Results Math (2023) 78:196
© 2023 The Author(s)
1422-6383/23/050001-16
published online July 31, 2023
https://doi.org/10.1007/s00025-023-01970-y

Results in Mathematics

Extremal Structure of Projective Tensor Products

Luis C. García-Lirola, Guillaume Grelier,
Gonzalo Martínez-Cervantes, and Abraham Rueda Zoca

Abstract. We prove that, given two Banach spaces X and Y and bounded, closed convex sets $C \subseteq X$ and $D \subseteq Y$, if a nonzero element $z \in \overline{\text{co}}(C \otimes D) \subseteq X\widehat{\otimes}_\pi Y$ is a preserved extreme point then $z = x_0 \otimes y_0$ for some preserved extreme points $x_0 \in C$ and $y_0 \in D$, whenever $K(X, Y^*)$ separates points of $X\widehat{\otimes}_\pi Y$ (in particular, whenever X or Y has the compact approximation property). Moreover, we prove that if $x_0 \in C$ and $y_0 \in D$ are weak-strongly exposed points then $x_0 \otimes y_0$ is weak-strongly exposed in $\overline{\text{co}}(C \otimes D)$ whenever $x_0 \otimes y_0$ has a neighbourhood system for the weak topology defined by compact operators. Furthermore, we find a Banach space X isomorphic to ℓ_2 with a weak-strongly exposed point $x_0 \in B_X$ such that $x_0 \otimes x_0$ is not a weak-strongly exposed point of the unit ball of $X\widehat{\otimes}_\pi X$.

Mathematics Subject Classification. Primary 46B08, 46B28.

Keywords. Banach space, projective tensor product, preserved extreme point, strongly exposed point.

Research partially supported by Agencia Estatal de Investigación and EDRF/FEDER "A way of making Europe" (MCIN/AEI/10.13039/501100011033) through Grants PID2021-122126NB-C32 and PID2021-122126NB-C31 (Rueda Zoca), by Fundación Séneca Región de Murcia 21955/PI/22. The research of García-Lirola was also supported by DGA project E48-23R. The research of G. Grelier was also supported by MICINN 2018 FPI fellowship with reference PRE2018-083703. The research of Abraham Rueda Zoca was also supported by Junta de Andalucía Grants FQM-0185 and PY20_00255.

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1. Introduction

One of the most celebrated and earlier results in Functional Analysis is Krein–Milman theorem. This result establishes that if K is a compact convex subset of a locally convex space then $K = \overline{\text{co}}(\text{ext}\,(K))$, where $\text{ext}\,(K)$ denotes the set of extreme points of K (see e.g. [16, Theorem 3.22]). An example of application is to the unit ball of a dual Banach space X^*, where it yields that $B_{X^*} = \overline{\text{co}}^{w^*}(\text{ext}\,(B_{X^*}))$. This result is of capital importance because, thanks to Hahn–Banach theorem, the structure of the geometry of a Banach space X is determined by the dual unit ball B_{X^*}. Thus, Krein–Milman theorem tells us that the set $\text{ext}\,(B_{X^*})$ codifies all the geometric information of the space.

The identification of the extreme points (and related notions as exposed, denting, or strongly exposed points) on particular classes of Banach spaces has attracted the attention of many researchers in functional analysis, especially in spaces where the definition of the norm is of high complexity, see e.g. [5,17,18] for duals of spaces of compact operators, [14] for Orlicz–Lorentz spaces, [11] for Kothe–Bochner spaces, or, more recently, [1–3,8,9] for Lipschitz-free spaces. In this note, we will focus on projective tensor products. Denoted by $X\widehat{\otimes}_\pi Y$, the projective tensor product is the completion of the algebraic tensor product $X \otimes Y$ endowed with the norm

$$\|z\|_\pi := \inf \left\{ \sum_{i=1}^n \|x_i\|\|y_i\| : z = \sum_{i=1}^n x_i \otimes y_i \right\},$$

where the infimum is taken over all such representations of z. Recall also that $B_{X\widehat{\otimes}_\pi Y} = \overline{\text{co}}(B_X \otimes B_Y)$.

In the analysis of the extremal structure of the projective tensor product we distinguish two lines. The first one is the exhaustive analysis of the extreme points in duals of operators spaces done by Collins and Ruess [5] and by Ruess and Stegall [18]. They established that, given two Banach spaces X and Y, the extreme points of the dual unit ball of the w^*-to-w continuous compact operators $K_{w^*}(X^*, Y)$ are the elements of the form $x^* \otimes y^*$, for $x^* \in \text{ext}\,(B_{X^*})$ and $y^* \in \text{ext}\,(B_{Y^*})$. As a consequence of a classical result of tensor product theory [19, Theorem 5.33], if X^* or Y^* has the Radon–Nikodym property and X^* or Y^* has the approximation property then

$$\text{ext}\left(B_{X^*\widehat{\otimes}_\pi Y^*}\right) = \text{ext}\,(B_{X^*}) \otimes \text{ext}\,(B_{Y^*})\,.$$

Little is known without the duality assumptions. Indeed, up to our knowledge, it is an open question whether every extreme point of $B_{X\widehat{\otimes}_\pi Y}$ must be of the form $x \otimes y$ for $x \in B_X$ and $y \in B_Y$. The situation clarifies for the stronger notions of denting points and strongly exposed points. Ruess and Stegall proved in [17] that

$$\text{strexp}\left(B_{X\widehat{\otimes}_\pi Y}\right) = \text{strexp}\,(B_X) \otimes \text{strexp}\,(B_Y)\,.$$

Furthermore, if x^* strongly exposes x in B_X and y^* strongly exposes y in B_Y, then $x^* \otimes y^*$ strongly exposes $x \otimes y$ in $B_{X\widehat{\otimes}_\pi Y}$. For denting points, D. Werner proved in [20] an analogous result in a more general framework:

$$\operatorname{dent}\left(\overline{\operatorname{co}}(C \otimes D)\right) = \operatorname{dent}(C) \otimes \operatorname{dent}(D)$$

whenever $C \subset X$ and $D \subset Y$ are closed bounded and absolutely convex subsets.

Motivated by the above results, in this note we study the notions of preserved extreme point and weak-strongly exposed point in projective tensor products. Recall that a point $x \in C$ is a *preserved extreme point* of C (also called *weak*-extreme point*) if it is an extreme point of $\overline{C}^{w^*} \subset X^{**}$; this is a stronger notion than being extreme but weaker than being denting (see e.g. [10]).

With this notation in mind, one of the main results of the present paper is the following one.

