Journal of Mathematical Psychology 116 (2023) 102806

Contents lists available at ScienceDirect

Journal of Mathematical Psychology

journal homepage: www.elsevier.com/locate/jmp

A characterization of two-agent Pareto representable orderings

Juan C. Candeal *

Departamento de Análisis Económico. Facultad de Economía y Empresa. Universidad de Zaragoza, Spain

ARTICLE INFO

Article history:
Received 20 July 2022
Received in revised form 27 June 2023
Accepted 1 August 2023
Available online 23 August 2023

Keywords:
Pareto preferences
Multi-utility representation
Poset dimension

ABSTRACT

Partial orders defined on a nonempty set X admitting a two-agent Pareto representation are characterized. The characterization is based upon the fulfillment of two axioms. The first one entails the existence, for any point $x \in X$, of a very particular decomposition of the points which are incomparable to x. The second one encodes a separability condition. Our approach is then applied to show that if the cardinality of X is, at most, 5, then a two-agent Pareto representation always exists whereas this need not be the case otherwise. The connection with the concept of the dimension of a poset is also discussed. Certain examples are also presented that illustrate the scope of our tools.

1. Introduction

This article is aimed at showing a characterization of those partial orderings $\precsim$, defined on a nonempty set X, that admit a *two-agent Pareto representation*. This means that the corresponding poset $(X, \precsim)$ can be embedded into the Euclidean plane equipped with usual (strong) *Pareto dominance relation*. This representation is a particular case of what nowadays is called a *multi-utility representation* of a binary relation based, at most, upon two utilities (also known as a *bi-utility representation*). A multi-utility representation turns out to be a generalization of the classical (single-)utility representation whenever the binary relation at hand is incomplete. The natural interpretation underlying such a kind of representation has to do with the way that an agent may use in order to evaluate distinct options or attributes. Relevant contributions to this topic in decision theory, some of them involving also continuity assumptions, are those by Bosi and Herden (2012), Evren and Ok (2011), Ok (2002) and, more recently, Hack et al. (2022, 2023). To the best of our knowledge a characterization of the existence of such a representation remains an open problem in the general case.[1]

For the particular case of two utilities, a characterization of the existence of a two-agent Pareto representation was shown in Candeal (2022) whenever the ground set X is countable (finite or denumerable). Here, we extend this characterization by covering an arbitrary, not necessarily countable, partially ordered set $(X, \precsim)$. The result relies on two axioms. The first one entails the existence, for any point $x \in X$, of a very particular decomposition of the points which are incomparable to x. This decomposition satisfies certain duality conditions and leads to the key concept of a *two-agent Pareto ordering*. The second one, as is very often encountered in utility theory, encodes a suitable *separability condition*.

Our approach also allows us to offer some new insights related to posets with *dimension* at most 2. This important concept was defined in the foundational paper of Dushnik and Miller (1941). They proved that any partial order defined on an arbitrary set X is the intersection of all the total orders extending it. Many years later, Donaldson and Weymark (1998) provided a similar result for the case of quasi-orders. Then, Sprumont (2001) studied the same problem but now considering the number of total orders on X to be fixed. Sprumont (2001) focused on the two-agent case studying what he called the regular case. Other interesting papers, all of them concerning the finite case, which make use of certain graph-theoretical and combinatorial deep results are those by Baker et al. (1972), Kelly (1977), Trotter (1992), Trotter and Moore (1976) and, more recently, in the context of economics Qi (2015, 2016). In the latter reference a motivating presentation of the Pareto problem, viewed as an inverse social decision problem, is described.

The article is organized as follows. Section 2 presents basic definitions and notations. Section 3 includes the main contributions of the paper. Firstly, we introduce the concept of a two-agent Pareto ordering. Then, a key technical result is stated; namely, Lemma 1. Roughly speaking, it tells that a two-agent Pareto ordering $\precsim$ has associated two total orders the intersection of which is $\precsim$. The first part of this lemma, concerning the existence of such total orders, was already used in Candeal (2022) whenever the ground set X is countable. Here, we provide a

* Correspondence to: Departamento de Análisis Económico. Facultad de Economía y Empresa. Universidad de Zaragoza., Gran Vía, 2. 50005 Zaragoza, Spain.
E-mail address: candeal@unizar.es.

[1] It is a remarkable fact that representations of this type for ordered structures, other than partial orders, has also been studied in the literature. For instance, bi-utility representations for interval orders and semiorders in Rébillé (2023).

https://doi.org/10.1016/j.jmp.2023.102806
0022-2496/

generalization of the latter result by showing that it also works for an arbitrary poset.

Secondly, and taking advantage of the conclusion of Lemma 1, the main result of the paper (Theorem 1) is presented; to wit, a neat characterization of those partial orders admitting a two-agent Pareto representation. Concretely, it is shown that a partial order is two-agent Pareto representable if and only if it is a separable two-agent Pareto ordering. Also, a second remarkable result (Theorem 2) is stated: if the cardinality of X is, at most, 5, then a two-agent Pareto representation always exists whereas this need not be the case otherwise.

Thirdly, we discuss to some extent how the results obtained are related to the concept of the dimension of a poset. Actually, it is noted that a poset $(X, \precsim)$ has, at most, dimension 2 if and only if $\precsim$ is a two-agent Pareto ordering. This, in particular, applies to the finite case providing a characterization of the, so-called, *bi-linear orders* as defined in Fishburn (1997). Section 3 ends by presenting certain examples that enlighten the scope of our tools, especially, in the finite case.

The main conclusions of the article along with a brief discussion about certain lines for future research are shown in Section 4.

