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Maintenance of systems with critical components. Prevention of early failures and wear-out

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ABSTRACT

We present a model for inspection and maintenance of a system under two types of failures. Early failures (type I), affecting only a proportion p of systems, are due to a weak critical component detected by inspection. Type II failures are the result of the system ageing and preventive maintenance is used against them. The two novelties of this model are: (1) the use of a defective distribution to model strong components free of defects and thus immune to early failures. (2) the removal of the weak critical part once it is detected with no other type of rejuvenation of the system which constitutes an alternative to the minimal repair. We study the conditions under which this model outperforms, from a cost viewpoint, other two classical age-replacement models. The analysis reveals that inspection is advantageous if the system can function with the critical component in the defective state for a long enough time. The proportion of weak units and the quality of inspections also determine the optimum policy. The results about the range of application of the model are useful for decision making in actual maintenance. A case study concerning the timing belt of a four-stroke engine illustrates the model.

1. Introduction

Wear-out failures arise after a long use that makes systems degrade and eventually fail. Preventive maintenance is regularly applied to avoid these risks that emerge as time goes by. The high costs derived from these procedures motivate the interest in maintenance optimization as well as the broad existing literature. The works of Alaswad and Xiang (2017) and de Jonge and Scarf (2020) contain thorough reviews on this issue.

A second type of failures occurs during the early life of the system. They are usually caused by hidden defects that appear because of design errors, mistakes during the production process or low quality controls. The research in Fernandez-Francos, Martinez-Rego, Fontenla-Romero, and Alonso-Betanzos (2013) is aimed at the early detection of defects in bearings. The works of Dourado and Viana (2021), and Li, Deng, Golilarz, and Guedes Soares (2021) model infant mortality and analyze early failures. The presence of hidden defects is traditionally traced in new components but they can also occur in reused units due to a defective rejuvenation. Recycling end-of-life products is becoming a usual practice in order to reduce the consumption of resources. Poorly refurbished components can constitute another source of concealed faults. Maintenance for second-hand systems in considered in Heydari (2021). Sometimes in order to extend the life-length of a system, it is only required that a critical component still work. In case it is defective, just a partial replacement can make the system keep on working. This is one of the assumptions of the model in this paper.

Many systems contain critical components which are crucial for the functionality of the system. Although its intrinsic value may be very low, a fatal breakdown of the whole system can occur when they fail, incurring high costs. Therefore, its maintenance requires special attention for a proper operating condition. When a hidden fault implies that a critical component is weak or undersized, then a catastrophic early failure is very likely to occur. Fragile or weak parts are usually removed by burn-in Cha (2010, 2014). This implies a period of simulated use prior to the real one. These procedures are costly and can eliminate good components or accelerate their degradation. Thus, Li, Liu, Wang, and Li (2019) include maintenance actions to reduce these negative side effects. Zhang, Ye, and Xie (2014) present an alternative inspection–replacement model in comparison with a joint burn-in and age replacement policy. Inspection to detect hidden failures has also been considered in Taghipour and Banjevic (2011) and more recently in Levitin, Xing, and Huang (2019) and Zhang, Shen, and Ma (2021).

The case study of this paper concerns the maintenance of the timing belt of a four-stroke engine. This is a central part of the valve train and manufacturers recommend changing it preventively after a period of

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time or an interval of use to avoid wear-out failures. The timing belt tensioner is a critical part since a faulty one results in the slack of the belt, losing its synchronism. Then, the pistons violently hit the valves and, consequently, the main and most expensive mechanical components of the engine result irreparably damaged. Visual inspections can reveal the actual state of the tensioner and whether its replacement is required before the whole system is renewed. Additional examples of critical components are present in common engineering systems: shaft bearings, the oil pump of a lubrication station, or safety systems that avoid the system damage when it is working under hard or unsuitable conditions.

Following these ideas, we present a model for inspection and maintenance of a system that may undergo two types of failures. Type I failures are due to a weak critical component and they occur in two stages according to the delay-time concept (Christer, 1987). Hence, there is a defective state previous to failure. This defective state is detected only by inspection and it results from an overload or random shock. The system can perform its intended function while it is in the period from defect to failure (delay-time). Recent applications of the delay time appear in Akcay, Topan, and van Houtum (2021), Azimpoor and Taghipour (2020), Heydari (2021) and Zhang et al. (2021). We assume that type I failures are more likely to appear during the early life of the system since they are due to intrinsic flaws. Therefore, inspections are restricted to the initial operating time as in Cavalcante, Lopes, and Scarf (2018, 2021). The current model includes a proportion of units free of defects and, thus, immune to early failures. To the best of our knowledge previous literature on maintenance modeling has not considered this idea of an "immune population" so it constitutes the first novelty of this paper. In so doing we describe flaws caused by random disruptions in the production process affecting only a fraction of the manufactured units. This assumption resembles, for example, car recalls of specific vehicles that occur when the manufacturer detects a faulty component that implies a diminished safety for drivers.

The second type of failures (type II) are due to wear-out and a preventive replacement is carried out at a scheduled time after the last inspection to avoid them. The similarities of this approach with burn-in have been highlighted in Scarf, Cavalcante, Dwight, and Gordon (2009). Different models to systems with two failure modes have been considered in Mituzani, Zhao, and Nakagawa (2021), Peng, Liu, Zhai, and Wang (2019) and Xiao, Yan, Kou, and Wu (2023).

As far as we know, the previous two-failure models have not dealt with the catastrophic effects of failures in critical components. In addition, the foregoing references state that the system in whole is replaced when an inspection indicates a defective state. In this model we assume that only the defective critical component is renewed but the condition in the rest of the system (age and use) remains as it was before the defect was detected. This approach makes perfect sense if the rest of the system is still in a good condition and at the same time, it is significantly more expensive than the critical component. Thus, the potential type I failure disappears with a noticeable cost reduction derived from replacing just one part instead of the system in whole. This partial renewal of the system constitutes the second contribution of this paper. This procedure can be seen as something in between the minimal repair and the total renewal. We consider that this assumption also presents potential applications in maintenance of second-hand systems whose life-length can be extended just replacing some of the old or defective critical parts. The study of Santos, Cavalcante, and Wu (2023) contains a bibliometric analysis and a review of maintenance policies that deal with reuse or remanufacturing as sustainable strategies.

The maintenance of many current systems is based on a large amount of data. This is so when the state of the system can be predicted from other observable variables. These systems tend to be continuously monitored with many sensors providing information about the latent variable and the need of maintenance. Regarding data acquisition and processing the review in Jardine, Lin, and Banjevic (2006) is a key reference. The reviews in Kan, Tan, and Mathew (2015), Liu, Yang, Zio, and Chen (2018) and Tahan, Tsoutsanis, Muhammad, and Karim (2017) summarize the research on maintenance engineering in different areas.

Nevertheless in other type of systems there are no diagnostic variables indicating the system malfunction or a high risk to collapse. This is so, for example, in systems that do not operate continuously as safety systems. Failures due to fatigue, degrading processes, or crack growth can be additional examples of systems providing "no symptom" of the failure proneness. In those cases, the system itself has to be inspected in order to check its state. The approach considered in this paper focuses on the latter and therefore it is not data-driven.

In the next section, we describe the mathematical model deriving the cost-rate function. Section 3 contains two additional age-based maintenance policies for comparison purposes. In both cases inspections are removed which can be useful when they are expensive or difficult to carry out as in telecommunications or spatial technology. In the first model, the risk of a weak critical component is ignored and the system in whole undergoes a preventive age replacement. The second one describes an initial intervention to replace the potential weak component to a new one free of defects in every system before the start-up. This action could resemble a sort of burn-in and it can be an alternative procedure to extend the life of second-hand systems, just replacing their critical parts. This second policy can be appropriate when the hidden faults are due to design errors, affecting a large proportion of the produced units. The study analyzes the consequences of removing inspections in the long-run cost. Section 4 contains the analysis of the optimum policies as well as the comparison between the model proposed in this paper and the two additional age-based policies. The analysis of the range of application of this model when comparing with the other two is presented in Section 5. This can also serve as a guide for maintenance of used systems with some new parts. Section 6 contains the conclusions.

2. Model building

We consider a system subject to two types of revealed failures namely, early failures (type I) and failures by use (type II). A revealed failure is the one that is detected at the very moment it takes place. The system contains a critical component which is responsible for the type I failures when that component is weak, or it has been poorly installed. In addition we assume that, as in real life, only a fraction of the units are weak. The rest are free of defects and thus they are immune to type I failures. Hence if the component is good, then the system only undergoes type II failures. This is the first novelty of the paper since previous research states a mixture of distributions with longer durations for the strong sub-population.

A defective state of the weak unit precedes the type I failure of the system and since defects are unrevealed, that is, detected only by inspection, the unit is periodically inspected at times kT, $k = 1, 2, \dots, N$. The traditional use of the delay-time model assumes both stages (from good to defective and from defective to failed) in the same component. In this paper we apply the delay-time concept to model the time until the weak component enters the defective state (first stage) and from this moment to the early and catastrophic failure of the system (second stage). To the best of our knowledge this application of the delay-time time for a critical component has not been previously considered in literature. We also analyze conditions in the ratio of the time span in the first stage to the delay time that make inspection a profitable strategy. This type of study is also new.

