\ \frac{a_{k-1}(\epsilon-1)}{\sqrt{(2\epsilon-k)k}}, & k>0 \end{cases} \tag{36}$$

and $P_k^{(a,b)}(x)$ is a (ordinary) Jacobi polynomial of degree k, $\Gamma(\cdot)$ is the usual Gamma function, and i is the imaginary unit (from the context it should not be considered as a summation index).

4.2. Case (29) (Pöschl-Teller type)

We have set $\epsilon = M + \beta_j I_j, \rho = d_j I_j$. We have for $\epsilon - \rho < 1/2, \epsilon > 0$ and $\epsilon - k - \rho < 1/2, \epsilon - k > 0$:

$$\begin{aligned}
W(x;\epsilon,\rho) &= \epsilon \coth(x) - \rho \operatorname{csch}(x), \quad x \in (0,\infty) \\
V(x;\epsilon,\rho) &= \epsilon^2 \coth^2(x) - \rho(2\epsilon+1)\coth(x)\operatorname{csch}(x) + (\rho^2+\epsilon)\operatorname{csch}^2(x) \\
\tilde{V}(x;\epsilon,\rho) &= \epsilon^2 \coth^2(x) - \rho(2\epsilon-1)\coth(x)\operatorname{csch}(x) + (\rho^2-\epsilon)\operatorname{csch}^2(x) \\
R(\epsilon,\rho) &= 2\epsilon+1 \\
E_k &= -k(k-2\epsilon) \\
\zeta_k(x;\epsilon,\rho) &= 2^\epsilon \sqrt{\frac{\Gamma(1/2-k+\epsilon+\rho)}{\Gamma(2(\epsilon-k))\Gamma(1/2+k-\epsilon+\rho)}}\, k!\, b_k(\epsilon) \\
&\quad (\cosh(x)-1)^{(-\epsilon+\rho)/2}(\cosh(x)+1)^{-(\epsilon+\rho)/2} P_k^{(-1/2-\epsilon-\rho,-1/2-\epsilon+\rho)}(-\cosh(x))
\end{aligned}$$

where

$$b_k(\epsilon) = \begin{cases} 1, & k=0 \\ \frac{b_{k-1}(\epsilon+1)}{\sqrt{k(2\epsilon-k)}}, & k>0 \end{cases} \tag{37}$$

4.3. Case (30) (Morse type)

We have set $\epsilon = M + \beta_j I_j, \rho = d_j I_j$. We have for $\epsilon > 0, \rho > 0$ and $\epsilon - k > 0$:

$$\begin{aligned}
W(x;\epsilon,\rho) &= \epsilon - \rho e^{-x}, \quad x \in (-\infty,\infty) \\
V(x;\epsilon,\rho) &= \rho^2 e^{-2x} - \rho(2\epsilon+1)e^{-x} + \epsilon^2 \\
\tilde{V}(x;\epsilon,\rho) &= \rho^2 e^{-2x} - \rho(2\epsilon-1)e^{-x} + \epsilon^2 \\
R(\epsilon,\rho) &= 2\epsilon+1 \\
E_k &= (2\epsilon-k)k \\
\zeta_k(x;\epsilon,\rho) &= (-1)^k 2^{\epsilon-k} e^{kx} \rho^{\epsilon-k} a_k(\epsilon) k! \frac{\exp(-\rho e^{-x}) e^{-\epsilon x}}{\sqrt{\Gamma(2(\epsilon-k))}} L_k^{(2\epsilon-2k)}(2\rho e^{-x})
\end{aligned}$$

where $a_k(\epsilon)$ is given by (36) and $L_k^{(a)}(x)$ is a (ordinary) Laguerre polynomial of degree k.

4.4. Case (31)

We have set $\epsilon = M + \beta_j I_j, \rho = d_j I_j$. We have for $\epsilon > 0, \rho < 0$ and $\epsilon - k > 0$:

$$\begin{aligned}
W(x;\epsilon,\rho) &= -\epsilon - \rho e^{x}, \quad x \in (-\infty,\infty) \\
V(x;\epsilon,\rho) &= \rho^2 e^{2x} + \rho(2\epsilon+1)e^{x} + \epsilon^2 \\
\tilde{V}(x;\epsilon,\rho) &= \rho^2 e^{2x} + \rho(2\epsilon-1)e^{x} + \epsilon^2 \\
R(\epsilon,\rho) &= 2\epsilon+1 \\
E_k &= (2\epsilon-k)k \\
\zeta_k(x;\epsilon,\rho) &= (-1)^k 2^{\epsilon-k} e^{-kx} \rho^{\epsilon-k} a_k(\epsilon) k! \frac{\exp(\rho e^{x}) e^{\epsilon x}}{\sqrt{\Gamma(2(\epsilon-k))}} L_k^{(2\epsilon-2k)}(-2\rho e^{x})
\end{aligned}$$

where $a_k(\epsilon)$ is given again by (36).

4.5. Case (32) (Radial harmonic oscillator type)

We have set $\epsilon = M + d_j I_j$ and $\rho = \frac{1}{2}\beta_j I_j$. We have for $\epsilon < \frac{1}{2}$, $\rho > 0$ and $\epsilon - k < \frac{1}{2}$:

$$
\begin{aligned}
W(x;\epsilon,\rho) &= \frac{\epsilon}{x} + \rho x, \quad x \in (0,\infty) \\
V(x;\epsilon,\rho) &= \rho^2 x^2 + \rho(2\epsilon - 1) + \frac{\epsilon(\epsilon+1)}{x^2} \\
\tilde{V}(x;\epsilon,\rho) &= \rho^2 x^2 + \rho(2\epsilon + 1) + \frac{\epsilon(\epsilon-1)}{x^2} \\
R(\epsilon,\rho) &= 4\rho \\
E_k &= 4\rho k \\
\zeta_k(x;\epsilon,\rho) &= \sqrt{\frac{2\rho^{1/2+k-\epsilon}}{\Gamma(1/2+k-\epsilon)}}\, k!(-2)^k c_k(\epsilon)\exp(-\rho x^2/2)x^{-\epsilon}L_k^{(-1/2-\epsilon)}(\rho x^2)
\end{aligned}
$$

where

$$
c_k(\epsilon) = \begin{cases} 1, & k = 0 \\ \frac{c_{k-1}(\epsilon-1)}{\sqrt{4\rho k}}, & k > 0 \end{cases} \tag{38}
$$

