Quantum general covariance

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Abstract

The structure of quantum interactions with fields of helicity two ("gravitons") is strongly constrained by three principles: positivity (Hilbert space), covariance, and locality of observables. To fulfil them simultaneously, some (non-observable) fields need to be non-local. We work with string-localized fields. The results then follow from the condition that entities closely related to observables, like the $\mathbb{S}$-matrix, be local and string-independent. They in particular reproduce the interactions dictated by general covariance in classical field theory. Graviton-matter couplings are consistent only when the graviton self-interaction is taken into account as well.

Science is not a collection of truths; it is a continuous exploration of mysteries

– Freeman Dyson

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Contents

1 Introduction

1.1 The background

Physics being above all an experimental science, the development of quantum gravity (QG), as compared to other branches of theoretical physics, has been hampered by a relative poverty of phenomenological development [1] and the widespread belief that QG becomes dominant only at Planck scales. On the other hand, there seems to be in the literature a large, if not total, consensus that the (Einstein equations and the) Hilbert and/or the Einstein's actions for general relativity can be re-derived by means of quantum field theory. The idea of coupling a massless helicity $|h| = 2$ particle ("graviton") to its own stress-energy-momentum tensor, and the program of iterating the process, go back to Kraichnan and Gupta in the 1950s. They were elaborated by Feynman [12, 13] and Deser [4, 6], with coworkers. A more sophisticated later version is [38] – leading to Einstein's unimodular action. Similar techniques were put to work in [19], in spite of prior pointed and detailed criticism of the whole idea by Padmanabhan [32].

To the same ends, the book [33] by Günter Scharf employs in the framework of "causal gauge invariance" the inductive development by Epstein and Glaser of Bogoliubov's recursive $\mathbb{S}$-matrix theory – that is, the functional analogue of the Dyson series [3, 9, 11]. It was based on previous partial work by Scharf and Wellmann [34], and on more advanced one of the same kind by Schorn [35]. In [33] the first two non-trivial terms in the Newton's constant expansion of the Einstein–Hilbert action for general relativity are recovered; standard lore [40] has it that this is sufficient to determine all higher-order terms, and so to the recover the whole theory.

Now, in the respect of the still elusive search for QG, there remains a chasm, since whereas *classical* linear or non-linear systems can always be set on a Hilbert space by use of Koopman operators [18, 23], that is structurally impossible in the setting of gauge theories. The purpose of the present article is to revisit the problem from the viewpoint and with the more rigorous tools of string-localized quantum field theory (SQFT), which manages as well to incorporate the Bogoliubov–Epstein–Glaser method under the guise of the *string-independence* (SI) condition. This turns out to be a fruitful strategy, leading to the concept of "quantum general covariance", exploited in this paper, dispelling the perceived inconsistency between general relativity and quantum field theory. The main aim of SQFT is to avoid the difficulties associated with "canonical quantization" of gauge theories, notably the use of state spaces with indefinite metrics, by working with interaction densities which are operators on a physical Hilbert space throughout – so that observables are elements of an operator algebra with a *positive* vacuum functional. This is possible with the help of *string-localized* fields – here above all the quantum counterpart $h_{\mu\nu}(x; c)$ of the metric deviation field or "graviton" – arising as integrals over the associated field strengths – here $F_{[\mu\kappa][\nu\lambda]}(x)$, essentially the linearized Riemann–Christoffel curvature tensor – see Eq. (C.7). For recent accounts of SQFT, consult [21, 28].

The dependence of the interaction density on the "string" is a total derivative, so that the action (an integral over the density) is string-independent, and so string-localization does not spoil locality of a perturbative QFT model, at least in first order. The latter property of the action appears to be a rather restrictive one; however, it is satisfied by all interactions among particles of the Standard Model. Here we intend to show that it can be satisfied also both for the self-interaction of gravitons and for the interactions between gravitons and matter.

On the face of it, the total-derivative property is sufficient to ensure SI of the action only at first order of perturbation theory. At higher orders, there will occur "obstructions" to SI. These must be well-localized, so that they have a chance to be canceled by higher-order interactions – then we say "the obstruction is resolved". Both the conditions on the interaction at the lower order ensuring that resolution, and the form of the higher-order interactions themselves, are therefore predictions of the approach. They do not refer to an underlying classical theory, nor do they assume gauge or diffeomorphism invariance, but arise as consistency conditions for local and covariant quantum field models on Hilbert space.[1]

1.2 Quantum general covariance

Remarkably, for all the interactions of interest, one finds oneself in the "lucky" situation of well-localized obstructions. In other words, quantum field theory in the SQFT dispensation has the power to predict on its own the structure of consistent interactions. In the presence of interacting massive vector bosons, like in the Standard Model (SM) that structure includes chirality of the weak interaction [20], the Lie-algebra makeup of cubic self-couplings of several vector bosons and the quartic self-coupling familiar from Yang–Mills theory [17], the need of a scalar field with its potential terms [28] – without ever invoking "gauge invariance" nor classical "Lagrangians".

The above is shown here to hold also true for gravity: the unique self-interactions obtained from the expansion of the classical diffeomorphism invariant action of general relativity at first and second order in terms of the Newton constant are coincident with the self-interactions of our $h^{\mu\nu}(x;c)$. So we reproduce the uniqueness of graviton self-interactions which is the general conclusion of the various approaches presented in the beginning. The main new feature in SQFT is the nature of the argument, totally based on quantum principles.

We find a second, even more remarkable result, concerning the couplings to matter. Free matter fields in a flat spacetime have conserved stress-energy tensors [25,26]. In perturbation theory, their time-ordered products fail to be conserved, i.e., Ward identities are violated. Now, the coupling of matter to a field of helicity 2 turns out to have an obstruction proportional to the violation of the Ward identity. This obstruction cannot be resolved on its own. However, on including the self-interaction of the graviton field, one finds an interference term yielding another obstruction. And both obstructions together can be resolved by a higher-order matter coupling. That the latter coincide with the second-order term of the expansion of the generally covariant matter coupling is in the vein of the above: consistency of the quantum couplings "predicts" general covariance.

This outcome (at second order of perturbation theory) reveals a kind of "predestined match" between matter and gravity. On the one hand, the violation of the matter's Ward identity knows nothing about gravity. On the other hand, the self-interaction of gravitons knows nothing about matter. Yet, the two taken together produce a sum of non-resolvable obstructions that together can be resolved – like a lock and a key. Consistency is only achieved when the key and the lock, apparently ignorant of each other, do meet. We are

[1]The need for unobservable fields assigned to infinite string-like spacelike regions for the *mathematical description* of charged particle states was recognized by Buchholz and Fredenhagen over forty years ago [5], see our Sect. 4. In spite of the apparent non-locality, the principle of locality of observables is held, and they derive localization properties of particle states which hold in all standard models of relativistic quantum theory.

tempted to call this mechanism, which does not require classical covariance: **"Quantum general covariance"**. The latter holds for matter consisting of an arbitrary number of scalar, photon and Dirac fields. We prove these facts here in due course.

1.3 Strategy of the paper

In order to arrive at our results, the overall strategy schematically is: (i) Start with a first order input interaction and compute its obstruction to SI. (ii) Make sure that the obstruction is resolvable. This is separately the case for the self-interaction of gravitons, but not for the coupling to matter. It is again true if both are taken together. (iii) Determine the respective second-order interactions that do the job. If necessary, continue to the next order.

We next elaborate a bit on how SQFT enters this strategy in the subject at hand – leaving the filling in of the technical details and proofs for the main body of the paper.

1. Consider the field-strength tensor $F_{[\mu\kappa][\nu\lambda]}(x)$ of the massless particle of helicity 2. It is defined on the Fock space over the direct sum of unitary representations of the Poincaré group corresponding to helicities $|h| = 2$. This Hilbert space therefore contains only physical graviton states. Its two-point function is given by

$$\langle\!\langle F_{[\mu\kappa][\nu\lambda]} F'_{[\rho\tau][\sigma\pi]} \rangle\!\rangle = \frac{1}{2}\big[\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\big]\partial_\kappa\partial_\lambda\partial'_\tau\partial'_\pi W_0(x-x') \\ \begin{matrix} -(\mu \leftrightarrow \kappa) & \quad -(\rho \leftrightarrow \tau) \\ -(\nu \leftrightarrow \lambda) & \quad -(\sigma \leftrightarrow \pi); \end{matrix} \tag{1.1}$$

where W_0 denotes the two-point function of a massless scalar field and $\eta_{\mu\nu}$ is the (mostly negative) Minkowski metric.

2. Represent $F_{[\mu\kappa][\nu\lambda]}$ as:

$$F_{[\mu\kappa][\nu\lambda]}(x) = [\partial_\mu\partial_\nu h_{\kappa\lambda}(x;c) - \partial_\kappa\partial_\nu h_{\mu\lambda}(x;c) - \partial_\mu\partial_\lambda h_{\kappa\nu}(x;c) + \partial_\kappa\partial_\lambda h_{\mu\nu}(x;c)].$$

Here $h_{\mu\nu}(x;c)$ is a traceless symmetric field tensor defined on the physical Fock space by "string integrations" I_c over F – see Eq. (2.4). This means that our graviton field $h_{\mu\nu}(x;c)$ is localized on a spacelike cone emanating from x and extending to infinity, with a directional profile given by a function $c(e)$ of the spacelike directions e.[2]

3. The prerequisite for a *string-independent* $\mathbb{S}$-matrix for gravity is a string-independent first-order action. That is, we require an interaction density $L_1(x;c)$ whose string variation $\delta_c L_1(x;c)$ is a total derivative:

$$\delta_c\big(L_1\big) = \partial_\mu Q_1^\mu. \tag{1.2}$$

[2]As an idealization, one may think of a narrow cone as a "(half-)string", i.e., a line extending from x to infinity. A gauge-theoretic motivation for the introduction of the strings will be given in App. A. But that motivation is not ours'. We need the field $h_{\mu\nu}(x;c)$ in order to formulate interactions on *Hilbert space*. The key point is that the physics ought not depend on c.

This structure is called an *"L-Q pair"* – see Sect. 2.2 below. In the cases at hand, L_1 is a Wick polynomial in $h(x;c)$ and its derivatives, and the Q_1^μ are Wick polynomials involving as well an auxiliary field w measuring the variation of h with respect to the string. The smallest possible UV scaling dimension of L_1 is *five* (reflecting the non-renormalizability of self-interactions with helicity 2 by power counting); with this dimension the form of L_1 is essentially unique, being cubic in h with two derivatives.

4. Property (1.2) ensures that the perturbative $\mathbb{S}$-matrix $\mathbb{S} = \mathbf{1} + i\kappa \int d^4x\, L_1(x;c) + \ldots$, where $\kappa = 4\sqrt{2\pi G}$ for G the Newton constant, be string-independent in first order.

   But SI in general is not assured to hold in second order, because $TL_1(x_1;c)L_1(x_2;c)$ in general fails to be a total derivative:

$$\delta_c\big(TL_1(x_1;c)L_1(x_2;c)\big) = T\partial_{x_1^\mu} Q_1^\mu(x_1;c)L_1(x_2;c) + (x_1 \leftrightarrow x_2) \tag{1.3}$$

$$\neq \partial_{x_1^\mu} TQ_1^\mu(x_1;c)L_1(x_2;c) + (x_1 \leftrightarrow x_2). \tag{1.4}$$

   The *difference* $O^{(2)}(x_1,x_2,c)$ between (1.3) and (1.4) is the *second-order obstruction* to SI of the $\mathbb{S}$-matrix.

5. The computation of the obstruction at the tree level turns out to be straightforward in terms of the propagators. The result must be of the form:

$$O^{(2)}(x_1,x_2;c) = i\delta_c\big(L_2(x_1;c)\big)\delta(x_1-x_2) + \text{derivatives} \tag{1.5}$$

   with L_2 being a quartic Wick polynomial.

6. If (1.5) holds, add L_2 to the interaction density:

$$L_{\text{int}}(x;c) = \kappa L_1(x;c) + \frac{\kappa^2}{2} L_2(x;c) + \cdots \tag{1.6}$$

   This ensures that in second order, the contribution from L_2 to the $\mathbb{S}$-matrix cancels (resolves) the obstruction $O^{(2)}$ up to derivatives. In other words, the $\mathbb{S}$-matrix including the *"induced" interaction* L_2 is string-independent in second order.

7. Proceed recursively: compute the obstruction in third order of the $\mathbb{S}$-matrix with interaction $\kappa L_1 + \frac{1}{2}\kappa^2 L_2$. It must be of the form:

$$O^{(3)}(x_1,x_2,x_3;c) = i^2\delta_c\big(L_3(x_1;c)\big)\delta(x_1-x_2)\delta(x_2-x_3) + \text{derivatives}. \tag{1.7}$$

   This determines L_3 (up to a derivative), and so on. In practice, these higher-order corrections are rarely required.

Note that (1.5) is a rather strong requirement, without which the construction would be worthless. The subtraction defining $O^{(2)}$ is convenient because (1.3) is supported everywhere in $M \times M$, whereas the difference is a priori supported on the set $\{x_1 \in x_2 + \mathbb{R}\,\mathrm{supp}\, c\}$. That Eq. (1.5) may be satisfied at all with *localized* support on the set $\{x_1 = x_2\}$ in all interactions of interest is the hallmark of SQFT. In order to decide on (1.5) and determine L_2 up to derivatives, it is sufficient to compute (1.3), up to derivatives. We exploit this simplicity in the subsequent analysis.

Our first main result is that the cubic graviton self-interaction L_1 satisfying (1.2) is unique up to a total divergence, and that its second-order obstruction can be resolved. Also L_1 and L_2 as determined by (1.5) reproduce the classical expansion of the Einstein action in powers of the metric deviation $h_{\mu\nu}$ defined by $\mathfrak{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}$, where the classical field $h^{\mu\nu}(x)$ is replaced by the string-localized quantum field $h^{\mu\nu}(x;c)$.

The second main result is that, given the stress-energy-momentum tensor $\Theta^{\mu\nu}$ of a matter field, the coupling $L_{1,\mathrm{mat}} = \frac{1}{2}h_{\mu\nu}(x;c)\Theta^{\mu\nu}$ to the latter – which is the unique possibility from the L-Q-pair condition discussed in item 3. above – has an obstruction that cannot be resolved; namely (1.5) fails. But this obstruction can be resolved when one adds the self-coupling of the graviton field: **the key opens the lock**. Moreover, the resulting second-order induced matter interaction reproduces the classical expansion of the generally covariant "matter Lagrangian".

1.4 Final introductory comments

An extra word is in order concerning renormalizability. The field strength F has scaling dimension 3, the string-localized field h – as any other boson potential in SQFT – has scaling dimension 1. In spite of the latter's good behaviour, the first-order interaction has dimension 5, as we shall see. Thus it is not UV power-counting renormalizable.[3] Our work in this paper does not need to address loop diagrams. Nevertheless, one faces an infinite series of higher-order interactions of increasing dimension already at tree level. This is not a surprise, given that also classical general covariance implies an infinite power series. Whether counterterms for UV-divergent loop diagrams can be absorbed in a renormalization of the unique coupling constant κ, or substantially change the structure of the higher-order interactions, cannot be assessed as of yet.

Our path to perturbative QG in a Minkowski background does not branch from standard gauge theory in the first place; rather it has to be regarded as an alternative to it. In the former, the BRST method is a crucial tool to eliminate unphysical degrees of freedom after the perturbation development has been done. Such tools we do not need, since from the beginning we work on a physical Hilbert space with the correct degrees of freedom. Nevertheless, it is instructive to discuss the relation between the two approaches. In App. A we show how SQFT modifies the BRST approach in such a way that the interaction density whose BRST variation is a divergence becomes invariant – and so BRST invariance becomes obsolete altogether. Quantum general covariance manifested by the "lock-key scenario" discussed above, is in fact a characteristic feature of the Hilbert space approach; whereas the non-resolvable obstructions of the matter interaction (Sect. 3.1) is removed in the BRST approach with the help of ghost fields, blurring the power of quantum principles.

[3] That power counting remains a meaningful notion in SQFT was shown in [16].

2 Preliminaries on string-localized fields

Standard lore has that the redundancy of degrees of freedom in gauge theories can be avoided if one allows the potentials of certain quantum fields to be non-local – see for example the discussion in the textbook [36, Ch. 22]. The mildest and better physically justified way of introducing such a non-locality is by using the *string-localized* potentials originally mooted in [29, 30]. String-localized potentials can be defined for arbitrary masses $m \geq 0$ and spins/helicities $s \geq 1$. Here we mainly deal with the massless potential field corresponding to helicity 2, baptized the *string-localized graviton potential*. In the following we list its relevant properties, introducing all the pertinent concepts and notations.