We consider an inspection policy restricted only to the initial running-in period of the system. Thus, we assume a maximum number of inspections, N. If N is such that a defect or type I failure after NT is rare, this limited inspection period implies a cost reduction without increasing significantly the risk of failure when comparing with an inspection developed during the whole lifetime. Therefore N and T are decision variables of the model. In addition inspections can be imperfect and so, the result may be a false positive (a defect is reported when the system is ok) or false negative (the inspection fails to detect an actual defect). After a positive inspection (true or not) a corrective maintenance is carried out. This corrective maintenance replaces the component by a strong one, removing the possibility of type I failure. Hence inspection and corrective maintenance emerge as an alternative to burn-in.

The most important consequence of false negatives is that an undetected defect can lead to a type I failure during middle stages of use or even after inspection ends. This can have a number of effects such as a significant reduction in the useful time of the system or the high cost of an out-of-warranty repair. We assume as in Akcay et al. (2021) a downtime cost while the defective state remains undetected. It is also considered in Liu, Zhao, Liu, and Do (2021) for a system that operates in the failed state. In the context of a manufacturing system this cost can be due to the unmet demand. Moreover, these consequences of false positives or false negatives are also crucial to determine the maximum number of inspections as well as the inspection frequency so that the risk of replacing good components or entering the defective state in the non inspection period is minimized.

Imperfect testing has been considered in foregoing papers Berrade, Cavalcante, and Scarf (2012) and Hao, Yang, and Berenguer (2020). Nevertheless there is a significant difference between the current model and those in previous literature because now a positive inspection (true or false) does not imply the renewal of the whole system but only the removal of the potential cause for an early failure. However the ageing and natural wear out of the system remain. This assumption resembles the actual procedures since maintainers tend to replace a failed unit instead of repairing it. As a result, used systems contain parts which are completely new. This assumption is different from both minimal repair and burn-in and it can be applied in systems with a critical component.

In order to prevent Type II failures we assume a maximum period of use, H, after the last inspection before the system is renewed. H is the third decision variable in the model.

Type I and Type II can be considered catastrophic failures in the sense that the system can no longer work after any of them. Therefore both lead to the replacement of the system and they are modeled as competing risks failures.

Then, the system is renewed in any of the following scenarios:

- on a type I failure.
- on a type II failure.
- preventively at $NT + H$.

whichever comes first.

The following list contains the notation used in the paper:

- X time to defect of a component with density, distribution and reliability functions $f_X(x)$, $F_X(x)$, $\bar{F}_X(x)$.
- Y time from defect to type I failure (delay-time) with density, distribution and reliability functions $f_Y(y)$, $F_Y(y)$, $\bar{F}_Y(y)$.
- Z time to type II failure with density, distribution and reliability functions $f_Z(z)$, $F_Z(z)$, $\bar{F}_Z(z)$.
- T inspection interval (decision variable).
- τ length of a renewal cycle.
- N maximum number of inspections in a renewal cycle (decision variable).
- K number of inspections in a renewal cycle ($K = 0, 1, 2 \ldots N$).
- α probability of a false positive inspection.
- β probability of a false negative inspection.
- H maximum usage time after the Nth inspection (decision variable).
- c_0 unitary cost of inspection.
- c_d cost per unit of time due to downtime.
- c_V cost of removing a defective component.
- c_F cost of replacement on failure.
- c_M cost of preventive replacement at $NT + H$ ($c_M < c_F$).
- $C(\tau)$ cost of a renewal cycle.
- $Q(T, N, H)$ objective cost function.

Fig. 1 illustrates a renewal cycle completed on failure whereas Fig. 2 represents the case of a system preventively replaced at $NT + H$.

Real data show that the major part of components produced by manufacturing systems are strong, being affected only by type II failures. Only a small proportion of them, p, are weak and may also undergo a type I failure. In previous papers Berrade et al. (2012), Cavalcante et al. (2018, 2021) and Scarf et al. (2009) the combination of weak and strong components is represented by a mixture of distributions as follows:

$$\bar{F}(x) = p\bar{F}_1(x) + (1-p)\bar{F}_2(x)$$

Where $\bar{F}_2(x)$ represents the population with longer time to the defective state.

Considering exponential distributions for $F_1(x)$ and $F_2(x)$ with scale parameter l_1 and l_2, respectively, we have

$$\bar{F}_X(x) = pe^{-x/l_1} + (1-p)e^{-x/l_2}$$

with $E[X] = pl_1 + (1-p)l_2$ the mean time to defective state of the mixture.

Nevertheless this representation is no longer valid in the current model since well-manufactured components present a larger lifetime than the system itself. Then, strong components do neither enter a defective state nor experience a type I failure. When a defective state is detected the component is replaced by a new and strong one and thus the system is totally free of experiencing a type I failure. Only type II failures remain. Thus, in order to model strong components without risk of an early failure, we assume the following limiting mixture for X:

$$\bar{F}_X(x) = pe^{-x/l_1} + (1-p)\lim_{l_2\to\infty} e^{-x/l_2} = pe^{-x/l_1} + (1-p) \tag{1}$$

The expression in (1) corresponds to a defective distribution, that is, $\bar{F}_X(\infty) = (1-p) > 0$. Defective distributions are considered by Feller (1968) (chapter 13). A particular characteristic of this sort of distributions is that $E[X] = \infty$, corresponding to strong components unaffected by type I failures. The works on survival analysis presented in Martínez and Achcar (2018) and Scudilio et al. (2019) use defective distributions to model a mixture with a proportion of cured patients. Scudilio et al. (2019) indicate that defective models have advantage of modeling the cure rate without adding any extra parameter in model. We have not found this idea previously used in maintenance modeling.

We also describe a different effect of false positive inspections than that in previous papers Berrade et al. (2012). Now if a false positive occurs the maintainer will incur an unnecessary cost substituting a strong component. However a false positive before a weak component enters a defective state would result in a beneficial outcome since an actual defect will be removed before showing up and the risk of a catastrophic failure eliminated.

In subsections from 2.1 to 2.4 we present the algebra that describes the model and the cost function.

We first obtain the probabilities of system replacement. Both renewals, after type I or type II failure, can occur between two consecutive inspections $((i-1)T, iT)$, $i = 1, 2 \ldots, N$ or during the period without inspections $(NT, NT + H)$.

2.1. Replacement after a type I failure

Probability of replacement on a type I failure that occurs in $(0, T)$:

$$P_I(1) = \int_0^T f_X(x)\left(\int_0^{T-x} f_Y(y)\bar{F}_Z(x+y)dy\right)dx \tag{2}$$

In the previous formula a type II failure does not occur before a type I failure in $(0, T)$.

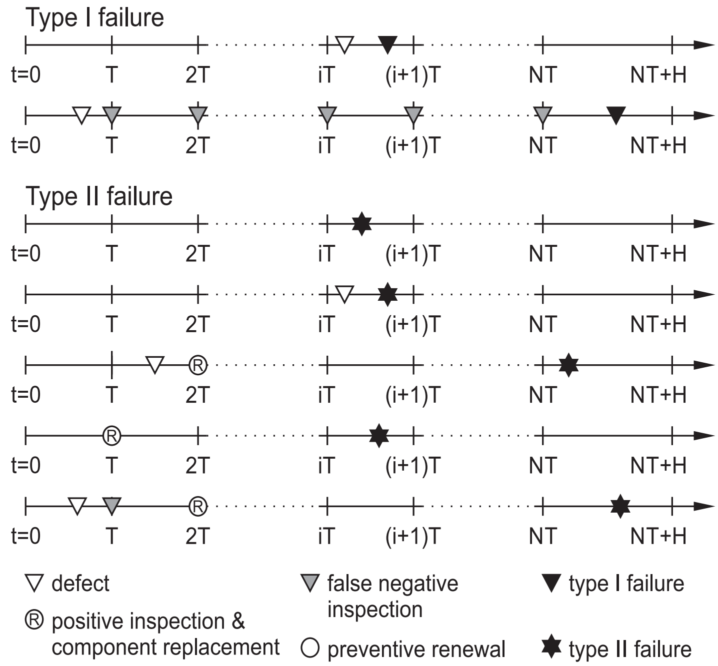

Fig. 1. Renewal cycle ending on a type I or a type II failure.

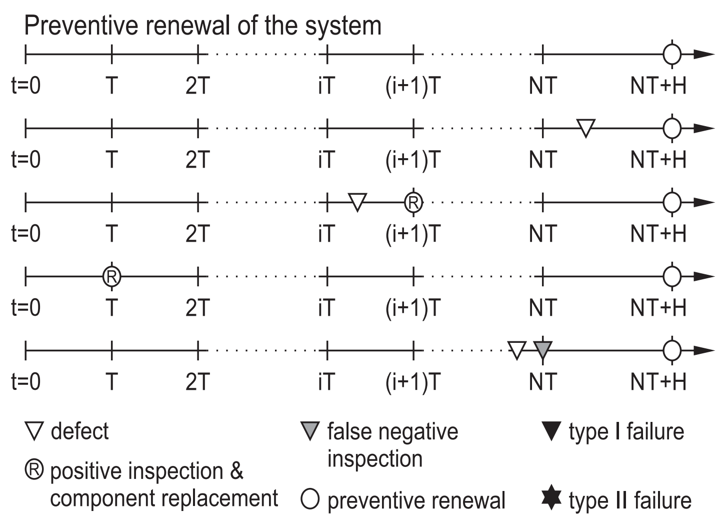

Fig. 2. Renewal cycle ending on preventive maintenance at $NT + H$.