4.6. Case (33) (Harmonic oscillator type)

We have set $\beta = \beta_j I_j$ and $\rho = d_j I_j$. We have, for $\beta > 0$:

$$
\begin{aligned}
W(x;\beta,\rho) &= \beta x + \rho, \quad x \in (-\infty,\infty) \\
V(x;\beta,\rho) &= \rho^2 + 2\rho\beta x + \beta(\beta x^2 - 1) \\
\tilde{V}(x;\beta,\rho) &= \rho^2 + 2\rho\beta x + \beta(\beta x^2 + 1) \\
R(\beta,\rho) &= 2\beta \\
E_k &= 2\beta k \\
\zeta_k(x;\beta,\rho) &= \left(\frac{\beta}{\pi}\right)^{1/4}\left(\frac{1}{\sqrt{k!2^k}}\right)\exp\left(-\frac{\beta}{2}\left(x+\frac{\rho}{\beta}\right)^2\right)H_k\left(\sqrt{\beta}\left(x+\frac{\rho}{\beta}\right)\right)
\end{aligned}
$$

where $H_k(x)$ is a (ordinary) Hermite polynomial of degree k.

4.7. Case (34) (Scarf 1 type)

We have set $\epsilon = M - \beta_j I_j$ and $\rho = d_j I_j$. We have, for $\epsilon < \frac{1}{2}$, $\frac{1}{2}(2\epsilon-1) < \rho < \frac{1}{2}(1-2\epsilon)$ and $\epsilon - k < \frac{1}{2}$, $\frac{1}{2}(2\epsilon - 2k - 1) < \rho < \frac{1}{2}(1 - 2\epsilon + 2k)$,

$$
\begin{aligned}
W(x;\epsilon,\rho) &= -\epsilon\tan(x) - \rho\sec(x), \quad x \in (-\pi/2,\pi/2) \\
V(x;\epsilon,\rho) &= \epsilon^2\tan^2(x) + \rho(2\epsilon+1)\tan(x)\sec(x) + (\rho^2+\epsilon)\sec^2(x) \\
\tilde{V}(x;\epsilon,\rho) &= \epsilon^2\tan^2(x) + \rho(2\epsilon-1)\tan(x)\sec(x) + (\rho^2-\epsilon)\sec^2(x) \\
R(\epsilon,\rho) &= -2\epsilon - 1 \\
E_k &= (k-2\epsilon)k \\
\zeta_k(x;\epsilon,\rho) &= 2^\epsilon k!\sqrt{\frac{\Gamma(1+2k-2\epsilon)}{\Gamma(1/2+k-\epsilon-\rho)\Gamma(1/2+k-\epsilon+\rho)}}\,d_k(\epsilon) \\
&\quad (1-\sin(x))^{-(\epsilon+\rho)/2}(1+\sin(x))^{-(\epsilon-\rho)/2}P_k^{(-1/2-\epsilon-\rho,-1/2-\epsilon+\rho)}(\sin(x))
\end{aligned}
$$

where

$$
d_k(\epsilon) = \begin{cases} 1, & k = 0 \\ \frac{d_{k-1}(\epsilon-1)}{\sqrt{k(k-2\epsilon)}}, & k > 0 \end{cases} \tag{39}
$$

4.8. Case (35)

We have set $\epsilon = M - \beta_j I_j$ and $\rho = d_j I_j$. We have, for $\epsilon < \frac{1}{2}$, $\frac{1}{2}(2\epsilon - 1) < \rho < \frac{1}{2}(1-2\epsilon)$ and $\epsilon - k < \frac{1}{2}$, $\frac{1}{2}(2\epsilon - 2k - 1) < \rho < \frac{1}{2}(1 - 2\epsilon + 2k)$,

$$\begin{aligned}
W(x;\epsilon,\rho) &= \epsilon\cot(x) + \rho\csc(x), \quad x \in (0,\pi) \\
V(x;\epsilon,\rho) &= \epsilon^2\cot^2(x) + \rho(2\epsilon+1)\cot(x)\csc(x) + (\rho^2+\epsilon)\csc^2(x) \\
\tilde{V}(x;\epsilon,\rho) &= \epsilon^2\cot^2(x) + \rho(2\epsilon-1)\cot(x)\csc(x) + (\rho^2-\epsilon)\csc^2(x) \\
R(\epsilon,\rho) &= -2\epsilon - 1 \\
E_k &= (k-2\epsilon)k \\
\zeta_k(x;\epsilon,\rho) &= 2^\epsilon k! \sqrt{\frac{\Gamma(1+2k-2\epsilon)}{\Gamma(1/2+k-\epsilon-\rho)\Gamma(1/2+k-\epsilon+\rho)}} d_k(\epsilon) \\
&\quad (1-\cos(x))^{-(\epsilon+\rho)/2}(1+\cos(x))^{-(\epsilon-\rho)/2} P_k^{(-1/2-\epsilon-\rho,-1/2-\epsilon+\rho)}(\cos(x))
\end{aligned}$$

where $d_k(\epsilon)$ is given again by (39).

4.9. Non-trivial examples in one and three parameters

The first non-trivial example might be to consider the case (34) with only one invariant, so $r = 1$, and only one parameter, so $n = 1$ and therefore $M = m_1$.

This is to say, a superpotential of the form

$$W(x;\epsilon,\rho) = -\epsilon\tan(x) - \rho\sec(x), \quad x \in (-\pi/2, \pi/2)$$

where $\epsilon = m_1 - \beta_1 I_1$ and $\rho = d_1 I_1$ and β_1, d_1 are constants. Let us remark that the *essential point* is that I_1 can be a non-trivial invariant of only one parameter, namely a periodic function of period 1 in m_1. This has the *essential consequence* that the shape invariant potentials of only one parameter shown in, for example, [2], can be *generalized* in the sense that some previously considered constants can be replaced by invariants of the *true parameter* m_1. Indeed this is perhaps our most important and path-breaking contribution.