2. Preliminaries

Let X be a nonempty set. The symbol $|X|$ will be used to denote its cardinality. A *binary relation* R on X is a nonempty subset of $X \times X$, and we will write $x\ R\ y$ instead of $(x, y) \in R$. A binary relation R on X is *reflexive* if $x\ R\ x$ for any $x \in X$, *complete* if either $x\ R\ y$ or $y\ R\ x$ or both for any $x, y \in X$, *antisymmetric* if $x\ R\ y$ and $y\ R\ x$ imply that x, y are identical for any $x, y \in X$, and *transitive* if $x\ R\ y$ and $y\ R\ z$ imply $x\ R\ z$ for any $x, y, z \in X$. If R is both reflexive and transitive, we will call it a *quasi-order* (or a *preorder*). An antisymmetric quasi-order is a *partial order*. (That is, a partial order is a reflexive, transitive and antisymmetric binary relation.) A complete quasi-order is also called a *preference* (or a *total preorder*). A complete partial order is a *total order*. An ordered pair (X, R) is called a *partially ordered set*, or simply, a *poset*, if R is a partial order on X. Throughout the paper, a typical partial order will be denoted by $\precsim$. In addition, $\prec$ will stand for the *strict part* of $\precsim$; i.e., $x \prec y$ if and only if $x \precsim y$ and $\neg(y \precsim x)$, for any $x, y \in X$. Two ordered pairs (X, R_1) and (Z, R_2) are said to be *order-isomorphic* or, simply, X and Z are *order-isomorphic*, provided that there is a one-to-one and onto map $f : X \longrightarrow Z$ such that $x\ R_1\ y$ if and only if $f(x)\ R_2\ f(y)$, for any $x, y \in X$.

Let there be given an ordered pair (X, R) and any two elements $x, y \in X$. We say that x and y are *comparable* in R, or simply, *comparable*, if either $x\ R\ y$ or $y\ R\ x$ or both. Accordingly, we say x and y are *incomparable* in R, or simply, *incomparable*, if x and y are not comparable in R. We will write $x \bowtie y$ if x and y are incomparable in R. Let $x \in X$. Then, as usual, the *lower contour set* and the *upper contour set* of x will be denoted by $L_R(x)$ and $G_R(x)$, respectively. That is, $L_R(x) := \{y \in X : y\ R\ x\}$ and $G_R(x) := \{y \in X : x\ R\ y\}$. In addition, we will use the notation $N_R(x)$ for the elements of X that are incomparable to x; i.e., $N_R(x) := \{y \in X : x \bowtie y\}$.

The usual partial order on the 2-dimensional Euclidean space $\mathbb{R}^2$ will be denoted by $\leq$; i.e., $a = (a_j) \leq b = (b_j)$ if and only if $a_j \leq b_j$, $j \in \{1, 2\}$, for any $(a_j), (b_j) \in \mathbb{R}^2$. On $\mathbb{R}^2$ we will consider the following two total orders denoted by $\leq_{lex_1}$ and $\leq_{lex_2}$: for any $(x, y), (u, v) \in \mathbb{R}^2$, $(x, y) \leq_{lex_1} (u, v) \iff x < u$, or $(x = u \wedge y \leq v)$, and $(x, y) \leq_{lex_2} (u, v) \iff y < v$, or $(y = v \wedge x \leq u)$. Note that $\leq_{lex_1}$ is the usual lexicographic order on $\mathbb{R}^2$ whereas for $\leq_{lex_2}$ the role of the coordinates is reversed.

3. Main results

The major goal of this section is to offer a characterization of those posets $(X, \precsim)$ that admit a two-agent Pareto representation in the general case; that is, without imposing any cardinality constraint on the ground set X. Roughly speaking, this means that the poset can be embedded into the plane with the usual partial order $\leq$. Our approach will also allow us to discuss in some detail the finite case; in particular, its connection with the fundamental concept of the dimension of a poset as introduced in the seminal paper by Dushnik and Miller (1941). We begin by stating the definition of a two-agent Pareto representable binary relation.

Definition 1. A binary relation R defined on X is *two-agent Pareto representable* whenever there is a function $u : X \to \mathbb{R}^2$ so that $x\ R\ y \Leftrightarrow u(x) \leq u(y)$, for any $x, y \in X$. In this case, $u = (u_j)_{j=1,2}$, where $u_j : X \to \mathbb{R}$ denotes the corresponding component function of u, is said to be a two-agent Pareto representation of R.

Remark 1. It should be noted that if a binary relation R is two-agent Pareto representable, then it is a quasi-order. Moreover, if, in addition, it is antisymmetric, then X is order-isomorphic to a subset of $\mathbb{R}^2$ endowed with its usual partial order $\leq$ and so R is, in fact, a partial order on X.

Now, a key definition is introduced. This definition abstracts certain properties of the usual partial order of $\mathbb{R}^2$ to the context of an arbitrary poset. As usual 2^X denotes the power set of X. Recall that $N_{\precsim}(x)$ denotes the set of objects incomparable to x.

Definition 2. A partial order $\precsim$ on X is said to be a *two-agent Pareto ordering* whenever there is a family of subsets $(N^1_{\precsim}(x), N^2_{\precsim}(x))_{x\in X} \subset 2^X \times 2^X$ so that, for any $x, y \in X$, the following conditions are met:

(i) $N_{\precsim}(x)$ partitions into two (possibly empty) subsets $N^1_{\precsim}(x)$ and $N^2_{\precsim}(x)$,

(ii) $x \in N^1_{\precsim}(y) \iff y \in N^2_{\precsim}(x)$,

(iii) $y \in N^1_{\precsim}(x) \implies N^1_{\precsim}(y) \subseteq N^1_{\precsim}(x)$.

Remark 2. (α_1) It is straightforward to see that conditions (ii) and (iii) of the previous definition entail $y \in N^2_{\precsim}(x) \implies N^2_{\precsim}(y) \subseteq N^2_{\precsim}(x)$, for any $x, y \in X$.

(α_2) Note also that conditions (ii) and (iii) of Definition 2 are also satisfied by the family of subsets $(L_{\precsim}(x), G_{\precsim}(x))_{x\in X} \subset 2^X \times 2^X$. However, condition (i) does not hold for this family because $L_{\precsim}(x) \cap G_{\precsim}(x) = \{x\}$, for any $x \in X$.

