Making Sullivan Algebras Minimal Through Chain Contractions

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Abstract. In this note, we provide an algorithm that, starting with a Sullivan algebra gives us its minimal model. More concretely, taking as input a (non-minimal) Sullivan algebra A with an ordered finite set of generators preserving the filtration defined on A, we obtain as output a minimal Sullivan algebra with the same rational cohomology as A. This algorithm is a kind of modified AT-model algorithm used, in the past, to compute a chain contraction providing other kinds of topological information such as (co)homology, cup products on cohomology and persistent homology.

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1. Introduction

Algebraic Topology consists, essentially, in the study of algebraic invariants associated with topological spaces. One of these invariants is the homotopy type of a space. At present, we are still far from having a complete algebraic description of homotopy. This could be the reason why the computability of the homotopy type of a space is still an open problem nowadays. However, this description exists within the framework of Rational Homotopy Theory ([2,3] is nowadays a standard reference for the theory). Two distinct approaches to Rational Homotopy Theory were given independently by Quillen [13] and Sullivan [15] at the end of the sixties. Following the classical Sullivan's approach [17], we associate to every rational homotopy type of 1-connected spaces (nilpotent spaces, in general) of finite type in a unique way (up to isomorphism) its minimal model, which is a commutative differential graded

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algebra. This association is functorial and encodes the homotopic properties of the space up to rationalization. From the theoretical point of view, we can always get the minimal model of a space but in practical examples, effective computations are not always possible. Observe that obtaining the minimal model is essential because it determines the rational homotopy type of the space and sometimes we are able to recognize the space from its minimal model. As an example, the loop space homology of a simply connected space X for which $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional is computed in [4] via the Hochschild cohomology of the corresponding minimal Sullivan algebra ΛW which is, at the same time, computed in terms of derivations of ΛW.

A weaker version of a minimal model is the one of a Sullivan algebra. The basic difference between both of them is that a minimal model is a Sullivan algebra such that its differential does not have linear terms. There are many situations that we get a Sullivan algebra instead of a minimal model. For example, if we have a fibration, we can associate it with a minimal extension or relative Sullivan algebra and we can get from it a model of the total space. Starting with a model of the base, we can extend it to obtain a Sullivan algebra model of the total space of the fibration, but this model, in general, is not minimal. If we want to try to recognize the total space, we can follow a standard procedure: Since a Sullivan algebra is always the tensor product of a minimal Sullivan algebra and a contractible one, then the idea is to eliminate the contractible part. In concrete examples, the procedure is "ad hoc". See, for example, [1], where the existence of minimal models of operads algebras is given by adapting Sullivan's step by step construction. In general, such "ad hoc" procedure is not effective and a computational approach would be of interest.

Our aim in this paper is to give an effective algorithm that starting with a non-minimal Sullivan algebra ends with its minimal model. The idea is to use a modification of the incremental algorithm for computing the AT-model of a chain complex [5–7]. Essentially the incremental algorithm is a sequence of chain contractions of modules that, starting with a chain complex, ends with its homology. The key idea is to get a sequence of chain contractions of algebras instead of modules, not finishing with zero differential but with a differential with no linear terms. We will prove that this is possible starting with a Sullivan algebra with a finite number of generators and will give several examples of effective computation of minimal models. An implementation of our method that runs in CoCalc (https://cocalc.com/) can be downloaded to test from the website http://grupo.us.es/cimagroup/downloads.htm.

The paper is organized as follows. Sections 1, 2 and 3 are devoted to introducing the background of the paper (DG-modules, AT-models, CDG-algebras and Sullivan algebras). In Sect. 4, as a main result of the paper, an algorithm is provided to compute minimal Sullivan algebras from Sullivan algebras not necessarily being minimal. Examples are given in Sect. 5. The paper ends with a section devoted to conclusions and future work.

2. Differential Graded Modules and AT-Models

In this section, we introduced the background for DG-modules needed to understand the paper and an algorithm for computing AT-models [5] which is a precursor of the algorithm presented here for computing minimal Sullivan algebras, as we will see later.

Let $\mathbb{K}$ be a field, taken henceforth as the ground ring. Denote by $M = \mathbb{K}\langle m_0, \ldots, m_n\rangle$ the $\mathbb{K}$-vector space M with basis $\{m_0, \ldots, m_n\} \subseteq M$. That is, for any $x \in M$, there exist $\lambda_0, \ldots, \lambda_n \in \mathbb{K}$ such that $x = \lambda_0 m_0 + \cdots + \lambda_n m_n$. For each i, $0 \leq i \leq n$, denote the coefficient λ_i by $\mathrm{coeff}(x, m_i)$. Finally, we say that $x \in M$ has index j and write $\mathrm{index}(x) = j$ if $x = m_j$.

Definition 1. A graded module (G-module) M is a direct sum of $\mathbb{K}$-vector spaces $\bigoplus_{n\in\mathbb{Z}} M^n$. We will suppose that $M^n = 0$ for $n < 0$. An element $x \in M$ is homogeneous if $x \in M^n$ for some n and, in such case, we say that x has degree n and write $|x| = n$.

Definition 2. A G-module morphism $f : M \to N$ of degree q (denoted by $|f| = q$) is a family of homomorphisms $\{f^n\}_{n\in\mathbb{Z}}$ such that $f^n : M^n \to N^{n+q}$ for all n.

Definition 3. Given two G-modules M and N, $M \otimes N$ is defined as the G-module

$$M \otimes N = \oplus_{n\in\mathbb{Z}}(M \otimes N)^n$$

where

$$(M \otimes N)^n := \oplus_{p+q=n}(M^p \otimes N^q)$$

and the degree of an element $a \otimes b \in M \otimes N$ is

$$|a \otimes b| = |a| + |b|.$$

Definition 4. Given two G-module morphisms, $f : M_1 \to N_1$ and $g : M_2 \to N_2$, the tensor product $f \otimes g$ is defined adopting the Koszul's convention as:

$$(f \otimes g)(x \otimes y) = (-1)^{|g|\cdot|x|} f(x) \otimes g(y).$$

Definition 5. A differential over a G-module M is a G-module morphism $d_M : M \to M$ such that $|d_M| = \pm 1$.