Then, if we take for example

$$\begin{aligned}
I_1(m_1) &= \sin^2(2\pi m_1) + \cos(2\pi m_1) + 1 \\
\epsilon &= m_1 - \beta_1 I_1 = m_1 - \beta_1(\sin^2(2\pi m_1) + \cos(2\pi m_1) + 1) \\
\rho &= d_1 I_1 = d_1(\sin^2(2\pi m_1) + \cos(2\pi m_1) + 1)
\end{aligned}$$

where β_1, d_1, m_1 and k are chosen so as to ensure $\epsilon < \frac{1}{2}$, $\frac{1}{2}(2\epsilon - 1) < \rho < \frac{1}{2}(1 - 2\epsilon)$ and $\epsilon - k < \frac{1}{2}$, $\frac{1}{2}(2\epsilon - 2k - 1) < \rho < \frac{1}{2}(1 - 2\epsilon + 2k)$. With these choices, it is immediate to see that with the change $m_1 \to m_1 - 1$ we obtain, clearly,

$$\begin{aligned}
\epsilon &\to \epsilon - 1 \\
\rho &\to \rho
\end{aligned}$$

but the spectrum of the potential is

$$E_k = (k - 2\epsilon)k = (k - 2(m_1 - \beta_1(\sin^2(2\pi m_1) + \cos(2\pi m_1) + 1)))k$$

which depends in a new non-trivial way on m_1 that is the only *true* parameter subject to translation. This dependence of the spectrum in m_1 has not been reported before as far as we know, and it is a new possibility for modifying the spectrum in a novel way. In other words, when writing the potential in the parameter m_1 the new dependencies on invariants of m_1 are observed, and that allows to introduce levels that switch up and down in different ways with m_1 if the invariant is a periodic function of m_1 of the type “square wave” or “sawtooth wave” for example. A new way of engineering the spectra does appear. In other words, our approach for this specific example mixes several ideas:

- The *true* parameter subject to translation is m_1, as $m_1 \to m_1 - 1$.

- In this example, the spectrum depends on ϵ and k only, and *not* on ρ.
- The change $-\epsilon = A$, attempted to try to put this example into the form of the Scarf trigonometric case of [2], when written into the form of the *true* parameter m_1 would be

$$-m_1 + \beta_1(\sin^2(2\pi m_1) + \cos(2\pi m_1) + 1) = A$$

  which is clearly not one-to-one (if $\beta_1 \neq 0$) and thus it would be an unfeasible change. And one can write multiplying to β_1 *any* periodic function of period 1 in m_1, and in any case the change from m_1 to A would be as well not one-to-one unless the invariant would be a constant, which is really the case reported before in [2] and other sources.
- In this paper we report only cases that are *similar* to the well-known cases shown for example in [2] but where some constants can be *generalized* to non-trivial invariants on the $m_1, \ldots m_n$, $n \geq 1$ *true* parameters subject to translation. It is not excluded that other newer cases might appear, combining non-trivial invariants and other different solutions to the shape invariance relations.
- That said, our impression is that the new cases reported in this paper and perhaps other solutions might be obtained with the approach of [2] by modifying it a little bit, that is, by considering a kind of a system of Euler difference-differential equations. And this newer modified approach might be able as well to recover the cases with more than one parameter subject to translation presented in this paper and perhaps another ones. If that (slightly modified) approach is able to provide such solutions, it could be the case that it might be considered as a superior approach to the obtention of shape invariant potentials, something that nowadays is in dispute [35].

As another non-trivial example with only one parameter m_1 we can recall the case (32) (Radial harmonic oscillator type), with $\epsilon = m_1 + d_1 I_1$, $\rho = \frac{1}{2}\beta_1 I_1$, where d_1, β_1 are constants and I_1 is an invariant such that $\epsilon < \frac{1}{2}$, $\rho > 0$, $\epsilon - k < \frac{1}{2}$ and

$$W(x; \epsilon, \rho) = \frac{\epsilon}{x} + \rho x, \quad x \in (0, \infty)$$

and for example,

$$\begin{aligned} I_1(m_1) &= \sin^2(2\pi m_1) + \cos(2\pi m_1) + 1 \\ \epsilon &= m_1 + d_1 I_1 = m_1 + d_1(\sin^2(2\pi m_1) + \cos(2\pi m_1) + 1) \\ \rho &= \frac{1}{2}\beta_1 I_1 = \frac{1}{2}\beta_1(\sin^2(2\pi m_1) + \cos(2\pi m_1) + 1) \end{aligned}$$

with d_1, β_1, m_1 chosen so as to ensure

$$\begin{aligned} & m_1 + d_1(\sin^2(2\pi m_1) + \cos(2\pi m_1) + 1) < \frac{1}{2} \\ & \frac{1}{2}\beta_1(\sin^2(2\pi m_1) + \cos(2\pi m_1) + 1) > 0 \end{aligned}$$

To this example it corresponds the spectrum

$$E_k = 4k\rho = 2k\beta_1(\sin^2(2\pi m_1) + \cos(2\pi m_1) + 1)$$

which depends clearly on the parameter subject to translation m_1 by means of a periodic function of period 1. Thus this is a new possibility of modifying the spectrum by, for example, levels that oscillate between two prescribed values and change with the true parameter m_1, as the periodic function to be included is completely arbitrary and subject to the quantum engineer's wishes or needs.

Another specific example of the case (34) could be as follows. Let us choose $r = 1$ (only one invariant), $n = 3$ (three parameters m_1, m_2, m_3), and

$$\begin{aligned} M &= \frac{1}{3}(m_1 + m_2 + m_3) \\ I_1(m_1, m_2, m_3) &= \sin(2\pi M) + \sin^2(M - m_1) + \sin^2(M - m_2) + \cos^2(M - m_3) \\ \epsilon &= M - \beta_1 I_1 \\ \rho &= d_1 I_1 \end{aligned}$$

where $\beta_1, d_1, m_1, m_2, m_3, k$ are chosen so as to ensure $\epsilon < \frac{1}{2}, \frac{1}{2}(2\epsilon - 1) < \rho < \frac{1}{2}(1 - 2\epsilon)$ and $\epsilon - k < \frac{1}{2}, \frac{1}{2}(2\epsilon - 2k - 1) < \rho < \frac{1}{2}(1 - 2\epsilon + 2k)$. Then, the spectrum of the potential is

$$E_k = (k - 2\epsilon)k = k\left(k - 2M + 2\beta_1\left(\sin(2\pi M) + \sin^2(M - m_1) + \sin^2(M - m_2) + \cos^2(M - m_3)\right)\right)$$

which depends in a new non-trivial way on M, m_1, m_2, m_3.