Probability of replacement on a type I failure that occurs during the interval $((i-1)T, iT)$, $i = 2 \ldots, N$:

$$P_I(i) = \sum_{j=1}^{i-1} (1-\alpha)^{j-1} \int_{(j-1)T}^{jT} f_X(x)\beta^{i-j} \left(\int_{(i-1)T-x}^{iT-x} f_Y(y)\bar{F}_Z(x+y)dy \right) dx + (1-\alpha)^{i-1} \int_{(i-1)T}^{iT} f_X(x) \left(\int_0^{iT-x} f_Y(y)\bar{F}_Z(x+y)dy \right) dx \tag{3}$$

The first term of the previous equation represents that the defect occurs in a previous interval than that of the type I failure and this defect remains undetected. In the second term both, defect and type I failure take place in the same interval. Note that the possibility of a false positive inspection is excluded with the term $(1-\alpha)$ because a false alarm implies an action that eliminates the element causing the potential defect and, consequently, the risk of a type I failure.

Hence, the probability of replacement on a type I failure during the period of inspections, $(0, NT)$ is $\sum_{i=1}^{N} P_I(i)$.

The expected length of a cycle that ends with replacement on a type I failure in $(0, T)$ is

$$E[\tau_I(1)] = \int_0^T f_X(x) \left(\int_0^{T-x} (x+y) f_Y(y)\bar{F}_Z(x+y)dy \right) dx \tag{4}$$

The expected length of a cycle that ends with replacement on a type I failure that occurs during the interval $((i-1)T, iT)$, $i = 2 \ldots, N$

$$E[\tau_I(i)] = \sum_{j=1}^{i-1} (1-\alpha)^{j-1} \int_{(j-1)T}^{jT} f_X(x)\beta^{i-j} \left(\int_{(i-1)T-x}^{iT-x} (x+y) f_Y(y)\bar{F}_Z(x+y)dy \right) dx + (1-\alpha)^{i-1} \int_{(i-1)T}^{iT} f_X(x) \left(\int_0^{iT-x} (x+y) f_Y(y)\bar{F}_Z(x+y)dy \right) dx \tag{5}$$

The expected downtime when replacement occurs on a type I failure in $(0, T)$

$$E[D_I(i)] = \int_0^T f_X(x) \left(\int_0^{T-x} y f_Y(y)\bar{F}_Z(x+y)dy \right) dx \tag{6}$$

The expected downtime when replacement occurs on a type I failure in $((i-1)T, iT)$, $i = 2\dots, N$ is given by

$$E[D_I(i)] = \sum_{j=1}^{i-1}(1-\alpha)^{j-1}\int_{(j-1)T}^{jT} f_X(x)\beta^{i-j}\left(\int_{(i-1)T-x}^{iT-x} y f_Y(y)\bar{F}_Z(x+y)dy\right)dx + (1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\left(\int_0^{iT-x} y f_Y(y)\bar{F}_Z(x+y)dy\right)dx \quad (7)$$

Probability of replacement on a type I failure that happens during the interval $(NT, NT+H)$:

$$P_{IH} = \sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\beta^{N-i+1}\left(\int_{NT-x}^{NT+H-x} f_Y(y)\bar{F}_Z(x+y)dy\right)dx + (1-\alpha)^{N}\int_{NT}^{NT+H} f_X(x)\left(\int_0^{NT+H-x} f_Y(y)\bar{F}_Z(x+y)dy\right)dx \quad (8)$$

In the first term the defect takes place during the inspection period but all the subsequent inspections until N fail to detect them. In the second term the defect occurs after the end of the inspection period. The latter term requires that inspection at NT does not result in a false positive. Thus, the probability of replacement on type I failure (P_I):

$$P_I = \sum_{i=1}^{N} P_I(i) + P_{IH} \quad (9)$$

The time span since the system is new until it is replaced is known as renewal cycle. The conditional length of a cycle when the defect occurs at $X = x$ and the delay-time is $Y = y$ and a type II failure has not occurred before, is $x + y$. Hence the expected length of a renewal cycle completed on a type I failure that occurs during the interval $(NT, NT+H)$ is given by

$$E[\tau_{IH}] = \sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\beta^{N-i+1} \times \left(\int_{NT-x}^{NT+H-x}(x+y)f_Y(y)\bar{F}_Z(x+y)dy\right)dx + (1-\alpha)^{N}\int_{NT}^{NT+H} f_X(x)\left(\int_0^{NT+H-x}(x+y)f_Y(y)\bar{F}_Z(x+y)dy\right)dx \quad (10)$$

The expected downtime if the system is replaced on a type I failure that occurs in $(NT, NT+H)$:

$$E[D_{IH}] = \sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\beta^{N-i+1}\left(\int_{NT-x}^{NT+H-x} y f_Y(y)\bar{F}_Z(x+y)dy\right)dx + (1-\alpha)^{N}\int_{NT}^{NT+H} f_X(x)\left(\int_0^{NT+H-x} y f_Y(y)\bar{F}_Z(x+y)dy\right)dx \quad (11)$$

and the expected length of a renewal cycle completed on a type I failure ($E[\tau_I]$):

$$E[\tau_I] = \sum_{i=1}^{N} E[\tau_I(i)] + E[\tau_{IH}] \quad (12)$$

The corresponding expected downtime in a system replaced after a type I failure:

$$E[D_I] = \sum_{i=1}^{N} E[D_I(i)] + E[D_{IH}] \quad (13)$$

2.2. Replacement after a type II failure

Probability of replacement on a type II failure that occurs in $(0, T)$:

$$P_{II}(1) = \int_0^T f_X(x)\left(\int_x^T f_Z(z)\bar{F}_Y(z-x)dz\right)dx + \int_0^T f_Z(z)\bar{F}_X(z)dz \quad (14)$$

Probability of replacement on a type II failure that occurs during the interval $((i-1)T, iT)$, $i = 2\dots, N$:

$$P_{II}(i) = \sum_{j=1}^{i-1}(1-\alpha)^{j-1}\int_{(j-1)T}^{jT} f_X(x)\beta^{i-j}\left(\int_{(i-1)T}^{iT} f_Z(z)\bar{F}_Y(z-x)dz\right)dx + (1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\left(\int_x^{iT} f_Z(z)\bar{F}_Y(z-x)dz\right)dx + \sum_{j=1}^{i-1}(1-\alpha)^{j-1}\int_{(j-1)T}^{jT} f_X(x)\sum_{k=0}^{i-1-j}\beta^k(1-\beta)\bar{F}_Y((j+k)T-x) \times \left(\int_{(i-1)T}^{iT} f_Z(z)dz\right)dx + \sum_{j=1}^{i-1}(1-\alpha)^{j-1}\alpha\bar{F}_X(jT)\int_{(i-1)T}^{iT} f_Z(z)dz + (1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_Z(z)\bar{F}_X(z)dz \quad (15)$$

The first two terms describe a competing risks situation where the risk of a type I failure is present but a type II failure takes place before a type I failure. In the first term the defect occurs in a previous interval than that of the type II failure and this defect remains undiscovered. In the second term the defect and the type II failure happen in the same interval.

The third and the fourth terms refer to the case where only the risk of a type II failure remains. In the third term a defect occurs in $((j-1)T, jT)$ but it is detected in one of the posterior inspections from jT to $(i-1)T$. This occurs with probability $\sum_{k=0}^{i-1-j}\beta^k(1-\beta)$. In the fourth term a false alarm leads to eliminate a potential type I failure. In the fifth term a type II failure is previous to a defect.

The conditional length of a cycle when a type II failure occurs at $Z = z$ before a type I failure is z. Thus the expected length of a renewal cycle completed on a type II failure that occurs in $(0, T)$ is:

$$E[\tau_{II}(1)] = \int_0^T f_X(x)\left(\int_x^T z f_Z(z)\bar{F}_Y(z-x)dz\right)dx + \int_0^T z f_Z(z)\bar{F}_X(z)dz \quad (16)$$

The expected length of a renewal cycle completed on a type II failure that occurs during the interval $((i-1)T, iT)$, $i = 2\dots, N$:

$$E[\tau_{II}(i)] = \sum_{j=1}^{i-1}(1-\alpha)^{j-1}\int_{(j-1)T}^{jT} f_X(x)\beta^{i-j}\left(\int_{(i-1)T}^{iT} z f_Z(z)\bar{F}_Y(z-x)dz\right)dx + (1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\left(\int_x^{iT} z f_Z(z)\bar{F}_Y(z-x)dz\right)dx + \sum_{j=1}^{i-1}(1-\alpha)^{j-1}\int_{(j-1)T}^{jT} f_X(x)\sum_{k=0}^{i-j-1}\beta^k(1-\beta)\bar{F}_Y((j+k)T-x) \times \left(\int_{(i-1)T}^{iT} z f_Z(z)dz\right)dx + \sum_{j=1}^{i-1}(1-\alpha)^{j-1}\alpha\bar{F}_X(jT)\int_{(i-1)T}^{iT} z f_Z(z)dz + (1-\alpha)^{i-1}\int_{(i-1)T}^{iT} z f_Z(z)\bar{F}_X(z)dz \quad (17)$$

In the first two terms the risk of a type I failure is present when the type II failure happens. The defect is eliminated on inspection in the third term and after a false alarm in the fourth term. The expected downtime and system replaced on a type II failure that occurs during the interval $(0, T)$:

$$E[D_{II}(1)] = \int_0^T f_X(x)\left(\int_x^T (z-x)f_Z(z)\bar{F}_Y(z-x)dz\right)dx \quad (18)$$

The expected downtime in a renewal cycle completed on a type II failure that occurs during the interval $((i-1)T, iT)$, $i = 2\dots, N$:

$$E[D_{II}(i)] = \quad (19)$$

$$\sum_{j=1}^{i-1}(1-\alpha)^{j-1}\int_{(j-1)T}^{jT} f_X(x)\beta^{i-j}\left(\int_{(i-1)T}^{iT}(z-x)f_Z(z)\bar{F}_Y(z-x)dz\right)dx+$$
$$(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\left(\int_{x}^{iT}(z-x)f_Z(z)\bar{F}_Y(z-x)dz\right)dx+$$
$$\sum_{j=1}^{i-1}(1-\alpha)^{j-1}\int_{(j-1)T}^{jT} f_X(x)\sum_{k=0}^{i-j-1}\beta^k(1-\beta)((j+k)T-x)\bar{F}_Y((j+k)T-x)$$
$$\times\left(\int_{(i-1)T}^{iT} f_Z(z)dz\right)dx$$

In the first term the defect remains undetected when the type II failure happens in a posterior interval. In the second term both, defect and type II failure, happen in the same interval. In the third term the defect is detected after several inspections, implying the end of the downtime period. The occurrence of a false alarm is excluded in the foregoing expression since the downtime starts when an actual defect takes place until it is detected or a failure happens whichever occurs first.