(α_3) Definition 2 claims the existence, for any $x \in X$, of a very particular decomposition of the set of objects incomparable to x, $N_{\precsim}(x)$, into two subsets satisfying certain duality conditions. Let us see two examples.

Firstly, let $\leq$ be the usual partial order on $\mathbb{R}^2$. Clearly, $\leq$ is a two-agent Pareto ordering by considering the following family of subsets: for any $(x, y) \in \mathbb{R}^2$, let $N^1_{\leq}(x, y) = \{(u, v) \in \mathbb{R}^2 : u < x, v > y\}$, and $N^2_{\leq}(x, y) = \{(u, v) \in \mathbb{R}^2 : u > x, v < y\}$. Note that $N_{\leq}(x, y)$ partitions into $N^1_{\leq}(x, y)$ and $N^2_{\leq}(x, y)$, and also that conditions (ii) and (iii) of Definition 2 are straightforwardly satisfied.

Secondly, consider the three-point set $X = \{a, b, c\}$ equipped with the following partial order: $a \precsim b$, $c \precsim b$, and the corresponding self relation of any point to meet reflexivity. Note that $N_{\precsim}(a) = \{c\}$, $N_{\precsim}(b) = \emptyset$ and $N_{\precsim}(c) = \{a\}$. Then, and as can be easily checked, a family that meets conditions (i) to (iii) of Definition 2 is the following:

(β_1) $N^1_{\precsim}(a) = \{c\}$ and $N^2_{\precsim}(a) = \emptyset$,

(β_2) $N^1_{\precsim}(b) = N^2_{\precsim}(b) = \emptyset$,

(β_3) $N^1_{\precsim}(c) = \emptyset$ and $N^2_{\precsim}(c) = \{a\}$.

Thus, $\precsim$ is a two-agent Pareto ordering on $\{a, b, c\}$. For other examples, see Remark 5 below.

The next lemma provides an important fact about a two-agent Pareto ordering $\precsim$ defined on a set X. It states that, associated with $\precsim$, there are two total orders which are closely related to $\precsim$. The existence of such total orders was already observed in Candeal (2022) whenever the ground set X was countable (finite or denumerable). Here, we provide a generalization of that result by proving that it also works for an arbitrary poset $(X, \precsim)$. In addition, we show that $\precsim$ is exactly the intersection of those total orders.

Lemma 1. *Let $\precsim$ be a two-agent Pareto ordering on X. Consider the following binary relations on X. For any $x, y \in X$:*
(1) $x \precsim_1 y \iff x \in L_{\precsim}(y) \cup N^1_{\precsim}(y)$,
(2) $x \precsim_2 y \iff x \in L_{\precsim}(y) \cup N^2_{\precsim}(y)$.

Then, $\precsim_1$ and $\precsim_2$, so-defined, are total orders on X. Moreover, $\precsim = \precsim_1 \bigcap \precsim_2$.

Proof. We will show only that $\precsim_1$ is a total order the argument for $\precsim_2$ being completely similar.

(α_1) $\precsim_1$ is transitive: Indeed, let there be given pairwise different elements $x, y, z \in X$ such that $x \precsim_1 y$ and $y \precsim_1 z$. Four cases may occur. First, $x \in L_{\precsim}(y)$ and $y \in L_{\precsim}(z)$. In this case, by transitivity of $\precsim$, it follows that $x \in L_{\precsim}(z)$ and so $x \precsim_1 z$. Second, $x \in L_{\precsim}(y)$ and $y \in N^1_{\precsim}(z)$. If $x \notin L_{\precsim}(z) \cup N^1_{\precsim}(z)$, then either $x \in G_{\precsim}(z)$ or $x \in N^2_{\precsim}(z)$. The case $x \in G_{\precsim}(z)$ leads to the contradiction $z \precsim y$ because $z \precsim x$ and $x \precsim y$. If $x \in N^2_{\precsim}(z)$, then, by condition (ii), $z \in N^1_{\precsim}(x)$ hence, by (iii), $N^1_{\precsim}(z) \subseteq N^1_{\precsim}(x)$. Now, because $y \in N^1_{\precsim}(z)$, it follows that $y \in N^1_{\precsim}(x)$ which contradicts the fact that $y \in G_{\precsim}(x)$. Thus, $x \in L_{\precsim}(z) \cup N^1_{\precsim}(z)$ and, therefore, $x \precsim_1 z$. Third, $x \in N^1_{\precsim}(y)$ and $y \in L_{\precsim}(z)$. It is entirely similar to the previous case. Fourth, $x \in N^1_{\precsim}(y)$ and $y \in N^1_{\precsim}(z)$. In this situation, it holds $x \in N^1_{\precsim}(z)$ because $y \in N^1_{\precsim}(z)$ entails, by condition (iii), $N^1_{\precsim}(y) \subseteq N^1_{\precsim}(z)$ and, by hypothesis, $x \in N_1(y)$. Therefore, $x \precsim_1 z$ and we are done.

(α_2) $\precsim_1$ is antisymmetric: Assume $x \precsim_1 y$ and $y \precsim_1 x$, for some $x, y \in X$. If $x \in L_{\precsim}(y)$ and $y \in L_{\precsim}(x)$, then, because $\precsim$ is antisymmetric, it holds that $x = y$. The possibility $x \in N^1_{\precsim}(y)$ and $y \in L_{\precsim}(x)$ leads to a contradiction because this would entail $x \in N^1_{\precsim}(y) \cap G_{\precsim}(y)$, and the latter subset is empty. Similarly, for the situation $x \in L_{\precsim}(y)$ and $y \in N^1_{\precsim}(x)$. Finally, the case $x \in N^1_{\precsim}(y)$ and $y \in N^1_{\precsim}(x)$ is not possible because, by condition (iii), it would imply $x \in N^1_{\precsim}(x)$, which is a contradiction.

