Topology and its Applications 339 (2023) 108576

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

On the Borromean arithmetic orbifolds

María Teresa Lozano [a,*,1], José María Montesinos-Amilibia [b]

[a] IUMA, Departamento de Matemáticas, Universidad de Zaragoza, Zaragoza 50009, Spain
[b] Departamento de Geometría y Topología, Universidad Complutense, Madrid 28040, Spain

ARTICLE INFO

Article history:
Received 25 July 2022
Received in revised form 14 September 2022
Accepted 2 May 2023
Available online 16 May 2023

Dedicated to Professor José Manuel Rodríguez Sanjurjo in his 70th birthday

MSC:
57M50
57M25
57M60
11E57
11H56

Keywords:
Arithmetic group
Orbifold
Knots
Borromean link

ABSTRACT

We revisit the fundamental groups G_{mnp} of the orbifolds B_{mnp}, where the underlying manifold is the 3-sphere S^3 and the Borromean rings are the singular set with isotropies of order m, n and p. We correct an omission in [2] and show that G_{mnp} is arithmetic if and only if (m, n, p) is one of the 12 triples $(3, 3, 3)$, $(3, 3, \infty)$, $(3, 4, 4)$, $(3, 4, \infty)$, $(3, 6, 6)$, $(3, \infty, \infty)$, $(4, 4, 4)$, $(4, 4, \infty)$, $(4, \infty, \infty)$, $(6, 6, 6)$, $(6, 6, \infty)$, (∞, ∞, ∞). The main purpose of the paper is to present each G_{mnp}, arithmetic, as a group of 4×4 matrices with entries in the ring of integers of a totally real number field K, and which are automorphs of a quaternary form F with entries in K of Sylvester type $(+, +, +, -)$.

1. Introduction

Assume F is a quaternary quadratic form: F is a symmetric 4×4 matrix with determinant different from 0. Assume that the entries of F belong to a totally real number field K. If F^σ is definite for every embedding σ of K in $\mathbb{R}$ different from the identity, but F itself is of Sylvester type $\langle +1, +1, +1, -1 \rangle$ (diagonal matrix), then the group

$$Aut^+ F = \left\{ U \in SL(4, K_0) \, : \, U^T F U = F \right\},$$

* Corresponding author.
E-mail address: tlozano@unizar.es (M.T. Lozano).

[1] M.T. Lozano partially supported by grants PID2020-114750GB-C31 and MTM2016-76868-C2-2-P and Gobierno de Aragón (Grupo de referencia Algebra y Geometría) cofunded by Feder 2014-2020 "Construyendo Europa desde Aragón".

https://doi.org/10.1016/j.topol.2023.108576

where K_0 is the ring of algebraic integers of K, acts proper and discontinuously on the Klein model of hyperbolic 3-space $\mathbb{H}^3$ defined as the interior of the quadric $x^T F x = 0$. The group Aut^+F is the group of orientation preserving automorphs of the form F, and $\mathbb{H}^3/Aut^+F$ is an orientable hyperbolic orbifold, complete and of finite volume.

Any subgroup of $SL(4,\mathbb{R})$ commensurable with such an Aut^+F is called *real-arithmetic*. In other words $G \leq SL(4,\mathbb{R})$ is real-arithmetic if and only if there is a basis b of $\mathbb{R}^4$ such that the elements of G, written in the basis b, are matrices with their entries in a totally real number field K, and, moreover, there exists a finite index subgroup G_0 of G such that $G_0 \leq Aut^+F$ is of finite index, where F is a form defined as above.

If such a thing happens we have proved in [5] that the basis b can be chosen in such a way that G itself is a subgroup of finite index of some $\hat{F}$ as defined above, though not necessarily $\hat{F}$ is the same as F ($\hat{F} = F$ if K_0 is a ring of principal ideals).

We remark that the orbifold $\mathbb{H}^3/G$ is an orbifold covering of finite degree of the orbifold $\mathbb{H}^3/Aut^+F$. The orbifold $\mathbb{H}^3/G$ is therefore complete and of finite volume. Moreover, it is known that $\mathbb{H}^3/G$ is compact if and only if either $K \neq \mathbb{Q}$, or if $K = \mathbb{Q}$, then F fails to represent 0 over $\mathbb{Q}$.