Theorem 1.1. *Let X and Y be Banach spaces such that $K(X, Y^*)$ is separating for $X\widehat{\otimes}_\pi Y$ (in particular, if the pair (X, Y^*) has the CAP). Let $C \subseteq X$, $D \subseteq Y$ be bounded closed convex subsets. If z is a preserved extreme point of $\overline{\operatorname{co}}(C \otimes D) \subset X\widehat{\otimes}_\pi Y$ then $z = x \otimes y$ for some $x \in C$ and $y \in D$. Moreover, if $z \neq 0$ then x and y are preserved extreme points of C and D respectively.*

As a particular case we get:

Corollary 1.2. *Let X and Y be Banach spaces such that $K(X, Y^*)$ is separating for $X\widehat{\otimes}_\pi Y$ (in particular, if the pair (X, Y^*) has the CAP). If z is a preserved extreme point of $B_{X\widehat{\otimes}_\pi Y}$, then $z = x \otimes y$ where x and y are preserved extreme points of B_X and B_Y respectively.*

Theorem 1.1 points out that, in order to look for preserved extreme points in projective tensor products, we only have to pay attention to basic tensors. We do not know whether the converse holds. However, we will prove a kind of converse for w-strongly exposed points.

A point $x \in C$ is said to be *exposed* if there exists $x^* \in X^*$ such that $x^*(x) > x^*(y)$ for all $y \in C\backslash\{x\}$. We also say that x^* exposes x in C. A point $x \in C$ is said *strongly exposed* (resp. *w-strongly exposed*) if there exists $x^* \in X^*$ exposing x and such that for all sequences $(x_n)_n \subset C$ such that $x^*(x_n) \underset{n}{\longrightarrow} x^*(x)$, it follows that $x_n \underset{n}{\longrightarrow} x$ (resp. $x_n \overset{w}{\underset{n}{\longrightarrow}} x$). Equivalently, the slices of C produced by x^* are a neighbourhood basis of x for the norm (resp. weak) topology in C.[1] In this case, we write $x \in \operatorname{strexp}(C)$ (resp. $x \in w\text{-}\operatorname{strexp}(C)$).

Theorem 1.3. *Let X and Y be Banach spaces such that $K(X, Y^*)$ is separating for $X\widehat{\otimes}_\pi Y$ (in particular, if the pair (X, Y^*) has the CAP). Let $C \subseteq X$ and*

[1]This notation should not be confused with the one in [17], where a point in B_{X^*} is called weak*-strongly exposed if it is strongly exposed by an element of X.

$D \subseteq Y$ be symmetric, bounded closed convex subsets. Assume that $x \otimes y \neq 0$ has a compact neighbourhood system for the weak topology in $\overline{\text{co}}(C \otimes D) \subset X \widehat{\otimes}_\pi Y$. Then the following are equivalent:

(i) *$x \otimes y$ is w-strongly exposed in $\overline{\text{co}}(C \otimes D)$.*
(ii) *x and y are w-strongly exposed in C and D, respectively.*

In particular, if $C \otimes D$ is weakly compact, then

$$w\text{-strexp}\,(\overline{\text{co}}(C \otimes D)) = w\text{-strexp}\,(C) \otimes w\text{-strexp}\,(D).$$

The assumption that $x \otimes y$ has a compact neighbourhood system in the above result might seem to be artificial but, surprisingly or not, it cannot be removed. Indeed, in Example 3.8 we find a Banach space X which is isomorphic to ℓ_2 satisfying that there exists a w-strongly exposed point $x_0 \in B_X$ and such that $x_0 \otimes x_0$ is not a w-strongly exposed point of $B_{X \widehat{\otimes}_\pi X}$.

2. Notation and Preliminary Results

Throughout the paper we will only deal with real Banach spaces. Let C be a bounded subset of a Banach space X. Given $x^* \in X^*$ and $\alpha > 0$, we denote

$$S(C, x^*, \alpha) = \{x \in C : x^*(x) > \sup x^*(C) - \alpha\}$$

the (open) slice of C produced by x^*.

We say that $x \in C$ is *extreme* if the condition $x = \frac{y+z}{2}$ with $y, z \in C$ implies $y = z$. We write $x \in \text{ext}\,(C)$.

A point $x \in C$ is a *preserved extreme point* (or a *w^*-extreme point*) if x is an extreme point of $\overline{C}^{w^*}$. It can be proved that x is a preserved extreme point if and only if the open slices containing x form a basis for x in the weak topology induced on C (see [15]). This characterization will be used twice in the proof of Theorem 1.1 without further mention. Notice that, in particular, every w-strongly exposed point is a preserved extreme point.

Let us write here the following lemma, which we will use systematically throughout the text. This is a well-known result (see e.g. [13, Lemma 7.21], a preprint version of [12]) but we include a proof for completeness.

Lemma 2.1. *Let X be a Banach space. Let A be a bounded subset of X and write $C = \overline{\text{co}}(A)$. Let $R := \sup_{x \in A} \|x\|$. Then, given $x^* \in X^*$, we have:*

(1) *$\sup_{x \in A} x^*(x) = \sup_{x \in C} x^*(x)$.*
(2) *Given $0 < \varepsilon < \frac{1}{2}$ we have that*

$$S(C, x^*, \varepsilon^2) \subseteq \text{co}(S(A, x^*, \varepsilon)) + 4R\varepsilon B_X.$$

Proof. (1) is clear. Let's prove (2). First, take an element of $S(\text{co}(A), x^*, \varepsilon^2)$, which is a (finite) convex combination of the form $\sum_{n \in \mathbb{N}} \lambda_n a_n$ where $a_n \in A$ for every n and

$$\sup x^*(A) - \varepsilon^2 < \sum_n \lambda_n x^*(a_n).$$

Put $J := \{n \in \mathbb{N} : \sup x^*(A) - \varepsilon < x^*(a_n)\}$. Then

$$\begin{aligned}\sup x^*(A) - \varepsilon^2 &< \sum_{n\in J} \lambda_n x^*(a_n) + \sum_{n\notin J} \lambda_n x^*(a_n)\\ &\leqslant \left(\sum_{n\in J} \lambda_n\right) \sup x^*(A) + (\sup x^*(A) - \varepsilon) \sum_{n\notin J} \lambda_n\\ &= \sup x^*(A) - \varepsilon \sum_{n\notin J} \lambda_n\end{aligned}$$

which allows to deduce that $\sum_{n\notin J} \lambda_n < \varepsilon$. Since $a_n \in S(A, x^*, \varepsilon)$ for each $n \in J$, the result follows from the following estimation:

$$\begin{aligned}\left\| \sum_{n\in J} \left(\frac{\lambda_n}{\sum_{n\in J} \lambda_n}\right) a_n - \sum_{n\in\mathbb{N}} \lambda_n a_n \right\| &\leqslant \left\| \sum_{n\in J} \left(\frac{\lambda_n}{\sum_{n\in J} \lambda_n} - \lambda_n\right) a_n \right\| + \left\| \sum_{n\notin J} \lambda_n a_n \right\|\\ &\leqslant \left| \frac{1}{\sum_{n\in J} \lambda_n} - 1 \right| \cdot \left\| \sum_{n\in J} \lambda_n a_n \right\| + \left\| \sum_{n\notin J} \lambda_n a_n \right\|\\ &\leqslant \frac{\varepsilon}{1-\varepsilon} R + \varepsilon R = \frac{2\varepsilon - \varepsilon^2}{1-\varepsilon} R.\end{aligned}$$