(α_3) $\precsim_1$ is complete: Let $x, y \in X$. If $\neg(x \precsim_1 y)$, then either $x \in G_{\precsim}(y)$ or $x \in N^2_{\precsim}(y)$. The first possibility amounts to $y \in L_{\precsim}(x)$. The second one entails, by condition (ii), $y \in N^1_{\precsim}(x)$. Thus, in either of the two cases it holds $y \precsim_1 x$.

Finally, and in order to show that $\precsim = \precsim_1 \bigcap \precsim_2$, note that, obviously, $x \precsim y$ implies $x \precsim_1 y$ and $x \precsim_2 y$, for any $x, y \in X$. Thus, $\precsim \subseteq \precsim_1 \bigcap \precsim_2$. Conversely, assume there are $x, y \in X$ such that $x \precsim_1 y$ and $x \precsim_2 y$. Then, by definition, it holds $x \in L_{\precsim}(y) \cup N^1_{\precsim}(y)$ and $x \in L_{\precsim}(y) \cup N^2_{\precsim}(y)$. Now, because $L_{\precsim}(y), N^1_{\precsim}(y)$ and $N^2_{\precsim}(y)$ are pairwise disjoint, it follows that $x \in L_{\precsim}(y)$; that is, $x \precsim y$. Thus, $\precsim_1 \bigcap \precsim_2 \subseteq \precsim$ and, therefore, $\precsim = \precsim_1 \bigcap \precsim_2$ which ends the proof. □

Remark 3. (i) It is crystal clear that if $\precsim$ is a total order on X, then $N_{\precsim}(x) = \emptyset$, for any $x \in X$, and so $\precsim$ is, trivially, a two-agent Pareto ordering. Obviously, in this case, $\precsim = \precsim_1 = \precsim_2$.
(ii) The nature of the two total orders $\precsim_1$ and $\precsim_2$ is an interesting question to study. For instance, if $X = \mathbb{R}^2$ and $\precsim = \leq$ is the usual partial order on $\mathbb{R}^2$, then $\precsim_1$ and $\precsim_2$ are the lexicographic total orders $\leq_{lex_1}$ and $\leq_{lex_2}$, respectively. Thus, $\leq = \leq_{lex_1} \bigcap \leq_{lex_2}$.

Now, a characterization of those partial orders that are two-agent Pareto representable is presented. To that end, the two total orders $\precsim_1$ and $\precsim_2$ provided in Lemma 1, associated with a partial order $\precsim$ on X, will play an important role. Note that, if both total orders $\precsim_1$ and $\precsim_2$ admit a utility function, say u_1 and u_2, respectively, then the pair (u_1, u_2) is a two-agent Pareto representation of $\precsim$. In particular, this happens whenever the ground set X is countable because it is well-known that any total order on a countable set is representable by a utility function (see, e.g., Bridges & Mehta, 1995). Nevertheless, in general, this is not the case. Indeed, for the usual partial order $\leq$ on the plane none of the two total orders $\precsim_1$ and $\precsim_2$ admit a utility function despite $\leq$ is, obviously, two-agent Pareto representable. This is because, as noted in Remark 3(ii), $\precsim_1 = \leq_{lex_1}$ and $\precsim_2 = \leq_{lex_2}$, and neither $\leq_{lex_1}$ nor $\leq_{lex_2}$ admit a utility function. Then, and as is usual in classical utility theory, a suitable separability condition will help us to overcome this drawback.

Definition 3. A two-agent Pareto ordering $\precsim$ on X is said to be *separable*[2] provided that there is a coinitial[3] countable subset $D \subseteq X$ in such a way that, for any $x, y \in X$, the following condition holds true:

$$x \prec_1 y \text{ or } x \prec_2 y \Longrightarrow \exists\, d, d' \in D\,;\, x \precsim_1 d \prec_1 d' \precsim_1 y \text{ or } x \precsim_2 d \prec_2 d' \precsim_2 y$$

We now present the main result of the paper.

Theorem 1. *A partial order $\precsim$ defined on X is two-agent Pareto representable if and only if it is a separable two-agent Pareto ordering. Moreover, if (u_1, u_2) is a two-agent Pareto representation of $\precsim$, then $(\phi_1 \circ u_1, \phi_2 \circ u_2)$ is convenient with ϕ_1 and ϕ_2 strictly increasing on $u_1(X)$ and $u_2(X)$, respectively.*

Proof. Suppose first that $u : X \to \mathbb{R}^2$, $u = (u_1, u_2)$, is a two-agent Pareto representation of $\precsim$. Then, for each $x \in X$, define the subsets $N^1_{\precsim}(x) := \{y \in X : (u_1(y) < u_1(x)) \wedge (u_2(x) < u_2(y))\}$ and $N^2_{\precsim}(x) := \{y \in X : (u_1(x) < u_1(y)) \wedge (u_2(y) < u_2(x))\}$.

It is straightforward to see that the family of subsets $(N^1_{\precsim}(x), N^2_{\precsim}(x))_{x \in X}$, so-defined, satisfies conditions (i) to (iii) of Definition 2. So, $\precsim$ is a two-agent Pareto ordering. In order to see that it is also separable, note that X is order-isomorphic to $u(X) =: \{(u_1(x), u_2(x)) : x \in X\} \subseteq \mathbb{R}^2$, endowed with the usual partial order $\leq$. In addition, $\precsim_1$ (respectively, $\precsim_2$) on X amounts to $\leq_{lex_1}$ (respectively, $\leq_{lex_2}$) on $u(X)$. Note that $\leq = \leq_{lex_1} \bigcap \leq_{lex_2}$ and, because $\leq_{lex_1} \bigcap \leq_{lex_2}$ satisfies the separability condition on $u(X)$, so does $\precsim$ on X.