In this paper we are interested in the orbifolds (hyperbolic) B_{mnp}, where the underlying manifold is the 3-sphere S^3, and the Borromean rings are the singular set and where the angles at the three components are $(\frac{2\pi}{m}, \frac{2\pi}{n}, \frac{2\pi}{p})$, where $m, n, p \in \mathbb{Z} \bigcup \{\infty\}$ are all > 2.

The group G_{mnp} acting in $\mathbb{H}^3$ and yielding $\mathbb{H}^3/G_{mnp} = B_{mnp}$ has been studied in [2] and a system of generators were given there.

We also have determined in [2] what groups G_{mnp} are real arithmetic (with an omission that we remedy here), but we did not present each group G_{mnp} as a finite index subgroup of Aut^+F for some quadratic form F as above.

Using the methods of [5] we obtain here these presentations of G_{mnp} as group of matrices with entries in the ring of integers of an appropriate number field K in case that G_{mnp} is real-arithmetic.

Note that K must be $\mathbb{Q}$ in case that at least one of m, n, p is ∞, since then B_{mnp} has finite volume, it is complete but not compact.

2. The group G_{mnp}

Assume m, n, p are integers > 2. Then the hyperbolic orbifold B_{mnp} is obtained from identification on the faces of a pyritohedron (a hyperbolic dodecahedron with hyperbolic right angles for all the dihedral angles but for three pairs if opposite edges which have angles $(\frac{2\pi}{m}, \frac{2\pi}{n}, \frac{2\pi}{p})$). The identification is made by the rotations g_m, g_n, g_p on these edges (see for example [6], [1], [2]). Therefore the fundamental group G_{mnp} of the hyperbolic orbifold B_{mnp} is generated by g_m, g_n, g_p. Using the Beltrami-Klein model of $\mathbb{H}^3$ and an analogous computation as the one made in [2], one obtains the following expression for the generators of the group $G_{mnp} \leq SO(4,\mathbb{R})$

$$g_m = \frac{1}{C(A-1)+1}\begin{pmatrix} -AC-C+1 & 0 & -2A & 2AC \\ 0 & AC-C+1 & 0 & 0 \\ 2C & 0 & -AC+C+1 & -2C \\ -2C & 0 & -2 & AC+C+1 \end{pmatrix}$$

$$g_n = \frac{1}{A(B-1)+1}\begin{pmatrix} -BA+A+1 & 2A & 0 & -2A \\ -2B & -BA-A+1 & 0 & 2BA \\ 0 & 0 & BA-A+1 & 0 \\ -2 & -2A & 0 & BA+A+1 \end{pmatrix}$$

$$g_p = \frac{1}{B(C-1)+1}\begin{pmatrix} CB-B+1 & 0 & 0 & 0 \\ 0 & -CB+B+1 & 2B & -2B \\ 0 & -2C & -CB-B+1 & 2CB \\ 0 & -2 & -2B & CB+B+1 \end{pmatrix}$$

where A, B, C are positive numbers such that

$$C(A-1) = \tan^2\frac{\pi}{m}$$
$$A(B-1) = \tan^2\frac{\pi}{n}$$
$$B(C-1) = \tan^2\frac{\pi}{p}$$

The matrices g_m, g_n, g_p are automorphs of the quadratic form

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} - t^2$$

3. The arithmetic groups G_{mnp}

It was shown in [2, Proposition 4.2] that if $G_{m,n,p}$ is quasiarithmetic, then $m, n, p \in \{3, 4, 6, \infty\}$. Moreover it was shown in [5] that G_{mnp} with $m, n, p \in \{3, 4, 6, \infty\}$ is integral if and only if m, n, p is one of the 12 triples quoted in the abstract. Hence it follows that precisely these 12 orbifolds are the arithmetics ones.