This shows that

$$S(\operatorname{co} A, x^*, \varepsilon^2) \subseteq S(\operatorname{co} A, x^*, \varepsilon) + \frac{2\varepsilon - \varepsilon^2}{1-\varepsilon} R B_X.$$

Finally,

$$S(C, x^*, \varepsilon^2) \subset \overline{S(\operatorname{co}(A), x^*, \varepsilon^2)} \subset S(\operatorname{co} A, x^*, \varepsilon) + 4\varepsilon R B_X$$

since $\frac{2\varepsilon-\varepsilon^2}{1-\varepsilon} < 4\varepsilon$ for $\varepsilon < 1/2$. □

Given two Banach spaces X, Y, we denote $L(X,Y)$, $K(X,Y)$ and $F(X,Y)$ the spaces of linear, compact, and finite-rank operators, respectively. Recall that $(X \widehat{\otimes}_\pi Y)^* = L(X, Y^*)$ isometrically. We refer the reader to [19] for basic properties of tensor products. We denote by τ_c the topology of compact convergence in $L(X,Y)$, i.e. the topology of uniform convergence on compact subsets of X. It is well known that X has the approximation property if $\overline{F(X,X)}^{\tau_c} = L(X,X)$, whereas it has the compact approximation property if $\overline{K(X,X)}^{\tau_c} = L(X,X)$. The definition of the approximation property was extended to pairs of Banach spaces by E. Blonde in [4] as follows: The pair (X,Y) is said to have the approximation property if $\overline{F(X,Y)}^{\tau_c} = L(X,Y)$. In a similar fashion we say that the pair (X,Y) has the compact approximation property (CAP for short) if $\overline{K(X,Y)}^{\tau_c} = L(X,Y)$ (see, for instance, [6]). Notice that for any set $S \subset (X \widehat{\otimes}_\pi Y)^* = L(X, Y^*)$, we have $\overline{S}^{\tau_c} \subset \overline{S}^{w^*}$. Since for a subspace $Z \subseteq X^*$ to separate points of X is equivalent to the equality $\overline{Z}^{w^*} = X^*$, we have the following lemma:

Lemma 2.2. *Let X, Y be two Banach spaces. If the pair (X, Y^*) has the CAP, then $K(X, Y^*)$ separates points of $X\widehat{\otimes}_\pi Y$.*

We will make use of the previous lemma throughout the text without further mention. We finish this section by recalling that the pair (X, Y^*) has the CAP if and only if the pair (Y, X^*) has the CAP. It is immediate that if X or Y has the compact approximation property or the approximation property then the pair (X, Y^*) has the CAP. As a consequence of [4, Example 4.2], for every $1 \leqslant p < 2 < q < \infty$ and every subspaces $X \subset \ell_q$ and $Y \subset \ell_p$, the pair (X, Y) has the CAP. Nevertheless, there are such subspaces X and Y failing the compact approximation property.

In order to prove our results about extremal structure, we need the following topological result which is of independent interest.

Theorem 2.3. *Let X and Y be two Banach spaces such that $K(X, Y^*)$ is separating for $X\widehat{\otimes}_\pi Y$ (in particular, if the pair (X, Y^*) has the CAP). Let $C \subseteq X$ and $D \subseteq Y$ be two bounded subsets. Then the weak-closure of $C \otimes D$ in $X\widehat{\otimes}_\pi Y$ is equal to $\overline{C}^w \otimes \overline{D}^w$, that is $\overline{C \otimes D}^w = \overline{C}^w \otimes \overline{D}^w$.*

Proof. First, given $x \in \overline{C}^w$, we have that the operator $i\colon Y \to X \otimes Y$ given by $y \mapsto x \otimes y$ is continuous. Thus, it is also weak-to-weak continuous, so

$$\{x\} \otimes \overline{D}^w = i(\overline{D}^w) \subseteq \overline{i(D)}^w = \overline{\{x\} \otimes D}^w.$$

This shows that $\overline{C}^w \otimes \overline{D}^w \subseteq \overline{\overline{C}^w \otimes D}^w$. Analogously, we get $\overline{C}^w \otimes D \subseteq \overline{C \otimes D}^w$ and so

$$\overline{C}^w \otimes \overline{D}^w \subseteq \overline{\overline{C \otimes D}^w}^w = \overline{C \otimes D}^w.$$

Now, given $z \in \overline{C \otimes D}^w$, take a net $(x_s \otimes y_s)$ in $C \otimes D$ such that $x_s \otimes y_s \to z$ weakly, and let us prove that $z = x \otimes y$ for certain $x \in \overline{C}^w$ and $y \in \overline{D}^w$. We denote by $\overline{C}^{w^*}$ and $\overline{D}^{w^*}$ respectively the closure of C and D in the w^* topology of X^{**} and Y^{**} respectively, which are w^*-compact because they are bounded.

Since $(x_s)_s \subset \overline{C}^{w^*}$ and $(y_s)_s \subset \overline{D}^{w^*}$ we can assume, up to taking a suitable subnet, that both $x_s \to x^{**}$ in the w^*-topology of X^{**} and $y_s \to y^{**}$ in the w^*-topology of Y^{**}. □

Claim 2.4. *For any compact operator $K\colon X \longrightarrow Y^*$, we have that*

$$K(x_s)(y_s) \to K^{**}(x^{**})(y^{**}).$$

Proof of the claim. First, recall that $K^{**}\colon X^{**} \longrightarrow Y^{***}$ is a compact operator which satisfies $K^{**}(X^{**}) \subseteq Y^*$. Fix $\varepsilon > 0$. We claim that there exists s_0 such that $|K(x_s)(y_s) - K^{**}(x^{**})(y^{**})| < \varepsilon$ for every $s \geqslant s_0$. Namely, we know that, since K^{**} is compact, $K^{**}(x^{**}) \in Y^*$ and $y_s \to y^{**}$ in the w^*-topology, there exists s_0 such that

$$\|K(x_s) - K^{**}(x^{**})\| < \varepsilon/(2R) \quad \text{and} \quad |K^{**}(x^{**})(y_s) - K^{**}(x^{**})(y^{**})| < \varepsilon/2$$