5. Previously known potentials with two parameters

There are two cases of previously well-known superpotentials in two parameters subject to translation (see, e.g., [6, 7, 11]), namely

$$\begin{aligned} W(x; m_1, m_2) &= m_1 \tanh(x) + m_2 \coth(x), \quad \text{(Pöschl} - \text{Teller II)} &(40)\\ W(x; m_1, m_2) &= -m_1 \tan(x) + m_2 \cot(x), \quad \text{(Pöschl} - \text{Teller I)} &(41)\end{aligned}$$

Let us show that both can be understood in the previous framework.

In fact, for (40) we set $r = 1$ and

$$\begin{aligned} G(x) &= 2\coth(2x)\\ v_1(x) &= 2\,\text{csch}\,(2x)\\ M &= \frac{1}{2}(m_1 + m_2)\\ I_1(m_1, m_2) &= \frac{1}{2}(m_2 - m_1)\end{aligned}$$

that satisfy

$$\begin{aligned} G'(x) + G(x)^2 &= 4\\ v_1'(x) + v_1(x)G(x) &= 0\end{aligned}$$

and then, according to (8),

$$\begin{aligned} W(x; m_1, m_2) &= MG(x) + I_1(m_1, m_2)v_1(x)\\ &= \frac{1}{2}(m_1 + m_2)2\coth(2x) + \frac{1}{2}(m_2 - m_1)2\,\text{csch}\,(2x) = m_1 \tanh(x) + m_2 \coth(x)\end{aligned}$$

This superpotential is of the type (29) re-scaling $x \to 2x$, $\beta_1 = 0$ and $d_1 = -1$. Since $\beta_1 = 0$, the spectrum does not depend on the invariant $I_1(m_1, m_2)$ but only on $M = \frac{1}{2}(m_1 + m_2) = \epsilon$. A different potential, with the same spectrum, can be obtained setting, for example, $I_1(m_1, m_2) = e^{m_2 - m_1} + 1$ instead of $I_1(m_1, m_2) = \frac{1}{2}(m_2 - m_1)$, and with this modification the superpotential no longer takes the form (40).

A similar thing can be said about (41). In fact, setting $r = 1$ and

$$\begin{aligned} G(x) &= 2\cot(2x)\\ v_1(x) &= 2\csc(2x)\\ M &= \frac{1}{2}(m_1 + m_2)\\ I_1(m_1, m_2) &= \frac{1}{2}(m_2 - m_1)\end{aligned}$$

that satisfy

$$\begin{aligned} G'(x) + G(x)^2 &= -4\\ v_1'(x) + v_1(x)G(x) &= 0\end{aligned}$$

and then, according to (8),

$$\begin{aligned} W(x;m_1,m_2) &= MG(x)+I_1(m_1,m_2)v_1(x) \\ &= \frac{1}{2}(m_1+m_2)2\cot(2x)+\frac{1}{2}(m_2-m_1)2\csc(2x) = -m_1\tan(x)+m_2\cot(x) \end{aligned}$$

This superpotential is of the type (35) re-scaling $x \to 2x$, and putting $\beta_1 = 0$, $d_1 = 1$. Since $\beta_1 = 0$, the spectrum does not depend again on the invariant $I_1(m_1,m_2)$ but only on $M = \frac{1}{2}(m_1+m_2) = \epsilon$. A different potential, with the same spectrum, can be obtained setting, for example, $I_1(m_1,m_2) = \ln((m_2-m_1)^2+1)$ instead of $I_1(m_1,m_2) = \frac{1}{2}(m_2-m_1)$, and with this modification the superpotential no longer takes the form (41).

6. Rational extensions

In the articles [36, 37] it has been established a compatibility condition that the previously known rational extensions [30, 31, 1, 2, 24, 25, 27, 26, 28, 29] should satisfy. Namely, if $W_0(x;m) = k_0(x)+mk_1(x)$ is a super-potential of the classical type [22, 4], where

$$k_1' + k_1^2 = \alpha \qquad (42)$$

$$k_0' + k_0 k_1 = \beta \qquad (43)$$

being α, β constants, then the extensions $W(x;m) = W_0(x;m)+W_{1+}(x;m)-W_{1-}(x;m)$, define shape invariant potentials with m subject to translation $m \to m-1$ if and only if they are satisfied the following two conditions [36, 37]:

$$W_{1+}^2 + W_{1+}' + W_{1-}^2 + W_{1-}' + 2W_0W_{1+} - 2W_0W_{1-} - 2W_{1+}W_{1-} = h \qquad (44)$$

$$W_{1-}(x;m) = W_{1+}(x;m-1) \qquad (45)$$

where (44) is evaluated on $(x;m)$ or $(x;m-1)$ and h is a function of x only. The appearance of h is due to the existence of symmetries [37] of W_{1+}, W_{1-}; in fact, it can be added to them the same arbitrary (differentiable) function of x only. When $n \geq 1$ parameter(s) subject to translation is/are involved, it is easy to see that the symmetries extend to the adding of an arbitrary differentiable function $f(x;J_k)$ to W_{1+} and W_{1-}, where J_k, $k = 1,\ldots,s \geq 1$ are also invariants under the change $m_i \to m_i - 1$, $i = 1,\ldots,n$. The invariants J_k do not need to coincide with the previous I_j.

Let us see how a number of cases fit in our approach, extracted and adapted from the cited literature. All of the following cases satisfy (44) evaluated at $(x;\epsilon,\rho)$, $(x;\epsilon,\rho,\ell)$ or $(x;\epsilon,\ell)$ where in addition $h(x) = 2f'(x;J_k)$ and (45) is satisfied with the change in the notation $W_{1-}(x;\epsilon,\rho) = W_{1+}(x;\epsilon-1,\rho)$, $W_{1-}(x;\epsilon,\rho,\ell) = W_{1+}(x;\epsilon-1,\rho,\ell)$ or $W_{1-}(x;\epsilon,\ell) = W_{1+}(x;\epsilon-1,\ell)$, depending on the case.

1. We have

$$\begin{aligned} W_0(x;\epsilon,\rho) &= \epsilon\coth(x) - \rho\,\mathrm{csch}\,(x) \\ W_{1+}(x;\epsilon,\rho) &= -\frac{2\rho\sinh(x)}{2\epsilon+1-2\rho\cosh(x)} + f(x;J_k) \\ W_{1-}(x;\epsilon,\rho) &= -\frac{2\rho\sinh(x)}{2\epsilon-1-2\rho\cosh(x)} + f(x;J_k) \end{aligned}$$

   where $\epsilon = M + \beta_j I_j$, $\rho = d_j I_j$, $M = \frac{1}{n}\sum_{i=1}^n m_i$, β_j, d_j are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1,\ldots,n$.