In this paper, all the differentials will have degree $+1$.

Definition 6. A DG-module (M, d_M) is a G-module M endowed with a differential $d_M : M \to M$ which satisfies that $d_M d_M = 0$.

Definition 7. A DG-module morphism $f : (M, d_M) \to (N, d_N)$ of degree q is a G-module morphism $f : M \to N$ of degree q such that $d_N f = (-1)^q f d_M$.

Definition 8. If (M, d_M) is a DG-module, then $(M \otimes M, d_{M\otimes M})$ is a DG-module with the differential given by:

$$d_{M\otimes M} = id_M \otimes d_M + d_M \otimes id_M.$$

An interesting family of morphisms is one of chain contractions between two DG-modules.

Definition 9. Given two DG-modules (M, d_M) and (N, d_N), a chain contraction from (M, d_M) to (N, d_N) is a triplet (f, g, ϕ) such that:

- $f : (M, d_M) \to (N, d_N)$ and $g : (N, d_N) \to (M, d_M)$ are DG-module morphisms of degree 0 satisfying that:
$$fg = id_N.$$
- $\phi : M \to M$ is a G-module morphism of degree -1 satisfying that:
$$f\phi = 0,\ \phi g = 0,\ \phi\phi = 0,$$
$$id_M - gf = \phi d_M + d_M \phi.$$

If a chain contraction between two DG-modules exists, then it is easy to see that both DG-modules have isomorphic (co)homology.

An AT-model for a DG-module is nothing more than a chain contraction from the DG-module to its homology.

Definition 10. An AT-model for a DG-module (M, d_M) is a chain contraction (f, g, ϕ) from (M, d_M) to a finitely-generated DG-module $(\mathbb{K}H, d_H)$ with set of generators H and null-differential, that is, $d_H = 0$.

Therefore, an AT-model (f, g, ϕ) for (M, d_M) satisfies that:

$$fd_M = d_M g = f\phi = \phi g = \phi\phi = 0,$$
$$id_M - gf = \phi d_M + d_M \phi,\ fg = id_H,$$
$$\phi d_M \phi = \phi,\ d_M \phi d_M = d_M.$$

Algorithm 1 computes an AT-model for (M, d_M). In this incremental algorithm, we start with a filtering or order of the generators with the condition that the differential of any generator is always a linear combination of the generators that appear previously in the filtering.

The complexity of Algorithm 1 is cubic in the number of generators of M (see [5]). The (co)homology of (M, d_M) is isomorphic to the one of $(\mathbb{K}H, d_H = 0)$ since (f, g, ϕ) is a chain contraction. Besides, Algorithm 1 can be used to compute more sophisticated topology information such as cup products on cohomology [9] or persistent homology [8].

Algorithm 1: AT-model for computing homology [5].

Input: A finitely-generated DG-module (M, d_M) with $M = \mathbb{K}\langle m_0, \ldots, m_n \rangle$ such that $d_M(m_i) \in \mathbb{K}\langle m_0, \ldots, m_{i-1} \rangle$ for $1 \leq i \leq n$

Initialize: $H := \{m_0\}$, $f(m_0) := m_0$, $g(m_0) := m_0$ and $\phi(m_0) := 0$

for $i = 1$ **to** n **do**

  Let $a = fd_M(m_i)$ and $b = m_i - \phi d_M(m_i)$

  if $a = 0$ **then**

    (a new homology class α_i is born when m_i is added)

    $H := H \cup \{m_i\}$, $f(m_i) := m_i$, $g(m_i) := b$ and $\phi(m_i) := 0$

  if $a \neq 0$ **then**

    let $j = \max\{\text{index}(m) : m \in H \text{ and } \text{coeff}(a, m) \neq 0\}$

    (the homology class α_j dies when m_i is added)

    $H := H \setminus \{m_j\}$, $f(m_i) := 0$ and $\phi(m_i) := 0$

    foreach $m \in \{m_0, \ldots, m_{i-1}\}$ **do**

      $f(m) := f(m) - \lambda a$ and

      $\phi(m) := \phi(m) + \lambda b$ where $\lambda := \frac{\text{coeff}(f(m), m_j)}{\text{coeff}(a, m_j)}$

Output: An AT-model (f, g, ϕ) for (M, d_M).

3. Graded (Differential) Algebras

In this section, we recall the notion of commutative differential graded algebras.

Definition 11. A G-algebra (A, μ_A) is a G-module A together with a G-module morphism $\mu_A : A \otimes A \to A$ of degree 0 such that:

$$\mu_A(\mu_A(x \otimes y) \otimes z) = \mu_A(x \otimes \mu_A(y \otimes z)) \ \ (\mu_A \text{ is associative})$$

and

$$\mu_A(1_A \otimes x) = x = \mu_A(x \otimes 1_A) \ \ (1_A \in A^0 \text{ is an identity}).$$

Definition 12. A CG-algebra (A, μ_A) is a G-algebra that is commutative in the graded sense, that is:

$$\mu_A(a_p \otimes a_q) = (-1)^{pq} \mu_A(a_q \otimes a_p) \ \text{ where } a_p \in A^p \ \text{ and } \ a_q \in A^q.$$

Definition 13. A DG-algebra (A, μ_A, d_A) is a G-algebra (A, μ_A) together with a G-module morphism $d_A : A \to A$ of degree $+1$ such that:

$$d_A d_A = 0 \ \ (d_A \text{ is a differential})$$

and

$$d_A \mu_A(x \otimes y) = \mu_A(d_A(x) \otimes y) + (-1)^{|x|} \mu_A(x \otimes d_A(y)) \ \ (d_A \text{ is a derivation}).$$

Definition 14. Given two DG-algebras (A, μ_A, d_A) and (B, μ_B, d_B), a DG-algebra morphism $f : (A, \mu_A, d_A) \to (B, \mu_B, d_B)$ is a DG-module morphism satisfying that

$$f \mu_A = \mu_B f.$$

Definition 15. A CDG-algebra is a CG- and DG-algebra.