for every $s \geqslant s_0$, where $R > 0$ is such that $D \subset RB_Y$. Then

$$|K(x_s)(y_s) - K^{**}(x^{**})(y^{**})| \leqslant \|K(x_s) - K^{**}(x^{**})\|\|y_s\| \\ + |K^{**}(x^{**})(y_s) - K^{**}(x^{**})(y^{**})| < \varepsilon$$

for every $s \geqslant s_0$ as desired. □

Now, we claim we can assume $x^{**} \neq 0$ and $y^{**} \neq 0$. Indeed, if $x^{**} = 0$ this would imply $0 \in \overline{C}^w$. Moreover, since $z(K) = (0 \otimes y^{**})(K) = 0$ holds for every $K \in K(X, Y^*)$, which is separating for $X \widehat{\otimes}_\pi Y$, we would get that $z = 0$ so, taking any $y \in D$, we have $z = 0 \otimes y \in \overline{C}^w \otimes \overline{D}^w$ and the proof would be finished. Henceforth, we assume $x^{**} \neq 0$ and $y^{**} \neq 0$ and, clearly, the above mentioned equality $z(K) = (x^{**} \otimes y^{**})(K)$ holding true for every $K \in K(X, Y^*)$ implies $z \neq 0$ too.

Claim 2.5. *$x^{**} \in X$ and $y^{**} \in Y$.*

Proof of the claim. Let us prove that x^{**} is w^*-continuous, the proof for y^{**} being completely analogous. Take $y^* \in S_{Y^*}$ such that $y^{**}(y^*) \neq 0$. Now we have that

$$x^{**}(x^*) = \frac{(x^{**} \otimes y^{**})(x^* \otimes y^*)}{y^{**}(y^*)} = \frac{(x^* \otimes y^*)(z)}{y^{**}(y^*)} \quad \forall x^* \in X^*.$$

Thus, to see that x^{**} is weak*-continuous it suffices to show that $(x_s^* \otimes y^*)(z) \to (x^* \otimes y^*)(z)$ whenever $x_s^* \overset{w^*}{\to} x^*$. This is a consequence of the fact that the operator $X^* \to L(X, Y^*)$ given by $x^* \mapsto x^* \otimes y^*$ is w^*-to-w^*-continuous as being the adjoint of the operator $X \widehat{\otimes}_\pi Y \to X$ given by $x \otimes y \mapsto y^*(y)x$. □

At this point we will save notation calling $x := x^{**} \in X$ and $y := y^{**} \in Y$. Now we have that $K(z) = K(x \otimes y)$ holds for every $K \in K(X, Y^*)$. Since $K(X, Y^*)$ is separating for $X \widehat{\otimes}_\pi Y$, we deduce that $z = x \otimes y$. Moreover, observe that $x_s \to x$ in the weak topology of X. Since $x_s \in C$ for every s we conclude that $x \in \overline{C}^w$. Analogously, $y \in D$, so $z = x \otimes y \in \overline{C}^w \otimes \overline{D}^w$, which finishes the proof. □

In spite of the fact that, under the approximation property, the tensor product of weakly closed sets is weakly closed (see e.g. Theorem 2.3), it is interesting to notice that if C and D are weakly compact, it does not follow that $C \otimes D$ is weakly compact in $X \widehat{\otimes}_\pi Y$ (for instance, if we take $C = D = B_{\ell_2}$, then the sequence $(e_n \otimes e_n)_n$ is equivalent to the ℓ_1-basis, c.f. e.g. [19, Example 2.10]).

3. Main Results

The aim of this section is to present the proof of Theorems 1.1 and 1.3. We start with the proof of Theorem 1.1.

Proof of Theorem 1.1. Since z is a preserved extreme point, there is a neighbourhood basis $\{S_\alpha\}$ of z for the weak-topology of $\overline{\text{co}}(C \otimes D)$ so that S_α is a slice for every α. Now, since S_α is a slice of $\overline{\text{co}}(C \otimes D)$ we can find $x_\alpha \otimes y_\alpha \in S_\alpha \cap (C \otimes D)$ for every α. Since S_α is a weak basis for the weak topology at z we get that $z \in \overline{\{x_\alpha \otimes y_\alpha\}}^w$, and now Theorem 2.3 and the fact that C and D are weakly closed imply that $z = x \otimes y$ for certain $x \in C$ and $y \in D$.

If $z \neq 0$ it is not difficult to prove that x and y are preserved extreme points of C and D. Indeed, if $S(\overline{\text{co}}(C \otimes D), T_\alpha, \beta_\alpha)$ is a neighbourhood system of $x \otimes y$ for the weak topology in $\overline{\text{co}}(C \otimes D)$, then the family of slices S'_α defined as

$$S'_\alpha := \{x' \in C : T_\alpha(x')(y) > 1 - \beta_\alpha\}$$

is a neighbourhood system of x for the weak topology of X. □

An immediate consequence of Theorem 1.1 is the following corollary.

Corollary 3.1. *Let X and Y be Banach spaces such that $K(X, Y^*)$ is separating for $X \widehat{\otimes}_\pi Y$ (in particular, if the pair (X, Y^*) has the CAP). Let $C \subseteq X$ and $D \subseteq Y$ be convex bounded subsets. If z is a w-strongly exposed point of $\overline{\text{co}}(C \otimes D)$ then $z = x \otimes y$ for some $x \in C$ and $y \in D$. Moreover, if $z \neq 0$ then x and y are w-strongly exposed points of C and D respectively.*

Proof. Theorem 1.1 provides points $x \in C$ and $y \in D$ such that $z = x \otimes y$. It remains to prove that x and y are w-strongly exposed points if $z \neq 0$. Let $T \in L(X, Y^*)$ w-strongly exposing $x \otimes y$ in $\overline{\text{co}}(C \otimes D)$, and define $f \in X^*$ by $f(v) := T(v)(y)$. It is immediate that f w-strongly exposes x in C. The argument for y is analogous. □

Now we will analyse a possible converse for Corollary 3.1. The first result we find is the following.