2.1 The string-localized graviton potential

Consider the massless free *field strength* tensor $F_{[\mu\kappa][\nu\lambda]}(x)$ of helicity 2 – essentially, as advertised, the linearized curvature tensor. It is defined in terms of the creation and annihilation operators for the physical helicities $h = +2$ and $h = -2$ on the Fock space over the corresponding unitary Wigner representation of the Poincaré group with $m = 0$ and $|h| = 2$. By construction it satisfies the wave equation and enjoys the symmetries and field equations:

$$F_{[\mu\kappa][\nu\lambda]} = -F_{[\kappa\mu][\nu\lambda]}, \quad F_{[\mu\kappa][\nu\lambda]} = F_{[\nu\lambda][\mu\kappa]}, \quad \eta^{\mu\nu} F_{[\mu\kappa][\nu\lambda]} = 0, \quad \partial^\mu F_{[\mu\kappa][\nu\lambda]} = 0, \qquad (2.1)$$
$$F_{[\mu\kappa][\nu\lambda]} + F_{[\kappa\nu][\mu\lambda]} + F_{[\nu\mu][\kappa\lambda]} = 0, \quad \partial_\rho F_{[\mu\kappa][\nu\lambda]} + \partial_\mu F_{[\kappa\rho][\nu\lambda]} + \partial_\kappa F_{[\rho\mu][\nu\lambda]} = 0,$$

its positive-definite two-point function being given already in (1.1). In order to introduce interactions one needs a potential $h_{\mu\nu}$, from which the field tensor $F_{[\mu\kappa][\nu\lambda]}$ arises as the double exterior derivative or linearized curvature. Now, the construction of a local pointlike potential as an operator-valued distribution on the same Fock space is impossible [39]. However, there is a *string-localized* potential satisfying (2.5), which does live on the same Fock space [29,30]. For its definition 2.2, we need the the operation of string integration.

Definition 2.1. *Let $H := \{e \in \mathbb{R}^{1+3} \mid e^2 < 0\}$ denote the open subset of spacelike vectors in Minkowski space and let $c \in C_c^\infty(H)$ be a smooth test function with compact support so that the cone $\{\mathbb{R}_+ \cdot \operatorname{supp} c\}$ is convex, satisfying the unit weight condition $\int_H d^4e\, c(e) = 1$. We define the* string integration I_c^μ *by its action on a generic (possibly operator-valued) distribution f on the Minkowski space:*

$$I_c^\mu f(x) := \int_H d^4e\, c(e) e^\mu \int_0^\infty ds\, f(x + se) \equiv \int_H d^4e\, c(e) \int_{C_{x,e}} f(y)\, dy^\mu, \qquad (2.2)$$

where the curve $C_{x,e}$ is the ray ("string") in direction e extending from x to spatial infinity.

It is obvious that

$$I_c^\mu \partial_\mu f(x) = \partial_\mu I_c^\mu f(x) = -f(x), \quad \text{or} \quad I_c^\mu \partial_\mu =: (I_c \partial) = -1, \qquad (2.3)$$

whenever derivates and integrals may be interchanged and there are no boundary terms at infinity.

Definition 2.2. *The* string-localized graviton potential $h_{\mu\nu}(x;c)$ *is defined via*

$$h_{\mu\nu}(x;c) := I_c^\kappa I_c^\lambda F_{[\mu\kappa][\nu\lambda]}(x). \tag{2.4}$$

Thus it "lives" on the Hilbert-Fock space of the F-tensor field [29, 30] and possesses the same two degrees of freedom as F – see [26, 30] for the general spin/helicity cases.

The scaling dimension of quantum fields and their propagators dictates the strength of UV divergences. Because the two-point function for F in (1.1) scales under $x \to \lambda x$ like λ^{-6}, the field F has scaling dimension 3; and because of two string-integrations in its definition (2.4), $h_{\mu\nu}(x;c)$ has scaling dimension 1, the same as the scaling dimension of the canonically quantized gauge potential $h_{\mu\nu}(x)$, with the difference that the former is defined on the Hilbert space, while the latter is not.

The cubic interaction coupling will consequently have scaling dimension 5, which is beyond the power-counting bound for renormalizability, as in all other approaches. This is to be expected since the gravitational coupling constant κ has a negative mass dimension.[4]

The $\mathbb{S}$-matrix constructed in terms of the string-localized potential will be manifestly unitary, because all involved fields act on a Hilbert space. The issue with the $\mathbb{S}$-matrix will rather be its string-independence – see Sect. 2.2.

It is also clear that $h_{\mu\nu}(c)$ depends on the choice of c, being localized along the string (or rather cone) $x + \mathbb{R}_+ \operatorname{supp} c$. However, as already pointed out, its double exterior derivative gives back the original point-localized F-tensor,

$$\partial_\mu\partial_\nu h_{\kappa\lambda}(x;c) - \partial_\mu\partial_\lambda h_{\kappa\nu}(x;c) - \partial_\kappa\partial_\nu h_{\mu\lambda}(x;c) + \partial_\kappa\partial_\lambda h_{\mu\nu}(x;c) = F_{[\mu\kappa][\nu\lambda]}(x), \tag{2.5}$$

which is independent of c. The string-localized potential satisfies the following relations, as consequences of the properties (2.1) of the F-tensor:

$$h_{\mu\nu} = h_{\nu\mu}, \quad \eta^{\mu\nu}h_{\mu\nu} = 0, \quad \partial^\mu h_{\mu\nu} = 0, \quad \Box h_{\mu\nu} = 0, \quad I_c^\mu h_{\mu\nu} = 0. \tag{2.6}$$

In the sequel we shall omit in the notation the dependence on c; it is sufficient to know that the dependence of h on c is such that when the profile $c(e)$ is varied, one has

$$\delta_c\big(h_{\mu\nu}\big) = \partial_\mu w_\nu + \partial_\nu w_\mu, \qquad \text{where} \quad w_\mu = (\delta_c I_c^\kappa)h_{\kappa\mu} \tag{2.7}$$

is another string-localized field. The precise form of the string-integrated differential operator $\delta_c I_c^\kappa$ will not be relevant.[5] By (2.6), the field w_μ is divergenceless:

$$\partial^\mu w_\mu = 0. \tag{2.8}$$

[4]Recall however the comment in Sect. 1.4 on prospects that "infinitely many UV renormalization constants" might be all fixed by the condition of string-independence of the $\mathbb{S}$-matrix.

[5]One has to vary the function c by a function of weight zero. That is, $\delta c(e) = \partial_{e^\tau} b^\tau(e)$. This gives

$$\delta_c I_c^\kappa f(x) = \int_H e^\kappa d^4e\, \partial_{e^\tau} b^\tau(e) \int_0^\infty ds\ f(x+se) = -\int_H d^4e\, b^\tau(e) \int_0^\infty ds\, \Big(f(x+se)\delta_\tau^\kappa + s\partial_\tau f(x+se)e^\kappa\Big).$$

That the string variation (2.7) be a derivative is essential in order to construct $\mathbb{S}$-matrices $T\exp\big(i\int d^4x\, L_{\mathrm{int}}(x;c)\big)$ that do not depend on the profile function c, from the interactions satisfying (1.2).

2.2 The L-Q pair formalism and the condition of string-independence

In string-localized QFT, the perturbative $\mathbb{S}$-matrix can be defined as a formal power series

$$\mathbb{S}[g;c] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots \int d^4x_n\, g(x_1)\dots g(x_n)\, S_n(x_1,\dots,x_n;c), \tag{2.9}$$

where g is a test function over Minkowski space and the S_n are time-ordered products of a normal-ordered interaction density, possibly plus local terms supported on the "small diagonal": $\Delta_n := (x_1 = x_2 = \cdots = x_n)$. Such local terms supported on Δ_n either correspond to new Wick-polynomials in the fields or to corrections to already existent ones. We call the new Wick-polynomials *induced interactions*, while the corrections to already existent ones correspond to a renormalization of their prefactor (that is, they are *local counterterms*).

The expressions S_n depend on the string-smearing function c via the contained string-localized fields. The first order term $S_1(x;c)$ is related to the $O(g)$-part of the interaction density,

$$S_1(x;c) \equiv iL_1(x;c). \tag{2.10}$$

At higher orders, the situation becomes more complicated. In general, we will have

$$S_2(x_1,x_2;c) = i^2\, T_{\mathrm{ren}}[L_1(x_1;c)L_1(x_2;c)] + iL_2(x_1;c)\delta(x_1-x_2), \tag{2.11}$$

where the renormalized time-ordered product $T_{\mathrm{ren}}[L_1(x_1;c)L_1(x_2;c)]$ contains the local counterterms and $L_2(x_1;c)$ is an induced interaction. Then the third order contribution will be a time-ordered product

$$\begin{aligned} S_3(x_1,x_2,x_3;c) = {}& iT_{\mathrm{ren}}[S_2(x_1,x_2;c)S_1(x_3;c)] + \text{symmetric} \\ & + iL_3(x_1;c)\delta(x_1-x_2)\delta(x_2-x_3), \end{aligned} \tag{2.12}$$

and so on. The full (unrenormalized) interaction density of the model would then be:

$$L[g;c](x) := \sum_{k=1}^{\infty} \frac{g(x)^k}{k!} L_k(x;c), \tag{2.13}$$

All that matters is that $\delta_c I_c^\kappa \partial_\kappa = 0$ by (2.3), hence

$$\delta_c\big(h_{\mu\nu}\big) = \delta_c(I_c^\kappa I_c^\lambda)F_{[\mu\kappa][\nu\lambda]} = (\delta_c I_c^\kappa)\partial_{[\mu}h_{\kappa]\nu} + (\mu\leftrightarrow\nu) = (\delta_c I_c^\kappa)\partial_\mu h_{\kappa\nu} + (\mu\leftrightarrow\nu) =: \partial_\mu w_\nu + \partial_\nu w_\mu.$$

If $\delta c(e)$ is compactly supported, then one may choose $b(e)$ to be compactly supported, and w_μ is string-localized in the convex hull of $\mathbb{R}_+\cdot\big(\mathrm{supp}(b)\cup\mathrm{supp}(c)\big)$.

with normal-ordered induced interactions L_k. For the interactions of the Standard Model, $L_k = 0$ for all $k > 2$, and so one does not have to look for induced couplings after third order. For graviton interactions, it is expected that the series does not terminate.

The dependence of scattering amplitudes on the test function c (or the existence of a preferred direction, if the support of c is very narrow) is not observed in scattering experiments. Hence, one requires that the $\mathbb{S}$-matrix be independent of the choice of c in the *adiabatic limit* $g(x) \uparrow \mathrm{const}$, where the test function g becomes a true coupling constant. That is,

$$\lim_{g\uparrow\mathrm{const}} \delta_c\big(\mathbb{S}[g;c]\big) \stackrel{!}{=} 0. \tag{2.14}$$

Requirement (2.14) is what we call the *SI condition* for the $\mathbb{S}$-matrix. Provided that the adiabatic limit can be performed, it is fulfilled if

$$\delta_c\big(S_n(x_1,\dots,x_n;c)\big) = i^n \sum_{k=1}^{n} \frac{\partial}{\partial x_k^\mu}\, Q_n^\mu(x_1,\dots,x_n;c), \quad \forall n \in \mathbb{N}, \tag{2.15}$$

so that after integration by parts, one finds

$$\delta_c(\mathbb{S}) = -\sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots \int d^4x_n \sum_{k=1}^{n} g(x_1)\dots(\partial_\mu g)(x_k)\dots g(x_n)\, Q_n^\mu \stackrel{g\uparrow\mathrm{const}}{\longrightarrow} 0.$$

As already indicated in (1.2), at first order in perturbation theory the SI condition implies the L-Q-pair condition $\delta_c\big(L_1(x;c)\big) = \partial_\mu Q_1^\mu(x;c)$. That is, the string variation of the interaction density must be a total divergence. In this paper we only consider tree graph contributions up to second order of perturbation theory. In particular, we will not have to deal with local counterterms, but only with induced interactions. By Wick's theorem, modulo loop graphs we have:

$$T_{\mathrm{ren}}[L_1(x_1;c)L_1(x_2;c)] = {:}L_1(x_1;c)L_1(x_2;c){:} + {:}\underbrace{L_1(x_1;c)L_1}(x_2;c){:} + \dots, \tag{2.16}$$

$$\text{where} \quad {:}\underbrace{L_1(x_1;c)L_1}(x_2;c){:} := \sum_{\varphi,\chi} {:}\frac{\partial L_1}{\partial\varphi}(x_1;c)\langle\!\langle T_{\mathrm{ren}}\varphi(x_1;c)\chi(x_2;c)\rangle\!\rangle \frac{\partial L_1}{\partial\chi}(x_2;c){:} \tag{2.17}$$

is the sum of all terms with one contraction. The fields φ, χ may depend on the string-smearing function c, and $\langle\!\langle T_{\mathrm{ren}}\varphi(x_1)\chi(x_2)\rangle\!\rangle$ denotes the renormalized time-ordered two-point function of φ and χ, whose precise meaning is explained in the next subsection 2.3. Under application of δ_c, the first term on the right-hand side of (2.16) is a total divergence by virtue of the L-Q-pair condition. Then the SI condition is satisfied at second-order tree level if

$$\delta_c\big({:}\underbrace{L_1(x_1;c)L_1}(x_2;c){:}\big) = \sum_{k=1}^{2} \frac{\partial}{\partial x_k^\mu}\, Q_{2,\mathrm{tree}}^\mu(x_1,x_2;c) + i\delta_c\big(L_{2,\mathrm{tree}}(x_1)\big)\delta(x_1-x_2). \tag{2.18}$$

For our present aims, the precise form of $Q_n^\mu, Q_{n,\mathrm{tree}}^\mu$ and other total divergences is not required.

2.3 Two-point functions and propagators

All two-point functions and propagators (i.e. time-ordered two-point functions) can be respectively expressed in terms of the two-point functions and propagators of scalar fields ϕ of mass $m \geq 0$:

$$\langle\!\langle \phi(x)\phi(x')\rangle\!\rangle = \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\theta(k^0)\,e^{-ik(x-x')} =: W_m(x-x'), \tag{2.19}$$

$$\langle\!\langle T\phi(x)\phi(x')\rangle\!\rangle = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k^2-m^2+i\varepsilon} =: D_{m,F}(x-x'). \tag{2.20}$$

The two-point function for the Maxwell field strength tensor (of helicity 1) is

$$\langle\!\langle F_{\mu\nu}(x)F_{\kappa\lambda}(x')\rangle\!\rangle = -\eta_{\mu\kappa}\partial_\nu\partial'_\lambda W_0(x-x') - (\mu\leftrightarrow\nu) - (\kappa\leftrightarrow\lambda). \tag{2.21}$$

The two-point function for the field strength tensor of helicity 2 was given in (1.1). It is of course consistent with the relations (2.1) and the wave equation. Likewise, the "kinematic" propagator is obtained by replacing the massless two-point function W_0 by the massless propagator $D_{0,F}$.

Similarly, the kinematic propagator for $h_{\mu\nu}$ is defined by applying the string integrations as in (2.2). The result is:

$$\langle\!\langle T_0 h_{\mu\nu}(x)h_{\rho\sigma}(x')\rangle\!\rangle = \tfrac{1}{2}\big[E'_{\mu\rho}E'_{\nu\sigma} + E'_{\mu\sigma}E'_{\nu\rho} - E_{\mu\nu}E''_{\rho\sigma}\big]D_{0,F}(x-x'), \tag{2.22}$$

where (with I'_c the string integrations w.r.t. x'):

$$\begin{aligned}
E_{\mu\nu} &:= \eta_{\mu\nu} + I_{c,\mu}\partial_\nu + I_{c,\nu}\partial_\mu + I_c^2\partial_\mu\partial_\nu,\\
E''_{\rho\sigma} &:= \eta_{\rho\sigma} + I'_{c,\rho}\partial'_\sigma + I'_{c,\sigma}\partial'_\rho + I'^2_c\partial'_\rho\partial'_\sigma,\\
E'_{\mu\rho} &:= \eta_{\mu\rho} + I'_{c,\mu}\partial'_\rho + I_{c,\rho}\partial_\mu + (I_cI'_c)\partial_\mu\partial'_\rho,
\end{aligned} \tag{2.23}$$

cf. [26] in this respect. Kinematic propagators of derivatives of $h_{\mu\nu}$ are obtained by applying such derivatives in Eq. (2.22).