4. Presentation of G_{mnp} arithmetic as groups of matrices with entries in the ring of integers of a number field

The entries of the generators g_m, g_n, g_p belong to the field $\mathbb{Q}(A, B, C)$. Denoting by R_{mnp} the ring of integers of the field, we consider the R_{mnp}-module L_{mnp} generated by the columns of the matrices I, g_m, g_n, g_p, where I is the identity matrix. In all cases we will see that L_{mnp} is generated by a base b_{mnp} formed by 4 vectors lineally independent over R_{mnp}, and L_{mnp} will be invariant under the action of the group G_{mnp}. This will imply that the matrices of the elements of G_{mnp} in the base b_{mnp} will have all its entries in R_{mnp} which is our final goal.

Claim 4.1. *Let e_1, e_2, e_3, e_4 denote the columns of I and let Z be the R_{mnp}-module generated by e_1, e_2, e_3, e_4. We note that the columns of g_m, g_n, g_p contain the vectors e_1, e_2, e_3. We also observe that adding the first and fourth columns of g_m we obtain the vector $(1, 0, 0, 1)^T$; and similarly, that adding the second (resp. third) and fourth columns of g_n (resp. g_p) we obtain the vector $(0, 1, 0, 1)^T$ (resp. $(0, 0, 1, 1)^T$). This implies that L_{mnp} is generated by Z together with the columns*

$$3 \quad and \quad 1\, or 4 \quad of \quad g_m;$$
$$1 \quad and \quad 2\, or 4 \quad of \quad g_n;$$
$$2 \quad and \quad 3\, or 4 \quad of \quad g_p.$$

Finally we remark that if any of A, B, C is already an algebraic integer, we can simplify even more. Indeed the first column of g_m minus C times the third yields:

$$\frac{1}{C(A-1)+1}\left\{\begin{pmatrix}-AC-C+1\\0\\2C\\-2C\end{pmatrix}-C\begin{pmatrix}-2A\\0\\-AC+C+1\\-2\end{pmatrix}\right\}=\begin{pmatrix}1\\0\\C\\0\end{pmatrix}$$

Hence if C (resp. A, B) is integral we can get rid of the column 1 (resp. 2, 3) of g_m (resp. g_n, g_p).

4.1. The group G_{333}

Here $A = B = C = (1+\sqrt{13})/2$ are all algebraic integers. The module L_{333} over the ring $\mathbb{Q}(\sqrt{13})_0$ of integers of the field $\mathbb{Q}(\sqrt{13})$ is generated by Z and the columns 3, 1 and 2 of g_m, g_n, g_p respectively:

$$v_m=\begin{pmatrix}-(1+\sqrt{13})/4\\0\\-1/2\\-1/2\end{pmatrix},v_n=\begin{pmatrix}-1/2\\-(1+\sqrt{13})/4\\0\\-1/2\end{pmatrix},v_p=\begin{pmatrix}0\\-1/2\\-(1+\sqrt{13})/4\\-1/2\end{pmatrix}$$

Now we have:

$$e_4=-e_1-2v_n-\frac{1}{2}(1+\sqrt{13})e_2$$
$$e_3=-e_4-2v_m-\frac{1}{2}(1+\sqrt{13})e_1$$

Hence L_{333} is generated by $\{e_1,e_2,v_m,v_n,v_p\}$. Now

$$xv_m+yv_n+zv_p+ue_1+ve_2=0$$

is true for $x=-(1+\sqrt{13})/2$, $y=(-1+\sqrt{13})/2$, $z=1$, $u=-2$, $v=2$. Hence L_{333} is free with base $\{v_m,v_n,e_1,e_2\}$.