Proposition 3.2. *Let X, Y be Banach spaces. Let $C \subseteq X$ and $D \subseteq Y$ be bounded and symmetric convex subsets. Let x_0 be a strongly exposed point of C and y_0 be a w-strongly exposed point of D. Then $x_0 \otimes y_0$ is a w-strongly exposed point of $\overline{\text{co}}(C \otimes D)$.*

Proof. By homogeneity, we may assume that $C \subseteq B_X$ and $D \subseteq B_Y$, so $R := \sup_{z \in C \otimes D} \|z\| \leqslant 1$. Assume that x^* strongly exposes x_0 in C and y^* w-strongly exposes y_0 in D. We may also assume that $\sup x^*(C) = 1 = \sup y^*(D)$. Let us prove that $x^* \otimes y^*$ w-strongly exposes $x_0 \otimes y_0$ in $\overline{\text{co}}(C \otimes D)$. To this end, pick $U := \bigcap_{i=1}^{n} S(\overline{\text{co}}(C \otimes D), T_i, \alpha_i)$ to be a relatively weakly open subset of $\overline{\text{co}}(C \otimes D)$ containing $x_0 \otimes y_0$, with $\|T_i\| = 1$ for each i, and let us prove that $S(\overline{\text{co}}(C \otimes D), x^* \otimes y^*, \beta) \subseteq U$ for a suitable β. Notice that

$$(x^* \otimes y^*)(x_0 \otimes y_0) = x^*(x_0)y^*(y_0) = 1 = \sup_{z \in \overline{\text{co}}(C \otimes D)} (x^* \otimes y^*)(z)$$

(here we use Lemma 2.1).

Since $x_0 \otimes y_0 \in U$, we have $T_i(x_0)(y_0) > \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z) - \alpha_i$ for every $1 \leqslant i \leqslant n$. Thus we can find $\varepsilon_0 > 0$ so that $T_i(x_0)(y_0) > \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z) - \alpha_i + \varepsilon_0$ for every i.

Since x^* strongly exposes x_0, there is $\delta' > 0$ such that $\operatorname{diam}(S(C, x^*, \delta')) < \frac{\varepsilon_0}{4}$. Moreover, notice that

$$y_0 \in \bigcap_{i=1}^{n} \left\{ y \in D : T_i(x_0)(y) > \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z) - \alpha_i + \varepsilon_0 \right\},$$

which is a relatively weakly open subset of D containing y_0. Since y_0 is weakly exposed by y^* we can find $\delta'' > 0$ such that

$$S(D, y^*, \delta'') \subseteq \bigcap_{i=1}^{n} \left\{ y \in D : T_i(x_0)(y) > \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z) - \alpha_i + \varepsilon_0 \right\}.$$

Take $\delta := \min\{\delta', \delta'', \varepsilon_0/4\}$. Consider finally the slice $S(\overline{\text{co}}(C\otimes D), x^*\otimes y^*, \eta^2)$, where $0 < \eta < \delta/4$. Let us prove that the previous slice is contained in U. To this end, notice that

$$\begin{aligned} S(\overline{\text{co}}(C\otimes D), x^*\otimes y^*, \eta^2) &\subseteq \text{co}(S(C\otimes D, x^*\otimes y^*, \eta)) + 4\eta B_{X\widehat{\otimes}_\pi Y} \\ &\subseteq \text{co}(S(C\otimes D, x^*\otimes y^*, \delta)) + \delta B_{X\widehat{\otimes}_\pi Y} =: A \end{aligned}$$

thanks to Lemma 2.1 and the choice of η. So, it suffices to prove that $A \subseteq U$. To this end, pick $x\otimes y \in S(C\otimes D, x^*\otimes y^*, \delta)$. This means that $x^*(x)y^*(y) > 1-\delta$, from where $x^*(x) > 1-\delta \geqslant 1-\delta'$ and $y^*(y) > 1-\delta \geqslant 1-\delta''$. By the definition of δ' and δ'' we get that $\|x-x_0\| < \frac{\varepsilon_0}{4}$ and $T_i(x_0)(y) > \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z)-\alpha_i+\varepsilon_0$ for every i. Hence

$$\begin{aligned} T_i(x)(y) \geqslant T_i(x_0)(y) - \|T_i\|\|x-x_0\|\|y\| &> \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z) - \alpha_i + \varepsilon_0 - \frac{\varepsilon_0}{4} \\ &= \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z) - \alpha_i + \frac{3\varepsilon_0}{4}. \end{aligned}$$

An easy convexity argument implies that

$$T_i(u) > \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z) - \alpha_i + \frac{3\varepsilon_0}{4} \quad \forall u \in \text{co}(S(C\otimes D, x^*\otimes y^*, \delta)).$$

Now, given $u \in A$, we have $u = v + w$ with $v \in \text{co}(S(C\otimes D, x^*\otimes y^*, \delta))$ and $\|w\| \leqslant \delta \leqslant \varepsilon_0/4$. Then,

$$\begin{aligned} T_i(u) = T_i(v) + T_i(w) &\geqslant \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z) - \alpha_i + \frac{3\varepsilon_0}{4} - \|w\| \\ &\geqslant \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z) - \alpha_i + \frac{\varepsilon_0}{2} > \sup_{z\in\overline{\text{co}}(C\otimes D)} T_i(z) - \alpha_i \end{aligned}$$

for each i. We conclude that $u \in U$, which proves that $A \subseteq U$ and the proof is finished. □

Note that in Proposition 3.2 we obtain a compact operator ($T\colon X \to Y^*$ given by $T(x) = x^*(x)y^*$) providing a neighbourhood basis for $x_0 \otimes y_0$ for the weak topology in $\overline{\text{co}}(C \otimes D)$. This motivates to consider the following notion.

Definition 3.3. Let X and Y be Banach spaces, and let $C \subseteq X$, $D \subseteq Y$ be two subsets. We say that $x \otimes y \in C \otimes D$ *has a compact neighbourhood system for the weak topology in* $\overline{\text{co}}(C \otimes D)$ if, given any weakly open subset U containing $x_0 \otimes y_0$, there are slices $S(\overline{\text{co}}(C \otimes D), T_i, \alpha_i)$ given by compact operators $T_i \in K(X, Y^*)$ such that

$$x_0 \otimes y_0 \in \bigcap_{i=1}^{n} S(\overline{\text{co}}(C \otimes D), T_i, \alpha_i) \subseteq U.$$

Remark 3.4. (a) The above definition has an easy interpretation in terms of nets: $x \otimes y$ has a compact neighbourhood system for the weak topology in $\overline{\text{co}}(C \otimes D)$ if, and only if, given a net $(z_\alpha)_\alpha \subset \overline{\text{co}}(C \otimes D)$, the condition $T(z_\alpha) \to T(x \otimes y)$ for every $T \in K(X, Y^*)$ implies $z_\alpha \to x \otimes y$ in the weak topology on $X \widehat{\otimes}_\pi Y$. Equivalently, $x \otimes y$ is a point of continuity of the formal identity

$$I\colon (\overline{\text{co}}(C \otimes D), w) \longrightarrow (\overline{\text{co}}(C \otimes D), \sigma(X \widehat{\otimes}_\pi Y, K(X, Y^*))).$$

(b) In the case that $K(X, Y^*)$ is separating for $X \widehat{\otimes}_\pi Y$ (in particular, if the pair (X, Y^*) has the CAP) and $C \otimes D$ is weakly compact, $\overline{\text{co}}(C \otimes D)$ is also weakly compact by Krein–Smulyan theorem (see e.g. [7, Theorem II. 2.11]) and $\sigma(X \widehat{\otimes}_\pi Y, K(X, Y^*))$ is Hausdorff because $K(X, Y^*)$ is separating for $L(X, Y^*)$. Thus, the identity map I above is a homeomorphism and so every $x \otimes y \in C \otimes D$ has a compact neighbourhood system.