The kinematic propagator (2.22) and its derivatives respect the symmetry of $h_{\mu\nu}$, but because of

$$\Box D_{0,F}(x-x') = -i\delta(x-x') \tag{2.24}$$

they do not respect the trace condition in (2.6) – consult (B.10) in this respect. This would seem to imply that the propagators reflect a non-existent trace degree of freedom – an unacceptable feature in a Hilbert space theory. Fortunately, the linear dependency among field components can be restored by exploiting the freedom of renormalization of propagators (see again App. B). The matter generally consists in adding string-integrals of derivatives of $\delta(x-x')$:

$$\begin{aligned}
\langle\!\langle T_{\rm ren}\phi(x)\chi(x')\rangle\!\rangle &= \langle\!\langle T_0\phi(x)\chi(x')\rangle\!\rangle + \langle\!\langle T_r\phi(x)\chi(x')\rangle\!\rangle,\\
\text{where}\qquad \langle\!\langle T_r\phi(x)\chi(x')\rangle\!\rangle &= \sum_{\underline{\mu},\underline{\nu},\underline{\lambda}} iC^{\phi,\chi}_{\underline{\mu},\underline{\nu},\underline{\lambda}} I_c^{\underline{\mu}} I'^{\underline{\nu}}_c\,\partial^{\underline{\lambda}}\delta(x-x'),
\end{aligned} \tag{2.25}$$

with numerical coefficients C. $\underline{\lambda}$ etc. are Lorentz multi-indices. The lengths $|\underline{\mu}|$ and $|\underline{\nu}|$ give the numbers of string-integrations, and $4 + |\underline{\lambda}| - |\underline{\mu}| - |\underline{\nu}|$ is the scaling dimension of the renormalization term. By the standard power counting argument, these numbers cannot exceed those of $\langle\langle T_0 \phi \chi' \rangle\rangle$. Since we do not want to break scale invariance of a massless theory already at the tree level, we shall actually demand *equality* of the scaling dimensions.

The coefficients in (2.25) can be adjusted so that the trace conditions are restored. Further constraints on these coefficients will arise from the condition that the second-order obstruction has the form (1.5) without string-integrated δ-functions. For the details, see App. B.

The renormalized propagators determine the obstructions to SI. Propagators for matter fields and their renormalizations are studied in Sect. 3.1.

3 Graviton couplings on Hilbert space

We set out now to derive up to second order of perturbation theory in the framework of SQFT the structure of graviton couplings to the stress-energy tensors of the *scalar, Maxwell and Dirac* "matter" fields, as well as the structure of the graviton self-coupling.

As already indicated, to describe spin one-half fermions and scalars the SQFT construction is unnecessary, their stress-energy tensors thus being point-localized from the outset. In the case of Maxwell fields, one might envisage coupling the string-localized graviton potential to a string-localized stress-energy tensor for helicity 1. It turns out however [15] that the L-Q-pair condition (1.2) can only hold when the string-localized graviton is coupled to the well-known *point-localized* stress energy-tensor of the Maxwell field.

Notation 3.1. *To simplify notation in what follows, we drop the colon-notation for Wick-ordering and write $A \equiv A(x)$ and $A' \equiv A(x')$ for generic fields A (possibly Wick polynomials), and φ, χ' for generic linear fields. It will be also convenient to write*

$$(AB)_{\mu\nu} := A_{\mu\kappa} B^{\kappa}_{\nu}, \quad ((AB)) := A_{\kappa\lambda} B^{\kappa\lambda}$$

for contractions of symmetric tensors A and B, and similarly for just one tensor or products of more than two tensors: i.e., $((A))$ is the trace of A. We will write $A \overset{\text{div}}{=} B$ when expressions A and B differ by a total divergence.

3.1 Graviton coupling to stress-energy tensors

For the moment, let $\Theta^{\mu\nu}_{\text{mat}}$ denote the (symmetric conserved, point-localized) stress-energy tensor associated with a generic "matter" field, and consider the coupling:

$$L_{1,\text{mat}} := \frac{1}{2}((h\Theta_{\text{mat}})). \tag{3.1}$$

In view of (2.7), clearly $L_{1,\text{mat}}$ of above satisfies the L-Q-pair condition:

$$\delta_c\big(L_{1,\text{mat}}\big) = \partial_\mu Q^\mu_{1,\text{mat}} \quad \text{with} \quad Q^\mu_{1,\text{mat}} = w_\nu\,\Theta^{\mu\nu}_{\text{mat}}. \tag{3.2}$$

From Eq. (2.17), the renormalized second order tree graph can be expanded as

$$\underbrace{L_{1,\text{mat}} L_{1,\text{mat}}} = \tfrac14 \Theta^{\mu\nu}_{\text{mat}} \langle\!\langle T_{\text{ren}} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle \Theta'^{\rho\sigma}_{\text{mat}} + \tfrac14 h_{\mu\nu} h'_{\rho\sigma} \sum_{\varphi,\chi'} \frac{\partial \Theta^{\mu\nu}_{\text{mat}}}{\partial\varphi} \langle\!\langle T_{\text{ren}} \varphi\chi'\rangle\!\rangle \frac{\partial \Theta'^{\rho\sigma}_{\text{mat}}}{\partial\chi'}. \tag{3.3}$$

We next have to keep an eye on the renormalization of the graviton propagators in App. B. By inspection of (2.22) and (2.23), we see that all string-dependent parts of the kinematic propagator contain uncontracted derivatives. Because the stress-energy tensors are conserved, this means that they only contribute divergences to the first term in (3.3). To ensure the same for the renormalized propagator $\langle\!\langle T_r hh'\rangle\!\rangle$, as well it must contain only terms with uncontracted derivatives, or string-independent terms. The latter cannot occur by the bounds discussed around Eq. (2.25). Thus a necessary condition for the string variation of expression (3.3) to satisfy Eq. (1.5) is to impose the form (B.11) as a renormalization condition on $\langle\!\langle T_r hh'\rangle\!\rangle$.

In summary, the string variation of the first term in (3.3) is a total divergence and we find:

$$\delta_c\Big(\underbrace{L_{1,\text{mat}} L_{1,\text{mat}}}\Big) \overset{\text{div}}{=} -\tfrac12 h'_{\rho\sigma} w_\nu\, \partial_\mu \sum_{\varphi,\chi'} \frac{\partial \Theta^{\mu\nu}_{\text{mat}}}{\partial\varphi} \langle\!\langle T_{\text{ren}} \varphi\chi'\rangle\!\rangle \frac{\partial \Theta'^{\rho\sigma}_{\text{mat}}}{\partial\chi'} + (x \leftrightarrow x').$$

That is to say, obstructions to second-order SI at tree level can only be caused by a violation of the **Ward identity** associated to the (point-localized) matter stress-energy tensors, that is:

$$\partial_\mu T_{\text{ren}} \Theta^{\mu\nu}_{\text{mat}} \Theta'^{\rho\sigma}_{\text{mat}}\Big|_{\text{tree}} = \partial_\mu \sum_{\varphi,\chi'} \frac{\partial \Theta^{\mu\nu}_{\text{mat}}}{\partial\varphi} \langle\!\langle T_{\text{ren}} \varphi\chi'\rangle\!\rangle \frac{\partial \Theta'^{\rho\sigma}_{\text{mat}}}{\partial\chi'} \neq 0. \tag{3.4}$$

We compute the obstructions to string-independence at second order tree level corresponding to the possible violation of the Ward identities for the following stress-energy tensors:

$$\Theta^{\mu\nu}_\phi = \partial^\mu\phi\partial^\nu\phi - \tfrac12\eta^{\mu\nu}\big(\partial_\kappa\phi\partial^\kappa\phi - m^2\phi^2\big) \qquad \text{of a scalar field } \phi(x); \tag{3.5}$$

$$\Theta^{\mu\nu}_F = -F^{\mu\kappa}F^\nu{}_\kappa + \tfrac14\eta^{\mu\nu}F^{\kappa\lambda}F_{\kappa\lambda} \qquad \text{of the Maxwell tensor } F_{\mu\nu}(x); \tag{3.6}$$

$$\Theta^{\mu\nu}_\psi = \tfrac{i}{4}\Big(\overline{\psi}\gamma^\mu \overset{\leftrightarrow}{\partial^\nu}\psi + \overline{\psi}\gamma^\nu \overset{\leftrightarrow}{\partial^\mu}\psi\Big) \qquad \text{of a Dirac field } \psi(x). \tag{3.7}$$

Notice first that the trace parts of the stress-energy tensors do not contribute to the tree graph, because $h_{\mu\nu}$ is traceless and the renormalizations are determined such that

$$\eta^{\mu\nu}\langle\!\langle T_{\text{ren}} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle = 0 = \eta^{\rho\sigma}\langle\!\langle T_{\text{ren}} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle.$$

We are going to show:

Proposition 3.2. *For the matter couplings $L_{1,\text{mat}} = \frac12 h_{\mu\nu}\Theta^{\mu\nu}_{\text{mat}}$ with either of* (3.5)–(3.7)*, there is a non-resolvable obstruction of the interaction $L_{1,\text{mat}}$, of the universal form:*

$$O_{2,\text{mat}}(x,x') = -i\,\Theta^{\mu\nu}_{\text{mat}}\, w^\kappa\big(\partial_\kappa h_{\mu\nu} - \partial_\mu h_{\kappa\nu} - \partial_\nu h_{\kappa\mu}\big)\,\delta(x-x'). \tag{3.8}$$

The proof proceeds by a case-by-case analysis in the next three subsections. The cases are mutually dissimilar enough that one is allowed to conjecture universality of the above form.

3.1.1 Graviton coupling to scalar fields

Having seen that the contractions $\langle\!\langle T_{\rm ren} hh' \rangle\!\rangle$ in $\underbracket{L_{1,\phi}L'_{1,\phi}}$ do not contribute to the obstruction, we study the contractions of the scalar fields. Since the string-localized graviton potential is traceless, in view of Eq. (3.3) those are:

$$\frac{1}{4} h_{\mu\nu} h'_{\rho\sigma} \sum_{\varphi,\chi'} \frac{\partial \Theta^{\mu\nu}_{\phi}}{\partial \varphi} \langle\!\langle T_{\rm ren} \varphi \chi' \rangle\!\rangle \frac{\partial {\Theta'}^{\rho\sigma}_{\phi}}{\partial \chi'} = h_{\mu\nu} \partial^\mu \phi \langle\!\langle T_{\rm ren}\, \partial^\nu \phi\, \partial'^\sigma \phi' \rangle\!\rangle h'_{\rho\sigma} \partial'^\rho \phi'. \tag{3.9}$$

Because the scaling dimension of the renormalized propagator on the right-hand side of (3.9) equals the spacetime dimension, it admits one free renormalization parameter:

$$\begin{aligned} \langle\!\langle T_{\rm ren} \partial^\nu \phi\, \partial'^\sigma \phi' \rangle\!\rangle &= \langle\!\langle T_0\, \partial^\nu \phi\, \partial'^\sigma \phi' \rangle\!\rangle + \langle\!\langle T_r \partial^\nu \phi\, \partial'^\sigma \phi' \rangle\!\rangle \\ &= \langle\!\langle T_0 \partial^\nu \phi \partial'^\sigma \phi' \rangle\!\rangle + i c_\phi \eta^{\nu\sigma} \delta(x - x'). \end{aligned} \tag{3.10}$$

The non-kinematic part of (3.9) can always be absorbed into an *induced interaction*:

$$L_{2,r,\phi} := c_\phi (hh)^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = c_\phi \big((\!(hh\Theta_\phi)\!) + (\!(hh)\!) L_{0,\phi} \big),$$

where $L_{0,\phi}$ is the quadratic (free) scalar "Lagrangian" visible in (3.5), so that

$$i L_{2,r,\phi} \delta(x - x') + i^2 h_{\mu\nu} \partial^\mu \phi\, \langle\!\langle T_r \partial^\nu \phi \partial'^\sigma \phi' \rangle\!\rangle\, h'_{\rho\sigma} \partial'^\rho \phi' = 0.$$

In conclusion, *only the kinematic part* can contribute a possible obstruction to SI. Since one has for it:

$$\partial_\mu \langle\!\langle T_0 \partial^\mu \phi \partial'^\sigma \phi' \rangle\!\rangle = \Box \partial'^\sigma D_{m,F}(x - x') = -m^2 \langle\!\langle T_0\, \phi \partial'^\sigma \phi' \rangle\!\rangle - i \partial'^\sigma \delta(x - x'),$$

we find

$$\begin{aligned} &\delta_c \Big(\underbracket{L_{1,\phi} L'_{1,\phi}}\big|_{T_0} \Big) = \delta_c \big(h_{\mu\nu} \partial^\mu \phi\, \langle\!\langle T_0 \partial^\nu \phi \partial'^\sigma \phi' \rangle\!\rangle\, h'_{\rho\sigma} \partial'^\rho \phi' \big) \\ &\overset{\rm div}{=} w_\mu \partial^\mu \phi\, h'_{\rho\sigma} \partial'^\rho \phi' \partial'^\sigma \cdot i\delta(x - x') + (x \leftrightarrow x') \overset{\rm div}{=}: i\delta_c \big(L_{2,0,\phi} \big) \delta(x - x') + O_{2,\phi}(x, x'), \end{aligned}$$

with the *induced interaction* density:

$$L_{2,0,\phi} = (hh)_{\mu\nu} \Theta^{\mu\nu}_\phi + \frac{1}{4} (\!(hh)\!) \big((\partial\phi\partial\phi) - m^2 \phi^2 \big) = (\!(hh\Theta_\phi)\!) + \frac{1}{2} (\!(hh)\!) L_{0,\phi},$$

plus the remaining *obstruction*:

$$O_{2,\phi}(x, x') = \Theta^{\mu\nu}_\phi w^\kappa \big(\partial_\mu h_{\kappa\nu} + \partial_\nu h_{\kappa\mu} - i \partial_\kappa h_{\mu\nu} \big) \delta(x - x'), \tag{3.11}$$

which cannot be resolved by adding an induced term with field content (ϕ, ϕ, h, h). In total, we collect:

$$\delta_c \Big(\underbracket{L_{1,\phi} L'_{1,\phi}} \Big) \overset{\rm div}{=} \delta_c \big((1 + c_\phi) (\!(hh\Theta_\phi)\!) + i \big(\tfrac{1}{2} + c_\phi \big) (\!(hh)\!) L_{0,\phi} \big) \delta(x - x') + O_{2,\phi}(x, x').$$

3.1.2 Graviton coupling to electromagnetic fields

Keeping in mind that that contractions $\langle\!\langle T_{\rm ren} hh' \rangle\!\rangle$ in $\underbracket{L_{1,\phi} L'_{1,\phi}}$ did not contribute to the obstruction, we now study the contractions for electromagnetic fields:

$$\tfrac{1}{4} h_{\mu\nu} h'_{\rho\sigma} \sum_{\varphi,\chi'} \frac{\partial \Theta_F^{\mu\nu}}{\partial \varphi} \langle\!\langle T_{\rm ren} \varphi \chi' \rangle\!\rangle \frac{\partial \Theta'^{\rho\sigma}_F}{\partial \chi'} = h_{\mu\nu} F^\mu{}_\kappa \langle\!\langle T_{\rm ren} F^{\nu\kappa} F'^{\sigma\lambda} \rangle\!\rangle h'_{\rho\sigma} F'^\rho{}_\lambda. \tag{3.12}$$

Similarly to the scalar case, the renormalized propagator of the Maxwell field has an ambiguity:

$$\begin{aligned} \langle\!\langle T_{\rm ren} F^{\nu\kappa} F'^{\sigma\lambda} \rangle\!\rangle &= \langle\!\langle T_0 F^{\nu\kappa} F'^{\sigma\lambda} \rangle\!\rangle + \langle\!\langle T_r F^{\nu\kappa} F'^{\sigma\lambda} \rangle\!\rangle \\ &= \langle\!\langle T_0 F^{\nu\kappa} F'^{\sigma\lambda} \rangle\!\rangle + i c_F \big(\eta^{\nu\sigma}\eta^{\kappa\lambda} - \eta^{\nu\lambda}\eta^{\kappa\sigma}\big)\delta(x-x'), \end{aligned}$$

with a free parameter c_F. As before, the non-kinematic part can be absorbed into an induced interaction

$$\begin{aligned} L_{2,r,F} &:= c_F \big((hh)^\rho_\mu F^{\mu\kappa} F_{\rho\kappa} - h_{\mu\nu} h_{\rho\sigma} F^{\mu\sigma} F^{\rho\nu} \big) \\ &= -c_F \big((\!(hh\Theta_F)\!) + (\!(hh)\!) L_{0,F} + h_{\mu\nu} h_{\rho\sigma} F^{\mu\sigma} F^{\rho\nu} \big), \end{aligned} \tag{3.13}$$

where $L_{0,F}$ denotes the quadratic (free) Maxwell Lagrangian, so that

$$i L_{2,r,F} \delta(x-x') + i^2 h_{\mu\nu} F^\mu{}_\kappa \langle\!\langle T_r F^{\nu\kappa} F'^{\sigma\lambda} \rangle\!\rangle h'_{\rho\sigma} F'^\rho{}_\lambda = 0. \tag{3.14}$$