Probability of replacement on a type II failure in $(NT, NT+H)$:

$$P_{IIH}= \tag{20}$$
$$\sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\beta^{N-i+1}\left(\int_{NT}^{NT+H} f_Z(z)\bar{F}_Y(z-x)dz\right)dx+$$
$$\sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\sum_{k=0}^{N-i}\beta^k(1-\beta)\bar{F}_Y((i+k)T-x)dx$$
$$\times\int_{NT}^{NT+H} f_Z(z)dz+$$
$$(1-\alpha)^N\int_{NT}^{NT+H} f_X(x)\left(\int_{x}^{NT+H} f_Z(z)\bar{F}_Y(z-x)dz\right)dx+$$
$$(1-\alpha)^N\int_{NT}^{NT+H} f_Z(z)\bar{F}_X(z)dz+\sum_{j=1}^{N}(1-\alpha)^{j-1}\alpha\bar{F}_X(jT)\int_{NT}^{NT+H} f_Z(z)dz$$

In the first term a defect occurs in $(0, NT)$ and the subsequent inspections fail to detect it. In the second term the defect is detected on a posterior inspection before a type I failure occurs and so the risk of a type I failure is no longer present. The term $\sum_{k=0}^{N-i}\beta^k(1-\beta)$ is the probability of that detection. In the third term the defect occurs after the end of the inspection period. The fourth case represents a type II failure that occurs before the defect. In the fifth term the risk of a type I failure is eliminated after a false alarm.

The expected length of a renewal cycle completed on a type II failure that occurs during the interval $(NT, NT+H)$:

$$E[\tau_{IIH}]= \tag{21}$$
$$\sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\beta^{N-i+1}\left(\int_{NT}^{NT+H} zf_Z(z)\bar{F}_Y(z-x)dz\right)dx+$$
$$\sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\sum_{k=0}^{N-i}\beta^k(1-\beta)\bar{F}_Y((i+k)T-x)dx$$
$$\times\int_{NT}^{NT+H} zf_Z(z)dz+$$
$$(1-\alpha)^N\int_{NT}^{NT+H} f_X(x)\left(\int_{x}^{NT+H} zf_Z(z)\bar{F}_Y(z-x)dz\right)dx+$$
$$(1-\alpha)^N\int_{NT}^{NT+H} zf_Z(z)\bar{F}_X(z)dz+\sum_{j=1}^{N}(1-\alpha)^{j-1}\alpha\bar{F}_X(jT)$$
$$\times\int_{NT}^{NT+H} zf_Z(z)dz$$

The corresponding expected downtime:

$$E[D_{IIH}]= \tag{22}$$
$$\sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\beta^{N-i+1}$$
$$\times\left(\int_{NT}^{NT+H}(z-x)f_Z(z)\bar{F}_Y(z-x)dz\right)dx+$$
$$\sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\sum_{k=0}^{N-i}\beta^k(1-\beta)((i+k)T-x)\bar{F}_Y((i+k)T-x)dx$$
$$\times\int_{NT}^{NT+H} f_Z(z)dz+$$
$$(1-\alpha)^N\int_{NT}^{NT+H} f_X(x)\left(\int_{x}^{NT+H}(z-x)f_Z(z)\bar{F}_Y(z-x)dz\right)dx$$

Thus, the probability of replacement on type II failure (P_{II}):

$$P_{II}=\sum_{i=1}^{N}P_{II}(i)+P_{IIH} \tag{23}$$

The expected length of a renewal cycle ending on a type II failure ($E[\tau_{II}]$):

$$E[\tau_{II}]=\sum_{i=1}^{N}E[\tau_{II}(i)]+E[\tau_{IIH}] \tag{24}$$

The expected downtime in a cycle completed on the renewal after a type II failure:

$$E[D_{II}]=\sum_{i=1}^{N}E[D_{II}(i)]+E[D_{IIH}] \tag{25}$$

2.3. Preventive replacement at $NT+H$, expected renewal cycle and number of inspections

We denote by P_m the probability of replacing the system preventively at $NT+H$. It follows that

$$P_m= \tag{26}$$
$$\sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\beta^{N-i+1}\bar{F}_Y(NT+H-x)\bar{F}_Z(NT+H)dx+$$
$$\sum_{i=1}^{N}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} f_X(x)\sum_{k=0}^{N-i}\beta^k(1-\beta)\bar{F}_Y((i+k)T-x)\bar{F}_Z(NT+H)dx+$$
$$(1-\alpha)^N\int_{NT}^{NT+H} f_X(x)\bar{F}_Y(NT+H-x)\bar{F}_Z(NT+H)dx+$$
$$(1-\alpha)^N\bar{F}_X(NT+H)\bar{F}_Z(NT+H)+\sum_{j=1}^{N}(1-\alpha)^{j-1}\alpha\bar{F}_X(jT)\bar{F}_Z(NT+H)$$

In the first term a defect happens, remaining undetected. In the second term the defect is detected and thus the risk of a type I failure is no longer present. In the third term the defect occurs after the end of the inspection period. In the fourth term no defect happens before the system is replaced at $NT+H$. In the last term a false alarm eliminates the possibility of a type I failure.

In the first three cases a downtime period is induced. Its expected value is given as follows:

$$E[D_{PM}]= \tag{27}$$
$$\sum_{i=1}^{N}(1-\alpha)^{i-1}\times$$
$$\left(\int_{(i-1)T}^{iT}(NT+H-x)f_X(x)\beta^{N-i+1}\bar{F}_Y(NT+H-x)\bar{F}_Z(NT+H)dx\right)+$$
$$\sum_{i=1}^{N}(1-\alpha)^{i-1}\times$$
$$\left(\int_{(i-1)T}^{iT} f_X(x)\sum_{k=0}^{N-i}\beta^k(1-\beta)((i+k)T-x)\bar{F}_Y((i+k)T-x)\bar{F}_Z(NT+H)dx\right)+$$
$$(1-\alpha)^N\int_{NT}^{NT+H}(NT+H-x)f_X(x)\bar{F}_Y(NT+H-x)\bar{F}_Z(NT+H)dx$$

Replacement after a type I failure, type II failure or preventively at $NT+H$, denoted by I, II, and m, respectively are mutually exclusive events. Then, the length of a cycle, τ, can be expressed as

$$\tau=(\tau\cap I)+(\tau\cap II)+(\tau\cap m)$$

The expected length of a renewal cycle turns out to be

$$E[\tau] = E[\tau \cap I] + E[\tau \cap II] + E[\tau \cap m] = E[\tau_I] + E[\tau_{II}] + (NT + H)P_m \tag{28}$$

with $E[\tau_I]$ and $E[\tau_{II}]$ given in (12) and (24) respectively.

Next, the distribution of the number of inspections in a cycle, K, is presented. K takes values on $\{0, 1, 2, \ldots, N\}$.

The probability that a true positive inspection occurs at iT, $i = 1, 2, \ldots, N$ is

$$p_1(i) = \sum_{j=1}^{i} (1-\alpha)^{j-1} \int_{(j-1)T}^{jT} f_X(x)\beta^{i-j}(1-\beta)\bar{F}_Y(iT-x)\bar{F}_Z(iT)dx \tag{29}$$

The probability that a false positive inspection occurs at iT $i = 1, 2, \ldots, N$ is

$$p_2(i) = (1-\alpha)^{i-1}\alpha \bar{F}_X(iT)\bar{F}_Z(iT) \tag{30}$$

$K = 0$ in case that a type I or a type II failure occurs in $(0, T)$. Moreover

$$P(K=0) = \int_0^T f_X(x)\left(\int_0^{T-x} f_Y(y)\bar{F}_Z(x+y)dy\right)dx + \int_0^T f_X(x)\left(\int_x^T f_Z(z)\bar{F}_Y(z-x)dz\right)dx + \int_0^T f_Z(z)\bar{F}_X(z)dz \tag{31}$$

$K = i, i = 1, 2, \ldots, N-1$ in any of the following cases:

- A renewal cycle is completed after a type I failure in $[iT, (i+1)T]$. The corresponding probability is $P_I(i+1)$ given in (3).
- A true positive inspection occurs at iT. The corresponding probability is $p_1(i)$ in (29).
- A false positive inspection occurs at iT. The associated probability is $p_2(i)$ in (30).
- A type II failure occurs in $[iT, (i+1)T]$ with no previous false positive inspection at $jT, j = 1, 2, \ldots, i-1$ and a type I failure has not occurred before. The probability is given as follows:

$$p_3(i) = \sum_{j=1}^{i}(1-\alpha)^{j-1}\int_{(j-1)T}^{jT} f_X(x)\beta^{i-j+1}\left(\int_{iT}^{(i+1)T} f_Z(z)\bar{F}_Y(z-x)dz\right)dx + (1-\alpha)^i\int_{iT}^{(i+1)T} f_X(x)\left(\int_x^{(i+1)T} f_Z(z)\bar{F}_Y(z-x)dz\right)dx + (1-\alpha)^i\int_{iT}^{(i+1)T} f_Z(z)\bar{F}_X(z)dz \tag{32}$$

It follows that

$$P(K=i) = P_I(i+1) + p_1(i) + p_2(i) + p_3(i), \quad i = 1, 2, \ldots, N-1 \tag{33}$$

$K = N$ if neither a type I nor type II failure nor a false alarm happens before NT. In addition if a defective state occurs in any interval $[(j-1)T, jT]$ $j = 1, 2, \ldots, N-1$, then it is not detected on posterior inspections at $jT, (j+1)T, \ldots, (N-1)T$. The last inspection will be at NT no matter if this inspection detects the defect or not and also if the defect happens in $[(N-1)T, NT]$. If the defect has not occurred before NT, inspection at NT will be the last one regardless it is a false positive or a true negative. Hence, it follows that

$$P(K=N) = \sum_{j=1}^{N}(1-\alpha)^{j-1}\int_{(j-1)T}^{jT} f_X(x)\beta^{N-j}\bar{F}_Y(NT-x)\bar{F}_Z(NT)dx + (1-\alpha)^{N-1}\bar{F}_X(NT)\bar{F}_Z(NT) \tag{34}$$

2.4. Expected cost of a cycle and objective function

The critical component is removed at iT if a positive inspection occurs no matter if it is a true positive or a false one. The corresponding probabilities that a positive inspection occurs are given in $p_1(i)$ in (29) and $p_2(i)$ in (30). Hence, the cost of removal a defective component, c_v, is incurred with probability $P(c_v)$ given below:

$$P(c_v) = \sum_{i=1}^{N}(p_1(i) + p_2(i)) \tag{35}$$

The expected cost of a cycle, $E[C(\tau)]$

$$E[C(\tau)] = c_0\sum_{i=1}^{N} iP(K=i) + c_d(E[D_I] + E[D_{II}] + E[D_{PM}]) + c_m P_m + c_F(P_I + P_{II}) + c_v P(c_v)$$

with $E[D_I]$, $E[D_{II}]$, $E[D_{PM}]$, P_m, P_I and P_{II} given in (13), (25), (27) (26), (9) and (23), respectively.

The objective cost function:

$$Q(T, N, H) = \frac{E[C(\tau)]}{E[\tau]}$$

In the following study $(N^\star, T^\star, H^\star)$ will denote the minimum cost policy.

3. Comparison with classical age replacement models

The classical model under which the system is replaced on failure or a specified age, whichever comes first, is know as age replacement. Next we consider the application of this policy in two different scenarios to show the cost reduction resulting from the use of the model presented in previous section (model 1).

3.1. Model 2: Age replacement under type I and type II failures

The following formulae correspond to the case $N = 0$, that is, there is no inspection to detect defective states. The system may undergo both, type I and type II failures but the former is ignored. There is only an age replacement with H the specified age for replacement. We aim at studying the values of the parameters under which this can be a profitable strategy. This model can be used when maintainers are only concerned with the natural wear-out caused by use as well as operational and environmental factors. In practice it can be considered as a "Maintenance by Operation" since the age can be given in terms of a number of working cycles or another measure representing an amount of operation.

The probabilities of renewal on a type I failure (P_I), type II failure (P_{II}) or preventively at H, (P_H), are as follows:

$$P_I = \int_0^H f_X(x)\left(\int_0^{H-x} f_Y(y)\bar{F}_Z(x+y)dy\right)dx \tag{36}$$

$$P_{II} = \int_0^H f_Z(z)\bar{F}_X(z)dz + \int_0^H f_X(x)\left(\int_x^H f_Z(z)\bar{F}_Y(z-x)dz\right)dx \tag{37}$$

$$P_H = \bar{F}_X(H)\bar{F}_Z(H) + \int_0^H f_X(x)\bar{F}_Y(H-x)\bar{F}_Z(H)dx \tag{38}$$

The expected length of a renewal cycle that is completed after a type I failure:

$$E[\tau_I] = \int_0^H f_X(x)\left(\int_0^{H-x}(x+y)f_Y(y)\bar{F}_Z(x+y)dy\right)dx \tag{39}$$

The expected length of a renewal cycle that is completed after a type II failure:

$$E[\tau_{II}] = \int_0^H zf_Z(z)\bar{F}_X(z)dz + \int_0^H f_X(x)\left(\int_x^H zf_Z(z)\bar{F}_Y(z-x)dz\right)dx \tag{40}$$

Thus, the expected length of a cycle:

$$E[\tau] = E[\tau_I] + E[\tau_{II}] + HP_H \tag{41}$$

The expected downtime when replacement occurs after a type I failure:

$$E[D_I] = \int_0^H f_X(x)\left(\int_0^{H-x} y f_Y(y)\bar{F}_Z(x+y)dy\right)dx \tag{42}$$

The expected downtime when replacement occurs after a type II failure:

$$E[D_{II}] = \int_0^H f_X(x)\left(\int_x^H (z-x) f_Z(z)\bar{F}_Y(z-x)dz\right)dx \tag{43}$$

The expected downtime when there is a preventive replacement at H:

$$E[D_H] = \int_0^H (H-x) f_X(x)\bar{F}_Y(H-x)\bar{F}_Z(H)dx \tag{44}$$

Hence, the cost function when there is no inspection turns out to be:

$$Q(H) = \frac{c_F(P_I + P_{II}) + c_M P_H + c_d(E[D_I] + E[D_{II}] + E[D_H])}{E[\tau_I] + E[\tau_{II}] + HP_H} \tag{45}$$

3.2. Model 3: Removal of the potential weak component in all the systems. Age replacement under type II failures

The cost of removal a defective component (c_V) is usually assumed to be greater than the cost of inspection (c_0). However sometimes the component is cheap but the work to inspect it implies complicated operations or disassemblies. Therefore it is highly interesting to study the conditions under which is profitable replacing the critical component, that can be weak or not, with a defect free unit before the system begins to operate. Under this assumption only failures of type II can occur and inspection is not required. The system is renewed after a type II failure or at age H whichever comes first. This is the classical age replacement model.

Model 3 also fits the idea of "Maintenance by a design change" since all the components potentially weak are removed and replaced by good ones. When the proportion of defective units is high due to a problem arising in design, a new design is required. The results show that, in fact, this is the optimal policy.

The length of a renewal cycle:

$$E[\tau] = \int_0^H \bar{F}_Z(z)dz$$

Now the cost of a renewal cycle includes the cost c_I derived from initial corrective action to change the component to a strong one:

$$E[C(\tau)] = c_I + c_M\bar{F}_Z(H) + c_F(1 - \bar{F}_Z(H))$$

Thus

$$Q(H) = \frac{c_I + c_M\bar{F}_Z(H) + c_F(1 - \bar{F}_Z(H))}{\int_0^H \bar{F}_Z(z)dz}$$

It is reasonable to assume that $c_0 < c_I \le c_0 + c_V$.

4. Numerical study

In this section we present the sensitivity analysis for the model presented in this paper and its comparison with the other two age replacement models described in Section 3.

The case study involves the timing belt of a four-stroke engine. Fig. 3 contains an sketch of the system comprising the tensioner and the chain. Many tensioners are hydraulically actuated by using the oil of the lubrication system. If so, a retaining ring keeps the tension of the chain when the engine is off. If it is weakly designed or its production tolerances are poorly controlled, the retainer could fail causing the destruction of the engine when starting the car. In addition, the tensioner also absorbs the dynamic vibrations of the chain by means of a damping chamber with a non-return valve. Valve leakages prevent the tensioner from carrying out this function, causing the chain to skip or at least its early ageing.

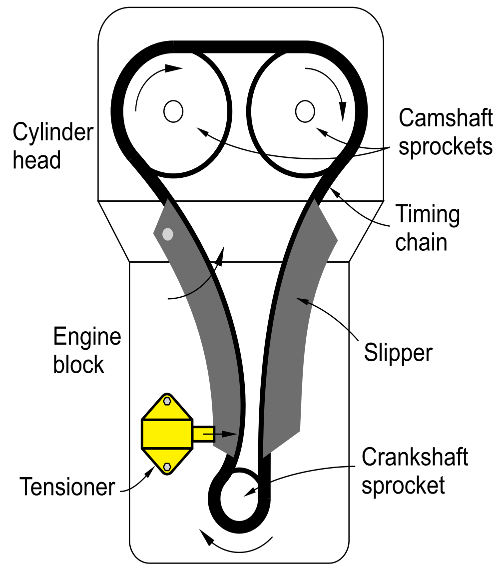

Fig. 3. Sketch of a timing belt.