Now we are ready to present the proof of Theorem 1.3.

Proof of Theorem 1.3. (i)$\Rightarrow$ (ii) follows from Corollary 3.1.

(ii)$\Rightarrow$(i). Write $R := \sup_{z \in C \otimes D} \|z\|$. Take $x_0^* \in X^*$ and $y_0^* \in Y^*$ w-strongly exposing x_0 and y_0 in C and D, respectively, with $x_0^*(x_0) = \sup x_0^*(C) = 1$, and $y_0^*(y_0) = \sup y_0^*(D) = 1$. Pick U to be a weak neighbourhood of $x_0 \otimes y_0$ in $\overline{\text{co}}(C \otimes D)$. By the assumption, we can assume that $U = \bigcap_{i=1}^{n} S(\overline{\text{co}}(C \otimes D), T_i, \alpha_i)$ for certain compact operators $T_1, \ldots, T_n\colon X \to Y^*$. Furthermore, we can assume $\sup_{\overline{\text{co}}(C \otimes D)} T_i = 1$ for every i. Let η small enough so that $T_i(x_0 \otimes y_0) > 1 - \alpha_i + \eta$ holds for every $1 \leqslant i \leqslant n$. Moreover, observe that $x_0 \in \bigcap_{i=1}^{n} \{z \in C : T_i(z)(y_0) > 1 - \alpha_i + \eta\}$, which is a relatively weakly open subset of C. Since x_0^* w-strongly exposes x_0 then there exists $\delta' > 0$ so that

$$x_0 \in S(C, x_0^*, \delta') \subseteq \bigcap_{i=1}^{n} \{z \in C : T_i(z)(y_0) > 1 - \alpha_i + \eta\}.$$

Now, for every $1 \leqslant i \leqslant n$, the set $T_i(S(C, x_0^*, \delta'))$ is a relatively compact subset of Y^*. Using the compactness condition on all the $T_i's$ we can find a finite set $x_1, \ldots, x_m \in S(C, x_0^*, \delta')$ so that $B(T_i(x_j), \frac{\eta}{8}), 1 \leqslant j \leqslant m$, is a covering of $T_i(S(C, x_0^*, \delta'))$ for every $1 \leqslant i \leqslant n$. Observe that $T_i(x_j)(y_0) > 1 - \alpha_i + \eta$ holds for every $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Consequently,

$$y_0 \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} \{y \in D : T_i(x_j)(y) > 1 - \alpha_i + \eta\}.$$

Since y_0^* w-strongly exposes y_0 we can find $\delta'' > 0$ so that

$$y_0 \in S(B_Y, y_0^*, \delta'') \subseteq \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} \{y \in D : T_i(x_j)(y) > 1 - \alpha_i + \eta\}.$$

We claim now that

$$S(C, x_0^*, \delta') \otimes S(D, y_0^*, \delta'') \subset \bigcap_{i=1}^{n} S\left(\overline{\text{co}}(C \otimes D), T_i, \alpha_i - \frac{\eta}{2}\right).$$

Indeed, let $x \in S(C, x_0^*, \delta')$ and $y \in S(D, y_0^*, \delta'')$. We have, for every $i \in \{1, \ldots, n\}$, an index $j_i \in \{1, \ldots, m\}$ such that $\|T_i(x) - T_i(x_{j_i})\| < \frac{\eta}{2}$. On the other hand, since $S(D, y_0^*, \delta'') \subseteq \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} \{y \in D : T_i(x_j)(y) > 1 - \alpha_i + \eta\}$ we have that, for every $1 \leqslant i \leqslant n$, $T_i(x_{j_i})(y) > 1 - \alpha_i + \eta$. Consequently

$$T_i(x)(y) \geqslant T_i(x_{j_i})(y) - \|T_i(x_{j_i}) - T_i(x)\| > 1 - \alpha_i + \eta - \frac{\eta}{2} = 1 - \alpha_i + \frac{\eta}{2}.$$

Take $\delta := \min\{\delta', \delta'', \frac{\eta}{8}, \frac{\eta}{8R}\}$ and consider $S := S(\overline{\text{co}}(C \otimes D), x_0^* \otimes y_0^*, \delta^2)$. Observe that $x_0 \otimes y_0 \in S$. Moreover,

$$S \subseteq \text{co}(S(C \otimes D, x_0^* \otimes y_0^*, \delta)) + 4R\delta B_{X \widehat{\otimes}_\pi Y}$$

in virtue of Lemma 2.1. Now, given $1 \leqslant i \leqslant n$, since $1 - \delta > \max\{1 - \delta', 1 - \delta''\}$ we conclude that every element $x \otimes y$ of $S(C \otimes D, x_0^* \otimes y_0^*, \delta)$ satisfies $x_0^*(x) > 1 - \delta'$ and $y_0^*(y) > 1 - \delta''$, so $T_i(x)(y) > 1 - \alpha_i + \frac{\eta}{2}$. Since T_i is a linear continuous functional on $X \widehat{\otimes}_\pi Y$ we conclude that $T_i(z) \geqslant 1 - \alpha_i + \frac{\eta}{2}$ holds for every $1 \leqslant i \leqslant n$ and every $z \in \text{co}(S(C \otimes D, x_0^* \otimes y_0^*, \delta))$. Henceforth, given $z \in S$ we can find $u \in \text{co}(S(C \otimes D, x_0^* \otimes y_0^*, \delta))$ and $v \in B_{X \widehat{\otimes}_\pi Y}$ so that $z = u + 4R\delta v$. Now, given $1 \leqslant i \leqslant n$ we get

$$T_i(z) = T_i(u) + 4\delta R T_i(v) \geqslant 1 - \alpha_i + \frac{\eta}{2} - 4R\delta > 1 - \alpha_i,$$

from where we conclude that $z \in \bigcap_{i=1}^{n} S(\overline{\text{co}}(C \otimes D), T_i, \alpha_i) = U$. This proves that $S \subseteq U$.

Summarising, we have proved that every relatively weakly open subset of $\overline{\text{co}}(C \otimes D)$ containing $x_0 \otimes y_0$ actually contains a slice $S(\overline{\text{co}}(C \otimes D), x_0^* \otimes y_0^*, \alpha)$.