Hence, like in the scalar case, the only possible obstruction to string independence can come from the kinematic part in (3.12). We have:

$$\partial_\mu \langle\!\langle T_0 F^{\mu\nu} F'^{\kappa\lambda} \rangle\!\rangle = \big(\eta^{\nu\lambda}\partial'^\kappa - \eta^{\nu\kappa}\partial'^\lambda\big) i\delta(x-x') = -i\big(\eta^{\nu\lambda}\partial^\kappa - \eta^{\nu\kappa}\partial^\lambda\big)\delta(x-x'). \tag{3.15}$$

A rather lengthy computation, requiring use of the Bianchi identity for the Maxwell field and (3.15) – cf. [15] in this repect – yields

$$\delta_c\Big(\underbracket{L_{1,F} L'_{1,F}}\big|_{T_0}\Big) = i\delta_c\big(h_{\mu\nu} h'_{\rho\sigma}\big) F^\mu{}_\kappa F'^\rho{}_\lambda \langle\!\langle T_0 F^{\nu\kappa} F'^{\sigma\lambda} \rangle\!\rangle \overset{\rm div}{=} \delta_c\big(L_{2,0,F}\big)\delta(x-x') + O_{2,F}(x,x'),$$

with the induced interaction density:

$$\begin{aligned} L_{2,0,F} &= -F_{\nu\lambda} F_\mu{}^\lambda (hh)^{\mu\nu} - \tfrac{1}{2} h_{\mu\nu} h_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} + \tfrac{1}{8} F_{\nu\lambda} F^{\nu\lambda} (\!(hh)\!) \\ &= (\!(hh\Theta_F)\!) + \tfrac{1}{2} (\!(hh)\!) L_{0,F} - \tfrac{1}{2} h_{\mu\nu} h_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} \end{aligned} \tag{3.16}$$

and the remaining obstruction:

$$O_{2,F}(x,x') = -i\Theta_F^{\mu\nu} w^\kappa \big(\partial_\kappa h_{\mu\nu} - \partial_\mu h_{\kappa\nu} - \partial_\nu h_{\kappa\mu}\big)\delta(x-x'), \tag{3.17}$$

totally analogous to the one in the scalar case. As in the latter, the obstruction $O_{2,F}$ cannot be resolved by an induced interaction with field content (F, F, h, h). In all, one finds:

$$\begin{aligned} \delta_c\big(\underbracket{L_{1,F} L'_{1,F}}\big) = \delta_c\Big(&(1-c_F)(\!(hh\Theta_F)\!) + \big(\tfrac{1}{2} - c_F\big)(\!(hh)\!) L_{0,F} \\ &- i\big(\tfrac{1}{2} + c_F\big) F_{\nu\rho} F_{\lambda\sigma} h^{\nu\lambda} h^{\rho\sigma}\Big)\delta(x-x') + O_{2,F}(x,x'). \end{aligned}$$

3.1.3 Graviton coupling to Dirac fields

The Fermi case is more involved than the scalar case and, differently from the previous Maxwell case, was not treated in [15]. For these reasons we give a slightly more detailed treatment here.

Since, as in the other cases, the contraction $\langle\!\langle T_{\rm ren} hh' \rangle\!\rangle$ does not contribute to the obstruction, we study only the contractions of the Dirac fields. As before, the obstruction is caused by the violation of the Ward identity corresponding to the fermion stress-energy tensor under the kinematic part of the time-ordered product,

$$\delta_c\Big(\underbrace{L_{1,\psi} L'_{1,\psi}}\big|_{T_0}\Big) = \delta_c\Bigg(\frac{1}{4} h_{\mu\nu} h'_{\rho\sigma} \sum_{\varphi,\chi'} \frac{\partial \Theta^{\mu\nu}_{\psi}}{\partial\varphi} \langle\!\langle T_0 \varphi\chi' \rangle\!\rangle \frac{\partial {\Theta'}^{\rho\sigma}_{\psi}}{\partial\chi'}\Bigg)$$
$$\stackrel{\rm div}{=} -\frac{1}{2} w_\nu h'_{\rho\sigma} \partial_\mu \sum_{\varphi,\chi'} \frac{\partial \Theta^{\mu\nu}_{\psi}}{\partial\varphi} \langle\!\langle T_0 \varphi\chi' \rangle\!\rangle \frac{\partial {\Theta'}^{\rho\sigma}_{\psi}}{\partial\chi'} + (x \leftrightarrow x'). \tag{3.18}$$

A somewhat protracted calculation yields:

$$\partial_\mu \sum_{\varphi,\chi'} \frac{d\Theta^{\mu\nu}_{\psi}}{d\varphi} \langle\!\langle T_0 \varphi\chi' \rangle\!\rangle \frac{d{\Theta'}^{\rho\sigma}_{\psi}}{d\chi'} = \frac{1}{16}\Big[\overline{\psi}\Big(-\overleftrightarrow{\partial}_\nu \gamma_\rho + i\gamma_\nu (i\partial^\mu\gamma_\mu + m)\gamma_\rho\Big)\delta(x-x')\overleftrightarrow{\partial}'_\sigma \psi'$$
$$+ \overline{\psi}' \overleftrightarrow{\partial}'_\sigma \delta(x-x')\Big(\gamma_\rho \overleftrightarrow{\partial}_\nu + \gamma_\rho i \overleftarrow{\partial}_\mu \gamma^\mu - m) i\gamma_\nu\Big)\psi\Big] + (\rho \leftrightarrow \sigma).$$

After several integrations by parts, and by use of the equations of motions of the fields, the string variation (3.18) is found to be:

$$\delta_c\Big(\underbrace{L_{1,\psi} L'_{1,\psi}}\big|_{T_0}\Big) \stackrel{\rm div}{=} \Big[w^\nu \partial_\nu h^{\rho\sigma}\Big(\overline{\psi}\gamma_\rho\partial_\sigma\psi - \partial_\sigma\overline{\psi}\gamma_\rho\psi\Big) + w^\nu h^{\rho\sigma}\Big(\overline{\psi}\gamma_\rho\partial_\sigma\partial_\nu\psi - \partial_\sigma\partial_\nu\overline{\psi}\gamma_\rho\psi\Big)$$
$$+ \frac{1}{4}\partial^\mu w^\nu h^{\rho\sigma}\Big(\overline{\psi}\gamma_\mu\gamma_\nu\gamma_\rho\partial_\sigma\psi + \partial_\sigma\overline{\psi}\gamma_\mu\gamma_\nu\gamma_\rho\psi\Big)$$
$$+ \frac{1}{2}\partial_\rho w^\nu h^{\rho\sigma}\partial_\sigma\overline{\psi}\gamma_\nu\psi - \frac{1}{2}\partial^\mu w_\rho h^{\rho\sigma}\partial_\sigma\overline{\psi}\gamma_\mu\psi\Big]\,\delta(x-x'). \tag{3.19}$$

Due to the many identities satisfied by the Dirac fields, the γ-matrices and the string-localized graviton field, there are just four linearly independent (up to total divergences) hermitean candidates for induced interactions with the field and derivative content $(\partial, \overline{\psi}, \psi, h, h)$:

$$L^1_{2,\psi} := (hh)_{\mu\nu}\Theta^{\mu\nu}_{\psi}, \qquad L^2_{2,\psi} := (hh)^{\rho\sigma}\partial_\sigma j_\rho, \tag{3.20}$$
$$L^3_{2,\psi} := \frac{i}{2}((hh))\overline{\psi}\gamma^\mu\overleftrightarrow{\partial}_\mu\psi = ((hh))\Theta^\mu_{\mu,\psi}, \qquad L^4_{2,\psi} := \frac{i}{2}(h\partial^\mu h)^{\rho\sigma}\overline{\psi}(\gamma_\mu\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho\gamma_\mu)\psi.$$

Yet another ponderous calculation allows to split (3.19) as

$$\delta_c\Big(\underbrace{L_{1,\psi} L'_{1,\psi}}\big|_{T_0}\Big) \stackrel{\rm div}{=} i\delta_c\big(L_{2,0,\psi}\big)\,\delta(x-x') + O_{2,\psi}(x,x')$$

with the induced interaction density:

$$L_{2,0,\psi} = \tfrac{3}{4}L^1_{2,\psi} - \tfrac{1}{8}L^4_{2,\psi}, \tag{3.21}$$

and the remaining non-resolvable obstruction:

$$O_{2,\psi}(x,x') = -i\Theta^{\mu\nu}_{\psi} w^{\kappa}\big(\partial_\kappa h_{\mu\nu} - \partial_\mu h_{\kappa\nu} - \partial_\nu h_{\kappa\mu}\big)\delta(x-x'). \tag{3.22}$$

Remark 3.3. *Similarly to the scalar and Maxwell cases, some of the fermion propagators have ambiguities. The ambiguous propagators are* $\langle\!\langle T_{\rm ren}\partial_\mu\overline{\psi}\psi'\rangle\!\rangle$, $\langle\!\langle T_{\rm ren}\overline{\psi}\partial'_\mu\psi'\rangle\!\rangle$ *and* $\langle\!\langle T_{\rm ren}\partial_\mu\overline{\psi}\partial'_\nu\psi'\rangle\!\rangle$. *Each of the first two has one free parameter corresponding to a correction* $\gamma_\mu\delta(x-x')$. *The third one has several free parameters, corresponding to corrections proportional to* $\delta(x-x')$ *and* $\partial_\rho\delta(x-x')$, *with several prefactors exhibiting combinations of* γ_ρ *and* $\eta_{\rho\sigma}$. *Since all these corrections correspond to induced interactions, they cannot resolve the obstruction* (3.22). *Therefore they are secondary to the present issue, and thus we omit further detail.*

This concludes the proof of Prop. 3.2. □

3.2 Graviton self-coupling

A cubic self-coupling of massless string-localized fields of helicity 2 without derivatives cannot satisfy the L-Q-pair condition for a string-independent action [14]. The next possibility to consider is cubic self-couplings with two derivatives. *A priori* and dropping all divergences, the general form of L_1 may consist of only three terms:

$$h^{\mu\nu}(\alpha_1\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} + \alpha_2\partial^\rho h^\sigma_\mu\partial_\sigma h_{\nu\rho} + \alpha_3\,\partial^\rho h^\sigma_\mu\partial_\nu h_{\sigma\rho}).$$

In fact Nature chooses $\alpha_1 = \frac{1}{2}, \alpha_2 = 1, \alpha_3 = 0$: a cubic self-coupling of massless string-localized fields of helicity 2, which satisfies the L-Q-pair condition, is *unique* up to a total divergence and a multiplicative constant [14].

Since there is the freedom of adding divergences, we will take the liberty to subtract the term $h^{\mu\nu}\partial^\rho h_{\sigma\mu}\partial_\rho h^\sigma_\nu$, because it entails Eq. (3.28) below, which is going to simplify computations, including the renormalizations. In fact, by adding divergences L_1 can be given many other forms with complementary conveniences. Our choice (3.23) for L_1 below *is identical* with the classical expansion of the Einstein action – which also has a certain freedom of adding divergences [33, Chap. 5] – when the classical metric deviation field $h_{\mu\nu}$ is replaced by the string-localized quantum field $h_{\mu\nu}(x;c)$ of helicity 2 – see App. C.

For a better grasp of the contraction scheme involved in the present problem, on regarding $(\partial_\mu h)_{\kappa\lambda} = \partial_\mu h_{\kappa\lambda}$ as a symmetric tensor, we rewrite L_1 in the notation of Convention 3.1 as:

$$L_1 = h^{\mu\nu}\big[\tfrac{1}{2}(\!(\partial_\mu h\partial_\nu h)\!) + \partial^\kappa h_{\mu\lambda}\partial^\lambda h_{\nu\kappa} - (\partial_\kappa h\partial^\kappa h)_{\mu\nu}\big]. \tag{3.23}$$

Turning to perturbation theory, let us write the Wick expansion as:

$$\underline{L_1\chi'} = V^{\mu\nu}\langle\!\langle T_{\rm ren} h_{\mu\nu}\chi'\rangle\!\rangle + W^{\kappa,\mu\nu}\langle\!\langle T_{\rm ren}\partial_\kappa h_{\mu\nu}\chi'\rangle\!\rangle, \tag{3.24}$$

where we have introduced the quadratic fields

$$\begin{aligned} V^{\mu\nu} &:= \tfrac{1}{2}(\!(\partial^\mu h\partial^\nu h)\!) + \partial_\kappa h^{\mu\lambda}\partial_\lambda h^{\nu\kappa} - (\partial_\kappa h\partial^\kappa h)^{\mu\nu}, \\ W^{\kappa,\mu\nu} &:= h^{\kappa\lambda}\partial_\lambda h^{\mu\nu} + (h\partial^\nu h)^{\mu\kappa} + (\partial^\mu hh)^{\kappa\nu} - \partial^\kappa(hh)^{\mu\nu}. \end{aligned}$$

We have to separately renormalize $\langle\!\langle T_{\rm ren} hh'\rangle\!\rangle$ and $\langle\!\langle T_{\rm ren}\partial hh'\rangle\!\rangle$, so as to respect the trace conditions. It turns out that the SI condition can be fulfilled with

$$\langle\!\langle T_{\rm ren}\partial_\kappa h_{\mu\nu}\chi'\rangle\!\rangle = \partial_\kappa\langle\!\langle T_{\rm ren} h_{\mu\nu}\chi'\rangle\!\rangle \tag{3.25}$$

both for $\chi' = h'$ and $\chi' = \partial' h'$. Thus, with an integration by parts,

$$\underline{L_1\chi'} \overset{\rm div}{=} U^{\mu\nu}\langle\!\langle T_{\rm ren} h_{\mu\nu}\chi'\rangle\!\rangle, \qquad \text{with} \quad U^{\mu\nu} := V^{\mu\nu} - \partial_\kappa W^{\kappa,\mu\nu}, \tag{3.26}$$

and in particular:

$$\underline{L_1 L_1'} \overset{\rm div}{=} U^{\mu\nu}\langle\!\langle T_{\rm ren} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle U'^{\rho\sigma}. \tag{3.27}$$

The field U satisfies:

$$\partial_\mu U^{\mu\nu} = \partial^\nu S \qquad \text{with} \quad S = \tfrac{1}{8}\Box(\!(hh)\!) - \tfrac{1}{2}\partial_\kappa\partial_\lambda(hh)^{\kappa\lambda}, \tag{3.28}$$

and there holds the identity

$$U^\mu_\mu - 2S = \tfrac{1}{2}\Box(\!(hh)\!). \tag{3.29}$$

Now we can compute the obstruction of the string-localized graviton self-interaction, exploiting features of the kinematic and renormalized propagators. The latter are expressed with the help of string integrated differential operators $a_{\mu,\rho\sigma}$, $b_{\rho,\sigma}$, c_ρ acting on the Feynman propagator, which are displayed in App. B. The method is a "cascade of integrations by parts", illustrating the power of the feature of SQFT that string-dependent parts are derivatives.

We begin with the kinematic propagator, of which we separate in the first step the string-independent part:

$$\langle\!\langle T_{0,*} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle = \tfrac{1}{2}\big[\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\big] D_{0,F}(x-x'). \tag{3.30}$$

It does coincide with the propagator that one would use in gauge theory, which is derived from an indefinite two-point function, and is not traceless.[6]

[6] In the gauge-theoretic setting there is no need for renormalization of the propagators, because the field cannot be set to zero on the indefinite metric Fock space; so there are no trace conditions.