$$T_3(e_1,e_2,e_3,e_4)=(e_1,e_2,-v_m.-v_n)$$

Denoting $\frac{1}{2}(1+\sqrt{13})$ by ϕ, we have

$$T_3=\begin{pmatrix}1&0&\phi/2&1/2\\0&1&0&\phi/2\\0&0&1/2&0\\0&0&1/2&1/2\end{pmatrix};$$

and

$$T_3^{-1}g_mT_3=\begin{pmatrix}-2&0&1-\phi&0\\3+\phi&1&\phi&0\\\phi&0&1&0\\-2\phi&0&-2&-1\end{pmatrix}$$

$$T_3^{-1}g_nT_3=\begin{pmatrix}0&\phi&-\phi&1\\0&1&0&0\\0&0&1&0\\-1&-\phi&0&-1\end{pmatrix}$$

$$g'_p(G_{333}) = T_3^{-1} g_p T_3 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 0 & -\phi \\ 0 & -\phi & 1 & 0 \\ 0 & -1+\phi & 0 & 1 \end{pmatrix}$$

The quadratic form in the new basis is

$$\begin{pmatrix} 2 & 0 & \phi & 1 \\ 0 & 2 & 0 & \phi \\ \phi & 0 & 2 & 0 \\ 1 & \phi & 0 & 2 \end{pmatrix}$$

It becomes definite by sending $\sqrt{13}$ to $-\sqrt{13}$.

4.2. The group $G_{33\infty}$

Here $A = 4$, $B = 7/4$, $C = 1$. Then $L_{33\infty}$ is generated by Z; the column 3 of g_m; 1 of g_n; and 2 and 3 of g_p. The column 2 of g_p is integral and we can suppress it. It is easy to see that $L_{33\infty}$ has basis $\{e_1, -v_n, e_3, v_p - 2e_1 - e_2\}$. Hence

$$T = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 7/8 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}, \text{ and } \quad T^{-1} g_m T = \begin{pmatrix} -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 \end{pmatrix},$$

$$T^{-1} g_n T = \begin{pmatrix} 0 & 1 & 0 & -2 \\ -1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T^{-1} g_p T = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 2 & -11 & 1 \end{pmatrix}$$

and the quadratic form is

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & -2 \\ 0 & 0 & 8 & 4 \\ 0 & -2 & 4 & 0 \end{pmatrix}$$

4.3. The group G_{443}

Here $A = \frac{1}{2}(1+\sqrt{3})$, $B = \sqrt{3}$, $C = 1+\sqrt{3}$, where B and C are algebraic integers. Here $\{v_m, e_2, e_3, v_p\}$ is a basis of L_{443}. We have

$$T = \begin{pmatrix} (1+\sqrt{3})/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & (1+\sqrt{3})/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}, \quad \text{and}$$

$$T^{-1} g_m T = \begin{pmatrix} -1-\sqrt{3} & 0 & -1 & 0 \\ 2+\sqrt{3} & 1 & 1 & 0 \\ 7+4\sqrt{3} & 0 & 1+\sqrt{3} & -1-\sqrt{3} \\ -2(2+\sqrt{3}) & 0 & -2 & 1 \end{pmatrix},$$

$$T^{-1}g_nT = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2+\sqrt{3} & 2+\sqrt{3} & 1 & 0 \\ -1-\sqrt{3} & -1-\sqrt{3} & 0 & 1 \end{pmatrix},$$

$$T^{-1}g_pT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -\sqrt{3} & -1 \end{pmatrix}$$

And the quadratic form is

$$\begin{pmatrix} (1+\sqrt{3})/2 & 0 & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 & 1/2\sqrt{3} \\ 0 & 0 & 1/(1+\sqrt{3}) & 1/2 \\ 0 & 1/2\sqrt{3} & 1/2 & 1/\sqrt{3} \end{pmatrix}$$

which becomes definite by replacing $-\sqrt{3}$ by $\sqrt{3}$.