In what follows X represents the time to defective state of a faulty tensioner. We assume that X follows the mixture given in (1) and therefore, only a small proportion of tensioners are affected by this problem which in turn leads to an early failure of the cambelt. The rest of the tensioners are strong and immune to this problem and hence the corresponding cambelts are only affected by use. The time from defective state of the tensioner to the failure of the system, Y, and the time to failure induced by age (type II), Z, follow Weibull distributions:

$$f_Y(y) = \frac{k_Y}{l_Y}\left(\frac{y}{l_Y}\right)^{k_Y-1} e^{-(y/l_Y)^{k_Y}}, \quad \bar{F}_Y(y) = e^{-(y/l_Y)^{k_Y}}$$

$$f_Z(z) = \frac{k_Z}{l_Z}\left(\frac{z}{l_Z}\right)^{k_Z-1} e^{-(z/l_Z)^{k_Z}}, \quad \bar{F}_Z(z) = e^{-(z/l_Z)^{k_Z}}$$

Without loss of generality, all the results are obtained under the assumption $k_Y = 2$, $k_Z = 1.5$ and $l_Z = 1$. The scale parameter of Z is thus considered the time measurement unit.

The parameters in the base case are given as follows: $l_X = 0.2$, $l_Y = 0.2$, $c_0 = 0.025$, $c_d = 0.1$, $c_V = 0.2$, $c_F = 4$, $c_I = 0.2$ and $c_M = 1$ (reference cost).

The expected values of X, Y and Z in the base case are respectively: $\mu_X = 0.2$, $\mu_Y = 0.177$, $\mu_Z = 0.9$

Next we study the dependence of the optimum policy on the parameters. The tables in this section contain the optimum policy $(N^\star, T^\star, H^\star)$ and the optimum cost $Q_1^\star$ for model 1 under different values of the parameters. In addition they show the corresponding optimum costs $Q_2^\star$ and $Q_3^\star$ for model 2 (age replacement under type I and type II failures) and model 3 (age replacement under type II failures), respectively. Both, the changing parameter and the optimum cost provided by the best model are highlighted in bold.

Table 1 reveals that increasing the inspection cost, c_0, leads to reduce both the number of inspections, $N^\star$, and the inspection frequency, $\frac{1}{T^\star}$. In addition the period of use without inspections, $H^\star$, also increases and so does the potential period of use, $N^\star T^\star + H^\star$. The reduced inspection and the longer usage time can compensate for the extra cost but they result in a higher risk of both types of failure previous to the preventive replacement. Some counterpart such as lower probabilities of inspection error can pay back for the extra cost reducing the chance of failure. There is no reward in return of higher inspection costs otherwise. A similar behavior is observed when the cost

Table 1
Minimum cost-rate policy under different costs , $c_M = 1$, $p = 0.1$, $\alpha = 0.05$, $\beta = 0.05$. First row is the base case.

Case	c_0	c_d	c_V	c_F	N^*	T^*	H^*	N^*T^*	$N^*T^* + H^*$	$Q_1{}^*$	$Q_2{}^*$	$Q_3{}^*$
1	0.025	0.1	0.2	4	3	0.114	0.583	0.343	0.926	**4.366**	4.538	4.424
2	**0.008**	0.1	0.2	4	4	0.097	0.506	0.389	0.895	**4.297**	4.538	4.424
3	**0.014**	0.1	0.2	4	3	0.111	0.574	0.333	0.907	**4.325**	4.538	4.424
4	**0.045**	0.1	0.2	4	2	0.135	0.680	0.269	0.950	**4.419**	4.538	4.424
5	0.025	**0.056**	0.2	4	3	0.114	0.583	0.343	0.926	**4.366**	4.537	4.424
6	0.025	**0.58**	0.2	4	3	0.113	0.587	0.341	0.928	**4.371**	4.548	4.424
7	0.025	**1**	0.2	4	3	0.113	0.591	0.339	0.930	**4.375**	4.558	4.424
8	0.025	**3.2**	0.2	4	3	0.110	0.608	0.330	0.938	**4.395**	4.606	4.424
9	0.025	**10**	0.2	4	4	0.092	0.595	0.368	0.963	4.456	4.752	**4.424**
10	0.025	0.1	**0.36**	4	2	0.132	0.677	0.264	0.941	**4.400**	4.538	4.651
11	0.025	0.1	**0.65**	4	1	0.161	0.804	0.161	0.966	**4.456**	4.538	5.037
12	0.025	0.1	**1.2**	4	1	0.164	0.834	0.164	0.998	**4.523**	4.538	5.714
13	0.025	0.1	**2.1**	4	1	0.174	0.875	0.174	1.049	4.631	**4.538**	6.749
14	0.025	0.1	0.2	**2.2**	1	0.162	1.864	0.162	2.025	**2.562**	2.599	2.654
15	0.025	0.1	0.2	**3**	2	0.132	1.023	0.264	1.287	**3.408**	3.494	3.485
16	0.025	0.1	0.2	**5.4**	3	0.111	0.336	0.333	0.669	**5.556**	5.899	5.603
17	0.025	0.1	0.2	**7.2**	3	0.099	0.210	0.298	0.507	**6.909**	7.521	6.961

Table 2
Minimum cost-rate policy under different parameter values $c_0 = 0.025$, $c_d = 0.1$, $c_V = 0.2$, $c_F = 4$, $c_M = 1$. First row is the base case.

Case	p	α	β	N^*	T^*	H^*	N^*T^*	$N^*T^* + H^*$	$Q_1{}^*$	$Q_2{}^*$	$Q_3{}^*$
1	0.1	0.05	0.05	3	0.114	0.583	0.343	0.926	**4.366**	4.538	4.424
2	**0.031**	0.05	0.05	1	0.168	0.715	0.168	0.883	**4.243**	4.250	4.424
3	**0.056**	0.05	0.05	1	0.163	0.739	0.163	0.902	**4.299**	4.357	4.424
4	**0.14**	0.05	0.05	3	0.111	0.606	0.334	0.940	**4.409**	4.707	4.424
5	**0.155**	0.05	0.05	4	0.099	0.548	0.398	0.946	**4.424**	4.771	4.424
6	0.1	**0.0086**	0.05	3	0.117	0.564	0.351	0.915	**4.347**	4.538	4.424
7	0.1	**0.015**	0.05	3	0.117	0.567	0.350	0.917	**4.350**	4.538	4.424
8	0.1	**0.028**	0.05	3	0.116	0.573	0.347	0.921	**4.356**	4.538	4.424
9	0.1	**0.09**	0.05	2	0.128	0.677	0.255	0.932	**4.378**	4.538	4.424
10	0.1	**0.16**	0.05	2	0.122	0.698	0.244	0.943	**4.396**	4.538	4.424
11	0.1	**0.29**	0.05	2	0.111	0.737	0.222	0.959	**4.424**	4.538	4.424
12	0.1	0.05	**0.0086**	2	0.133	0.656	0.265	0.922	**4.359**	4.538	4.424
13	0.1	0.05	**0.015**	2	0.132	0.658	0.265	0.922	**4.360**	4.538	4.424
14	0.1	0.05	**0.028**	2	0.132	0.660	0.264	0.924	**4.362**	4.538	4.424
15	0.1	0.05	**0.09**	3	0.112	0.594	0.335	0.930	**4.374**	4.538	4.424
16	0.1	0.05	**0.16**	3	0.107	0.613	0.322	0.936	**4.387**	4.538	4.424
17	0.1	0.05	**0.25**	3	0.100	0.648	0.300	0.948	**4.414**	4.538	4.424

of removal a hidden defect, c_V increases. The relevance of inspection is smaller since the ratio c_M/c_V decreases with c_V, and the removal of the hidden defect is less advantageous when compared to the preventive replacement. On the contrary, when the cost of replacement on failure increases, inspection becomes more important and both, the number and frequency of inspection, increase too. In addition the interval without inspection, $H^\star$ is reduced and so does the entire potential time of use $N^\star T^\star + H^\star$. Thus, all the efforts are directed to prevent the system from failing. When comparing with the other two models, model 1 provides the minimum cost (in bold) until c_d exceeds a threshold or the ratio c_M/c_V drops below a limiting value. Replacing the weak unit (model 3) is more profitable when c_d is high whereas neglecting inspection (model 2) can be fruitful when replacing the whole system is significantly cheaper than the removal of a defective component ($c_M \ll c_V$).

Regarding the downtime cost, c_d, it is 32 times greater in case 8 than in case 1 but the optimum policies are quite insensitive to those changes. In case 9 where c_d multiplies by 100 the base case, the results are as expected, that is, more and more frequent inspections are required although $H^\star$ remain robust respect to the base case only increasing by 2% the case base value. Nevertheless for a practical application of inspection models, very high values of c_d are not reasonable. If c_d represents, for example, the cost-rate for the unmet demand, large values would not remain unnoticed for a long period. Then, the maintainer would advert the malfunction of the system without the need of inspections.

In Table 2 the results concerning changes in p, α and β are presented. The larger p, the greater both $N^\star$ and the inspection frequency. The inspection period $N^\star T^\star$ is not strictly monotonic because $T^\star$ decreases but $N^\star$ is not decreasing. Nevertheless $N^\star T^\star$ tends to increase whereas the non-inspection period is reduced. Summarizing, the larger the proportion of weak components the more crucial inspection is.

$N^\star$, $T^\star$ and $H^\star$ are robust under changes in α when inspection is nearly perfect, that is, α below 0.05. For α above 0.05, an increasing α implies less inspections and the inspection period $N^\star T^\star$ also decreases to prevent from unnecessary replacements of robust components. On the contrary $H^\star$ increases to compensate for this over maintenance. When the probability of a false negative increases, so do both, the number of inspections and their frequency to increment the chance to detect the defective state. $N^\star T^\star + H^\star$ increases to compensate for the additional cost incurred due to inspections. Again model 1 is more efficient than the other two, leading to the minimum cost. Thus, inspection is an advantageous procedure in the explored range of the parameters.