The following definition is taken from [10].

Definition 16. Let (f, g, ϕ) be a chain contraction from a DG-algebra (A, μ_A, d_A) to a DG-algebra (B, μ_B, d_B). We say that ϕ is an algebra homotopy if

$$\mu_A(1_A \otimes \phi + \phi \otimes gf) = \phi\mu_A.$$

And this last definition is taken from [14].

Definition 17. We say that a chain contraction (f, g, ϕ) from a DG-algebra (A, μ_A, d_A) to a DG-algebra (B, μ_B, d_B) is a full algebra contraction if f and g are DG-algebra morphisms and ϕ is an algebra homotopy.

Examples of full algebra contractions are given, for example, in [11], using the "tensor trick".

4. Sullivan Algebras

We recall basic results and definitions from Rational Homotopy Theory for which [2] is a standard reference.

A topological space is rational if all their homotopy groups are $\mathbb{Q}$-vector spaces. Rational homotopy can be seen as classical homotopy theory over the rational spaces. Following this theory, each space can be functorially associated with a rational space that satisfies that their homotopy groups are vector spaces on the rational numbers. This process that associates a rational space with each space is called rationalization. This rationalization suppresses the torsion part of the homotopy groups and is, therefore, a first approximation to the given space. From this point of view, the rational homotopy of a space is nothing more than the classical homotopy of its associated rational space. Sullivan showed in [16,17] that the rational homotopy type of a simply connected space is faithfully represented by graded, differential and commutative algebras (CDGAs). In particular, Sullivan defined a functor F that associates each space X with a CDGA $(F(X), d)$ on the rational numbers, with the property of inducing isomorphisms between the respective cohomologies (the one of the algebra $(F(X), d)$ and the one of the space X) with rational coefficients. In general, the algebra $(F(X), d)$ is huge and difficult to compute. This is the reason why Sullivan proposed to build a smaller algebra (A, d_A) from the algebra $(F(X), d)$ called a "minimal Sullivan algebra" which is a graded and free algebra over a certain graded vector space. This minimal model is unique except for isomorphisms and encodes the rational homotopy type of X.

From now on, the ground ring $\mathbb{K}$ is the field of the rational numbers $\mathbb{Q}$. Besides, we will work with DG-algebras which are free over graded vector spaces, that is, of the form ΛV, being V a graded vector space $V = \{V^p\}_{p\geq 1}$. Denote by $\Lambda^{\geq 2}V$ the G-module generated by elements of ΛV obtained as the product of two or more elements of V. Let $V = \mathbb{Q}\langle m_1, \dots, m_n\rangle$. Then, for any $x \in \Lambda V$, $x = \lambda_1 m_1 + \cdots + \lambda_n m_n + b$ for some $b \in \Lambda^{\geq 2}V$. The expression

$\mathrm{coeff}(x, m_i)$ will denote the coefficient λ_i. Finally, we say that $x \in \Lambda V$ has index j and write $\mathrm{index}(x) = j$ if $x = m_j$.

We will refer such an algebra by $(\Lambda V, \mu_{\Lambda V}, d_{\Lambda V})$ or simply by ΛV when no confusion can arise.

Definition 18. A DG-algebra ΛV is contractible if for some $U \subset V$ the inclusions of $\Lambda(U \oplus dU)$ in ΛV extends to an isomorphism:

$$\Lambda(U \oplus dU) \xrightarrow{\simeq} \Lambda V$$

where $dU = \{d_{\Lambda V}(u) : u \in U\}$.

Definition 19. A Sullivan algebra $(\Lambda V, \mu_{\Lambda V}, d_{\Lambda V})$ is a kind of CDG-algebra:

(1) It is a CG-algebra which is free over a graded vector space V.

(2) There is an increasing filtration of subspaces in V:

$$V(0) \subset V(1) \subset \cdots$$

with $V = \bigcup_{k \geq 0} V(k)$, such that

$$d_{\Lambda V}(x) = 0 \text{ for any } x \in V(0) \text{ and}$$
$$d_{\Lambda V}(x) \in \Lambda V(k-1) \text{ for any } x \in V(k), \text{ with } 1 \leq k.$$

Essentially, the difference of a non-minimal Sullivan algebra with a minimal one is that the differential of the non-minimal Sullivan algebra has linear terms in the generators of V. In other words, being minimal means that the Sullivan algebra has no linear part in the generators of V.

Definition 20. The Sullivan algebra $(\Lambda V, \mu_{\Lambda V}, d_{\Lambda V})$ is *minimal* if $d_{\Lambda V}(x) \in \Lambda^{\geq 2} V$ for all $x \in V$.

A crucial property of minimal Sullivan algebras is the following.

Theorem 1 (Section 12 of [2]). *For every rational homotopy type of spaces, there is a unique (up to isomorphism) minimal Sullivan algebra.*

An essential property of Sullivan algebras is that they are isomorphic to a minimal algebra tensor product a contractible algebra and, therefore, removing the contractible part, we obtain its minimal model.

Theorem 2 (Theorem 14.9 of [2]). *Every Sullivan algebra ΛV is isomorphic to $\Lambda W \otimes \Lambda(U \oplus dU)$ where ΛW is a minimal Sullivan algebra and $\Lambda(U \oplus dU)$ is contractible.*

5. Making Sullivan Algebras Minimal

Essentially, what is done in the literature for calculating a minimal Sullivan algebra (see, for example, [2]) is a change of basis in a way that the algebra is divided in a contractible part and a non-contractible part, and the differential of the non-contractible part has no linear terms. Therefore, the non-contractible part is a minimal Sullivan algebra such that the inclusion to the given algebra induces an isomorphism in cohomology, indicating that it is the minimal model of the given algebra.