The string-dependent part is of the form (B.5):

$$\langle\!\langle T_{0,c}\, h_{\mu\nu} h'_{\rho\sigma} \rangle\!\rangle = \frac{1}{2}\big[\partial_{(\mu} a_{\nu),\rho\sigma} + a'_{\mu\nu,(\rho}\partial_{\sigma)}\big] D_{0,F}(x-x'), \tag{3.31}$$

where $\partial_{(\mu} a_{\nu),\rho\sigma}$ stands for the symmetric sum $\partial_\mu a_{\nu,\rho\sigma} + \partial_\nu a_{\mu,\rho\sigma}$. The string-integrated differential operators $a_{\nu,\rho\sigma}$ and $a_{\mu\nu,\rho}$ are detailed in (B.6) and (B.7). This gives:

$$\begin{aligned} T_0\, \underline{L_1 L'_1} &\overset{\text{div}}{=} U^{\mu\nu} \langle\!\langle T_{0,*} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle U'^{\rho\sigma} + U^{\mu\nu}\langle\!\langle T_{0,c} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle U'^{\rho\sigma} \\ &= U^{\mu\nu} D_{0,F} U'_{\mu\nu} - \frac{1}{2} U^\mu_\mu D_{0,F} U'^\rho_\rho + \frac{1}{2} U^{\mu\nu}\big(\partial_{(\mu} a_{\nu),\rho\sigma} + a'_{\mu\nu,(\rho}\partial_{\sigma)}\big) D_{0,F} U'^{\rho\sigma}. \end{aligned} \tag{3.32}$$

Of the last term, we compute the contribution of $\partial_{(\mu} a_{\nu),\rho\sigma}$:

$$\begin{aligned} U^{\mu\nu}\frac{1}{2}\partial_{(\mu} a_{\nu),\rho\sigma} D_{0,F} U'^{\rho\sigma} &\overset{\text{div}}{=} -\partial_\mu U^{\mu\nu} a_{\nu,\rho\sigma} D_{0,F} U'^{\rho\sigma} = -\partial^\nu S a_{\nu,\rho\sigma} D_{0,F} U'^{\rho\sigma} \\ &\overset{\text{div}}{=} S a_{\nu,\rho\sigma}\partial^\nu D_{0,F} U'^{\rho\sigma} = S(\eta_{\rho\sigma} + b_{\rho\sigma}\Box + c_{(\rho}\partial_{\sigma)}) D_{0,F} U'^{\rho\sigma} \\ &\overset{\text{div}}{=} S D_{0,F} U'^\rho_\rho - iS b_{\rho\sigma}\delta(x-x') U'^{\rho\sigma} + 2S c_\rho D_{0,F}\partial'^\rho S' \\ &\overset{\text{div}}{=} S D_{0,F} U'^\rho_\rho - iS b_{\rho\sigma}\delta(x-x') U'^{\rho\sigma} + 2S c_\rho \partial^\rho D_{0,F} S'. \end{aligned} \tag{3.33}$$

In the second line, we have used the identity (B.8). Likewise, we now use from (B.9)

$$c_\rho\partial^\rho = -1 + \frac{1}{4}\big(I'^2 - (II')\big)\Box.$$

Adding the corresponding contribution from $a'_{\mu\nu,(\rho}\partial_{\sigma)}$ in (3.32), we get

$$T_0\, \underline{L_1 L'_1} \overset{\text{div}}{=} U_{\rho\sigma} D_{0,F} U'^{\rho\sigma} - \frac{1}{2}(U^\mu_\mu - 2S) D_{0,F}(U'^\rho_\rho - 2S') - 2S D_{0,F} S' \tag{3.34}$$

$$- iS b_{\rho\sigma}\delta(x-x') U'^{\rho\sigma} - iU^{\mu\nu} b'_{\mu\nu}\delta(x-x') S' - \frac{i}{2} S(I-I')^2 \delta(x-x') S'. \tag{3.35}$$

The second line (3.35) contains string-integrated δ functions – which we must *not have* by condition (1.5). Lemma B.1 in App. B proves that the renormalization part of the propagator, that is $T_r\, \underline{L_1 L'_1}$, exactly cancels (3.35). Thus the obstruction of the self-interaction is the string variation of the first line (3.34).

The string variations of the fields U and S are computed to be:

$$\delta_c\big(U^{\mu\nu}\big) = \Box\big(2(\partial^\mu w_\kappa + \partial_\kappa w^\mu) h^{\kappa\nu}\big) - \Box\big(w_\kappa(\partial^\kappa h^{\mu\nu} - \partial^\mu h^{\kappa\nu} - \partial^\nu h^{\mu\kappa})\big) + \partial^{(\mu} K^{\nu)}; \tag{3.36}$$

$$\delta_c\big(S\big) = \frac{1}{4}\delta_c\big(\Box(\!(hh)\!)\big) + \partial_\mu K^\mu; \quad \text{with} \tag{3.37}$$

$$K^\mu = -\Box(w_\kappa h^{\kappa\mu}) - h^{\kappa\lambda}\partial_\kappa\partial_\lambda w^\mu, \quad \partial_\mu K^\mu = \delta_c\big(S - \tfrac{1}{4}\Box(\!(hh)\!)\big). \tag{3.38}$$

When the string variation of (3.34) is computed, there arise K-terms not containing a wave operator, hence producing undesired bulk contributions. Happily, these bulk terms cancel each other:

$$\begin{aligned} &\partial_{(\mu} K_{\nu)} D_{0,F} U'^{\mu\nu} + (x \leftrightarrow x') \overset{\text{div}}{=} -2K_\nu D_{0,F}\partial'^\nu S' + (x \leftrightarrow x') \\ &\overset{\text{div}}{=} 2\partial_\nu K^\nu D_{0,F} S' + (x \leftrightarrow x') = 2\delta_c\big(S D_{0,F} S'\big) + iS\delta_c\big((\!(hh)\!)\big)\delta(x-x'), \end{aligned} \tag{3.39}$$

by (3.38). Thus, the obstruction $\delta_c\big(T_{\rm ren}\, \underbrace{L_1 L_1'}\big)$ of the self-interaction is (up to divergences)

$$- 2\big(\delta_c\big((hh)^{\mu\nu}\big) - w_\kappa(\partial^\kappa h^{\mu\nu} - \partial^\mu h^{\kappa\nu} - \partial^\nu h^{\mu\kappa})\big)U_{\mu\nu} + \delta_c\big(((hh))\big)S$$
$$+ \tfrac{1}{8}\delta_c\big(((hh))\Box((hh))\big), \tag{3.40}$$

multiplying $i\delta(x - x')$.

To establish SI at second order, it remains to show that (3.40) is the string variation of a quartic induced interaction L_2:

$$(3.40) \overset{!}{=} \delta_c(L_2). \tag{3.41}$$

Theorem 3.4. *The second order obstruction of the graviton self-interaction is resolved by the induced interaction*

$$L_2 \overset{\rm div}{=} -2(\partial_\nu hhh\partial_\mu h)^{\mu\nu} - \tfrac{1}{2}((h\partial_\mu h))((h\partial^\mu h)) - 2((h\partial_\mu h\partial_\nu h))h^{\mu\nu}; \tag{3.42}$$

where L_2 is uniquely determined up to total divergences.

Proof. After the long preparations leading to (3.40), the proof is routine: work out (3.40) with (3.36) and (3.37), make the most general Ansatz to solve (3.41), and compare coefficients. This leads to a highly overdetermined system of twenty-four equations for eight unknowns, with a unique solution. One obtains a large number of structures $O(w, \partial h\partial h\partial h)$. To control linear dependences (up to divergences) one first detaches all derivatives from the factor w^κ by an integration by parts. The remaining linear dependences arise from $\partial_\kappa w^\kappa = 0$ implying $(w\partial)X \overset{\rm div}{=} 0$, and from $\Box w^\kappa = 0$ implying $w^\kappa \Box X \overset{\rm div}{=} 0$. One gets seven different contraction schemes of the tensors h, with various placements of the derivatives. Our basis of choice is:

$$\begin{array}{lll}
w^\kappa h_{\kappa\lambda}((hh)) : & S^1 = w^\kappa \cdot h_{\kappa\lambda}((\partial^\lambda\partial_\alpha h\partial^\alpha h)), & S^2 = w^\kappa \cdot \partial_\alpha h_{\kappa\lambda}((\partial^\alpha h\partial^\lambda h)), \\
w^\kappa \cdot h_{\kappa\lambda}(hh)^{\alpha\beta} : & S^3 = w^\kappa \cdot h_{\kappa\lambda}(\partial^\lambda\partial_\alpha h\partial_\beta h)^{\alpha\beta}, & S^4 = w^\kappa \cdot \partial_\alpha h_{\kappa\lambda}(\partial_\beta h\partial^\lambda h)^{\alpha\beta}, \\
 & S^5 = w^\kappa \cdot \partial_\alpha h_{\kappa\lambda}(\partial_\beta\partial^\lambda hh)^{\alpha\beta}, & S^6 = w^\kappa \cdot \partial_\alpha\partial_\beta h_{\kappa\lambda}(h\partial^\lambda h)^{\alpha\beta}, \\
w^\kappa \cdot (hh)_{\kappa\lambda}h^{\alpha\beta} : & S^7 = w^\kappa \cdot (h\partial_\alpha\partial_\beta h)_{\kappa\lambda}\partial^\lambda h^{\alpha\beta}, & S^8 = w^\kappa \cdot (\partial_\alpha h\partial_\beta h)_{\kappa\lambda}\partial^\lambda h^{\alpha\beta}, \\
 & S^9 = w^\kappa \cdot (\partial_\alpha\partial_\beta hh)_{\kappa\lambda}\partial^\lambda h^{\alpha\beta}, & S^{10} = w^\kappa \cdot (\partial_\alpha\partial^\lambda h\partial_\beta h)_{\kappa\lambda}h^{\alpha\beta}, \\
 & S^{11} = w^\kappa \cdot (\partial_\alpha\partial_\beta\partial^\lambda hh)_{\kappa\lambda}h^{\alpha\beta}, & S^{12} = w^\kappa \cdot (\partial^\lambda h\partial_\alpha\partial_\beta h)_{\kappa\lambda}h^{\alpha\beta}, \\
w^\kappa \cdot (hhh)_{\kappa\lambda} : & S^{13} = w^\kappa \cdot (h\partial_\alpha\partial^\lambda h\partial^\alpha h)_{\kappa\lambda}, & S^{14} = w^\kappa \cdot (\partial^\lambda h\partial_\alpha h\partial^\alpha h)_{\kappa\lambda}, \\
 & S^{15} = w^\kappa \cdot (\partial_\alpha h\partial^\lambda h\partial^\alpha h)_{\kappa\lambda}, & S^{16} = w^\kappa \cdot (\partial_\alpha\partial^\lambda hh\partial^\alpha h)_{\kappa\lambda}, \\
w^\kappa \cdot ((\partial_\kappa hhh)) : & S^{17} = w^\kappa \cdot ((\partial_\kappa\partial_\alpha h\partial^\alpha hh)), & \\
w^\kappa \cdot ((\partial_\kappa hh))h^{\alpha\beta} : & S^{18} = w^\kappa \cdot ((\partial_\kappa h\partial_\alpha\partial_\beta h))h^{\alpha\beta}, & S^{19} = w^\kappa \cdot ((\partial_\kappa\partial_\alpha h\partial_\beta h))h^{\alpha\beta}, \\
 & S^{20} = w^\kappa \cdot ((\partial_\kappa\partial_\alpha\partial_\beta hh))h^{\alpha\beta}, & \\
w^\kappa \cdot (\partial_\kappa hhh)^{\alpha\beta} : & S^{21} = w^\kappa \cdot (\partial_\kappa h\partial_\beta h\partial_\alpha h)^{\alpha\beta}, & S^{22} = w^\kappa \cdot (\partial_\kappa h\partial_\alpha\partial_\beta hh)^{\alpha\beta}, \\
 & S^{23} = w^\kappa \cdot (\partial_\kappa\partial_\beta h\partial_\alpha hh)^{\alpha\beta}, & S^{24} = w^\kappa \cdot (\partial_\kappa\partial_\alpha hh\partial_\beta h)^{\alpha\beta}.
\end{array}$$

In this basis,

$$(3.40) = -4[S^1 - S^5 - S^6 + S^9 + S^{10} + S^{11} + S^{12} + S^{13}$$
$$+ S^{14} + S^{15} + S^{16} + S^{17} + S^{21} + S^{22} + S^{23} + S^{24}] \tag{3.43}$$

On the right-hand side of (3.41), there are eight linearly independent (up to divergences) candidate structures for the induced interaction $O(h^4)$ with two derivatives:

$$
\begin{aligned}
&L_2^1 := ((h\partial^\mu h))((h\partial_\mu h)), && L_2^2 := ((\partial_\mu h\partial_\nu h))(hh)^{\mu\nu},\\
&L_2^3 := ((hh))(\partial_\nu h\partial_\mu h)^{\mu\nu}, && L_2^4 := ((h\partial_\mu h\partial_\nu h))h^{\mu\nu},\\
&L_2^5 := ((h\partial_\mu hh\partial^\mu h)), && L_2^6 := (\partial_\nu hhh\partial_\mu h)^{\mu\nu},\\
&L_2^7 := (h\partial_\mu h\partial_\nu hh)^{\mu\nu}, && L_2^8 := (h\partial_\nu h\partial_\mu hh)^{\mu\nu}.
\end{aligned}
$$

One computes

$$
\begin{aligned}
&\delta_c\big(L_2^1\big) \overset{\text{div}}{=} 8S^1,\\
&\delta_c\big(L_2^2\big) \overset{\text{div}}{=} -2[S^1+S^2-2S^4-2S^5-4S^6+S^{18}+S^{19}],\\
&\delta_c\big(L_2^3\big) \overset{\text{div}}{=} -4[S^1+2S^3+S^{18}+2S^{19}+S^{20}],\\
&\delta_c\big(L_2^4\big) \overset{\text{div}}{=} 2[S^7+S^8+S^9+S^{10}+S^{11}+S^{12}+S^{17}],\\
&\delta_c\big(L_2^5\big) \overset{\text{div}}{=} -8[S^{15}+S^{16}],\\
&\delta_c\big(L_2^6\big) \overset{\text{div}}{=} -2[S^5+S^6+S^7+S^8-S^{13}-S^{14}-S^{15}-S^{16}-S^{21}-S^{22}-S^{23}-S^{24}],\\
&\delta_c\big(L_2^7\big) \overset{\text{div}}{=} 2[2S^3+S^4+S^5+2S^{10}+S^{11}+S^{12}+S^{13}-S^{21}-S^{22}],\\
&\delta_c\big(L_2^8\big) \overset{\text{div}}{=} 2[S^3+S^4+S^5+S^6+S^8+S^9+S^{10}+S^{11}+S^{16}+S^{21}+S^{23}+2S^{24}].
\end{aligned}
$$

Comparing the coefficients of S^a with the ones in (3.43), the SI condition (3.41) gives 24 equations for the eight coefficients of L_2^n with the unique solution:

$$
L_2 \overset{\text{div}}{=} -\tfrac{1}{2}L_2^1 - 2L_2^4 - 2L_2^6 = L_2
$$

in (3.42). This completes the proof. □

As a byproduct of the analysis of string variations of interaction structures with the field content $hhhh$ and two derivatives – namely, uniqueness of the solution (3.42) – we have found:

Corollary 3.5. *Any interaction density with field and derivative content* $(\partial,\partial,h,h,h,h)$*, which is string-independent on its own, must itself be a total divergence.*

There is another side-message in the proof of Theorem 3.4: the effectiveness and elegance of the computations shows that SQFT – technically complicated as it might appear at first sight – is hardly more difficult than causal perturbation theory in gauge theory, as in [33]. One diligently ought to know how to take advantage of systematic cancellations before starting actual computations. By Lemmata B.1 and B.2 in App. B, the necessary renormalization of propagators only serves to cancel terms with string-dependent δ-functions.

By inspection of (3.42), we conclude:

Theorem 3.6. *The induced graviton self-interaction* (3.42) *as a (Wick) polynomial in the string-localized quantum field $h^{\mu\nu}(x;c)$ coincides with the second-order term of the classical Einstein action* (C.6) *as a polynomial in the classical metric deviation field $h^{\mu\nu}(x)$, defined by* (C.2).

Recall that the induced quantum coupling was determined only up to divergences, and notice the last line in (C.6) is a divergence. Recall as well that (3.23) was distinguished by the L-Q-pair condition as a consistency condition based on quantum principles. We thus are entitled to regard (3.23) and (3.42) as "predictions" from quantum theory on the classical self-interaction for helicity 2.

3.3 Cancellation of the matter obstructions

We have seen in Prop. 3.2 that the matter obstruction to second-order string-independence in the graviton-matter couplings is always of the form

$$\delta_c\Big(\underline{L_{1,\mathrm{mat}}L'}_{1,\mathrm{mat}}\Big) \overset{\mathrm{div}}{=} i\delta_c\big(L_{2,0,\mathrm{mat}} + L_{2,r,\mathrm{mat}}\big)\delta(x-x') + O_{2,\mathrm{mat}}(x,x'). \tag{3.44}$$

where $O_{2,\mathrm{mat}}$ is the non-resolvable obstruction given in (3.8).

We will now show the second main result of this paper. To wit, that these obstructions are *exactly cancelled* if we take the graviton self-interaction into account. Since the contractions of the graviton field with matter fields do vanish, we do not need to specify the concrete form of $\Theta^{\mu\nu}_{\mathrm{mat}}$. We just assume that it is conserved, point-localized (thus string-independent) and symmetric. The cue is that (3.8) features the characteristic structure

$$O_{2,\mu\nu} := w^\kappa\big(\partial_\kappa h_{\mu\nu} - \partial_\mu h_{\kappa\nu} - \partial_\nu h_{\kappa\mu}\big) \tag{3.45}$$

in the sector of helicity 2, which is not separately a string variation. The same structure also appears in the string variation (3.36) of the field $U_{\mu\nu}$.