2. We have

$$\begin{aligned} W_0(x;\epsilon,\rho,\ell) &= \epsilon\coth(x) - \rho\,\mathrm{csch}\,(x) \\ W_{1+}(x;\epsilon,\rho,\ell) &= \frac{(\ell-2\rho-1)\sinh(x)}{2}\,\frac{P_{\ell-1}^{(1/2+\epsilon-\rho,-1/2-\epsilon-\rho)}(\cosh(x))}{P_{\ell}^{(-1/2+\epsilon-\rho,-3/2-\epsilon-\rho)}(\cosh(x))} + f(x;J_k) \\ W_{1-}(x;\epsilon,\rho,\ell) &= \frac{(\ell-2\rho-1)\sinh(x)}{2}\,\frac{P_{\ell-1}^{(-1/2+\epsilon-\rho,1/2-\epsilon-\rho)}(\cosh(x))}{P_{\ell}^{(-3/2+\epsilon-\rho,-1/2-\epsilon-\rho)}(\cosh(x))} + f(x;J_k) \end{aligned}$$

where $\epsilon = M + \beta_j I_j$, $\rho = d_j I_j$, $M = \frac{1}{n}\sum_{i=1}^{n} m_i$, β_j, d_j are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1, \ldots, n$.

3. We have

$$\begin{aligned}
W_0(x;\epsilon,\rho,\ell) &= 2(\ell+\epsilon)\coth(2x) + 2\rho\,\mathrm{csch}\,(2x)\\
W_{1+}(x;\epsilon,\rho,\ell) &= -(2\rho-\ell+1)\sinh(2x)\frac{P_{\ell-1}^{(-1/2-\ell-\epsilon-\rho,1/2+\ell+\epsilon-\rho)}(\cosh(2x))}{P_{\ell}^{(-3/2-\ell-\epsilon-\rho,-1/2+\ell+\epsilon-\rho)}(\cosh(2x))} + f(x;J_k)\\
W_{1-}(x;\epsilon,\rho,\ell) &= -(2\rho-\ell+1)\sinh(2x)\frac{P_{\ell-1}^{(1/2-\ell-\epsilon-\rho,-1/2+\ell+\epsilon-\rho)}(\cosh(2x))}{P_{\ell}^{(-1/2-\ell-\epsilon-\rho,-3/2+\ell+\epsilon-\rho)}(\cosh(2x))} + f(x;J_k)
\end{aligned}$$

where $\epsilon = M + \beta_j I_j$, $\rho = d_j I_j$, $M = \frac{1}{n}\sum_{i=1}^{n} m_i$, β_j, d_j are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1, \ldots, n$.

4. We have

$$\begin{aligned}
W_0(x;\epsilon,\rho) &= \frac{\epsilon}{x} + \rho x\\
W_{1+}(x;\epsilon,\rho) &= -\frac{4\rho x}{2\epsilon+1-2\rho x^2} + f(x;J_k)\\
W_{1-}(x;\epsilon,\rho) &= -\frac{4\rho x}{2\epsilon-1-2\rho x^2} + f(x;J_k)
\end{aligned}$$

where $\epsilon = M + d_j I_j$, $\rho = \frac{1}{2}\beta_j I_j$, $M = \frac{1}{n}\sum_{i=1}^{n} m_i$, β_j, d_j are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1, \ldots, n$.

5. We have

$$\begin{aligned}
W_0(x;\epsilon,\rho,\ell) &= \frac{\epsilon}{x} + \rho x\\
W_{1+}(x;\epsilon,\rho,\ell) &= 2x\rho\frac{L_{\ell-1}^{(-1/2-\epsilon)}(-\rho x^2)}{L_{\ell}^{(-3/2-\epsilon)}(-\rho x^2)} + f(x;J_k)\\
W_{1-}(x;\epsilon,\rho,\ell) &= 2x\rho\frac{L_{\ell-1}^{(1/2-\epsilon)}(-\rho x^2)}{L_{\ell}^{(-1/2-\epsilon)}(-\rho x^2)} + f(x;J_k)
\end{aligned}$$

where $\epsilon = M + d_j I_j$, $\rho = \frac{1}{2}\beta_j I_j$, $M = \frac{1}{n}\sum_{i=1}^{n} m_i$, β_j, d_j are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1, \ldots, n$.

6. We have

$$\begin{aligned}
W_0(x;\epsilon,\ell) &= \frac{\epsilon+\ell}{x} - x\\
W_{1+}(x;\epsilon,\ell) &= 2x\frac{L_{\ell-1}^{(1/2+\ell+\epsilon)}(-x^2)}{L_{\ell}^{(-1/2+\ell+\epsilon)}(-x^2)} + f(x;J_k)\\
W_{1-}(x;\epsilon,\ell) &= 2x\frac{L_{\ell-1}^{(-1/2+\ell+\epsilon)}(-x^2)}{L_{\ell}^{(-3/2+\ell+\epsilon)}(-x^2)} + f(x;J_k)
\end{aligned}$$

where $\epsilon = M + d_j I_j$, $M = \frac{1}{n}\sum_{i=1}^{n} m_i$, d_j are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1, \ldots, n$.

7. We have

$$
\begin{aligned}
W_0(x;\epsilon,\ell) &= \frac{\epsilon+\ell}{x} - x\\
W_{1+}(x;\epsilon,\ell) &= 2\ell x \frac{{}_1F_1\left(\begin{array}{c}1-\ell\\ 3/2+\ell+\epsilon\end{array}\Big| -x^2\right)}{(1/2+\ell+\epsilon)\,{}_1F_1\left(\begin{array}{c}-\ell\\ 1/2+\ell+\epsilon\end{array}\Big| -x^2\right)} + f(x;J_k)\\
W_{1-}(x;\epsilon,\ell) &= 2\ell x \frac{{}_1F_1\left(\begin{array}{c}1-\ell\\ 1/2+\ell+\epsilon\end{array}\Big| -x^2\right)}{(-1/2+\ell+\epsilon)\,{}_1F_1\left(\begin{array}{c}-\ell\\ -1/2+\ell+\epsilon\end{array}\Big| -x^2\right)} + f(x;J_k)
\end{aligned}
$$

where ${}_1F_1\left(\begin{array}{c}a\\ b\end{array}\Big| x\right)$ denotes the confluent hypergeometric function, $\epsilon = M + d_jI_j$, $M = \frac{1}{n}\sum_{i=1}^n m_i$, d_j are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1,\ldots,n$.