Our main goal in this section is to provide Algorithm 2 for computing a minimal Sullivan algebra from a Sullivan algebra not necessarily being minimal. It is precisely the fact that the Sullivan algebra and its minimal model have the same cohomology (since they are only different in a contractible part), what made us relate this problem to another apparently different in principle: The incremental algorithm for computing an "AT-model" of a chain complex (see Algorithm 1).

Algorithm 2: Making a Sullivan algebra minimal

Input: A Sullivan algebra ΛV, being
$V(0) \subset V(1) \subset \cdots \subset V(\kappa) = V$ a filtration, $V(0) = \mathbb{Q}\langle n_0, \dots n_\ell \rangle$
and $V = V(0) \oplus \mathbb{Q}\langle m_1, \dots, m_n \rangle$ satisfying that if $1 \leq i < i' \leq n$
then there exist k, with $1 \leq k \leq \kappa$, such that $m_i, m_{i'} \notin V(k+1)$
and either (a): $m_i, m_{i'} \in V(k) \setminus V(k-1)$, or (b): $1 < k$,
$m_i \in V(k-1)$ and $m_{i'} \in V(k) \setminus V(k-1)$.

Initialize: the set of generators of $U \oplus dU$, $Y := \{(0,0)\}$,
$B := \{n_0, \dots n_\ell\}$, W will always be the vector space spanned by B,
$f := id_{V(0)}$, $g := id_{V(0)}$, $\phi := 0$ and $d_{\Lambda W} := 0$

for $i = 1$ **to** n **do**

  let $a := f d_{\Lambda V}(m_i)$ and $b := m_i - \phi d_{\Lambda V}(m_i)$

  if $a \in \Lambda^{\geq 2} W$ **then**

    (a new generator m_i is added to B)
    $B := B \cup \{m_i\}$, $f(m_i) := m_i$, $g(m_i) := b$, $\phi(m_i) := 0$ and
    $d_{\Lambda W}(m_i) := a$

  if $a \notin \Lambda^{\geq 2} W$ **then**

    let $j = \max\{\text{index}(m) : m \in B \text{ and } \text{coeff}(a, m) \neq 0\}$
    (m_j is removed from B and (m_i, m_j) is added to Y)
    $B := B \setminus \{m_j\}$, $f(m_i) := 0$, $\phi(m_i) := 0$ and
    $Y := Y \cup \{(m_i, m_j)\}$
    foreach $m \in \{n_0, \dots, n_\ell\} \cup \{m_1, \dots, m_{i-1}\}$ **do**

      $f(m) := f(m) - \lambda a$ and $\phi(m) := \phi(m) + \lambda b$ where
      $\lambda := \frac{\text{coeff}(f(m), m_j)}{\text{coeff}(a, m_j)}$

  $f \mu_{\Lambda V} := \mu_{\Lambda W}(f \otimes f)$, $g \mu_{\Lambda W} := \mu_{\Lambda V}(g \otimes g)$,
  $\phi \mu_{\Lambda V} := \mu_{\Lambda V}(id_{\Lambda V} \otimes \phi + \phi \otimes gf)$ and
  $d_{\Lambda W} \mu_{\Lambda W} := f d_{\Lambda V} \mu_{\Lambda V}(g \otimes g)$

Output: A full algebra contraction (f, g, ϕ) from ΛV to ΛW.
The generators $U \oplus dU$ of the contractible part.

The following result ensures that Algorithm 2 makes a Sullivan algebra minimal.

Theorem 3. *Given a Sullivan algebra $(\Lambda V, \mu_{\Lambda V}, d_{\Lambda V})$, Algorithm 2 produces a full algebra contraction (f, g, ϕ) from ΛV to ΛW. Moreover, ΛW is a minimal Sullivan algebra.*

Proof. Denote the output of Algorithm 2 at the step i by Y_i, B_i, W_i, f_i, g_i and ϕ_i.

First, it is straightforward to see that (f_0, g_0, ϕ_0) is a full algebra contraction from $\Lambda V(0)$ to ΛW_0.

Now, by induction, suppose that $(f_{i-1}, g_{i-1}, \phi_{i-1})$ is a full algebra contraction from ΛV_{i-1} to ΛW_{i-1} where $V_{i-1} := V(0) \oplus \mathbb{Q}\langle m_1, \ldots, m_{i-1}\rangle$. Let us prove that (f_i, g_i, ϕ_i) is a full algebra contraction from ΛV_i to ΛW_i where $V_i := V_{i-1} \oplus \mathbb{Q}\langle m_i\rangle$. To start with, let us prove the following properties by induction:

1. $f_i\phi_i = 0$
2. $\phi_i g_i = 0$
3. $\phi_i\phi_i = 0$
4. $f_i g_i = id_{\Lambda W_i}$
5. $id_{\Lambda V_i} - g_i f_i = \phi_i d_{\Lambda V_i} + d_{\Lambda V_i}\phi_i$
6. $f_i d_{\Lambda V_i} = d_{\Lambda W_i} f_i$
7. $d_{\Lambda V_i} g_i = g_i d_{\Lambda W_i}$

Suppose first that $f_{i-1}d_{\Lambda V_i}(m_i) \in \Lambda^{\geq 2}W_{i-1}$. In this case, it is enough to prove the properties above only for m_i:

1. $f_i\phi_i(m_i) = 0$ since $\phi_i(m_i) = 0$.
2. $\phi_i g_i(m_i) = \phi_i(m_i - \phi_{i-1}d_{\Lambda V_i}(m_i)) = \phi_i(m_i) - \phi_{i-1}\phi_{i-1}d_{\Lambda V_i}(m_i) = 0$ since $\phi_i(m_i) = 0$ and $\phi_{i-1}\phi_{i-1} = 0$ by induction.
3. $\phi_i\phi_i(m_i) = 0$ since $\phi_i(m_i) = 0$.
4. $f_i g_i(m_i) = f_i(m_i - \phi_{i-1}d_{\Lambda V_i}(m_i)) = f_i(m_i) - f_{i-1}\phi_{i-1}d_{\Lambda V_i}(m_i) = f_i(m_i) = m_i$ since $f_{i-1}\phi_{i-1} = 0$ by induction.
5. $\phi_i d_{\Lambda V_i}(m_i) + d_{\Lambda V_i}\phi_i(m_i) = \phi_{i-1}d_{\Lambda V_i}(m_i) = m_i - g_i(m_i) = m_i - g_i f_i(m_i)$.
6. $d_{\Lambda W_i} f_i(m_i) = d_{\Lambda W_i}(m_i) = f_{i-1}d_{\Lambda V_i}(m_i) = f_i d_{\Lambda V_i}(m_i)$.
7. $g_i d_{\Lambda W_i}(m_i) = g_i f_{i-1} d_{\Lambda V_i}(m_i) = g_i f_i d_{\Lambda V_i}(m_i) = d_{\Lambda V_i}(m_i) - \phi_i d_{\Lambda V_i} d_{\Lambda V_i}$ $(m_i) - d_{\Lambda V_i}\phi_i d_{\Lambda V_i}(m_i) = d_{\Lambda V_i}(m_i) - d_{\Lambda V_i}\phi_i d_{\Lambda V_i}(m_i) = d_{\Lambda V_i}(m_i - \phi_{i-1}d_{\Lambda V_i}(m_i)) = d_{\Lambda V_i} g_i(m_i)$.

Second, suppose that $f_{i-1}d_{\Lambda V_i}(m_i) \notin \Lambda^{\geq 2}W_{i-1}$. Let $m \in \{n_0, \ldots, n_\ell\} \cup \{m_1, ..., m_{i-1}\}$, then:

1. $f_i\phi_i(m_i) = 0$ since $\phi_i(m_i) = 0$;
   $f_i\phi_i(m) = f_i(\phi_{i-1}(m) + \lambda(m_i - \phi_{i-1}d_{\Lambda V_i}(m_i)) = f_{i-1}\phi_{i-1}(m) + \lambda f_i(m_i) - \lambda f_{i-1}\phi_{i-1}d_{\Lambda V_i}(m_i) = 0$ since $f_i(m_i) = 0$ and $f_{i-1}\phi_{i-1} = 0$ by induction.
2. $\phi_i g_i(m) = \phi_{i-1}g_{i-1}(m) = 0$ since $\phi_{i-1}g_{i-1} = 0$ by induction.
3. $\phi_i\phi_i(m_i) = 0$ since $\phi_i(m_i) = 0$;
   $\phi_i\phi_i(m) = \phi_i(\phi_{i-1}(m) + \lambda(m_i - \phi_{i-1}d_{\Lambda V_i}(m_i)) = \phi_{i-1}\phi_{i-1}(m) + \lambda\phi_i(m_i)$ $-\lambda\phi_{i-1}\phi_{i-1}d_{\Lambda V_i}(m_i) = 0$ since $\phi_i(m_i) = 0$ and $\phi_{i-1}\phi_{i-1} = 0$ by induction.
4. $f_i g_i(m) = f_{i-1}g_{i-1}(m) = m$ since $f_{i-1}g_{i-1} = id_{\Lambda W_{i-1}}$ by induction.
5. $\phi_i d_{\Lambda V_i}(m_i) + d_{\Lambda V_i}\phi_i(m_i) = \phi_{i-1}d_{\Lambda V_i}(m_i) + m_i - \phi_{i-1}d_{\Lambda V_i}(m_i) = m_i - g_i f_i(m_i)$; $\phi_i d_{\Lambda V_i}(m) + d_{\Lambda V_i}\phi_i(m) = \phi_{i-1}d_{\Lambda V_i}(m) + d_{\Lambda V_i}(\phi_{i-1}(m) + \lambda(m_i - \phi_{i-1}d_{\Lambda V_i}(m_i)) = m - g_{i-1}f_{i-1}(m) + \lambda g_{i-1}f_{i-1}d_{\Lambda V_i}(m_i) = m - g_i f_i(m)$.
6. $f_i d_{\Lambda V_i}(m_i) = f_{i-1}d_{\Lambda V_i}(m_i) - f_{i-1}d_{\Lambda V_i}(m_i) = 0 = d_{\Lambda W_i} f_i(m_i)$; $d_{\Lambda W_i} f_i(m)$ $= d_{\Lambda W_i}(f_{i-1}(m) - \lambda f_{i-1}d_{\Lambda V_i}(m_i)) = f_{i-1}d_{\Lambda V_i}(m) = f_i d_{\Lambda V_i}(m)$.
7. $g_i d_{\Lambda W_i}(m) = g_{i-1}d_{\Lambda W_{i-1}}(m) = d_{\Lambda V_{i-1}} g_{i-1}(m) = d_{\Lambda V_i} g_i(m)$.

Now, it is easy to see that $(\Lambda W_i, d_{\Lambda W_i})$ is a DG-module, that is, $d_{\Lambda W_i}$ is a differential. It is enough to prove it for m_i:

$$d_{\Lambda W_i} d_{\Lambda W_i}(m_i) = d_{\Lambda W_i} f_{i-1} d_{\Lambda V_i}(m_i) = d_{\Lambda W_i} f_i d_{\Lambda V_i}(m_i) = f_i d_{\Lambda V_i} d_{\Lambda V_i}(m_i) = 0.$$

Therefore, we conclude that the output (f, g, ϕ) of Algorithm 2 is a chain contraction from ΛV to ΛW.