4.4. The group $G_{34\infty}$

Here $A = 4$, $B = 5/4$, $C = 1$.

$$T = \begin{pmatrix} 2 & 0 & 0 & 3 \\ 0 & 5/4 & 0 & -1 \\ 1/2 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and}$$

$$T^{-1}g_mT = \begin{pmatrix} -1 & 1 & -1 & -3 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T^{-1}g_nT = \begin{pmatrix} -31 & 8 & 0 & -56 \\ 16 & -4 & 0 & 29 \\ 16 & -4 & 1 & 28 \\ 20 & -5 & 0 & 36 \end{pmatrix}$$

$$T^{-1}g_pT = \begin{pmatrix} 1 & -60 & 135 & -60 \\ 0 & 31 & -70 & 32 \\ 0 & 30 & -69 & 32 \\ 0 & 40 & -90 & 41 \end{pmatrix}$$

And the quadratic form is

$$\begin{pmatrix} 20 & -10 & 10 & 30 \\ -10 & 5 & 0 & -20 \\ 10 & 0 & 20 & 0 \\ 30 & -20 & 0 & 61 \end{pmatrix}$$

4.5. The group G_{366}

Here $A = \frac{1}{2}(3+\sqrt{13})$, $B = \frac{1}{6}(3+\sqrt{13})$, $C = \frac{1}{2}(-1+\sqrt{13})$. A and C are algebraic integers.

$$T = \begin{pmatrix} (3+\sqrt{13})/4 & -1/2 & 0 & 0 \\ 0 & (3+\sqrt{13})/4 & 1 & 0 \\ 1/2 & 0 & 0 & 0 \\ 1/2 & 3/2 & 0 & 1 \end{pmatrix}, \quad \text{and}$$

$$T^{-1}g_mT = \begin{pmatrix} 1 & 1-\sqrt{13} & 0 & (3+\sqrt{13})/2 \\ 3+\sqrt{13} & -9-2\sqrt{13} & 0 & -5-\sqrt{13} \\ (-11-3\sqrt{13})/2 & 14+4\sqrt{13} & 1 & 7+2\sqrt{13} \\ (-11-3\sqrt{13})/2 & 14+4\sqrt{13} & 0 & 8+2\sqrt{13} \end{pmatrix},$$

$$T^{-1}g_nT =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3+\sqrt{13} & -1 & (-9-3\sqrt{13})/2 & (9+3\sqrt{13})/2 \\ (-11-3\sqrt{13})/2 & (3+\sqrt{13})/2 & (13+3\sqrt{13})/2 & (-11-3\sqrt{13})/2 \\ (-9-3\sqrt{13})/2 & 3 & (9+3\sqrt{13})/2 & (-7-3\sqrt{13})/2 \end{pmatrix},$$

$$T^{-1}g_pT = \begin{pmatrix} 1 & 0 & (3-3\sqrt{13})/2 & (5+\sqrt{13})/2 \\ 0 & 1 & (-15-3\sqrt{13})/2 & 7+2\sqrt{13} \\ 0 & (-3-\sqrt{13})/2 & 11+3\sqrt{13} & (-25-7\sqrt{13})/2 \\ 0 & 0 & 9+3\sqrt{13} & 10-3\sqrt{13} \end{pmatrix}$$

And the quadratic form is

$$\begin{pmatrix} (1+\sqrt{13})/6 & -1 & 0 & -1/2 \\ -1 & (-3+\sqrt{13})/2 & 3/2 & -3/2 \\ 0 & 3/2 & 3(-3+\sqrt{13})/2 & 0 \\ -1/2 & -3/2 & 0 & -1 \end{pmatrix}$$

4.6. The group $G_{3\infty\infty}$

Here $A = 4$, $B = 1$, $C = 1$.

$$T = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}, \quad \text{and}$$

$$T^{-1}g_mT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & -1 \end{pmatrix},$$

$$T^{-1}g_nT = \begin{pmatrix} 9 & 40 & 0 & -4 \\ -2 & -7 & 0 & 0 \\ 2 & 8 & 1 & 0 \\ -4 & -16 & 0 & 1 \end{pmatrix},$$

$$T^{-1}g_pT = \begin{pmatrix} 1 & 8 & 8 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & -4 & 1 \end{pmatrix}$$

And the quadratic form is

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 2 & 0 & 2 & 4 \end{pmatrix}$$

4.7. The group G_{444}

This group, G_{444}, is the universal group studied in [1].
Here $A = B = C = (1+\sqrt{5})/2 = \varphi$ which is an algebraic integer, the golden ratio.