8. We have

$$
\begin{aligned}
W_0(x;\epsilon,\rho) &= -\epsilon\tan(x) - \rho\sec(x)\\
W_{1+}(x;\epsilon,\rho) &= \frac{2\rho\cos(x)}{2\epsilon+1+2\rho\sin(x)} + f(x;J_k)\\
W_{1-}(x;\epsilon,\rho) &= \frac{2\rho\cos(x)}{2\epsilon-1+2\rho\sin(x)} + f(x;J_k)
\end{aligned}
$$

where $\epsilon = M - \beta_jI_j$, $\rho = d_jI_j$, $M = \frac{1}{n}\sum_{i=1}^n m_i, \beta_j, d_j$ are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1,\ldots,n$.

9. We have

$$
\begin{aligned}
W_0(x;\epsilon,\rho,\ell) &= 2(\epsilon+\ell)\cot(2x) - 2\rho\csc(2x)\\
W_{1+}(x;\epsilon,\rho,\ell) &= -(2\rho+\ell-1)\sin(2x)\frac{P_{\ell-1}^{(-1/2-\ell-\epsilon+\rho,1/2+\ell+\epsilon+\rho)}(\cos(2x))}{P_{\ell}^{(-3/2-\ell-\epsilon+\rho,-1/2+\ell+\epsilon+\rho)}(\cos(2x))} + f(x;J_k)\\
W_{1-}(x;\epsilon,\rho,\ell) &= -(2\rho+\ell-1)\sin(2x)\frac{P_{\ell-1}^{(1/2-\ell-\epsilon+\rho,-1/2+\ell+\epsilon+\rho)}(\cos(2x))}{P_{\ell}^{(-1/2-\ell-\epsilon+\rho,-3/2+\ell+\epsilon+\rho)}(\cos(2x))} + f(x;J_k)
\end{aligned}
$$

where $\epsilon = M - \beta_jI_j$, $\rho = d_jI_j$, $M = \frac{1}{n}\sum_{i=1}^n m_i, \beta_j, d_j$ are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1,\ldots,n$.

10. We have

$$
\begin{aligned}
W_0(x;\epsilon,\rho,\ell) &= 2\epsilon\cot(2x) + 2\rho\csc(2x)\\
W_{1+}(x;\epsilon,\rho,\ell) &= -\ell(2\rho+\ell-1)\sin(2x)\frac{\Gamma(1/2+\epsilon+\rho)\,{}_2F_1\left(\begin{array}{c}1-\ell,\ell+2\rho\\ 3/2+\epsilon+\rho\end{array}\Big| \sin^2(x)\right)}{\Gamma(3/2+\epsilon+\rho)\,{}_2F_1\left(\begin{array}{c}-\ell,-1+\ell+2\rho\\ 1/2+\epsilon+\rho\end{array}\Big| \sin^2(x)\right)} + f(x;J_k)\\
W_{1-}(x;\epsilon,\rho,\ell) &= -\ell(2\rho+\ell-1)\sin(2x)\frac{\Gamma(-1/2+\epsilon+\rho)\,{}_2F_1\left(\begin{array}{c}1-\ell,\ell+2\rho\\ 1/2+\epsilon+\rho\end{array}\Big| \sin^2(x)\right)}{\Gamma(1/2+\epsilon+\rho)\,{}_2F_1\left(\begin{array}{c}-\ell,-1+\ell+2\rho\\ -1/2+\epsilon+\rho\end{array}\Big| \sin^2(x)\right)} + f(x;J_k)
\end{aligned}
$$

where ${}_2F_1\left(\begin{array}{c}a,b\\ c\end{array}\Big| x\right)$ is the hypergeometric function, $\epsilon = M-\beta_jI_j$, $\rho = d_jI_j$, $M = \frac{1}{n}\sum_{i=1}^n m_i, \beta_j, d_j$ are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1,\ldots,n$.

11. We have

$$\begin{aligned} W_0(x;\epsilon,\rho,\ell) &= \epsilon\tanh(x) + i\rho\,\mathrm{sech}\,(x) \\ W_{1+}(x;\epsilon,\rho,\ell) &= \frac{1}{2}i(\ell-2\rho-1)\cosh(x)\frac{P_{\ell-1}^{(-\rho+\epsilon+1/2,-\rho-\epsilon-1/2)}(i\sinh(x))}{P_{\ell}^{(-\rho+\epsilon-1/2,-\rho-\epsilon-3/2)}(i\sinh(x))} + f(x;J_k) \\ W_{1-}(x;\epsilon,\rho,\ell) &= \frac{1}{2}i(\ell-2\rho-1)\cosh(x)\frac{P_{\ell-1}^{(-\rho+\epsilon-1/2,-\rho-\epsilon+1/2)}(i\sinh(x))}{P_{\ell}^{(-\rho+\epsilon-3/2,-\rho-\epsilon-1/2)}(i\sinh(x))} + f(x;J_k) \end{aligned}$$

where $\epsilon = M + \beta_j I_j$, $\rho = d_j I_j$, $M = \frac{1}{n}\sum_{i=1}^n m_i$, β_j, d_j are constants and I_j, J_k are invariants in the m_i under the change $m_i \to m_i - 1$, $i = 1,\ldots,n$.

The group theory approach of [38] can be achieved easily substituting therein by our current quantities $F(x) \to G(x)$, $G(x) \to I_j v_j$, $a \to \alpha$, $b \to \beta_j I_j$, $U(x;m) \to W_{1+}(x;\epsilon,\rho) - W_{1-}(x;\epsilon,\rho)$, $U(x;m) \to W_{1+}(x;\epsilon,\rho,\ell) - W_{1-}(x;\epsilon,\rho,\ell)$ or $U(x;m) \to W_{1+}(x;\epsilon,\ell) - W_{1-}(x;\epsilon,\ell)$, depending on the case.

7. More generalizations

In [22, 4] there are solutions to the shape invariance condition (1) for only one parameter m subject to $m \to m-1$ of the form

$$W(x;m) = \frac{q}{m} + m\,k_1(x)$$

where q is a constant. The function $k_1(x)$ must satisfy again the differential equation (9) or (42), whose solutions have been summarized earlier in this paper but otherwise fully discussed in [4].