To prove that ΛW is a CDG-algebra, we have to prove the following properties:

1. $d_{\Lambda W}$ is derivation.
2. $1_{\Lambda W} := f(1_{\Lambda V})$ is an identity.

Using that (f, g, ϕ) is a chain contraction and that, by construction, we have that $d_{\Lambda V}\mu_{\Lambda V} = f d_{\Lambda V}\mu_{\Lambda V}(g \otimes g)$, $f\mu_{\Lambda V} = \mu_{\Lambda W}(f \otimes f)$, $g\mu_{\Lambda W} = \mu_{\Lambda V}(g \otimes g)$ and $\phi\mu_{\Lambda V} = \mu_{\Lambda V}(id_{\Lambda V} \otimes \phi + \phi \otimes gf)$, then:

1. $d_{\Lambda W}\mu_{\Lambda W}(x \otimes y) = f d_{\Lambda V}\mu_{\Lambda V}(g(x) \otimes g(y)) = f\mu_{\Lambda V}(d_{\Lambda V} g(x) \otimes g(y)) + (-1)^{|g(x)|} f\mu_{\Lambda V}(g(x) \otimes d_{\Lambda V} g(y)) = \mu_{\Lambda W}(f d_{\Lambda V} g(x) \otimes f g(y)) + (-1)^{|g(x)|}$ $\mu_{\Lambda W}(f g(x) \otimes f d_{\Lambda V} g(y)) = \mu_{\Lambda W}(d_{\Lambda W} f g(x) \otimes f g(y)) + (-1)^{|x|}\mu_{\Lambda W}(f g(x)$ $\otimes d_{\Lambda W} f g(y)) = \mu_{\Lambda W}(d_{\Lambda W}(x) \otimes y) + (-1)^{|x|}\mu_{\Lambda W}(x \otimes d_{\Lambda W}(y))$ since $d_{\Lambda V}$ is a derivation and $|g| = 0$.
2. $\mu_{\Lambda W}(1_{\Lambda W} \otimes x) = \mu_{\Lambda W}(f(1_{\Lambda V}) \otimes f g(x)) = f\mu_{\Lambda V}(1_{\Lambda V} \otimes g(x)) = f g(x)) = x$ since $1_{\Lambda V}$ is an identity and $fg = 1_{\Lambda W}$.

Therefore, we can conclude that ΛW is a CDG-algebra and (f, g, ϕ) is a full algebra contraction from ΛV to ΛW.

Finally, ΛW is a Sullivan algebra considering the filtration of V restricted to W. Besides, ΛW is minimal by induction. Trivially, W_0 is minimal since $d_{\Lambda W_0} = 0$. Suppose that ΛW_{i-1} is minimal. Now, $d_{\Lambda W_i}$ is updated when $f_{i-1} d_{\Lambda V}(m_i) \in \Lambda^{\geq 2} W_{i-1}$ and in this case, $d_{\Lambda W_i}(m_i) = f_{i-1} d_{\Lambda V_i}(m_i) \in \Lambda^{\geq 2} W_i$. □

The following property also holds.

Proposition 1. *The Sullivan algebra ΛV is isomorphic to $\Lambda W \otimes \Lambda(U \oplus dU)$ where W and $U \oplus dU$ are the outputs of Algorithm 2 and $\Lambda(U \oplus dU)$ is contractible.*

Proof. Observe that g is one-to-one due to $fg = id_{\Lambda W}$ then, we can obtain a base of generators $g(W) \sqcup Z$ for ΛV, with the property that ΛZ is contractible (since ΛV and ΛW have the same rational cohomology, due to the chain contraction from ΛV to ΛW). Besides, by construction, ΛZ is isomorphic to $\Lambda(U \oplus dU)$ where $U \oplus dU$ is generated by the set Y of pairs $\{(m_i, m_j)\}_{i>j}$ obtained when running Algorithm 2 on V. □

As a consequence of Theorem 3, we can ensure the result shown below.

Corollary 4. *Let ΛV be a Sullivan algebra isomorphic to $\Lambda W \oplus \Lambda(U \oplus dU)$, being ΛW minimal and $U \oplus dU$ contractible. If ΛV is finitely generated then so is ΛW.*

6. Some Examples of Computation of Minimal Sullivan Algebras

Below we show some examples of the computation of minimal Sullivan algebras using Algorithm 2. For the sake of simplicity, from now on, given an algebra ΛV and two elements $a, b \in V$, the element $\mu_{\Lambda V}(a \otimes b)$ will be denoted by ab when no confusion can arise.

Example 1. Consider the Sullivan algebra ΛV where $V = \{V^p\}_{p \geq 1}$ being:

$$V^1 = \mathbb{Q}\langle a_1, b_1, c_1 \rangle,\ V^2 = \mathbb{Q}\langle v_2 \rangle,\ V^3 = \mathbb{Q}\langle u_3 \rangle,$$

the differential $d_{\Lambda V}$ defined on V being as follows:

$$d_{\Lambda V}(a_1) = v_2,\ d_{\Lambda V}(b_1) = 0 = d_{\Lambda V}(c_1),\ d_{\Lambda V}(u_3) = v_2^2,$$

and filtration:

$$V(0) = \mathbb{Q}\langle b_1, c_1, v_2 \rangle \subset V(1) = V.$$

Then, Algorithm 2 runs as follows.

Initially, $Y := \{(0,0)\}$, $B := \{b_1, c_1, v_2\}$, $f := id_{\Lambda V(0)}$, $g := id_{\Lambda V(0)}$, $\phi := 0$ and $d_{\Lambda W} := 0$.