Moreover,

$$(x_0^* \otimes y_0^*)(x_0 \otimes y_0) = \sup x_0^*(C) \sup y_0^*(D) = \sup(x_0^* \otimes y_0^*)(\overline{\text{co}}(C \otimes D)).$$

Thus, $x_0^* \otimes y_0^*$ w-strongly exposes $x_0 \otimes y_0$. □

Remark 3.5. The hypothesis of symmetry of C and D in Theorem 1.3 and Proposition 3.2 is needed. Indeed, let $C \subseteq X$ be any bounded subset with more than one point such that 0 is a strongly exposed point of C, and let $D \subseteq Y$ be a bounded set with a strongly exposed point $y \in D$ satisfying that $-y \in D$ too. In spite of 0 and y being strongly exposed, the basic tensor $0 \otimes y = 0 \in \overline{\text{co}}(C \otimes D)$ is not an extreme point, since

$$0 \otimes y = 0 = \frac{1}{2}(x \otimes y + x \otimes (-y)),$$

for any $x \in C \setminus \{0\}$. Similarly, the result in [20, Theorem 1] about the denting points of $\overline{\text{co}}(C \otimes D)$ does not hold when C or D are non-symmetric.

At this point one can wonder whether the assumption of the existence of the compact neighbourhood system can be removed in Theorem 1.3. We will show that the answer is negative. Let us consider first the following example, where the set is not symmetric.

Example 3.6. Consider $X = Y = \ell_2$, let $K := \overline{\text{co}}\{e_n : n \in \mathbb{N}\}$ and $f := \sum_{k=1}^{\infty} 2^{-k} e_k^*$. It is known that 0 is w-strongly exposed by f in K. Indeed, assume $(x_n)_{n=1}^{\infty} \subset K$ and $\lim_{n\to\infty}\langle f, x_n\rangle = 0$. Since $K \subset \ell_2$, each x_n can be expressed as $x_n = \sum_{k=1}^{\infty} a_k^n e_k$ with $a_k \geqslant 0$ and $\sum_{k=1}^{\infty} a_k^n \leqslant 1$. Therefore

$$0 = \lim_{n\to\infty}\langle f, x_n\rangle = \lim_{n\to\infty}\sum_{k=1}^{\infty} 2^{-k} a_k^n \geqslant \lim_{n\to\infty} 2^{-k} a_k^n.$$

This means that $\lim_{n\to\infty}\langle e_k, x_n\rangle = 0$ for each $k \in \mathbb{N}$ and so $x_n \xrightarrow{w} 0$. However, $f \otimes f$ does not weak-strongly exposes 0 in $K \otimes K \subseteq \ell_2 \widehat{\otimes}_\pi \ell_2$. Even more, 0 is not weakly strongly exposed in $K \otimes K$, i.e. there is no bilinear form $B\colon \ell_2 \times \ell_2 \to \mathbb{R}$ such that $z_k \to 0$ weakly whenever $z_k \in K \otimes K$ satisfies that $B(z_k) \to 0$.

In order to prove that, take a bilinear form B. For every $n \in \mathbb{N}$, the sequence $\{B(e_n, e_k)\}_{k\in\mathbb{N}} \to 0$ since e_k is weakly null in ℓ_2. Thus there is a subsequence $(e_{k_n})_n$ of $(e_n)_n$ satisfying that $B(e_n, e_{k_n}) \to 0$. Observe that $e_n \otimes e_{k_n} \in K \otimes K$ for every $n \in \mathbb{N}$.

However, $(e_n \otimes e_{k_n})_n$ does not converge weakly to 0 since it is isometrically equivalent to the ℓ_1 basis; this follows by the same argument as for the diagonal $e_n \otimes e_n$, see e.g. [19, Example 2.10].

Remark 3.7. The above argument also proves that 0 is not weakly strongly exposed in $\overline{\text{co}}(K \otimes K)$. It also follows that $f \otimes f$ exposes 0 in $\overline{\text{co}}(K \otimes K)$. Indeed, since f exposes 0 in K we have that, given any $z \in K$, $f(z) = 0$ if and only if $z = 0$. Consequently, given $x \otimes y \in K \otimes K$ we have that $(f \otimes f)(x \otimes y) = f(x)f(y) = 0$ implies that either x or y equals 0 and so $x \otimes y = 0$. From this, and the fact that $\{e_n \otimes e_m\}_{n,m}$ is a Schauder basis for $\ell_2 \widehat{\otimes}_\pi \ell_2$, it follows that $(f \otimes f)(z) = 0$ if and only if $z = 0$ for every $z \in \overline{\text{co}}(K \otimes K)$.

In order to find an example showing that Theorem 1.3 does not hold without the assumption of the existence of a compact neighbourhood system, we will use the set K from Example 3.6 to construct an equivalent renorming $|\cdot|$ on ℓ_2 satisfying that the new unit ball $B_{(\ell_2,|\cdot|)}$ has a w-strongly exposed point x_0 such that $x_0 \otimes x_0$ is not w-strongly exposed in $B_{(\ell_2,|\cdot|)\widehat{\otimes}_\pi(\ell_2,|\cdot|)}$.

Example 3.8. Set an equivalent norm $|\cdot|$ on ℓ_2 so that the new unit ball is $\overline{\text{co}}((K - e_1) \cup (-K + e_1) \cup \frac{1}{8} B_{\ell_2})$, where K is the set described in Example 3.6. We claim that $-e_1 \in B_{(\ell_2,|\cdot|)}$ is w-strongly exposed by $f := \sum_{k=1}^\infty 2^{-k} e_k^*$. Observe that $f(-e_1) = -\frac{1}{2}$. Call $A := K - e_1$ for simplicity. Since f is linear, it is clear that

$$\sup_{z \in B_{(\ell_2,|\cdot|)}} |f(z)| = \sup_{z \in A \cup -A \cup \frac{1}{8} B_{\ell_2}} |f(z)|.$$

Observe that the above supremum equals $\sup_{z \in A} |f(z)|$ since $|f(z)| \leqslant \frac{1}{8}$ on $\frac{1}{8} B_{\ell_2}$ and by a symmetry argument.

On the other hand, given $z \in A$ we have $z = v - e_1$ for $v \in K$. Now $f(v - e_1) = -\frac{1}{2} + f(v) \geqslant -\frac{1}{2}$ since $f \geqslant 0$ on K. This proves that $\|f\| = 1/2 = |f(-e_1)|$. Observe, moreover, that $f(v - e_1) \leqslant 0$ holds for every $v \in K$. In order to see that f w-strongly exposes $-e_1$ it remains to prove that if $f(z_n) \to -\frac{1}{2}$ with $(z_n)_n \subset B_{(\ell_2,|\cdot|)}$ then $z_n \to -e_1$ weakly. In order to do so, by a density argument, we can assume with no loss of generality that $z_n \in \text{co}(A \cup -A \cup \frac{1}{8} B_{\ell_2})$. For every n, we can write

$$z_n = \alpha_n a_n + \beta_n(-a'_n) + \gamma_n x_n$$

for $a_n, a'_n \in A, x_n \in \frac{1}{8} B_{\ell_2}$ and $\alpha_n, \beta_n, \gamma_n \in [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for every n.