We propose the following generalization to $m_1, m_2, \ldots, m_n$ parameters subject to translation $m_i \to m_i - 1$, $i = 1,\ldots,n$, with $n \geq 1$:

$$W(x;\epsilon,\rho) = \frac{\rho}{\epsilon} + \epsilon\,G(x)$$

where $\epsilon = M + d_j I_j = \frac{1}{n}\sum_{i=1}^n m_i + d_j I_j$, d_j are constants, I_j are invariants as before where $j = 1\ldots,r \geq 1$, $\rho = I$ is another invariant and $G(x)$ is a solution of (9). The non-constant solutions for $G(x)$ can be reduced to the basic forms (12), (13), (20), (24) and (25)

Let us describe in what follows the corresponding superpotential, partner potentials, remainder of the shape invariance condition, and eigenenergies and normalized eigenstates of the potential $V(x;\epsilon,\rho)$ for each case.

For these cases no rational extensions are known.

7.1. Case (12) (Rosen-Morse 2 type)

We have set $\epsilon = M + d_j I_j$, $\rho = I$. We have, for $\epsilon > \rho/\epsilon$, $\epsilon + \rho/\epsilon > 0$ and $\epsilon - k > \rho/(\epsilon-k)$, $\epsilon - k + \rho/(\epsilon-k) > 0$,

$$\begin{aligned} W(x;\epsilon,\rho) &= \epsilon\tanh(x) + \frac{\rho}{\epsilon}, \quad x \in (-\infty,\infty) \\ V(x;\epsilon,\rho) &= \epsilon^2\tanh^2(x) + 2\rho\tanh(x) - \epsilon\,\mathrm{sech}^2(x) + \rho^2/\epsilon^2 \\ \tilde{V}(x;\epsilon,\rho) &= \epsilon^2\tanh^2(x) + 2\rho\tanh(x) + \epsilon\,\mathrm{sech}^2(x) + \rho^2/\epsilon^2 \\ R(\epsilon,\rho) &= 1 + 2\epsilon - \frac{(2\epsilon+1)\rho^2}{\epsilon^2(\epsilon+1)^2} \\ E_k &= k(k-2\epsilon)\left(\frac{\rho^2}{(k-\epsilon)^2\epsilon^2} - 1\right) \\ \zeta_k(x;\epsilon,\rho) &= 2^{1/2+k-\epsilon}k!\sqrt{\frac{\Gamma(2\epsilon-2k)}{\Gamma\left(\epsilon-k+\frac{\rho}{k-\epsilon}\right)\Gamma\left(\epsilon-k-\frac{\rho}{k-\epsilon}\right)}}e_k(\epsilon,\rho) \\ & (1-\tanh(x))^{\frac{1}{2}(\epsilon-k+\rho/(\epsilon-k))}(1+\tanh(x))^{\frac{1}{2}(\epsilon-k-\rho/(\epsilon-k))} \\ & P_k^{(\epsilon-k+\rho/(\epsilon-k),\epsilon-k-\rho/(\epsilon-k))}(\tanh(x)) \end{aligned}$$

where

$$e_k(\epsilon,\rho) = \begin{cases} 1, & k=0 \\ \frac{(2\epsilon-k)e_{k-1}(\epsilon-1,\rho)}{\epsilon\sqrt{k(2\epsilon-k)-\rho^2/(k-\epsilon)^2+\rho^2/\epsilon^2}}, & k>0 \end{cases} \quad (46)$$

7.2. Case (13) (Eckart type)

We have set $\epsilon = M + d_j I_j$, $\rho = I$. We have, for $\epsilon < \frac{1}{2}$, $\epsilon + \rho/\epsilon > 0$ and $\epsilon - k < \frac{1}{2}$, $\epsilon - k + \rho/(\epsilon - k) > 0$,

$$\begin{aligned}
W(x;\epsilon,\rho) &= \epsilon\coth(x) + \frac{\rho}{\epsilon}, \quad x \in (0,\infty) \\
V(x;\epsilon,\rho) &= \epsilon^2\coth^2(x) + 2\rho\coth(x) + \epsilon\,\mathrm{csch}^2(x) + \rho^2/\epsilon^2 \\
\tilde{V}(x;\epsilon,\rho) &= \epsilon^2\coth^2(x) + 2\rho\coth(x) - \epsilon\,\mathrm{csch}^2(x) + \rho^2/\epsilon^2 \\
R(\epsilon,\rho) &= 1 + 2\epsilon - \frac{(2\epsilon+1)\rho^2}{\epsilon^2(\epsilon+1)^2} \\
E_k &= k(k-2\epsilon)\left(\frac{\rho^2}{(k-\epsilon)^2\epsilon^2} - 1\right) \\
\zeta_k(x;\epsilon,\rho) &= 2^{1/2+k-\epsilon}k!\sqrt{\frac{\Gamma(1+k-\epsilon+\rho/(\epsilon-k))}{\Gamma(1+2k-2\epsilon)\,\Gamma\left(\epsilon-k-\frac{\rho}{k-\epsilon}\right)}}e_k(\epsilon,\rho) \\
&\quad (\coth(x)-1)^{\frac{1}{2}(\epsilon-k+\rho/(\epsilon-k))}(\coth(x)+1)^{\frac{1}{2}(\epsilon-k-\rho/(\epsilon-k))} \\
&\quad P_k^{(\epsilon-k+\rho/(\epsilon-k),\epsilon-k-\rho/(\epsilon-k))}(\coth(x))
\end{aligned}$$

where $e_k(\epsilon,\rho)$ is given by (46).