Now, $f d_{\Lambda V}(a_1) = f(v_2) = v_2 \notin \Lambda^{\geq 2} W$. Then B and the image of the morphisms f, g and ϕ are updated as follows:

V	$d_{\Lambda V}$	W	$d_{\Lambda W}$	f	g	ϕ
b_1	0	b_1	0	b_1	b_1	0
c_1	0	c_1	0	c_1	c_1	0
v_2	0			0		a_1
a_1	v_2			0		0
u_3	v_2^2					

Finally, $f d_{\Lambda V}(u_3) = f(v_2^2) = f(v_2) f(v_2) = 0$ since $f(v_2) = 0$. Then B and the image of the morphisms f, g and ϕ are updated as follows:

V	$d_{\Lambda V}$	W	$d_{\Lambda W}$	f	g	ϕ
b_1	0	b_1	0	b_1	b_1	0
c_1	0	c_1	0	c_1	c_1	0
v_2	0			0		a_1
a_1	v_2			0		0
u_3	v_2^2	u_3	0	u_3	$u_3 - a_1 v_2$	0

Therefore, ΛW with set of generators $B = \{b_1, c_1, u_3\}$ is a minimal Sullivan algebra. Besides, $U \oplus dU$ is generated by $Y = \{(a_1, v_2)\}$, obtained when running Algorithm 2 on V. We obtain that ΛV is isomorphic to $\Lambda W \otimes \Lambda(U \oplus dU)$ being ΛW a minimal Sullivan algebra and $\Lambda(U \oplus dU)$ contractible. The basis provided by the "ad hoc" method can be obtained using the morphism g. This way, the new basis of generators of ΛV is:

$$\{g(b_1) = b_1,\ g(c_1) = c_1,\ g(u_3) = u_3 - a_1v_2, a_1, v_2\}$$

with differential:

$$d_{\Lambda V} g(b_1) = 0,\ d_{\Lambda V} g(c_1) = 0,\ d_{\Lambda V} g(u_3) = 0,\ d_{\Lambda V}(a_1) = v_2,\ d_{\Lambda V}(v_2) = 0.$$

Example 2. If we consider the Sullivan algebra ΛV where V is the same as the one given in Example 1, the differential $d_{\Lambda V}$ defined on V as follows:

$$d_{\Lambda V}(a_1) = d_{\Lambda V}(b_1) = d_{\Lambda V}(c_1) = v_2,\ d_{\Lambda V}(v_2) = 0,\ d_{\Lambda V}(u_3) = v_2^2$$

and filtration

$$V(0) = \mathbb{Q}\langle v_2\rangle \subset V(1) = V,$$

then Algorithm 2 produces the same minimal Sullivan algebra and the same full algebra contraction as in Example 1.

Example 3. Consider the Sullivan algebra ΛV where $V = \{V^p\}_{p\geq 1}$ being:

$$V^1 = \mathbb{Q}\langle a_1, b_1, c_1, x_1, y_1\rangle,\ V^2 = \mathbb{Q}\langle v_2, p_2, q_2, r_2\rangle,\ V^3 = \mathbb{Q}\langle u_3\rangle,$$

the differential $d_{\Lambda V}$ defined on V as follows:

$$\begin{aligned}
&d_{\Lambda V}(a_1) = d_{\Lambda V}(b_1) = d_{\Lambda V}(c_1) = d_{\Lambda V}(v_2) = 0,\\
&d_{\Lambda V}(x_1) = v_2 - 2a_1b_1 + 2b_1c_1,\ d_{\Lambda V}(y_1) = v_2 - 2a_1c_1 - 2b_1c_1,\ d_{\Lambda V}(p_2) = 2v_2a_1,\\
&d_{\Lambda V}(q_2) = 2v_2b_1,\ d_{\Lambda V}(r_2) = 2v_2c_1,\ d_{\Lambda V}(u_3) = v_2^2,
\end{aligned}$$

and filtration:

$$V(0) = \{a_1, b_1, c_1, v_2\} \subset V(1) = V.$$

Then, the output of Algorithm 2 is:

V	W	$d_{\Lambda W}$	f	g	ϕ
a_1	a_1	0	a_1	a_1	0
b_1	b_1	0	b_1	b_1	0
c_1	c_1	0	c_1	c_1	0
v_2			$2a_1b_1 - 2b_1c_1$		x_1
x_1			0		0
y_1	y_1	$2a_1b_1 - 2a_1c_1 - 4b_1c_1$	y_1	$y_1 - x_1$	0
p_2	p_2	$-4a_1b_1c_1$	p_2	$p_2 - 2a_1x_1$	0
q_2	q_2	0	q_2	$q_2 - 2b_1x_1$	0
r_2	r_2	$4a_1b_1c_1$	r_2	$r_2 - 2c_1x_1$	0
u_3	u_3	0	u_3	$u_3 - x_1v_2$ $-2a_1b_1x_1 + 2b_1c_1x_1$	0

Therefore, ΛW with $W = \mathbb{Q}\langle a_1, b_1, c_1, y_1, p_1, q_1, r_1 u_3\rangle$ is a minimal Sullivan algebra with the same (co)homology than ΛV. Besides, $U \oplus dU$ is generated by $Y = \{(x_1, v_2)\}$, obtained when running Algorithm 2, then ΛV is isomorphic to $\Lambda W \otimes \Lambda(U \oplus dU)$ being ΛW a minimal Sullivan algebra and $\Lambda(U \oplus dU)$ contractible. The basis of the minimal Sullivan algebra provided by the "ad hoc" method can be obtained using morphism g:

$$\begin{aligned}
&\big\{\, g(a_1) = a_1,\ g(b_1) = b_1,\ g(c_1) = c_1,\\
&\ g(y_1) = y_1 - x_1,\ g(p_2) = p_2 - 2a_1x_1,\ g(q_2) = q_2 - 2b_1x_1,\\
&\ g(r_2) = r_2 - 2c_1x_1,\ g(u_3) = u_3 - x_1v_2 - 2a_1b_1x_1 + 2b_1c_1x_1 \big\}
\end{aligned}$$

with differential:

$$\begin{aligned}
&d_{\Lambda V} g(a_1) = d_{\Lambda V} g(b_1) = d_{\Lambda V} g(c_1) = d_{\Lambda V} g(q_2) = d_{\Lambda V} g(u_3) = 0,\\
&d_{\Lambda V} g(y_1) = 2g(a_1)g(b_1) - 2g(a_1)g(c_1) - 4g(b_1)g(c_1),\\
&d_{\Lambda V} g(p_2) = -4g(a_1)g(b_1)g(c_1),\ d_{\Lambda V} g(r_2) = 4g(a_1)g(b_1)g(c_1).
\end{aligned}$$