Let us compute the interference term between the self-coupling (3.23) and the matter coupling (3.1) to an arbitrary conserved and symmetric stress-energy tensor. We use again (3.25), implying (3.26). Thus:

$$T_{\mathrm{ren}}\,\underline{L_1L'}_{1,\mathrm{mat}} \overset{\mathrm{div}}{=} \tfrac{1}{2}U^{\mu\nu}\langle\!\langle T_{\mathrm{ren}}h_{\mu\nu}h'_{\rho\sigma}\rangle\!\rangle\Theta'^{\rho\sigma}. \tag{3.46}$$

For the kinematic propagator of the form (B.4)+(B.5), we can drop the contribution from $a'_{\mu\nu,(\rho}\partial_{\sigma)}$ because Θ is conserved. Therefore

$$\begin{aligned}
T_0\,\underline{L_1L'}_{1,\mathrm{mat}} &\overset{\mathrm{div}}{=} \tfrac{1}{2}\big[(U_{\rho\sigma} - \tfrac{1}{2}U^\mu_\mu\eta_{\rho\sigma})D_{0,F}\Theta'^{\rho\sigma} + \tfrac{1}{2}U^{\mu\nu}(\partial_{(\mu}a_{\nu),\rho\sigma})D_{0,F}\Theta'^{\rho\sigma}\big]\\
&\overset{\mathrm{div}}{=} \tfrac{1}{2}\big[(U_{\rho\sigma} - \tfrac{1}{2}U^\mu_\mu\eta_{\rho\sigma})D_{0,F}\Theta'^{\rho\sigma} - \partial^\nu S a_{\nu,\rho\sigma}D_{0,F}\Theta'^{\rho\sigma}\big]\\
&\overset{\mathrm{div}}{=} \tfrac{1}{2}\big[(U_{\rho\sigma} - \tfrac{1}{2}U^\mu_\mu\eta_{\rho\sigma})D_{0,F}\Theta'^{\rho\sigma} + S(\eta_{\rho\sigma} + b_{\rho\sigma}\Box)D_{0,F}\Theta'^{\rho\sigma}\big]\\
&\overset{\mathrm{div}}{=} \tfrac{1}{2}\big[(U_{\rho\sigma} - (\tfrac{1}{2}U^\mu_\mu - S)\eta_{\rho\sigma})D_{0,F}\Theta'^{\rho\sigma} - iSb_{\rho\sigma}\delta(x-x')\Theta'^{\rho\sigma}\big].
\end{aligned} \tag{3.47}$$

The string variation of the first term in (3.47) is given by (3.36) with (3.29), giving

$$\delta_c\big(\tfrac{1}{2}(U_{\rho\sigma} - (\tfrac{1}{2}U^\mu_\mu - S)\eta_{\rho\sigma})D_{0,F}\Theta'^{\rho\sigma}\big)$$
$$\overset{\text{div}}{=} \frac{i}{2}\big[\big(\delta_c\big(\tfrac{1}{4}(\!(hh)\!)\eta_{\rho\sigma} - (hh)_{\rho\sigma}\big) + O_{2,\rho\sigma}\big)\Theta^{\rho\sigma}\delta(x-x')\big]. \tag{3.48}$$

Taking into account the term $x \leftrightarrow x'$, we have to add twice the above expression to the matter obstruction (3.44). The non-resolvable matter obstruction is cancelled, and the remaining terms are cancelled by

$$L_{2,\text{mat}} = \frac{1}{4}(\!(hh)\!)(\!(\Theta)\!) - (\!(hh\Theta)\!) + L_{2,0,\text{mat}} + L_{2,r,\text{mat}}. \tag{3.49}$$

There still remains the second term with string-integrated δ-functions in (3.47). When we discard ∂_ρ, ∂_σ in the string differential operators $b_{\rho\sigma}$ given in (B.9), it becomes

$$-\frac{i}{2}S\big(I_\rho I_\sigma - \tfrac{1}{2}I^2\eta_{\rho\sigma}\big)\delta(x-x')\Theta'^{\rho\sigma}. \tag{3.50}$$

Lemma B.2 in App. B asserts that this term is cancelled by the contribution of the renormalization part of the propagators. We conclude:

Theorem 3.7. *The non-resolvable obstruction* (3.8) *to string-independence at second order of the graviton-matter interaction is exactly cancelled if the graviton self-coupling is taken into account. The SI condition is satisfied with* (3.49).

One may rephrase Theorem 3.7 as follows: the obstruction to SI arising from the violation of the Ward identity (3.4) is resolved if the graviton self-interaction is taken into account.

Because the matter-matter obstructions for several free fields are additive, and the same is true for the obstructions of the interference term with the self-coupling, one has

Corollary 3.8. *The result of Theorem 3.7 is true for the coupling to any number of scalar, Maxwell and Dirac fields, with the respective sum of induced matter interactions.*

It remains to compare (3.49) to the classical expressions obtained from the expansion of the classical generally covariant Lagrangians. This can be done "by inspection" of the three cases. The expansions are sketched in App. C.

A perfect match with (3.49) is found in all three cases if all matter renormalization constants are zero ($L_{2,r,\text{mat}} = 0$). Indeed, unlike for helicity 2, there is no physical reason to renormalize the matter propagators with spin or helicity ≤ 1. E.g., for the scalar field:

$$L_{2,\phi} = \frac{1}{4}(\!(hh)\!)(\!(\Theta_\phi)\!) - (\!(hh\Theta_\phi)\!) + (\!(hh\Theta_\phi)\!) + \frac{1}{2}(\!(hh)\!)L_{0,\phi}, \tag{3.51}$$

coincident with (C.9).

We have thus established:

Theorem 3.9. *The induced matter interactions (3.49) as (Wick) polynomials in the string-localized quantum field $h^{\mu\nu}(x;c)$ and the quantum matter fields coincide, when the kinematical propagators are chosen for the matter fields, with the respective second-order terms of the classical generally covariant scalar, Maxwell and Dirac Lagrangians as polynomials in the classical metric deviation field $h_{\mu\nu}(x)$ and matter fields.*

Theorem 3.7 is what we called the lock-key situation, in fact the central result of this paper. We emphasize again that the self-interaction of gravitons knows nothing about matter; the violation of the matter's Ward identity knows nothing about the inner workings of gravity, either. Yet the obstructions are resolved together, like a lock and its key. This is what we call **quantum general covariance**. Theorems 3.6 and 3.9 are the precise formulation of the assertion "consistency of the quantum couplings predicts general covariance" in the Introduction.

4 Conclusion and further discussion

For perturbative QFT of helicity 2 on a Minkowski background we have established here a remarkable coincidence between the structure of quantum interactions – dictated by the quantum principles: *Hilbert space (positivity), covariance, locality of observables* – and of *classical* "Lagrangians" – dictated by classical general covariance. It pertains not only to the self-interaction of particles of spin 2 ("gravitons"), but also to matter couplings that go necessarily through the stress-energy tensor.

Now, the quantum principles are well-known to exclude each other for helicity ≥ 1 in standard axiomatic setups of QFT. This long-standing barrier has been conquered by admitting string-localized free fields and interactions, and imposing the condition of string-independence on the $\mathbb{S}$-matrix.

There are of course fundamental differences between classical Lagrangians and interaction densities in Quantum Field Theory. The former are the starting point of a variational principle to derive the equations of motion. One may freely add terms that are total divergences *without using the equations of motion*. E.g., the classical expansion of the Einstein Lagrangian as a power series in the metric deviation field – see [33]. In contrast, the (perturbative) $\mathbb{S}$-matrix of QFT is defined as the time-ordered exponential of the action, i.e., the integral over the *interaction density*

$$\int d^4x\, L_{\text{int}}(x), \tag{4.1}$$

not including a free Lagrangian. This interaction density is a functional of the free fields, that are autonomously defined on the Fock space over Wigner's unitary representations of the Poincaré group, automatically operating on a Hilbert space. The equations of motion of the free fields are consequences of their construction. (There is a similar split in the path integral approach: the "free Lagrangian" only defines the integration measure, while the integrand contains only the interaction Lagrangian.) Thus one may freely add terms to the interaction density that are total derivatives as quantum operators, i.e., when the free equations of motion are used, without changing the action (4.1). The last term in (3.23) is an example. It must not be discarded in classical field theory.

Now, adding a divergence does affect the $\mathbb{S}$-matrix, because the time-ordered exponential of the action is changed. A thorough analysis can be found in [28]. There, the point was to add divergences as to turn a non-renormalizable point-localized interaction into a renormalizable string-localized one, or a gauge-theoretic interaction on an indefinite Fock space into a string-localized interaction in an embedded Hilbert space, as in BRST theory.

The analysis is not restricted to such cases. The general tenor is that if two interactions $L_1 = K_1 + \partial_\mu V_1^\mu$ differ by a divergence, then the induced interactions will differ by $L_2 - K_2$, such that

$$\frac{1}{2}\big(\partial_\mu T V_1^\mu (L_1' + K_1') - T\partial_\mu V_1^\mu (L_1' + K_1')\big) + (x \leftrightarrow x') = i(L_2 - K_2)(x)\delta(x - x'). \tag{4.2}$$

In other words, the effect of adding a divergence in first order is compensated by an induced term in second order, and so on, which itself is not a divergence. It is then established that the resulting $\mathbb{S}$-matrices (including the induced higher-order interactions) will be the same. Thus, physics cannot tell the difference between those different choices of the interaction density.

To be sure, the story is a bit subtler. Formal expressions like $\int d^4x\, L(x)$ exist as quantum operators only with a cutoff function $g(x)$, and the "adiabatic limit" $g(x) \uparrow 1$ is particularly laborious in theories with massless fields. It is therefore important that the equality of $\mathbb{S}$-matrices hold for arbitrary cutoff functions – as shown in [28] – before the limit is taken.

4.1 On the usefulness of string-localized fields

Free string-localized quantum fields of any spin and helicity were introduced in [29, 30], not least in order to fill the gap of the (bosonic and fermionic) "infinite spin" fields in the list of quantum fields associated with Wigner's classification of particles. String-localized *charged* fields had been heuristically introduced much earlier [7, 22, 24, 37] in attempts to formulate Quantum Electrodynamics in terms of gauge-invariant charged fields. They were indirectly discovered in the axiomatic framework of algebraic quantum field theory (AQFT) by Buchholz and Fredenhagen [5] in their analysis of charged states.

AQFT does not address charged fields because they are not quantum observables. On the other hand, it has been shown that the localization properties of charged states are such as if they were created from the vacuum by either local or string-localized charged fields. An actual construction of string-localized Dirac fields going beyond the perturbative framework was presented in [27], mediated by an interaction that involves a string-localized free Maxwell potential – the helicity 1 analogue of the present work. The interacting Dirac field does carry a "string of electric flux", whose direction is described by a profile function $c(e)$ as in the present work, and whose "unit weight" constraint secures the correct total flux as required by the Gauss Law. Due to the infrared singularity of photons, different profile functions "cannot coexist" – they do belong to different superselection sectors. Translating into helicity 2, one expects that matter fields interacting with quantum gravitons carry a "string of curvature", that depends on the profile function $c(e)$ but ensures that the Komar mass of the curvature field is that of the matter particle. A spatially constant profile function (i.e., a completely diffuse string) would give a quantum state corresponding to the rotationally symmetric Schwarzschild metric. One also expects a similar infrared superselection structure, not addressed in this paper. Details of this picture are under investigation. These issues can be assessed only because in SQFT charged fields interacting with helicity 1 potentials (QED) or matter fields interacting with helicity 2 potentials fields are constructed on a Hilbert space. In contrast, in gauge-theoretic settings the interacting charged, resp. matter fields are not BRST invariant, hence they cannot be defined on the BRST Hilbert space.

At any rate, the construction of interacting matter fields along the lines of [27] is beyond the scope of the present paper. Suffice here to mention that *ab initio* string-localized *free* potentials as those in (2.4) serve as a tool to construct "inherently" string-localized *interacting* fields.

As before stated, the main result of our paper is that the quantum principles impose on interactions with string-local – thus positive – fields of helicity 2 the same structure as general covariance imposes on classical Lagrangians. This not only extends a multitude of analogous cases with helicity (or spin) 1: QED, Yang-Mills, weak interaction, Higgs models... It is also in line with the fact that classical field theory can only be a limit in an actual quantum world,[7] and may also be taken as Nature's cue that QG might exist beyond perturbative QFT on flat background as an autonomous quantum theory.

A Comparison of SQFT to the BRST formalism

A comment on the relation between gauge theory and SQFT is in order. In the former framework one starts with a canonically quantized *point-localized* tensor field $h_{\mu\nu}(x)$ – for instance in the Feynman gauge, with the propagator given below in Eq. (B.4) – defined on a space with indefinite metric ("Krein space"). It has spurious degrees of freedom, because $\eta^{\mu\nu}h_{\mu\nu} = 0$ and $\partial^\mu h_{\mu\nu} = 0$ cannot hold as operator identities. The BRST method has the related purposes of extracting a passage from the Krein space to a Hilbert space and eliminating spurious degrees of freedom. The BRST operator is a nilpotent fermionic operator Q,[8] whose kernel is a positive-semidefinite subspace of the Krein space. For its construction, one has to extend the Krein space by a ghost Fock space. The BRST variation is the graded commutator:

$$s(A) = i[Q, A]_\pm .$$

[7] Rather than, reversely, "to quantize" as an attempt to get back the full gamut from its limit.

[8] Not to be confused with our Q_1 in the L-Q-pair condition.

Then $\mathcal{H} := \mathrm{Ker}(Q)/\mathrm{Ran}(Q)$ is a Hilbert space, and only the operators $A \in \mathrm{Ker}(s)$ are well-defined on $\mathcal{H}$. These are the local observables of the theory. The BRST variation of the basic field $h_{\mu\nu}$ in this setting is[9] is a gauge transformation:

$$s(h_{\mu\nu}) = \tfrac{1}{2}\big(\partial_\mu u_\nu + \partial_\nu u_\mu\big) \tag{A.1}$$

where u_μ is the ghost field. Thus $h_{\mu\nu}$ is not an observable, but its field strength – defined as in (2.5) – is an observable. In fact, on the BRST Hilbert space $\mathcal{H}$ it is isomorphic to our original field F of helicity 2 on the Wigner Fock space, with only two degrees of freedom according to the helicities $|h| = 2$.

Let us now discuss how SQFT and BRST are related. This becomes most transparent in the approaches by Scharf [33] and Dütsch [8], who have achieved similar results concerning self-interactions of helicity 1 and 2, namely the necessity of quartic interaction terms, by imposing "perturbative gauge invariance" in the BRST setting. The cubic self-interaction density $L_1(x)$ from gauge theory is not BRST-invariant, but

$$s\big(L_1(x)\big) = \partial_\mu T_1^\mu(x), \tag{A.2}$$

where T_1^μ involves the ghost fields. The subsequent analysis is quite similar to ours (in fact it was a blueprint for ours): the $\mathbb{S}$-matrix is BRST-invariant in first order, but not in second order for the same reason as in (1.4). The second-order BRST obstruction has to be cancelled by a second-order interaction L_2. Which is possible because it has a form analogous to (1.5).

One could avoid the BRST obstruction if one had a BRST-invariant interaction density. Assuming that there is a field ϕ on the Krein space such that $s(\phi_\mu) = -\frac{1}{2}u_\mu$, then it is feasible to define the BRST-invariant potential

$$h^\phi_{\mu\nu}(x) = h_{\mu\nu}(x) + \partial_\mu\phi_\nu + \partial_\nu\phi_\mu.$$

In view of (A.1), ϕ_μ must be some primitive of $h_{\mu\nu}$. Such a primitive can be constructed with the help of string integrations over $h_{\mu\nu}$:

$$\phi_\mu = I_c^\nu h_{\mu\nu} + \tfrac{1}{2}\partial_\mu I_c^\kappa I_c^\lambda h_{\kappa\lambda}.$$

(To verify this claim, one needs Eq. (2.3).) Writing the interaction L_1 in terms of $h^\phi_{\mu\nu}$, one may discard derivatives – that are responsible for the right-hand side of (A.2) – to get a BRST-invariant interaction. The fields F and h^ϕ being BRST-invariant, they "descend" to the BRST Hilbert space $\mathcal{H}$. The restriction of $h^\phi_{\mu\nu}$ to $\mathcal{H}$ coincides with $h_{\mu\nu}(x;c)$ on the Wigner Fock space. In particular, $\eta^{\mu\nu}h^\phi_{\mu\nu}$ and $\partial^\mu h^\phi_{\mu\nu}$ are null fields on the Krein space (their two-point functions vanish). Finally, the BRST-invariant interaction coincides with the string-localized

[9] In Eq. (5.1.8) of [33] one actually finds the formula: $s(h^{\mu\nu}) = \frac{1}{2}\big(\partial^\mu u^\nu + \partial^\nu u^\mu - \eta^{\mu\nu}\partial_\kappa u^\kappa\big)$. One may pass from this form to ours by redefining $h_{\mu\nu}$ as $h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}((h))$ and idetifying u with our w. For the quantum field $h_{\mu\nu}(c)$ there is no difference because $((h)) = 0$ and moreover $(\partial w) = 0$. Notice that both fields have the same Feynman propagator (B.4).