For the converse, consider the partial order $\precsim$ restricted to the countable set D, denoted by $\precsim|_D$. Obviously, $\precsim|_D$ satisfies conditions (i) to (iii) of Definition 2. Thus, as noted in the paragraph just before Theorem 1, $\precsim|_D$ is two-agent Pareto representable. Let $v = (v_1, v_2)$ be a two-agent Pareto representation of $\precsim|_D$. We may assume, without loss of generality, that v_1 and v_2 are bounded. Note, in addition, that, for any distinct $d_1, d_2 \in D$ such that $d_1 \precsim d_2$, it holds $v_1(d_1) < v_1(d_2)$ and $v_2(d_1) < v_2(d_2)$.

Now, define $u : X \to \mathbb{R}^2$ in the following way: for any $x \in X$ $u_1(x) = \sup_{d \precsim_1 x, d \in D} v_1(d)$, and $u_2(x) = \sup_{d \precsim_2 x, d \in D} v_2(d)$.

Let us see that $u = (u_1, u_2)$ is a two-agent Pareto representation of $\precsim$ on X. First of all, note that u_1, u_2 are well-defined because D bounds by below X and v_1, v_2 are bounded. Let $x, y \in X$

[2] The separability condition as is was provided by an anonymous referee who found a shortcoming in its original formulation.

[3] A subset Y of a partially ordered set $(X, \precsim)$ is said to be *coinitial in* X or, simply, coinitial, if Y bounds by below X; i.e., for every $x \in X$ there is $y \in Y$ such that $y \precsim x$. In a similar way the concept of a *cofinal* set is defined.

such that $x \precsim y$. Then, $x \precsim_1 y$ and, obviously, by definition it holds that $u_1(x) \leq u_1(y)$. A similar argument applies to see that $u_2(x) \leq u_2(y)$. Suppose, now, that $u_1(x) \leq u_1(y)$ and $u_2(x) \leq u_2(y)$ hold true for some $x, y \in X$. Let us see that $x \precsim_1 y$ and $x \precsim_2 y$, hence it will follow that $x \precsim y$ because, by Lemma 1, $\precsim = \precsim_1 \bigcap \precsim_2$. Assume, otherwise, that $y \prec_1 x$. Then, by definition of u_1, there are no $d, d' \in D$ such that $y \precsim_1 d \prec_1 d' \precsim_1 x$ because, otherwise, it would follow that $u_1(y) \leq u_1(d) = v_1(d) < v_1(d') = u_1(d') \leq u_1(x)$, which contradicts the fact that $u_1(x) \leq u_1(y)$. Note that, a fortiori, it would have $u_1(y) = u_1(x)$. Now, by the separability condition, there would exist $d, d' \in D$ so that $y \precsim_2 d \prec_2 d' \precsim_2 x$. But this possibility entails $u_2(y) < u_2(x)$, which contradicts the fact that $u_2(x) \leq u_2(y)$. Therefore, $x \precsim_1 y$. Exchanging the roles of $\precsim_1$ and $\precsim_2$ in the latter argument, and using the separability condition again, we get at $x \precsim_2 y$.

Finally, the last sentence in the statement of Theorem 1, involving the uniqueness argument, can be easily proved. Thus, the proof is ended. □

Remark 4. (i) If X is countable, then the statement of Theorem 1 simplifies because the separability condition is taken for granted. If, in addition, X is finite, then a sharper information can be provided as will be elaborated in Theorem 2 and Remark 5 below. However, the separability condition cannot dispensed with in the statement of Theorem 1 as the next example shows. Let there be given $X = \mathbb{R}^2$ and $\precsim = \leq_{lex_1}$, the usual lexicographic order. Note that $\leq_{lex_1}$ is a two-agent Pareto ordering because it is a total order (see Remark 3(i)). Now, it is not two-agent Pareto representable because for a total order the two-agent Pareto representation amounts to the usual utility representation. Of course, it is not separable either.
(ii) If $\precsim$ is a total order on X, then the separability condition agrees with the usual *order-separable condition* due to Jaffray (see Jaffray (1975), and Debreu (1954, 1964) or Bridges and Mehta (1995) for other equivalent order separability conditions). Thus, Theorem 1 extends the classical Debreu's result about the characterization of the existence of a utility function for total orders. In addition, note that, for the usual partial order $\leq$ on $\mathbb{R}^2$, a countable subset D that provides the separability condition given in Definition 3 is $\mathbb{Q}^2$.
(iii) The fact of D being a coinitial set in X can be replaced with that of being a cofinal set. To see this, it suffices to change the definitions of the functions that appear in the proof of Theorem 1 by the following ones: for any $x \in X$
$u_1(x) = \inf_{x \precsim_1 d, d \in D} v_1(d)$, and
$u_2(x) = \inf_{x \precsim_2 d, d \in D} v_2(d)$.

The next corollary presents a straightforward consequence of Theorem 1. Recall that if X is endowed with a quasi-order $\precsim$, then $X/\sim$ denotes the quotient set under the equivalence relation $\sim$ given by: $x \sim y$ if and only if $(x \precsim y) \wedge (y \precsim x)$. It should be noted that $X/\sim$ is a partially ordered set under the relation, denoted by $\precsim^q$, given by: $[x] \precsim^q [y]$ if and only if $x \precsim y$. Here, $[x]$ stands for the equivalence class of x; i.e., $[x] := \{y \in X : y \sim x\}$.

Corollary 1. *A quasi-order $\precsim$ defined on a nonempty set X is two-agent Pareto representable if and only if $\precsim^q$ is two-agent Pareto representable on $X/\sim$.*

As already mentioned in Remark 4(i), the case X finite deserves a special consideration. In this context, the existence of a two-agent Pareto representation is linked with the dimension of a partially ordered set. The term dimension of a poset $(X, \precsim)$, which will be denoted by $\dim(X, \precsim)$, refers to the minimum number of total orders on X needed to express $\precsim$ as the intersection of such total orders. This is a very interesting, and widely studied, problem in the literature which arises from the seminal paper by Dushnik and Miller (1941).