And with respect to the contractible part, write $v_2' := v_2 - 2a_1b_1$. Then $d_{\Lambda V}(x_1) = v_2'$ and the new basis obtained for ΛV is:

$$\big\{g(a_1),\ g(b_1),\ g(c_1),\ g(y_1),\ g(p_2), g(q_2),\ g(r_2),\ g(u_3),\ x_1,\ v_2'\big\}.$$

Example 4. Consider the Sullivan algebra ΛV where $V = \{V^p\}_{p \geq 1}$ being:

$$\begin{aligned}
&V^1 = \mathbb{Q}\langle x_1\rangle,\ V^2 = \mathbb{Q}\langle v_2, w_2\rangle,\ V^3 = \mathbb{Q}\langle x_3\rangle,\ V^4 = \mathbb{Q}\langle v_4, w_4\rangle,\\
&V^5 = \mathbb{Q}\langle x_5\rangle,\ V^6 = \mathbb{Q}\langle x_7\rangle
\end{aligned}$$

with the differential $d_{\Lambda V}$ defined on V as follows:

$$\begin{aligned}
&d_{\Lambda V}(x_1) = v_2 + w_2,\ d_{\Lambda V}(x_3) = v_4 + w_4 + v_2w_2,\ d_{\Lambda V}(x_5) = v_4w_2 + v_2w_4\\
&d_{\Lambda V}(x_7) = v_4w_4,\ d_{\Lambda V}(v_2) = d_{\Lambda V}(w_2) = d_{\Lambda V}(v_4) = d_{\Lambda V}(w_4) = 0,
\end{aligned}$$

and filtration:

$$V(0) = \mathbb{Q}\langle v_2, w_2, v_4, w_4\rangle \subset V(1) = V.$$

Then, the output of Algorithm 2 is:

V	W	$d_{\Lambda W}$	f	g	ϕ
v_2	v_2	0	v_2	v_2	0
w_2			$-v_2$		x_1
v_4	v_4	0	v_4	v_4	0
w_4			$v_2^2 - v_4$		$-v_2x_1 + x_3$
x_1			0		0
x_3			0		0
x_5	x_5	$v_2^3 - 2v_2x_4$	x_5	$v_2^2x_1 - v_4x_1 - v_2x_3 + x_5$	0
x_7	x_7	$v_2^2v_4 - v_2^4$	x_7	$v_2v_4x_1 - v_4x_3 + x_7$	0

Therefore, ΛW with $W = \mathbb{Q}\langle v_2, v_4, x_5, x_7\rangle$ is a minimal Sullivan algebra with the same (co)homology than ΛV. Besides, $U \oplus dU$ is generated by $Y = \{(x_1, w_2), (x_3, w_4)\}$, obtained when running Algorithm 2 on V, then ΛV is isomorphic to $\Lambda W \otimes \Lambda(U \oplus dU)$ being ΛW a minimal Sullivan algebra and $\Lambda(U \oplus dU)$ contractible. The basis of the minimal Sullivan algebra provided by the "ad hoc" method can be obtained using morphism g:

$$\begin{aligned}&\big\{\, g(v_2) = v_2,\ g(v_4) = v_4,\\ &\quad g(x_5) = v_2^2 x_1 - v_4 x_1 - v_2 x_3 + x_5,\ g(x_7) = v_2 v_4 x_1 - v_4 x_3 + x_7 \big\}\end{aligned}$$

with differential:

$$\begin{aligned}&d_{\Lambda V} g(v_2) = d_{\Lambda V} g(v_4) = 0,\\ &d_{\Lambda V} g(x_5) = g(v_2)^3 - 2g(v_2)g(x_4),\ d_{\Lambda V} g(x_7) = g(v_2)^2 g(v_4) - g(v_2)^4.\end{aligned}$$

With respect to the contractible part, denote $w_2' = w_2 + v_2$ and $w_4' = w_4 + v_4 + v_2 w_2$, then the new basis obtained for ΛV is:

$$\big\{g(v_2),\ g(v_4),\ g(x_5),\ g(x_7),\ x_1,\ x_3,\ w_2',\ w_4'\big\}.$$

7. Conclusions and Future Works

In this paper, we provide an algorithm that computes the minimal Sullivan algebra from a Sullivan algebra not necessarily being minimal. The algorithm has been implemented, validated and tested with examples. The implementation made runs in CoCalc (https://cocalc.com/) and can be downloaded from http://grupo.us.es/\discretionary-cimagroup/downloads.htm.

As a future work, we plan to have a close look at the work made by Marco and Manero in [12] through the prism of chain contractions. Since they used the "ad hoc" approach of changing basis step by step, we believe it will be interesting to compute chain homotopies between the algebra and its minimal representation meanwhile the minimal representation is constructed. Algebraic properties of such homotopies will be also studied. Observe that, on one hand, their method can be applied to algebras not-necessarily being Sullivan whereas our algorithm can only be applied to Sullivan algebras, on the other hand, our algorithm produces the entire minimal Sullivan algebra when the input algebra is finitely generated whereas their method produces the minimal Sullivan algebra up to a fixed degree.

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