$$g_m = \begin{pmatrix} -\varphi & 0 & -\varphi & \varphi+1 \\ 0 & 1 & 0 & 0 \\ \varphi & 0 & 0 & -\varphi \\ -\varphi & 0 & -1 & \varphi+1 \end{pmatrix},$$

$$g_n = \begin{pmatrix} 0 & \varphi & 0 & -\varphi \\ -\varphi & -\varphi & 0 & \varphi+1 \\ 0 & 0 & 1 & 0 \\ -1 & -\varphi & 0 & \varphi+1 \end{pmatrix},$$

$$g_p(G_{444}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \varphi & -\varphi \\ 0 & -\varphi & -\varphi & \varphi+1 \\ 0 & 1 & -\varphi & \varphi+1 \end{pmatrix}$$

And the quadratic form is

$$\langle 1, 1, 1. - \varphi \rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\varphi \end{pmatrix}$$

4.8. The group $G_{44\infty}$

Here $A = 2$, $B = 3/2$, $C = 1$.

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and}$$

$$T^{-1}g_mT = \begin{pmatrix} -11 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 \\ -12 & 1 & -1 & 2 \end{pmatrix},$$

$$T^{-1}g_nT = \begin{pmatrix} 0 & 1 & 0 & -2 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T^{-1}g_pT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -2 & 3 \\ 0 & 2 & -5 & 6 \end{pmatrix}$$

And the quadratic form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & -2 \end{pmatrix}$$

4.9. The group $G_{4\infty\infty}$

Here $A = 2$, $B = 1$, $C = 1$.

$$g_m = \begin{pmatrix} -1 & 0 & -2 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix},$$

$$g_n = \begin{pmatrix} 0 & 4 & 0 & -4 \\ -2 & -3 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ -2 & -4 & 0 & 5 \end{pmatrix},$$

$$g_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & -2 & -1 & 2 \\ 0 & -2 & -2 & 3 \end{pmatrix}$$

And the quadratic form is

$$\langle 1, 2, 2, -2 \rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

4.10. The group G_{666}

Here $A = B = C = (3 + \sqrt{21})/6$. We consider the basis $(-v_m, -v_n, e_2, e_4)$ of L_{666}.

$$T_6 = \begin{pmatrix} (3+\sqrt{21})/4 & -1/2 & 0 & 0 \\ 0 & (3+\sqrt{21})/4 & 1 & 0 \\ -1/2 & 0 & 0 & 0 \\ 3/2 & 3/2 & 0 & 1 \end{pmatrix}, \quad \text{and}$$

$$T_6^{-1} g_m T_6 = \begin{pmatrix} -1 & 3+\sqrt{21} & 0 & (3+\sqrt{21})/2 \\ -3-\sqrt{21} & 11+2\sqrt{21} & 0 & 5+\sqrt{21} \\ (15+3\sqrt{21})/2 & -18-4\sqrt{21} & 1 & -9-2\sqrt{21} \\ (15+3\sqrt{21})/2 & -18-4\sqrt{21} & 0 & -8-2\sqrt{21} \end{pmatrix},$$

$$T_6^{-1} g_n T_6 =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3+\sqrt{21} & -1 & (-3-\sqrt{21})/2 & (3+\sqrt{21})/2 \\ (-15-3\sqrt{21})/2 & (3+\sqrt{21})/2 & (7+\sqrt{21})/2 & (-5-\sqrt{21})/2 \\ (-9-3\sqrt{21})/2 & 3 & (3+\sqrt{21})/2 & (-1-\sqrt{21})/2 \end{pmatrix},$$

$$g'p(G_{666}) = T_6^{-1} g_p T_6 =$$

$$= \begin{pmatrix} -4-\sqrt{21} & 0 & (3+\sqrt{21})/2 & (-5-\sqrt{21})/2 \\ -11-4\sqrt{21} & 1 & (15+3\sqrt{21})/2 & -9-2\sqrt{21} \\ 33+7\sqrt{21} & (-3-\sqrt{21})/2 & -13-3\sqrt{21} & (33+7\sqrt{21})/2 \\ 36+8\sqrt{21} & 0 & -15-3\sqrt{21} & 19+4\sqrt{21} \end{pmatrix}$$