The results in Table 3 refer to changes in the scale parameters l_X and l_Y. The value of l_z is assumed to be constant all over the study. Note that when the scale parameter increases, so does the corresponding expected time. In addition a greater value of l_X or l_Y implies a longer mean value of X and Y, respectively. The larger either l_X or l_Y, the inspection is relaxed with less inspections and lower inspection frequency. This result is as expected provided that the system is more reliable and for the same reason the optimum cost, $Q_1^\star$, decreases with l_Y. In particular an increasing l_Y implies that the system is more robust to keep on working while it is in the defective state.

However $Q_1^\star$ is not monotonic with l_X. This is also illustrated in Fig. 5b. The ratio l_X/l_Y explains this pattern. When l_X/l_Y is significantly below 1, the time until the component enters the defective state

Table 3
Minimum cost-rate policy under different scale parameters $c_0 = 0.025$, $c_d = 0.1$, $c_V = 0.2$, $c_F = 4$, $c_M = 1$, $l_Z = 1$, $p = 0.1$, $\alpha = 0.05$, $\beta = 0.05$. First row is the base case.

Case	l_x	l_y	N*	T*	H*	N*T*	N*T* + H*	Q_1*	Q_2*	Q_3*
1	0.2	0.2	3	0.114	0.583	0.343	0.926	**4.366**	4.538	4.424
2	**0.034**	0.2	2	0.057	0.795	0.114	0.909	**4.290**	4.628	4.424
3	**0.062**	0.2	2	0.079	0.762	0.157	0.920	**4.315**	4.611	4.424
4	**0.11**	0.2	2	0.103	0.724	0.206	0.930	**4.344**	4.584	4.424
5	**0.36**	0.2	2	0.159	0.573	0.319	0.892	**4.362**	4.465	4.424
6	**0.65**	0.2	2	0.191	0.481	0.382	0.863	**4.331**	4.371	4.424
7	**1.2**	0.2	1	0.249	0.597	0.249	0.846	**4.280**	4.281	4.424
8	**1.3**	0.2	1	0.255	0.591	0.255	0.846	4.274	**4.271**	4.424
9	**1.5**	0.2	1	0.267	0.578	0.267	0.845	4.263	**4.254**	4.424
10	0.2	**0.034**	3	0.036	0.916	0.108	1.025	4.565	4.620	**4.424**
11	0.2	**0.062**	3	0.056	0.827	0.167	0.994	4.499	4.606	**4.424**
12	0.2	**0.11**	3	0.081	0.720	0.242	0.962	4.432	4.581	**4.424**
13	0.2	**0.36**	2	0.175	0.532	0.351	0.882	**4.302**	4.467	4.424
14	0.2	**0.65**	1	0.265	0.585	0.265	0.850	**4.246**	4.333	4.424
15	0.2	**1.2**	1	0.300	0.559	0.300	0.859	**4.209**	4.210	4.424
16	0.2	**1.3**	1	0.308	0.552	0.308	0.860	4.206	**4.199**	4.424
17	0.2	**1.5**	1	0.329	0.534	0.329	0.862	4.202	**4.183**	4.424

is smaller than the time it remains in this state before the system fails. Even though the component is weak the failure is not imminent after the defect. As l_X increases and the ratio l_X/l_Y increases approaching to 1, the larger mean time to defect $E[X]$ induces a less frequent inspection and this greater value of $T^\star$ offers less chance to detect the defective state in the same delay-time. The reduction of the delay-time relative to l_X implies a higher risk of a type I failure and thus the optimum cost increases (cases 1–4). The relevance of the relative value of l_X to l_Y can also be noticed since $Q_1^\star$ is smaller in cases 2, 3, and 4 where l_X/l_Y is less than one than in cases 10 and 11 and 12 for which the corresponding values of l_X and l_Y are interchanged.

When l_X/l_Y is greater than 1, the component enters later the defective state and the optimum cost starts to decrease (cases 6–9). However, once defective, the failure occurs soon and inspection appear to be less useful. In fact model 2 without inspection becomes the optimum procedure in cases 8 and 9. Cases 10, 11 and 12 where $l_Y < l_X$ also support this idea since model 3, initial substitution of the component being it defective or not, presents the lower optimum cost. This is so because the delay-time is so short that there is not enough time to carry out inspections to detect the defective states of the critical component previous to failure.

Condition $l_X > 1.2$ (cases 8 and 9) and $l_Y > 1.2$ (cases 16 and 17) makes $X + Y$ comparable to Z and therefore the type I failure looses its characteristic of early failure to turn into a second failure due to wear-out. This long values, leading to model 2 as the best one, are represented in Fig. 5a and b with the shaded gray area. The numerical study in this paper shows that it is worth implementing an inspection model as long as the delay-time is long enough. Zhang et al. (2014) indicate that current manufacturing presents such high quality levels that a defective product can function under aggravated operational conditions. This statement provide an additional value to our approach based on inspections.

Fig. 5b also shows a parallel behavior of the time of preventive maintenance $N^\star T^\star + H^\star$, first increasing to compensate for the higher cost incurred, and then decreasing when type I is no longer an early failure but a second failure due to age. The smaller $N^\star T^\star + H^\star$ implies an earlier preventive replacement which is the best way to avoid the long-term failures.

5. Range of application of model 1

The study of this section focuses on the analysis of the region of the parameters where model 1 outperforms models 2 and 3 with the aim to provide a practical guide for actual maintenance. The shaded area in the figures indicates the intervals where models 2 or 3 are cheaper alternatives than model 1.

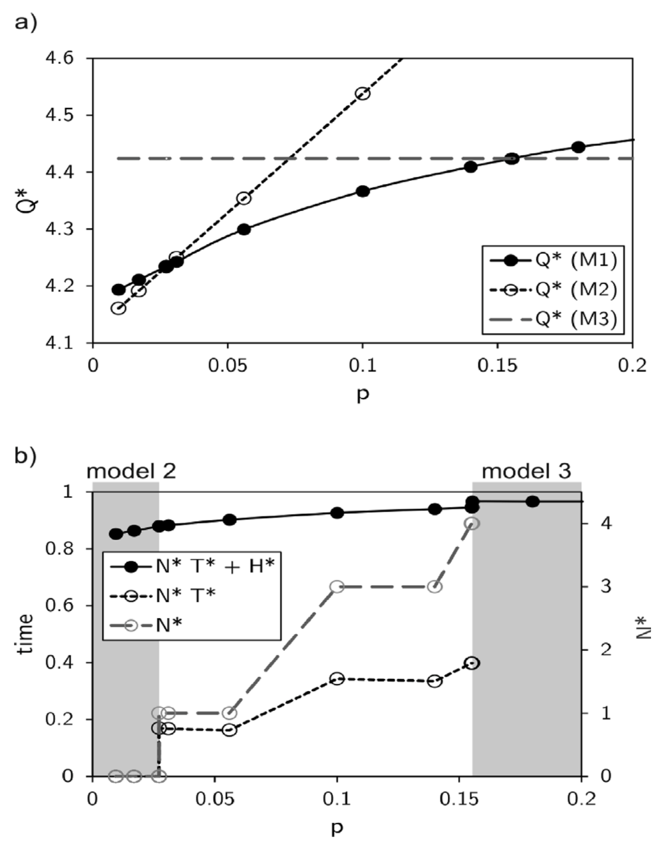

Fig. 4. Comparison of models 1, 2 and 3 under different values of p.

Fig. 4a describes the corresponding costs of the three models when the proportion, p, of weak components changes. When the proportion of weak components is really small ($p < 1\%$), model 2 provides the lower optimum cost, implying that inspection is not necessary. Such a small p would occur due to an accident or brief affair during the production process. If so, the maintainer could neglect the problem of early failures. Nevertheless ignoring this event for larger values of p ($p > 0.05$) can result in larger costs. It is reasonable that $p > 0.15$ can be the result of a poor design, or a deficient refurbishing and for these cases the assumption of model 3, investment in a new design or an improvement in the refurbishing, to obtain stronger components is the best choice. Therefore inspection to detect hidden defects is profitable for $1\% < p < 15\%$. These results suggest that in general there exist p_0