In relation with the results presented in the current article, we may conclude that a two-agent Pareto ordering $\precsim$ has, at most, dimension 2. Even further, a partial order $\precsim$, which is not a total order, has dimension 2 if and only if it is a two-agent Pareto ordering. In fact, and according to Lemma 1, we have been able to identify two total orders the intersection of which gives rise to $\precsim$.[4] In addition, and as established in Theorem 1, note that, when X is countable, a partial order is two-agent Pareto representable if and only if it is a two-agent Pareto ordering. This brief discussion is summarized in the corollary that follows.

Corollary 2. *For a poset $(X, \precsim)$ the following statements are equivalent:*
(i) $\precsim$ is a two-agent Pareto ordering,
(ii) $\dim(X, \precsim) \leq 2$.
Moreover, if X is countable, then we can add another equivalent statement:
(iii) $\precsim$ is two-agent Pareto representable.

According to Corollary 2, when X is finite or denumerable, the conditions given in Definition 2 turn out to be the nub of the matter to ensure the existence of a two-agent Pareto representation. These conditions suggest a kind of routine, or procedure, to determine when a partial order admits such a representation and, if feasible, how to obtain it (see Remark 5 below). Before, a somewhat surprising result is shown. Recall that $|X|$ denotes the cardinality of X.

Theorem 2. *Assume $|X| \leq 5$. Then, any partial order $\precsim$ on X is two-agent Pareto representable. However, the result fails if $|X| > 5$.*

Proof. Instead of providing a direct proof of this fact, which is tedious and lengthy, we will use a deep result due to Hiraguchi (1951).[5] Let $\precsim$ be a partial order on a finite set X. Then, Hiraguchi's inequality states that $\dim(X, \precsim) \leq |X|/2$, whenever $|X| \geq 4$. Thus, according to Hiraguchi's inequality, if X is endowed with a partial order and $|X| \leq 5$, then $\dim(X, \precsim) \leq 2$; hence it can be embedded into $\mathbb{R}^2$ with the usual partial order or, equivalently, it is two-agent Pareto representable. Nevertheless, if $|X| = 6$, then there is a partial order $\precsim$ on X such that $\dim(X, \precsim) = 3$; that is, it can be embedded into $\mathbb{R}^3$ but not into $\mathbb{R}^2$. Such a partial order is given in Sprumont (2001, Example 1 on page 438).[6] Let $X = \{x_1, \ldots, x_6\}$ and the partial order on X given by: $x_1 \precsim x_1, \ldots, x_6 \precsim x_6,\ x_1 \precsim x_4,\ x_1 \precsim x_5,\ x_2 \precsim x_5,\ x_2 \precsim x_6,\ x_3 \precsim x_4$, and $x_3 \precsim x_6$.

Let us prove that $\precsim$ is not two-agent Pareto representable because it fails to be a two-agent Pareto ordering. Indeed, suppose that, for each $x_i \in X$, there is a partition, $\{N^1_{\precsim}(x_i), N^2_{\precsim}(x_i)\}$, of $N_{\precsim}(x_i)$ so that conditions (ii) and (iii) of Definition 2 are met. We begin by arguing with the distinct possibilities that may occur for the subset $N^2_{\precsim}(x_1)$, being the argument for the subset $N^1_{\precsim}(x_1)$ completely analogous. To that end, we distinguish among the following cases:
(α_1) $N^2_{\precsim}(x_1)$ is a singleton. Assume $N^2_{\precsim}(x_1) = \{x_2\}$. This situation is impossible because if it were the case, then $N^1_{\precsim}(x_1) = \{x_3, x_6\}$. Moreover, by condition (ii), $x_1 \in N^1_{\precsim}(x_2)$. But then, by condition (iii), we would have $N^1_{\precsim}(x_1) \subseteq N^1_{\precsim}(x_2)$; hence $x_6 \in N^1_{\precsim}(x_2)$ which

4 According to Fishburn (1997) a partial order on a finite set with dimension, at most, 2 is said to be a *bi-linear order*. Thus, we have obtained a characterization of bi-linear orders for arbitrary, not necessarily finite, posets.

5 Alternative proofs of Hiraguchi's theorem were given by Bogart (1973) and Trotter (1975).

6 This example had appeared earlier in the literature. For instance, in Fishburn (1997, fig.2a on page 359) its corresponding Hasse diagram is shown.

contradicts the fact that $x_2 \precsim x_6$. The cases $N^2_{\precsim}(x_1) = \{x_3\}$ or $\{x_6\}$ are discussed in a similar way leading also to a contradiction.
(α_2) $N^2_{\precsim}(x_1)$ is a two-point set. Assume $N^2_{\precsim}(x_1) = \{x_2, x_3\}$. If this is the case, then $N^1_{\precsim}(x_1) = \{x_6\}$. Then, by condition (i), $x_1 \in N^2_{\precsim}(x_6)$ and, because by condition (iii) $N^2_{\precsim}(x_1) \subseteq N^2_{\precsim}(x_6)$, it follows that $N^2_{\precsim}(x_6) = \{x_1, x_4, x_5\}$ and $N^1_{\precsim}(x_6) = \emptyset$. But then $N^2_{\precsim}(x_1) = \{x_2, x_3\} \nsubseteq \{x_1, x_4, x_5\} = N^2_{\precsim}(x_6)$, which contradicts condition (iii). The cases $N^2_{\precsim}(x_1) = \{x_2, x_6\}$ and $N^2_{\precsim}(x_1) = \{x_3, x_6\}$ are treated similarly leading also to a contradiction.
(α_3) $N^2_{\precsim}(x_1)$ is a three-point set. The only possibility now is $N^2_{\precsim}(x_1) = \{x_2, x_3, x_6\}$, and so $N^1_{\precsim}(x_1) = \emptyset$. In this situation, necessarily, it occurs that either $N^2_{\precsim}(x_2) = \{x_3\}$ or $\emptyset$. Thus, in any case, $x_4 \in N^1_{\precsim}(x_2)$. Note, also, that, by condition (iii), $N^2_{\precsim}(x_3) = \{x_2\}$ or $\emptyset$. Now, if $x_3 \in N^1_{\precsim}(x_2)$, then $N^2_{\precsim}(x_3) = \{x_2\}$, hence $N^1_{\precsim}(x_3) = \{x_1, x_5\}$, and so $x_3 \in N^2_{\precsim}(x_5)$. But this is impossible because $N^2_{\precsim}(x_3) = \{x_2\} \notin N^2_{\precsim}(x_5)$. Therefore, $N^1_{\precsim}(x_2) = \{x_1, x_4\}$ and $N^2_{\precsim}(x_2) = \{x_3\}$. But this leads to a contradiction again because, in that case, it would happen that $N^2_{\precsim}(x_3) = \emptyset$, hence $N^1_{\precsim}(x_3) = \{x_1, x_2, x_5\}$, which violates the fact that $x_4 \in N^1_{\precsim}(x_2)$ and $x_4 \notin N^1_{\precsim}(x_3)$. Thus, the proof is complete. □