7.3. Case (20) (Coulomb type)

We have set $\epsilon = M + d_j I_j$, $\rho = I$. We have, for $\epsilon \neq 0$, $\epsilon < 1/2$, $\rho/\epsilon > 0$ and $\epsilon - k \neq 0$, $\epsilon - k < 1/2$, $\rho/(\epsilon - k) > 0$,

$$\begin{aligned}
W(x;\epsilon,\rho) &= \frac{\epsilon}{x} + \frac{\rho}{\epsilon}, \quad x \in (0,\infty) \\
V(x;\epsilon,\rho) &= \frac{2\rho}{x} + \frac{\epsilon(\epsilon+1)}{x^2} + \frac{\rho^2}{\epsilon^2} \\
\tilde{V}(x;\epsilon,\rho) &= \frac{2\rho}{x} + \frac{\epsilon(\epsilon-1)}{x^2} + \frac{\rho^2}{\epsilon^2} \\
R(\epsilon,\rho) &= -\frac{(2\epsilon+1)\rho^2}{\epsilon^2(\epsilon+1)^2} \\
E_k &= k(k-2\epsilon)\frac{\rho^2}{(k-\epsilon)^2\epsilon^2} \\
\zeta_k(x;\epsilon,\rho) &= (-1)^k k!\sqrt{-\frac{(k-\epsilon)^2}{\rho}\left(\frac{\rho}{\epsilon-k}\right)^{2\epsilon-2k}\Gamma(2k-2\epsilon)}\,p_k(\epsilon,\rho)(2x)^{-\epsilon}\exp\left(\rho x/(k-\epsilon)\right)L_k^{(-1-2\epsilon)}\left(\frac{2\rho x}{\epsilon-k}\right)
\end{aligned}$$

where

$$p_k(\epsilon,\rho) = \begin{cases} 1, & k=0 \\ \frac{(2\epsilon-k)}{\epsilon}\sqrt{\frac{(k-\epsilon)^2\epsilon^2}{k(k-2\epsilon)\rho^2}}\,p_{k-1}(\epsilon-1,\rho), & k>0 \end{cases} \quad (47)$$

7.4. Case (24) (Rosen-Morse 1 type)

We have set $\epsilon = M + d_j I_j$, $\rho = I$. We have, for $\epsilon \neq 0$, $\epsilon < 1/2$ and $\epsilon - k \neq 0$, $\epsilon - k < 1/2$,

$$\begin{aligned}
W(x;\epsilon,\rho) &= -\epsilon\tan(x) + \rho/\epsilon, \quad x \in (-\pi/2, \pi/2) \\
V(x;\epsilon,\rho) &= \epsilon^2\tan^2(x) - 2\rho\tan(x) + \epsilon\sec^2(x) + \rho^2/\epsilon^2 \\
\tilde{V}(x;\epsilon,\rho) &= \epsilon^2\tan^2(x) - 2\rho\tan(x) - \epsilon\sec^2(x) + \rho^2/\epsilon^2 \\
R(\epsilon,\rho) &= -1 - 2\epsilon - \frac{(2\epsilon+1)\rho^2}{\epsilon^2(\epsilon+1)^2} \\
E_k &= k(k-2\epsilon)\left(\frac{\rho^2}{(k-\epsilon)^2\epsilon^2} + 1\right) \\
\zeta_k(x;\epsilon,\rho) &= (-i)^k k! \frac{\left|\Gamma\left(1+k-\epsilon+\frac{i\rho}{k-\epsilon}\right)\right|}{\sqrt{\pi}\Gamma(1+2k-2\epsilon)} u_k(\epsilon,\rho)(2\cos(x))^{k-\epsilon}\exp\left(\frac{\rho x}{k-\epsilon}\right) \\
&\quad P_k^{(\epsilon-k+i\rho/(\epsilon-k),\epsilon-k-i\rho/(\epsilon-k))}(-i\tan(x))
\end{aligned}$$

where

$$u_k(\epsilon,\rho) = \begin{cases} 1, & k = 0 \\ \frac{(2\epsilon-k)u_{k-1}(\epsilon-1,\rho)}{\epsilon\sqrt{k(k-2\epsilon)-\rho^2/(k-\epsilon)^2+\rho^2/\epsilon^2}}, & k > 0 \end{cases} \tag{48}$$

7.5. Case (25)

We have set $\epsilon = M + d_j I_j$, $\rho = I$. Finally, we have, for $\epsilon \neq 0$, $\epsilon < 1/2$ and $\epsilon - k \neq 0$, $\epsilon - k < 1/2$,

$$\begin{aligned}
W(x;\epsilon,\rho) &= \epsilon\cot(x) + \rho/\epsilon, \quad x \in (0, \pi) \\
V(x;\epsilon,\rho) &= \epsilon^2\cot^2(x) + 2\rho\cot(x) + \epsilon\csc^2(x) + \rho^2/\epsilon^2 \\
\tilde{V}(x;\epsilon,\rho) &= \epsilon^2\cot^2(x) + 2\rho\cot(x) - \epsilon\csc^2(x) + \rho^2/\epsilon^2 \\
R(\epsilon,\rho) &= -1 - 2\epsilon - \frac{(2\epsilon+1)\rho^2}{\epsilon^2(\epsilon+1)^2} \\
E_k &= k(k-2\epsilon)\left(\frac{\rho^2}{(k-\epsilon)^2\epsilon^2} + 1\right) \\
\zeta_k(x;\epsilon,\rho) &= (-i)^k k! \frac{\left|\Gamma\left(1+k-\epsilon+\frac{i\rho}{k-\epsilon}\right)\right|}{\sqrt{\pi}\Gamma(1+2k-2\epsilon)} u_k(\epsilon,\rho)(2\sin(x))^{k-\epsilon}\exp\left(\frac{\rho(2x-\pi)}{2(k-\epsilon)}\right) \\
&\quad P_k^{(\epsilon-k+i\rho/(\epsilon-k),\epsilon-k-i\rho/(\epsilon-k))}(i\cot(x))
\end{aligned}$$

where $u_k(\epsilon,\rho)$ is given again by (48).

8. Conclusions

We have found in this paper a way of generalizing the previously known cases of shape invariant potentials (and their rational extensions) to the inclusion of an arbitrary number $r \geq 1$ of quantities that are invariant under the change of $n \geq 1$ parameters subject to translation. When $n = 1$ the invariants are essentially periodic functions in only one parameter, with period 1, and have not been observed before to the best of our knowledge. When $n \geq 1$ the non-trivial invariants may enter in the expression of quantities of physical significance, as it is for example the spectrum of the problem. This means that the spectrum and perhaps other meaningful physical quantities could be engineered to a great extent. The natural consequence is that this opens the door to a possible multitude of quantum applications, like for example quantum computing and physics of bilayer graphene [9]. Also, other fundamental questions might be relevant. For example, how a modification of the approach in [2] could generate the new solutions found here or perhaps another ones. Also, whether these new found cases satisfy the SWKB formalism [10] or not. All these would be probably interesting questions for future research.

Author contributions

Arturo Ramos: Conceptualization, formal analysis, funding acquisition, investigation, methodology, resources, software, validation, visualization, writing-original draft, writing-review & editing.

Competing interests statement

The author declares to have no competing interests concerning the research carried out in this article.

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