$L_1(x;c)$ in (3.23) up to a divergence. In this way, one precisely arrives at the setup of the present paper, where the task is to establish SI rather than BRST invariance.

The great difference is that we proceeded without the detour through Krein space and ghosts, from which BRST invariance has to bring us back home to the physical Hilbert space. A second major advantage of the string-localized approach is that *interacting fields* (not considered in the present paper) do exist on our Hilbert space, whereas in general they are *not* BRST-invariant in the gauge theoretic setting [28]. Whether string-localized quantum field theory is the "unkown formulation" for which the BRST fomalism has been an "efficient placeholder" (R. Stora) hangs nevertheless on unsolved problems of renormalization of loop graphs with internal stringlike lines.

B Graviton propagators and their renormalization

Let us split all propagators into their string-independent part and the string-dependent ones:

$$\langle\langle T\phi\chi'\rangle\rangle = \langle\langle T_*\phi\chi'\rangle\rangle + \langle\langle T_c\phi\chi'\rangle\rangle. \tag{B.1}$$

Recall the kinematic propagator (2.22). It is composed of the string-integrated differential operators E, E', and E'' of (2.23) which for the purposes of Sect. 3.2 and Sect. 3.3 we rewrite conveniently as

$$E_{\mu\nu} = \eta_{\mu\nu} + a_{(\mu}\partial_{\nu)}, \qquad E''_{\rho\sigma} = \eta_{\rho\sigma} + a'_{(\rho}\partial_{\sigma)}, \qquad E'_{\mu\rho} = \eta_{\mu\rho} + (b'_\mu\partial_\rho + b_\rho\partial_\mu), \tag{B.2}$$

with (using $\partial' = -\partial$)

$$\begin{aligned} a_\mu &= I_\mu + \tfrac{1}{2}I^2\partial_\mu, & a'_\rho &= -I'_\rho + \tfrac{1}{2}I'^2\partial_\rho, \\ b_\rho &= I_\rho - \tfrac{1}{2}(II')\partial_\rho, & b'_\mu &= -I'_\mu - \tfrac{1}{2}(II')\partial_\mu. \end{aligned} \tag{B.3}$$

This yields the decomposition

$$\langle\langle T_{0,*}h_{\mu\nu}h'_{\rho\sigma}\rangle\rangle = \tfrac{1}{2}\big[\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\big]D_{0,F}, \tag{B.4}$$

$$\langle\langle T_{0,c}h_{\mu\nu}h'_{\rho\sigma}\rangle\rangle = \tfrac{1}{2}\big[\partial_{(\mu}a_{\nu),\rho\sigma} + a'_{\mu\nu,(\rho}\partial_{\sigma)}\big]D_{0,F} \tag{B.5}$$

with

$$a_{\nu,\rho\sigma} = \eta_{\nu(\sigma}b_{\rho)} + \tfrac{1}{2}b'_\nu b_{(\rho}\partial_{\sigma)} + b_\rho b_\sigma\partial_\nu - a_\nu(\eta_{\rho\sigma} + \tfrac{1}{2}a'_{(\rho}\partial_{\sigma)}), \tag{B.6}$$

$$a'_{\mu\nu,\sigma} = \eta_{\sigma(\nu}b'_{\mu)} + \tfrac{1}{2}b_\sigma b'_{(\mu}\partial_{\nu)} + b'_\mu b'_\nu\partial_\sigma - a'_\sigma(\eta_{\mu\nu} + \tfrac{1}{2}a_{(\mu}\partial_{\nu)}). \tag{B.7}$$

$\langle\langle T_{0,*}hh\rangle\rangle$ is the propagator as one would use in gauge theory. The salient feature of $\langle\langle T_{0,r}hh\rangle\rangle$, seen in (B.5), is that each term contains an uncontracted derivative, which plays a major role in the computations.

It ensues, with $-(I\partial) = (I'\partial) = 1$,

$$a_{\nu,\rho\sigma}\partial^\nu = \eta_{\rho\sigma} + b_{\rho\sigma}\Box + c_{(\rho}\partial_{\sigma)}, \qquad a'_{\mu\nu,\sigma}\partial^\sigma = \eta_{\mu\nu} + b'_{\mu\nu}\Box + c'_{(\mu}\partial_{\nu)}, \tag{B.8}$$

where

$$\begin{aligned} b_{\rho\sigma} &= b_\rho b_\sigma - \tfrac{1}{4}(II')b_{(\rho}\partial_{\sigma)} - \tfrac{1}{2}I^2(\eta_{\rho\sigma} + \tfrac{1}{2}a'_{(\rho}\partial_{\sigma)}), \qquad & c_\rho &= \tfrac{1}{2}(b_\rho + a'_\rho), \\ b'_{\mu\nu} &= b'_\mu b'_\nu - \tfrac{1}{4}(II')b'_{(\mu}\partial_{\nu)} - \tfrac{1}{2}I^2(\eta_{\mu\nu} + \tfrac{1}{2}a_{(\mu}\partial_{\nu)}), \qquad & c'_\mu &= \tfrac{1}{2}(b'_\mu + a_\mu). \end{aligned} \tag{B.9}$$

These relations become instrumental in the proof of Lemma B.1 underlying Theorem 3.4.

The kinematic propagator violates the trace conditions:

$$\eta^{\mu\nu}\langle\!\langle T_0 h_{\mu\nu}h'_{\rho\sigma}\rangle\!\rangle = \big[(I_\rho - (II')\partial_\rho)(I_\sigma - (II')\partial_\sigma) - \tfrac{1}{2}I^2E''_{\rho\sigma}\big]\Box D_{0,F}(x-x'), \tag{B.10}$$

where $\Box D_{0,F} = -i\delta(x-x')$. It therefore requires a renormalization exploiting the freedom (2.25), so as to cancel these traces. Due to the good short distance scaling of $h_{\mu\nu}$, this renormalization cannot have a string-independent part. In order to allow the cancellations in Lemma B.1 and Lemma B.2, it must be of the special form

$$\langle\!\langle T_r h_{\mu\nu}h'_{\rho\sigma}\rangle\!\rangle = p_{\mu\nu,\rho\sigma}\, i\delta(x-x') \qquad \text{with} \quad p_{\mu\nu,\rho\sigma} = \partial_{(\mu}p_{\nu),\rho\sigma} + p'_{\mu\nu,(\rho}\partial_{\sigma)} \tag{B.11}$$

exhibiting an uncontracted derivative in each term.

There is a six-parameter family of renormalizations of the form (B.11) that restore the trace conditions:

$$\begin{aligned} p_{\nu,\rho\sigma} =& \big[\tfrac{1}{2}I'_\nu(I_\rho I_\sigma - \tfrac{1}{2}I^2\eta_{\rho\sigma}) + \tfrac{1}{8}((II')^2 - \tfrac{1}{2}I^2I'^2)\eta_{\nu(\rho}\partial_{\sigma)} \\ &\quad + \tfrac{1}{8}\big(I^2I'_\nu I'_{(\rho}\partial_{\sigma)} - 2(II')I'_\nu I_{(\rho}\partial_{\sigma)} + I'^2I_\nu I_{(\rho}\partial_{\sigma)}\big)\big] \end{aligned} \tag{B.12}$$

$$\begin{aligned} &+ c_1\big[(I_\nu + I'_\nu)I_{(\rho}I'_{\sigma)} + (II')\big(\tfrac{1}{2}I'_\nu I_{(\rho}\partial_{\sigma)} - \tfrac{1}{2}I_\nu I'_{(\rho}\partial_{\sigma)}\big)\big] \\ &+ c_2(II')\big[(I_\nu + I'_\nu)\eta_{\rho\sigma} - I_\nu I'_{(\rho}\partial_{\sigma)} + I'_\nu I_{(\rho}\partial_{\sigma)}\big] \\ &+ c_3\big[I^2\eta_{\nu(\rho}I'_{\sigma)} - \tfrac{1}{2}I'^2I_\nu I_{(\rho}\partial_{\sigma)} - \tfrac{1}{2}I^2I'_\nu I'_{(\rho}\partial_{\sigma)}\big] \\ &+ c_4\big[I'^2\eta_{\nu(\rho}I_{\sigma)} + \tfrac{1}{2}I'^2I_\nu I_{(\rho}\partial_{\sigma)} + \tfrac{1}{2}I^2I'_\nu I'_{(\rho}\partial_{\sigma)}\big] \\ &+ c_5(II')\big[\eta_{\nu(\rho}I_{\sigma)} - \tfrac{1}{2}I_\nu I'_{(\rho}\partial_{\sigma)} - \tfrac{1}{2}I'_\nu I_{(\rho}\partial_{\sigma)}\big] \\ &+ c_6(II')\big[\eta_{\nu(\rho}I'_{\sigma)} + \tfrac{1}{2}I_\nu I'_{(\rho}\partial_{\sigma)} + \tfrac{1}{2}I'_\nu I_{(\rho}\partial_{\sigma)}\big], \end{aligned} \tag{B.13}$$

and $p'_{\mu\nu,\rho}$ given by the replacements $\sigma \to \mu$, $\nu \leftrightarrow \rho$, $I' \leftrightarrow I$ and $\partial \to \partial' = -\partial$.

Lemma B.1. *The renormalization part $T_r\, \underbracket{L_1 L'_1}$ cancels the second line (3.35) of $T_0\, \underbracket{L_1 L'_1}$ for all values of the renormalization constants c_n.*

Proof: We repeatedly apply the identities (using only integrations by part w.r.t. x')

$$I_{(\rho}\partial_{\sigma)} i\delta(x-x')U'^{\rho\sigma} \overset{\text{div}}{=} 2I_\sigma i\delta(x-x')\partial'^\sigma S' \overset{\text{div}}{=} 2I_\sigma \partial^\sigma i\delta(x-x')S' = -2i\delta(x-x')S',$$
$$I'_{(\rho}\partial_{\sigma)} i\delta(x-x')U'^{\rho\sigma} \overset{\text{div}}{=} 2I'_\sigma i\delta(x-x')\partial'^\sigma S' \overset{\text{div}}{=} 2I'_\sigma \partial^\sigma i\delta(x-x')S' = 2i\delta(x-x')S',$$
$$\partial_\rho\partial_\sigma i\delta(x-x')U'^{\rho\sigma} \overset{\text{div}}{=} i\delta(x-x')\partial'_\rho\partial'_\sigma U'^{\rho\sigma} = i\delta(x-x')\Box' S' \overset{\text{div}}{=} i\Box\delta(x-x')S'. \tag{B.14}$$

We first reduce (3.35) further by working out $b_{\rho\sigma} i\delta(x-x')U'^{\rho\sigma}$ with (B.9) and (B.14):

$$b_{\rho\sigma} i\delta(x-x')U'^{\rho\sigma} \overset{\text{div}}{=} (I_\rho I_\sigma - \tfrac{1}{2}I^2\eta_{\rho\sigma})i\delta(x-x')U'^{\rho\sigma}$$
$$+ \tfrac{1}{2}\big[\big(3(II') + I^2\big) + \big((II')^2 - \tfrac{1}{2}I^2 I'^2\big)\Box\big] i\delta(x-x')S'.$$

Thus, (3.35) becomes

$$(3.35) \overset{\text{div}}{=} -S(I_\rho I_\sigma - \tfrac{1}{2}I^2\eta_{\rho\sigma})i\delta(x-x')U'^{\rho\sigma} + (x \leftrightarrow x')$$
$$- S\big[(I+I')^2 + ((II')^2 - \tfrac{1}{2}I^2 I'^2)\Box\big] i\delta(x-x')S'. \tag{B.15}$$

On the other hand, we compute the contribution from $\partial_{(\mu}p_{\nu),\rho\sigma}$ to $T_r\, \underbrace{L_1 L'_1}$:

$$U^{\mu\nu}\partial_{(\mu}p_{\nu),\rho\sigma}\, i\delta(x-x')U'^{\rho\sigma} \overset{\text{div}}{=} 2S\partial^\nu p_{\nu,\rho\sigma}\, i\delta(x-x')U'^{\rho\sigma}.$$

Inserting only (B.12) for $p_{\nu,\rho\sigma}$, we get

$$S(I_\rho I_\sigma - \tfrac{1}{2}I^2\eta_{\rho\sigma})i\delta(x-x')U'^{\rho\sigma} + S\cdot\tfrac{1}{2}\big[(I+I')^2 + \big((II')^2 - \tfrac{1}{2}I^2 I'^2\big)\Box\big] i\delta(x-x')S'.$$

Adding the corresponding contribution from $p'_{\mu\nu,(\rho}\partial_{\sigma)}$, we get an exact cancellation of (B.15).

Computing $p^i_{\nu,\rho\sigma}\partial^\nu$ for the six structures in (B.13) and applying then (B.14), one very quickly sees that, upon adding the corresponding term from $p'^i_{\mu\nu,\rho}$, one gets 0 for all values of the renormalization constants c_n. This proves the Lemma. □

Lemma B.2. *The renormalization part $T_r\, \underbrace{L_1 L'_1}_{1,\text{mat}}$ cancels the part (3.50) of $T_0\, \underbrace{L_1 L'_1}_{1,\text{mat}}$ for arbitrary values of the renormalization constants $c_1, \dots, c_6$.*

Proof. We compute

$$T_r\, \underbrace{L_1 L'_1}_{1,\text{mat}} \overset{\text{div}}{=} \tfrac{1}{2}U^{\mu\nu}\langle\!\langle T_r h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle \Theta'^{\rho\sigma} \overset{\text{div}}{=} \tfrac{1}{2}U^{\mu\nu}\partial_{(\mu}p_{\nu),\rho\sigma} i\delta(x-x')\Theta'^{\rho\sigma}. \tag{B.16}$$

The operators $p_{\nu,\rho\sigma}$ are given in (B.12), (B.13), and the contributions from $p'_{\mu\nu,(\rho}\partial_{\sigma)}$ can be dropped because Θ is conserved. This gives

$$T_r\, \underbrace{L_1 L'_1}_{1,\text{mat}} \overset{\text{div}}{=} i\tfrac{1}{2}U^{\mu\nu}\partial_{(\mu}p_{\nu),\rho\sigma}\delta(x-x')\Theta'^{\rho\sigma} \overset{\text{div}}{=} -\partial_\mu U^{\mu\nu} p_{\nu,\rho\sigma} i\delta(x-x')\Theta'^{\rho\sigma}$$
$$\overset{\text{div}}{=} -i\partial^\nu S p_{\nu,\rho\sigma}\delta(x-x')\Theta'^{\rho\sigma} \overset{\text{div}}{=} iSp_{\nu,\rho\sigma}\partial^\nu\delta(x-x')\Theta'^{\rho\sigma}. \tag{B.17}$$

We may also drop all derivatives ∂_ρ, ∂_σ within $p_{\nu,\rho\sigma}$. This leaves

$$\widetilde{p}_{\nu,\rho\sigma} = \frac{1}{2}I'_\nu I_\rho I_\sigma - \frac{1}{4}I^2 I'_\nu \eta_{\rho\sigma} + c_1(I_\nu + I'_\nu)I_{(\rho}I'_{\sigma)} + c_2(II')(I_\nu + I'_\nu)\eta_{\rho\sigma} \tag{B.18}$$

$$+ c_3 I^2 \eta_{\nu(\rho}I'_{\sigma)} + c_4 I'^2 \eta_{\nu(\rho}I_{\sigma)} + c_5(II')\eta_{\nu(\rho}I_{\sigma)} + c_6(II')\eta_{\nu(\rho}I'_{\sigma)}. \tag{B.19}$$

Contracting (B.18) with ∂^ν, we see that the contributions from c_1 and c_2 vanish identically and the terms $c_3, \ldots, c_6$ do not contribute after integration by parts due to the conservation of $\Theta'^{\rho\sigma}$. The first two terms in (B.18) cancel (3.50). This proves the Lemma. □

C Expansion of the classical Einstein action and matter couplings

We may most efficiently choose as starting point an action $\mathbb{S}_E$ for gravity introduced by Einstein himself in [10] – consult also [2] – instead of the more popular Einstein–Hilbert action. The aforesaid action contains, through the Riemann-Christoffel symbols $\Gamma^{\bullet}_{\bullet\bullet}$, *only first derivatives* of the metric tensor g, which is the fundamental field. It differs from the latter action by a divergence, and is therefore classically equivalent. It is of the form:

$$\frac{1}{2}\kappa^2\,\mathbb{S}_E = \int d^4x\,\sqrt{-g(x)}\,g^{\mu\nu}(x)\big(\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}\big)(x) =: \frac{1}{2}\kappa^2 \int d^4x\,S_E(x). \tag{C.1}$$