As a straightforward consequence of the previous theorem the following result is reached.

Corollary 3. *Let $\precsim$ be a quasi-order on X such that $|X/\sim| \leq 5$. Then, $\precsim$ is two-agent Pareto representable.*

Remark 5. (i) The non two-agent Pareto representable partial order shown in the last part of the proof of Theorem 2 has the following representation in $\mathbb{R}^3$: $v(x_1) = (1, 0, 0)$, $v(x_2) = (0, 0, 1)$, $v(x_3) = (0, 1, 0)$, $v(x_4) = (1, 1, 0)$, $v(x_5) = (1, 0, 1)$ and $v(x_6) = (0, 1, 1)$.

(ii) We now illustrate how the approach based on the conditions of Definition 2 can be used to generate two-agent Pareto representable orderings. In addition, this procedure will enable us to obtain a two-agent Pareto representation by simply evaluating the cardinality of the lower contour sets of the total orders $\precsim_1$ and $\precsim_2$, respectively. To offer an example, suppose $X = \{a, b, c, d\}$. Let us begin with an arbitrary selection; to wit, $L_{\precsim}(a) = \{a\}$, $G_{\precsim}(a) = \{a, c\}$ and $N_{\precsim}(a) = \{b, d\}$. Choose, $N^1_{\precsim}(a) = \{b\}$ and $N^2_{\precsim}(a) = \{d\}$.

Then, by applying the conditions of Definition 2, we obtain:
(α_1) $a \in N^2_{\precsim}(b)$ and $a \in N^1_{\precsim}(d)$,
(α_2) $N^1_{\precsim}(b) \subseteq N^1_{\precsim}(a)$ and $N^2_{\precsim}(d) \subseteq N^2_{\precsim}(a)$.

Thus, the corresponding conclusions are:
(β_1) $N^1_{\precsim}(b) = \emptyset$ and $d \in N^2_{\precsim}(b)$,
(β_2) $b \in N^1_{\precsim}(d)$ and $N^2_{\precsim}(d) = \emptyset$.

Note, in addition, that $d \notin N^1_{\precsim}(c)$ and $b \notin N^2_{\precsim}(c)$.

Assume now that $c \in G_{\precsim}(b)$ and $c \bowtie d$.[7] Then, necessarily, it holds that:
(γ_1) $b \in L_{\precsim}(c)$ and $N^1_{\precsim}(c) = \emptyset$,
(γ_2) $N^2_{\precsim}(c) = \{d\}$ and $c \in N^1_{\precsim}(d)$.

Thus, the process has been completed with the following family of subsets:
(1) $L_{\precsim}(a) = \{a\}$, $G_{\precsim}(a) = \{a, c\}$, $N^1_{\precsim}(a) = \{b\}$ and $N^2_{\precsim}(a) = \{d\}$,
(2) $L_{\precsim}(b) = \{b\}$, $G_{\precsim}(b) = \{b, c\}$, $N^1_{\precsim}(b) = \emptyset$ and $N^2_{\precsim}(b) = \{a, d\}$,
(3) $L_{\precsim}(c) = \{a, b, c\}$, $G_{\precsim}(c) = \{c\}$, $N^1_{\precsim}(c) = \emptyset$ and $N^2_{\precsim}(c) = \{d\}$,
(4) $L_{\precsim}(d) = \{d\}$, $G_{\precsim}(d) = \{d\}$, $N^1_{\precsim}(d) = \{a, b, c\}$ and $N^2_{\precsim}(d) = \emptyset$.

Note that, by simply imposing the initial conditions $L_{\precsim}(a) = \{a\}$, $G_{\precsim}(a) = \{a, c\}$, $N^1_{\precsim}(a) = \{b\}$ and $N^2_{\precsim}(a) = \{d\}$, together with $c \in G_{\precsim}(b)$ and $c \bowtie d$, all other subsets of the family are completely determined by the conditions of Definition 2 and Remark 2(b). The previous family of subsets gives rise to the following relation $\precsim$: $a, b \precsim c$; $a, b, c \bowtie d$ and $a \bowtie b$, together with the corresponding self relation for any point in order to meet reflexivity. Clearly, it is a partial order on X. The total orders $\precsim_1$ and $\precsim_2$, associated with $\precsim$, are:
(1) $b \precsim_1 a \precsim_1 c \precsim_1 d$,
(2) $d \precsim_2 a \precsim_2 b \precsim_2 c$.

Therefore, by calculating the cardinality of the lower contour sets, associated with $\precsim_1$ and $\precsim_2$, respectively, the following representation is obtained:

For the first component function v_1 of the two-agent Pareto representation we have $v_1(a) = |L_{\precsim_1}(a)| = 2$, $v_1(b) = |L_{\precsim_1}(b)| = 1$, $v_1(c) = |L_{\precsim_1}(c)| = 3$ and $v_1(d) = |L_{\precsim_1}(d)| = 4$.