And the quadratic form is

$$\begin{pmatrix} (-3+\sqrt{21})/2 & -3 & 0 & -3/2 \\ -3 & (-3+\sqrt{21})/2 & 3/2 & -3/2 \\ 0 & 3/2 & (-3+\sqrt{21})/2 & 0 \\ -3/2 & -3/2 & 0 & -1 \end{pmatrix}$$

4.11. The group $G_{66\infty}$

Here $A = 4/3$, $B = 5/4$ and $C = 1$

$$T = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & -5/2 & 1 & 0 \\ -1/2 & 0 & 0 & 0 \\ 3/2 & -3 & 0 & 1 \end{pmatrix}, \quad \text{and}$$

$$T^{-1} g_m T = \begin{pmatrix} -1 & -15 & 0 & 3 \\ 2 & 11 & 0 & -2 \\ 5 & 25 & 1 & -5 \\ 9 & 25 & 1 & -5 \end{pmatrix},$$

$$T^{-1} g_n T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & -1 \\ -5 & -5 & 1 & 0 \\ -6 & -4 & 1 & 0 \end{pmatrix},$$

$$T^{-1} g_p T = \begin{pmatrix} -9 & 5 & 4 & -5 \\ 10 & -4 & -4 & 5 \\ 20 & -5 & -9 & 10 \\ 50 & -25 & -20 & 26 \end{pmatrix}$$

And the quadratic form is

$$\begin{pmatrix} 10 & 75 & 0 & -15 \\ 75 & -10 & -20 & 30 \\ 0 & -20 & 8 & 0 \\ -15 & 30 & 0 & -10 \end{pmatrix}$$

4.12. The group $G_{\infty\infty\infty}$

Here $A = B = C = 1$. We consider the basis $(-v_m, -v_n, e_2, e_4)$ of L_{666}.

$$g_m = \begin{pmatrix} -1 & 0 & -2 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & -2 \\ -2 & 0 & -2 & 3 \end{pmatrix},$$

$$g_n = \begin{pmatrix} 1 & 2 & 0 & -2 \\ -2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ -2 & -2 & 0 & 3 \end{pmatrix}$$

$$g_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & -2 & -1 & 2 \\ 0 & -2 & -2 & 3 \end{pmatrix}$$

And the quadratic form is

$$\langle 1, 1, 1, -1 \rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

which is the monodromy of the hyperbolic structure of finite volume of the complement of the Borromean rings in S^3, compare [4].

5. The 3-fold cyclic symmetry of B_{mmm}

Let L be the rational link 10/3. Consider the orbifold L_{3m} whose underlying manifold is S^3, singular set L; and isotropies 3 and m in the components of L. There is a 3-fold cyclic orbifold covering

$$p_m : B_{mmm} \longrightarrow L_{3m}$$

whose group of covering transformations is generated by the 3-fold rotation defined by

$$R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The group G_{mmm} together with R generate a group, say $G(L_{3m})$, which is the fundamental group of the hyperbolic orbifold L_{3m}, $m > 2$.

5.1. $m = 4$

The group $G(L_{34})$ is generated by R and

$$g_p(G_{444}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \varphi & -\varphi \\ 0 & -\varphi & -\varphi & \varphi+1 \\ 0 & 1 & -\varphi & \varphi+1 \end{pmatrix}$$

which are automorphs of the diagonal quadratic form $\langle 1, 1, 1, -\varphi\rangle$. The group $G(L_{34})$ is therefore arithmetic (proved in [3]) and universal (for every closed 3-manifold M there is a subgroup $G(L_{3m})_M$ of finite index of $G(L_{3m})$ such that M is homeomorphic to the quotient of the hyperbolic 3-space under the action of $G(L_{3m})_M$; in other words, M is the underlying manifold of a hyperbolic orbifold covering of L_{34} with finite degree, see [1]).