Here, $-g \equiv -\det g > 0$ and $\kappa^2 = 32\pi G$. Since among other things we undertook to prove that classical general relativity can be determined from perturbative quantum field theory, it makes sense to expand the former in terms of κ. For our purposes, we remove the factor κ^2 in (C.1) by the change of variables such that $\Gamma^\alpha_{\mu\nu}$ are $O(\kappa)$. Specifically, it is convenient to introduce the Goldberg variable

$$\mathfrak{g}^{\mu\nu} := \sqrt{-g}g^{\mu\nu}; \quad \mathfrak{g}_{\mu\nu} := \frac{g_{\mu\nu}}{\sqrt{-g}}. \tag{C.2}$$

A straightforward computation renders $S_E(x)$ in (C.1) by means of the $\mathfrak{g}^{\mu\nu}$ [33, Eq. (5.5.20)]:

$$\kappa^2 S_E(x) = -\mathfrak{g}_{\alpha\beta}\,\partial_\nu\mathfrak{g}^{\alpha\mu}\,\partial_\mu\mathfrak{g}^{\beta\nu} + \frac{1}{2}\mathfrak{g}_{\alpha\rho}\,\mathfrak{g}_{\beta\sigma}\,\mathfrak{g}^{\mu\nu}\partial_\mu\mathfrak{g}^{\rho\beta}\partial_\nu\mathfrak{g}^{\alpha\sigma}$$
$$- \frac{1}{4}\mathfrak{g}_{\mu\nu}\,\mathfrak{g}_{\rho\sigma}\,\mathfrak{g}^{\alpha\beta}\partial_\alpha\mathfrak{g}^{\mu\nu}\partial_\beta\mathfrak{g}^{\rho\sigma}. \tag{C.3}$$

In order to make contact with SQFT theory on Minkowski space, we parametrize

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}, \tag{C.4}$$

where the dynamical field (the "metric deviation") $h^{\mu\nu}$ is small only in the sense of *scattering theory*, that is, it goes to zero at large distances. That is precisely what is required in order to eventually apply the string-independence condition. Therefore we develop $\kappa^2 S_E(x)$ following Eq. (C.3), with the help of the inverse metric, of the form: $\mathfrak{g}_{\alpha\beta}(x) = \eta_{\alpha\beta} - \kappa h_{\alpha\beta}(x) + \kappa^2 h_{\alpha\rho}(x)h^\rho_\beta(x) - \cdots$.

Collecting in the expression on the right of (C.3) terms of order κ, one obtains [33, Ch. 5.5] three-graviton self-couplings with two derivatives:

$$S_E^{(1)}(x) = \kappa h^{\mu\nu}\big[\tfrac{1}{2}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} + \partial^\beta h_{\mu\alpha}\partial^\alpha h_{\nu\beta} - \partial^\rho h^\alpha_\mu \partial_\rho h_{\nu\alpha} - \tfrac{1}{4}\partial_\mu h^\rho_\rho\,\partial_\nu h^\rho_\rho + \tfrac{1}{2}\partial_\alpha h^{\mu\nu}\partial^\alpha h^\rho_\rho\big].$$

Collecting as well terms of order κ^2, one obtains four-graviton couplings with two derivatives:

$$\begin{aligned} S_E^{(2)}(x) = -\kappa^2\big[& h_{\alpha\rho}h^\rho_\beta\,\partial_\nu h^{\alpha\mu}\partial_\mu h^{\beta\nu} - \tfrac{1}{4}h^{\mu\nu}h_{\rho\sigma}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\rho\sigma} - h^{\mu\nu}h_{\alpha\rho}\partial_\mu h^\rho_\sigma \partial_\nu h^{\alpha\sigma} \\ &+ h_{\rho\beta}h^\beta_\sigma \partial_\mu h^{\alpha\rho}\partial^\mu h^\sigma_\alpha + \tfrac{1}{2}h_{\alpha\rho}h_{\beta\sigma}\partial_\mu h^{\alpha\sigma}\partial^\mu h^{\beta\rho} - \tfrac{1}{2}h_{\rho\beta}h^\beta_\sigma\,\partial_\alpha h^{\rho\sigma}\partial^\alpha h^\rho_\rho + \tfrac{1}{2}h_{\mu\nu}\partial_\alpha h^{\mu\nu}h^{\alpha\beta}\partial_\beta h^\rho_\rho\big]. \end{aligned}$$

Take now the momentuous step of substituting the *SQFT graviton field* $h^{\mu\nu}(x;c)$ for $h^{\mu\nu}(x)$. This yields the immediate dividend that the two terms involving the trace h^ρ_ρ in $S_E^{(1)}$ do vanish. In the understanding that from now on h denotes the string-localized graviton, we replace $S_E^{(n)}(x)$ by $\frac{\kappa^n}{n!}L_n[h]$ for $n = 1, 2$ – recall the expansion convention (1.6):

$$\begin{aligned} L_1[h] &= h^{\mu\nu}\big[\tfrac{1}{2}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} + \partial^\beta h_{\mu\alpha}\partial^\alpha h_{\nu\beta} - \partial^\rho h^\alpha_\mu\partial_\rho h_{\nu\alpha}\big](x;c) \\ &\equiv \big[\tfrac{1}{2}(\!(\partial_\mu h\partial_\nu h)\!)h^{\mu\nu} + (\partial_\nu hh\partial_\mu h)^{\mu\nu} - (\!(h\partial_\mu h\partial^\mu h)\!)\big](x;c), \end{aligned} \tag{C.5}$$

$$\begin{aligned} \tfrac{1}{2}L_2[h] &= \big[-h_{\alpha\rho}h^\rho_\beta\,\partial_\nu h^{\alpha\mu}\partial_\mu h^{\beta\nu} - \tfrac{1}{4}h_{\mu\nu}\partial_\alpha h^{\mu\nu}h_{\rho\sigma}\partial^\alpha h^\rho\sigma - h^{\mu\nu}h_{\alpha\rho}\partial_\mu h^\rho_\sigma\partial_\nu h^{\alpha\sigma} \\ &\qquad + h_{\rho\beta}h^\beta_\sigma\partial_\mu h^{\alpha\rho}\partial^\mu h^\sigma_\alpha + \tfrac{1}{2}h_{\alpha\rho}h_{\beta\sigma}\partial_\mu h^{\alpha\sigma}\partial^\mu h^{\beta\rho}\big](x;c) \\ &\equiv \big[-(\partial_\nu hhh\partial_\mu h)^{\mu\nu} - \tfrac{1}{4}(\!(h\partial_\mu h)\!)(\!(h\partial^\mu h)\!) - (\!(h\partial_\mu h\partial_\nu h)\!)h^{\mu\nu} \\ &\qquad + (\!(h\partial_\mu h\partial^\mu hh)\!) + \tfrac{1}{2}(\!(h\partial_\mu hh\partial^\mu h)\!)\big](x;c). \end{aligned} \tag{C.6}$$

Of course these are (3.23) and (3.42) again, modulo divergences; in particular, the last line of (C.6) is separately a divergence. We remark that this is as well similar to the result of imposing the Hilbert gauge condition:

$$h^\mu_\mu \equiv \eta^{\mu\nu}h_{\mu\nu} = 0, \qquad \partial^\mu h_{\mu\nu} = 0,$$

because $h_{\mu\nu}(x;c)$ satisfies these relations as operator identities, see (2.6).

Note that because the Riemann–Christoffel symbols

$$\Gamma^\tau_{\kappa\lambda} := g^{\tau\mu}\Gamma_{\mu\kappa\lambda} := \tfrac{1}{2}g^{\tau\mu}\big(\partial_\kappa g_{\lambda\mu} + \partial_\lambda g_{\kappa\mu} - \partial_\mu g_{\kappa\lambda}\big)$$

are obviously $O(\kappa)$, also the Riemann tensor is $O(\kappa)$, specifically

$$R^\tau_{\kappa\nu\lambda} := \partial_\nu\Gamma^\tau_{\lambda\kappa} - \partial_\lambda\Gamma^\tau_{\nu\kappa} + \Gamma^\tau_{\nu\mu}\Gamma^\mu_{\lambda\kappa} - \Gamma^\tau_{\lambda\mu}\Gamma^\mu_{\nu\kappa} = \tfrac{1}{2}\kappa\eta^{\tau\mu}F_{[\mu\kappa][\nu\lambda]} + O(\kappa^2), \tag{C.7}$$

where F is defined in terms of the classical field h in the analogous way as the quantum field strength F is recovered from the string-localized potential $h(c)$ in (2.5). The Ricci tensor and scalar are $O(\kappa)$ as well – which is why we work with the Einstein rather than Einstein-Hilbert Lagrangian.

For the comparison of the classical matter couplings with the induced ones from SQFT in Theorem 3.9, one has to expand the generally covariant matter Lagrangians. Because the comparison requires the substitution of the string-localized quantum field $h_{\mu\nu}(x;c)$ for the classical field $h_{\mu\nu}(x)$, we shall drop $(\!(h)\!)$ and $\partial_\mu h^{\mu\nu}$ wherever they arise in the classical expansion. Then to $O(\kappa^2)$:

$$\sqrt{-g} = 1 - \tfrac{\kappa^2}{4}(\!(hh)\!), \quad g^{\mu\nu} = (1 + \tfrac{\kappa^2}{4}(\!(hh)\!))\eta^{\mu\nu} + \kappa h^{\mu\nu}. \tag{C.8}$$

This gives the expansion of the generally covariant scalar Lagrangian:

$$\begin{aligned} L_{g,\phi} &= \frac{\sqrt{-g}}{2}\big(g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - m^2\chi^2\big) \\ &= L_{0,\phi} + \tfrac{\kappa}{2}(\!(h\Theta_\phi)\!) + \tfrac{\kappa^2}{2}\Big[\tfrac{1}{2}(\!(hh)\!)L_{0,\phi} + \tfrac{1}{4}(\!(hh)\!)(\!(\Theta_\phi)\!)\Big] + \dots \end{aligned} \tag{C.9}$$

and of the generally covariant Maxwell Lagrangian:

$$\begin{aligned} L_{g,F} &= \sqrt{-g}\big(-\tfrac{1}{4}g^{\mu\nu}g^{\kappa\lambda}F_{\mu\kappa}F_{\nu\lambda}\big) \\ &= L_{0,F} + \tfrac{\kappa}{2}(\!(h\Theta_F)\!) + \tfrac{\kappa^2}{2}\big[\tfrac{1}{2}(\!(hh)\!)L_{0,F} - \tfrac{1}{2}h^{\mu\nu}h^{\kappa\lambda}F_{\mu\kappa}F_{\nu\lambda}\big] + \dots \end{aligned} \tag{C.10}$$

The case of the Dirac field is more intricate. The generally covariant Lagrangian is

$$L_{g,\psi} = \sqrt{-g}\cdot\overline{\psi}\big(\tfrac{i}{2}(\gamma_g^\mu D_\mu - \overleftarrow{D}_\mu\gamma_g^\mu) - m\big)\psi \tag{C.11}$$

where the covariant γ-matrices satisfy $\gamma_g^\mu\gamma_g^\nu + \gamma_g^\nu\gamma_g^\mu = 2g^{\mu\nu}$, and the covariant derivative is defined with the spin connection. These are defined in terms of a real tetrad field $e^\mu_\alpha(x)$ such that

$$g^{\mu\nu}(x) = e^\mu_\alpha(x)e^\nu_\beta(x)\eta^{\alpha\beta}. \tag{C.12}$$

The tetrad gives γ_g^μ in terms of "flat" Dirac matrices satisfying $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\eta^{\alpha\beta}$:

$$\gamma_g^\mu := e^\mu_\alpha\gamma^\alpha \tag{C.13}$$

as well as the coefficients of the spin connection:

$$\omega_{\mu;\alpha\beta} = -\omega_{\mu;\beta\alpha} := g_{\kappa\lambda}\, e^\kappa_\alpha D_\mu e^\lambda_\beta = g_{\kappa\lambda}\, e^\kappa_\alpha\big(\partial_\mu e^\lambda_\beta + \Gamma^\lambda_{\mu\nu}e^\nu_\beta\big). \tag{C.14}$$

Then, the covariant derivatives of the spinors are

$$D_\mu\psi = \partial_\mu\psi + \tfrac{1}{8}\omega_{\mu;\alpha\beta}\,[\gamma^\alpha,\gamma^\beta]\psi, \qquad D_\mu\overline{\psi} = \partial_\mu\overline{\psi} - \tfrac{1}{8}\omega_{\mu;\alpha\beta}\,\overline{\psi}[\gamma^\alpha,\gamma^\beta]. \tag{C.15}$$

Now, the tetrad as given by [31] is the "square root" of $g^{\mu\nu}$ given in (C.8). Its expansion is

$$e^\mu_\alpha = \delta^\mu_\alpha + \tfrac{\kappa}{2}h^\mu_\alpha + \tfrac{\kappa^2}{8}\big((\!(hh)\!)\delta^\mu_\alpha - (hh)^\mu_\alpha\big) + O(\kappa^3), \tag{C.16}$$

from which one obtains

$$\omega_{\mu;\alpha\beta} = \frac{\kappa}{2} \cdot \partial_{[\alpha} h_{\beta]\mu} + O(\kappa^2). \tag{C.17}$$

The first-order term of $L_{g,\psi}$ has contributions from the kinetic terms involving the derivatives of the Dirac field, and from $\omega_{\mu;\alpha\beta}$. The latter contribution vanishes by (C.17) together with the properties of the γ-matrices and the symmetry of $h_{\beta\mu}$. The former contribution easily gives $L_{1,\psi} = \frac{1}{2}(\!(h\Theta_\psi)\!)$. $L_{2,\psi}$ has contributions from $\sqrt{-g}$, from the kinetic terms, and from $\omega_{\mu;\alpha\beta}$. The first of these is $-\frac{1}{4}(\!(hh)\!) \cdot L_{0,\psi}$. The kinetic contribution is easily obtained as $\frac{\kappa^2}{8}\big((\!(hh)\!)(\!(\Theta_\psi)\!) - (\!(hh\Theta_\psi)\!)\big)$. Only the last contribution, involving (by (C.13)) the expansion of $e^\mu_\nu \omega_{\mu;\alpha\beta}$ to second order, requires a lengthy computation, with the result

$$\frac{i}{8}\big[e^\mu_\nu \omega_{\mu;\alpha\beta}\big]^{(2)} \cdot \overline{\psi}(\gamma^\nu\gamma^\alpha\gamma^\beta + \gamma^\alpha\gamma^\beta\gamma^\nu)\psi \overset{\text{div}}{=} \frac{i}{32} \cdot (h\partial_\nu h)_{\beta\alpha} \cdot \overline{\psi}(\gamma^\alpha\gamma^\beta\gamma^\nu - \gamma^\nu\gamma^\beta\gamma^\alpha)\psi.$$

Thus, to second order and modulo derivatives,

$$\begin{aligned} L_{g,\psi} \overset{\text{div}}{=} L_{0,\psi} + \frac{\kappa}{2}(\!(h\Theta_\psi)\!) + \frac{\kappa^2}{2}\Big[- \frac{1}{2}(\!(hh)\!) L_{0,\psi} \\ + \frac{1}{4}(\!(hh)\!)(\!(\Theta_\psi)\!) - \frac{1}{4}(\!(hh\Theta_\psi)\!) + \frac{i}{16}(h\partial_\nu h)_{\beta\alpha}\overline{\psi}(\gamma^\alpha\gamma^\beta\gamma^\nu - \gamma^\nu\gamma^\beta\gamma^\alpha)\psi)\Big]. \end{aligned} \tag{C.18}$$

Notice that $L_{0,\psi} = 0$ when the free quantum Dirac field is substituted for ψ, and the second line becomes $-\frac{1}{4}L^1_{2,\psi} + \frac{1}{4}L^3_{2,\psi} - \frac{1}{8}L^4_{2,\psi}$ in terms of $L^n_{2,\psi}$ as in (3.20).

In all three cases one reads off $L_{1,\text{mat}} = \frac{1}{2}(\!(h\Theta_{\text{mat}})\!)$, with Θ_{mat} given as in (3.5)–(3.7), and $L_{2,\text{mat}} = [\dots]$ being the expression in the bracket. These expressions coincide with (3.49) with $L_{2,r,\text{mat}} = 0$, when $h_{\mu\nu}(x;c)$ is substituted for the classical field $h_{\mu\nu}(x)$ and the free quantum matter field for the classical matter field. This completes the proof of Theorem 3.9.

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