Observe that

$$f(z_n) = \alpha_n f(a_n) + \beta_n f(-a'_n) + \gamma_n f(x_n) \geqslant \alpha_n f(a_n) + \gamma_n(-1/8)$$

since $f(-a'_n) \geqslant 0$ for every $n \in \mathbb{N}$ and since $|f(x_n)| \leqslant 1/8$. Since $f(z_n) \to -1/2$ the unique possibility is that $\alpha_n \to 1$ (which implies $\beta_n \to 0$ and $\gamma_n \to 0$). Moreover, it is immediate that $f(a_n) \to -1/2$. Since f w-strongly exposes $-e_1$ in A we conclude that $a_n \to -e_1$ weakly, so $z_n \to -e_1$ weakly, as desired.

Finally, if we consider $e_1 \otimes e_1$, we get that it is not w-strongly exposed in $B_{(\ell_2,|\cdot|)\widehat{\otimes}_\pi(\ell_2,|\cdot|)}$. As in Example 3.6, given any bilinear and continuous form B, we can find a strictly increasing sequence $(k_n)_n$ such that $B(e_n, e_{k_n}) \to 0$, so $B(-e_1 + e_n, -e_1 + e_{k_n}) \to B(e_1, e_1)$. However, if $k_1 < k_2 < \cdots$ we have that $\{e_n \otimes e_{k_n}\}$ is equivalent to the ℓ_1 basis since $(\ell_2, |\cdot|)$ and ℓ_2 are isomorphic (it follows for instance from [19, Proposition 2.3]) and therefore $((-e_1 + e_n) \otimes (-e_1 + e_{k_n}))_n$ is not weakly convergent to $e_1 \otimes e_1$.

We end the paper with some open questions.

Question 3.9. *Is $x \otimes y$ a preserved extreme point of $B_{X\hat{\otimes}_\pi Y}$ whenever x and y are preserved extreme points of B_X and B_Y?*

Question 3.10. *Is every (preserved) extreme point of $B_{X\hat{\otimes}_\pi Y}$ a basic tensor?*

Funding Funding for open access publishing: Universidad de Granada/CBUA.

Declarations

Conflict of interest The authors have not disclosed any competing interests.

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References

[1] Aliaga, R.J.: Extreme points in Lipschitz-free spaces over compact metric spaces. Mediterr. J. Math. **19**, 32 (2022)

[2] Aliaga, R.J., Guirao, A.J.: On the preserved extremal structure of Lipschitz-free spaces. Studia Math. **245**(1), 1–14 (2019)

[3] Aliaga, R.J., Pernecká, E., Petitjean, C., Procházka, A.: Supports in Lipschitz-free spaces and applications to extremal structure. J. Math. Anal. Appl. **489**(1), 124128 (2020)

[4] Bonde, E.: The approximation property for a pair of Banach spaces. Math. Scan. **57**, 375–385 (1985)

[5] Collins, H.S., Ruess, W.: Weak compactness in spaces of compact operators and of vector-valued functions. Pac. J. Math. **106**(1), 45–71 (1983)

[6] Dantas, S., Martínez-Cervantes, G., Jung, M.: On the existence of non-norm-attaining operators. J. Inst. Math. Jussieu. https://doi.org/10.1017/S1474748021000311

[7] Distel, J., UHL, J.J.: Vector Measures. American Mathematical Society Providence, Rhode Island (1977)

[8] García-Lirola, L., Petitjean, C., Procházka, A., Rueda Zoca, A.: Extremal structure and duality of Lipschitz free spaces. Mediterr. J. Math. **15**, 69 (2018)

[9] García-Lirola, L., Procházka, A., Rueda Zoca, A.: A characterisation of the Daugavet property in spaces of Lipschitz functions. J. Math. Anal. Appl. **464**(1), 473–492 (2018)

[10] Guirao, A., Montesinos, V., Vicente, Zizler, V.: On preserved and unpreserved extreme points. In: Descriptive Topology and Functional Analysis, Springer Proceedings in Mathematics & Statistics, vol. 80, pp. 163–193. Springer, Cham (2014)

[11] Hudzik, H., Mastyło, M.: Strongly extreme points in Köthe–Bochner spaces. Rocky Mt. J. Math. **23**(3), 899–909 (1993)

[12] Kadets, V., Martín, M., Merí, J., Pérez, A.: Spear operators between Banach spaces. In: Lecture Notes in Mathematics, vol. 2205. Springer (2018)

[13] Kadets, V., Martín, M., Merí, J., Pérez, A.: Spear operators between Banach spaces. arXiv:1701.02977

[14] Kamińska, A.: Extreme points in Orlicz–Lorentz spaces. Arch. Math. (Basel) **55**(2), 173–180 (1990)

[15] Lin, B.L., Lin, P.K., Troyanski, S.L.: Characterizations of denting points. Proc. Am. Math. Soc. **15**(3), 526–528 (1988)

[16] Rudin, W.: Functional Analysis. McGraw-Hill, New York (1973)

[17] Ruess, W.M., Stegall, C.P.: Exposed and denting points in duals of operator spaces. Isr. J. Math. **53**(2), 163–190 (1986)

[18] Ruess, W.M., Stegall, C.P.: Extreme points in duals of operator spaces. Math. Ann. **261**(4), 535–546 (1982)

[19] Ryan, R.A.: Introduction to tensor products of Banach spaces. In: Springer Monographs in Mathematics. Springer, London (2002)

[20] Werner, D.: Denting points in tensor products of Banach spaces. Proc. Am. Math. Soc. **101**, 122–126 (1987)

Luis C. García-Lirola
Departamento de Matemáticas
Universidad de Zaragoza
50009 Zaragoza
Spain
e-mail: luiscarlos@unizar.es
URL: https://personal.unizar.es/luiscarlos/

Guillaume Grelier
Departamento de Matemáticas, Campus de Espinardo
Universidad de Murcia
30100 Murcia
Spain
e-mail: g.grelier@um.es

Gonzalo Martínez-Cervantes
Departamento de Matemáticas, Facultad de Ciencias
Universidad de Alicante
03080 Alicante
Spain
e-mail: gonzalo.martinez@ua.es

Abraham Rueda Zoca
Departamento de Análisis Matemático, Facultad de Ciencias
Universidad de Granada
18071 Granada
Spain
e-mail: abrahamrueda@ugr.es
URL: https://arzenglish.wordpress.com

Received: December 4, 2022.
Accepted: July 10, 2023.

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