5.2. $m = \infty$

Similarly, the group $G(L_{3\infty})$, generated by the automorphs R and

$$g_p(G_{\infty\infty\infty}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & -2 & -1 & 2 \\ 0 & -2 & -2 & 3 \end{pmatrix}$$

of the quadratic form $\langle 1, 1, 1, -1\rangle$ is arithmetic ([3]).

5.3. $m = 3$

The group $G(L_{33})$, generated by the automorphs R and $g_p(G_{333})$, is conjugate by the matrix T_{333} (which is not integral in the ring of integers of $Q(\sqrt{13})$, $\phi = \frac{1}{2}(1 + \sqrt{13})$ into the group $\widehat{G}(L_{33})$ generated by the integral automorphs

$$\widehat{R_3} = T_{333}^{-1} R T_{333} = \begin{pmatrix} 0 & (1+\sqrt{13})/2 & 0 & -2 \\ 1 & (1-\sqrt{13})/2 & 0 & 2 \\ 0 & 2 & 0 & (1-\sqrt{13})/2 \\ 0 & -2 & 1 & (1+\sqrt{13})/2 \end{pmatrix}$$

and $g'_p(G_{333})$ (see §4.1) of the quadratic form

$$\begin{pmatrix} 2 & 0 & \phi & 1 \\ 0 & 2 & 0 & \phi \\ \phi & 0 & 2 & 0 \\ 1 & \phi & 0 & 2 \end{pmatrix}$$

The group $\widehat{G}(L_{33})$ is arithmetic (see [3]).

5.4. $m = 6$

An analogous remark applies to $G(L_{36})$. Here $\widehat{G}(L_{36})$, is generated by

$$\widehat{R_6} = T_6^{-1} R T_6 = \begin{pmatrix} 0 & (-3-\sqrt{21})/2 & 0 & 2 \\ 1 & (5+\sqrt{21})/2 & 0 & -2 \\ 0 & -5-\sqrt{21} & 0 & (5+\sqrt{21})/2 \\ 0 & 5+\sqrt{21} & 1 & (-3-\sqrt{21})/2 \end{pmatrix}$$

and $g'_p(G_{666})$ (see §4.10), which are integral. Thus also $\widehat{G}(L_{36})$ is arithmetic.

5.5. *Final remark*

It follows from [5, Theorems 2.2 and 3.2] that the group $G(L_{3m})$ is a real integral group for all $m > 2$. Then the generators R and g_m can be conjugated, via some no necessary integral matrix T_m, to a pair of integral matrices,

$$\widehat{R_m} = T_m^{-1} R T_m, \qquad g'_p(G_{mmm}) = T_m^{-1} g_m(G(mmm)) T_m$$

for all $m > 2$.

Acknowledgements

We thank the referee's remarks and suggestions.

References

[1] H. Hilden, M.T. Lozano, J.M. Montesinos-Amilibia, W.C. Whitten, On universal groups and three manifolds, Invent. Math. 87 (1987) 441–456.
[2] H. Hilden, M.T. Lozano, J.M. Montesinos-Amilibia, On the Borromean orbifolds: geometry and arithmetic, in: B. Apanasov, W.D. Neumann, A.W. Reid, L. Siebenmann (Eds.), Topology'90, Walter de Gruyter, Berlin, New York, 1992, pp. 133–167.
[3] H. Hilden, M.T. Lozano, J.M. Montesinos-Amilibia, On the arithmetic 2-bridge knots and link orbifolds and a new knot invariant, J. Knot Theory Ramif. 4 (1) (1995) 81–114.
[4] J.M. Montesinos-Amilibia, On integral quadratic forms having commensurable groups of automorphisms, Hiroshima Math. J. 43 (2013) 371–411.
[5] M.T. Lozano, J.M. Montesinos-Amilibia, Integral geometric orbifolds, J. Knot Theory Ramif. 30 (12) (2021) 2141010, https://doi.org/10.1142/S0218216521410108 (28 pages).
[6] W. Thurston, The Geometry and Topology of 3-Manifolds, Lecture Notes, Princeton University, 1978.