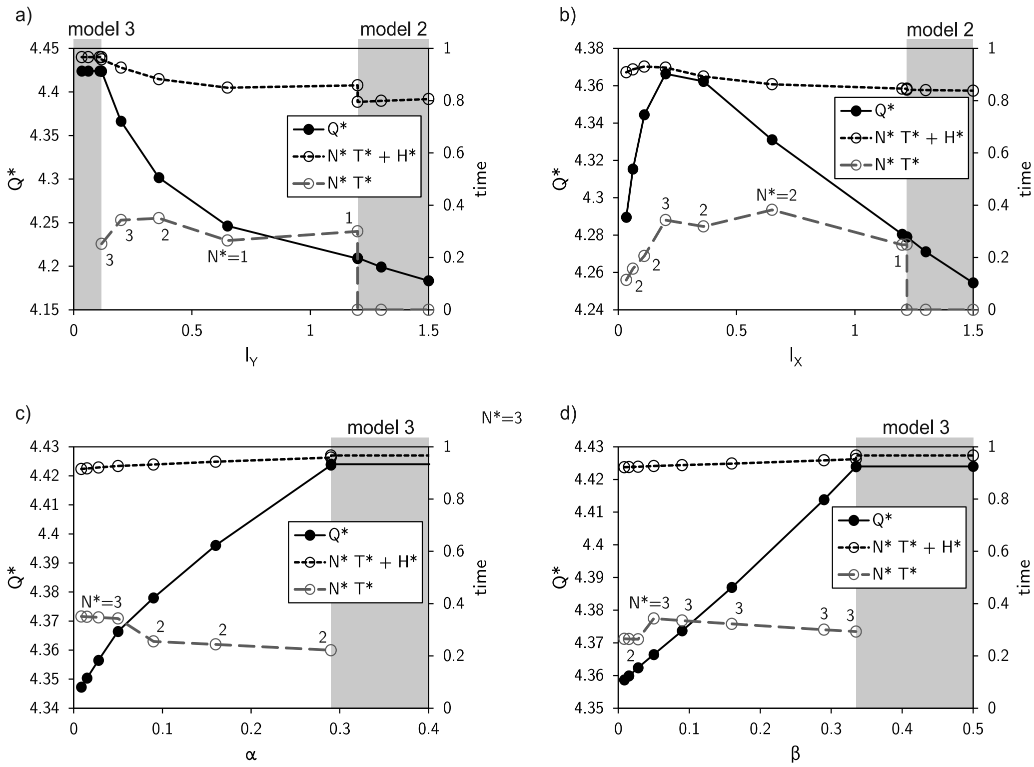

Fig. 5. Comparison of models 1, 2 and 3 under different values of l_X, l_Y, α and β.

and p_1, depending on the rest of the parameters, such that if $p_0 < p < p_1$, then model 1 outperforms models 2 and 3.

Fig. 4b contains the time for preventive maintenance, $N^\star T^\star + H^\star$, for the three models ($H^\star$ in the case of models 2 and 3). When p is small enough to ignore inspection, an slightly earlier preventive replacement serves as protection. On the contrary, in those cases where model 3 applies, the removal of weak components lead to a small increase in $N^\star T^\star + H^\star$, that is a postponed preventive maintenance. For $1\% < p < 15\%$, in the central zone of the graph where model 1 is better than the other two, $N^\star T^\star$ approaches to $N^\star T^\star + H^\star$ as p increases. In practice this means that the more likely weak units are the longer the inspection period has to be.

Fig. 5a, reflecting the effect of changes in the scale factor of the delay-time indicates as before that small values of l_Y associated with an immediate type I failure after the defective state are not appropriate to apply an inspection policy since there is not enough time for defect detection. If so, the initial change of the component in model 2 is more suitable. When l_Y is very large, the system can work while the critical component is defective for a long time. Hence inspection is not required since there is no longer a problem of early failures but a second source of failures due to use. Therefore the age replacement proposed in model 2 provides the optimum policy.

Fig. 5c and d reveal the importance of the quality of inspections. In fact if the probability of false positives or false negatives is beyond the 30%, there is no point in inspecting the system and model 3 is more profitable. $N^\star T^\star$ decreases with α and it is not monotonic with β, although the inspection period also tends to decrease for $\beta > 0.05$.

The effect of changes in the cost is described in Fig. 6. Fig. 6a and d indicate that model 1 outperforms the other two in a wide range of values of the cost of failure and the downtime cost. Considering the interval of application for p ($1\% < p < 15\%$) and α and β below 30%, inspection along with preventive maintenance turns out to be the most economic way to maintain a system like this. Fig. 6b reveals that it is profitable to replace the component, being it weak or not, by another one free or defects or correctly refurbished before the system starts to operate in those cases where the incurred cost of this operation is very low. According to Fig. 6c, the inspection period tends to decrease with c_0 with model 3 being the better alternative when the inspection costs are significantly high.

6. Conclusions

We model the inspection and maintenance of a system with a weak critical component that can cause the early failure of the system. A faulty tensioner in the timing belt of a four-stroke engine serves as a case study. When components are new, weak units are consequence of an undersized design or a defective production. In the case of recycled units, fragile parts can be the result of a low quality refurbishing. A novel modeling of the mixture of weak and strong components which are immune to early failures by means of a defective distribution reduces the number of parameters in the model. In addition we propose a new use of the delay-time concept to model the early failure of a system caused by a critical component. We consider an initial period of inspection during the running-in period to detect weak units as an alternative to burn-in. One of the novelties of the model is that once a positive inspection occurs the defective component is removed and so is the risk of an early failure. There is no other rejuvenation in the rest of the system. This assumption is significantly different from minimal repair and mimics real maintenance when failed or defective units are replaced by new ones instead of being repaired. This model is an approach to used systems containing new parts. Inspections are not perfect with false positives leading to an unnecessary component replacement and false negatives that fail to detect actual defects, increasing the risk of a type I failure. We also consider the possibility that a cost is incurred while a defect remains undetected. We also take into account failures occurring as the system gets older or degrades and a preventive replacement is scheduled to avoid them. The results in this paper give insight about the balance between the length of the inspection period and the time for preventive maintenance. We also analyze the conditions under which this policy that combines inspection and maintenance is advantageous when compared to those

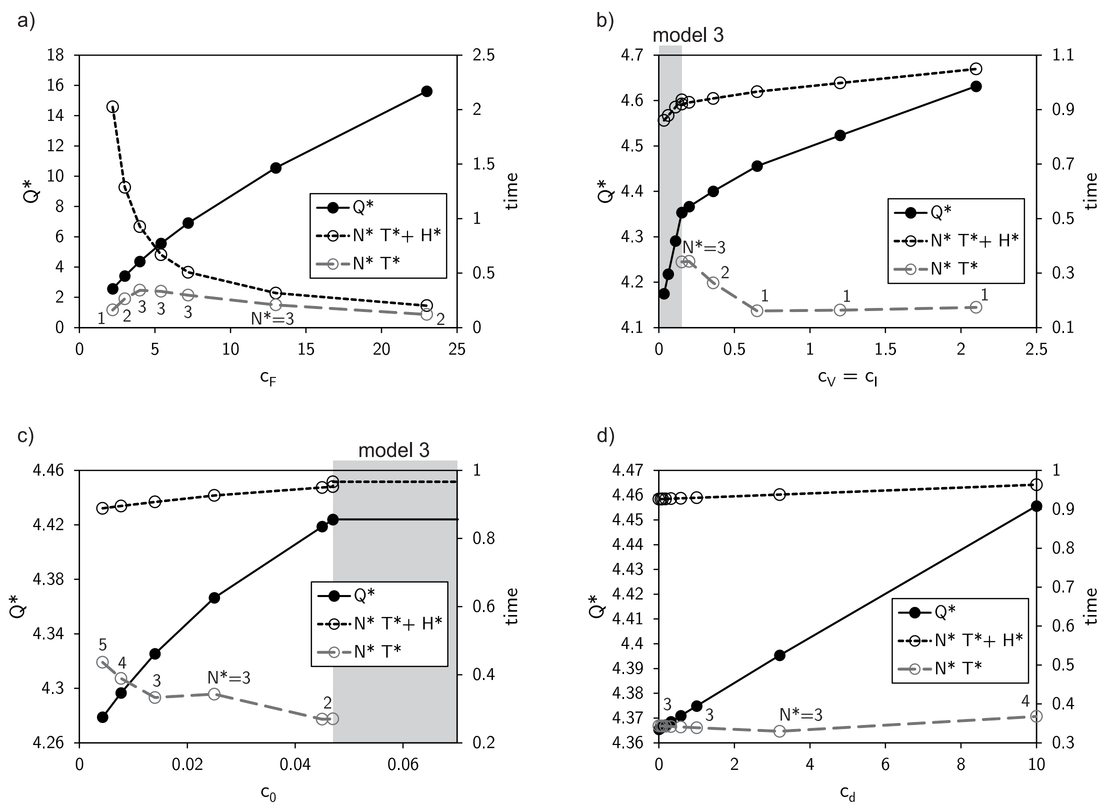

Fig. 6. Comparison of models 1, 2 and 3 under different values of the costs.

limited to preventive age replacement. The study provides a guide for maintainers.

The results in the cases considered show that if the proportion of weak units, p, is very low, early failures are so unlikely that the minimum cost is obtained by an age replacement policy without inspections. On the contrary if that proportion is over a threshold, weak units are more extended as a result of a problem in design or manufacturing and, thus, the initial replacement of the component is profitable. This analysis suggests that in general there exist, p_0 and p_1 dependent on the rest of the parameters, that make initial inspection and preventive maintenance the optimum strategy for $p_0 < p < p_1$. Nevertheless, in order to implement an inspection policy, it is preferable a system that can keep on working for a long time with a critical component that becomes defective early than the opposite case. That is, inspection is less profitable when the defect arises later but the system fails immediately afterwards. As far as we know this study of the ratio of the two random stages comprising the delay-time model is also new.

Regarding future research, an interesting analysis follows if the assumption of independence between type I and type II failures is dropped. Stochastically dependent failures are present for example in a system consisting of a filter and a bypass valve. Thus, there are further applications of the delay-time model where the time to defect occurs in a critical component and the delay time accounts for the catastrophic time to failure of the whole system.

CRediT authorship contribution statement

M.D. Berrade: Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Supervision, Project administration, Funding acquisition. **E. Calvo:** Conceptualization, Methodology, Formal analysis, Investigation, Software, Validation. **F.G. Badía:** Conceptualization, Methodology, Formal analysis, Investigation, Funding acquisition.

Data availability

No data was used for the research described in the article.

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