Similarly, for the second component function v_2 we obtain $v_2(a) = |L_{\precsim_2}(a)| = 2$, $v_2(b) = |L_{\precsim_2}(b)| = 3$, $v_2(c) = |L_{\precsim_2}(c)| = 4$ and $v_2(d) = |L_{\precsim_2}(d)| = 1$.

Thus, by attaching v_1 and v_2 in a single vector-valued function v and letting $v(a) = (2, 2)$, $v(b) = (1, 3)$, $v(c) = (3, 4)$ and $v(d) = (4, 1)$ a two-agent Pareto representation of $\precsim$ is reached.

(iii) As stated in Theorem 1, a two-agent Pareto representation of a partial order, if feasible, need not be unique. The procedure described just above, enables us to obtain another representation by choosing an initial distinct partition of $N_{\precsim}(a)$. Let now $N^1_{\precsim}(a) = \{b, d\}$ and $N^2_{\precsim}(a) = \emptyset$. Consider the following family of subsets of X:
(1) $L_{\precsim}(a) = \{a\}$, $G_{\precsim}(a) = \{a, c\}$, $N^1_{\precsim}(a) = \{b, d\}$ and $N^2_{\precsim}(a) = \emptyset$,
(2) $L_{\precsim}(b) = \{b\}$, $G_{\precsim}(b) = \{b, c\}$, $N^1_{\precsim}(b) = \{d\}$ and $N^2_{\precsim}(b) = \{a\}$,
(3) $L_{\precsim}(c) = \{a, b, c\}$, $G_{\precsim}(c) = \{c\}$, $N^1_{\precsim}(c) = \{d\}$ and $N^2_{\precsim}(c) = \emptyset$,
(4) $L_{\precsim}(d) = \{d\}$, $G_{\precsim}(d) = \{d\}$, $N^1_{\precsim}(d) = \emptyset$ and $N^2_{\precsim}(d) = \{a, b, c\}$.

Then, it is easy to check that the family of pairs $(N^1_{\precsim}(x), N^2_{\precsim}(x))_{x\in\{a,b,c,d\}}$ satisfies conditions (i) to (iii) of Definition 2. The partial order induced is the same as that of Remark 5(ii). Now, the total orders $\precsim_1$ and $\precsim_2$, associated with $\precsim$, are:
(1) $d \precsim_1 b \precsim_1 a \precsim_1 c$,
(2) $a \precsim_2 b \precsim_2 c \precsim_2 d$.

Thus, the two-agent Pareto representation, derived from $\precsim_1$ and $\precsim_2$, is now: $v'(a) = (3, 1)$, $v'(b) = (2, 2)$, $v'(c) = (4, 3)$ and $v'(d) = (1, 4)$. Note that, when comparing the two representations of the same partial order $\precsim$ obtained, v and v', there are no strictly increasing real-valued functions ϕ_1, ϕ_2 such that $(v'_1, v'_2) = (\phi_1 \circ v_1, \phi_2 \circ v_2)$.

(iv) Another example of a six-point poset which is not two-agent Pareto representable is the so-called *chevron* (see, e.g., Fishburn and Trotter (1999)). It is described in the following way: Let $X = \{x_1, \ldots, x_6\}$ and the partial order on X given by: $x_1 \precsim x_1, \ldots, x_6 \precsim x_6$, $x_1 \precsim x_2$, $x_2 \precsim x_3$, $x_1 \precsim x_3$, $x_1 \precsim x_4$, $x_4 \precsim x_5$, $x_1 \precsim x_5$, $x_6 \precsim x_3$, and $x_6 \precsim x_5$.

According to Corollary 2, suppose, by way of contradiction, that there is a family $(N^1_{\precsim}(x), N^2_{\precsim}(x))_{x\in X} \subset 2^X \times 2^X$ that makes $\precsim$ to be a two-agent Pareto ordering. First of all, note that, because $G_{\precsim}(x_1) = \{x_1, x_2, x_3, x_4, x_5\}$ and $L_{\precsim}(x_1) = \{x_1\}$, it follows that either $(N^1_{\precsim}(x_1) = \{x_6\}$ and $N^2_{\precsim}(x_1) = \emptyset)$, or $(N^1_{\precsim}(x_1) = \emptyset$ and $N^2_{\precsim}(x_1) = \{x_6\})$. Assume the first situation holds true, the argument for the second case being entirely analogous. Then, since $N^1_{\precsim}(x_6) \subseteq N^1_{\precsim}(x_1) = \{x_6\}$, it follows that $N^1_{\precsim}(x_6) = \emptyset$, hence $N^2_{\precsim}(x_6) = \{x_1, x_2, x_4\}$.

Now, because $N^2_{\precsim}(x_2) \subseteq N^2_{\precsim}(x_6)$, it holds that $x_5 \notin N^2_{\precsim}(x_2)$, hence $x_5 \in N^1_{\precsim}(x_2)$, and so $x_2 \in N^2_{\precsim}(x_5)$. Once again, since $N^2_{\precsim}(x_2) \subseteq N^2_{\precsim}(x_5)$ and $L_{\precsim}(x_5) = \{x_1, x_4, x_5, x_6\}$, it, necessarily, follows that $N^2_{\precsim}(x_2) = \emptyset$. This is because neither x_1 nor x_4 can belong to $N^2_{\precsim}(x_2)$. Therefore, $N^1_{\precsim}(x_2) = \{x_1, x_4, x_5, x_6\}$. But then we reach a contradiction because $x_1 \in N^1_{\precsim}(x_2)$ and $x_2 \notin N^2_{\precsim}(x_1)$. Thus, $\precsim$ is not a two-agent Pareto representable ordering. Actually, it can be embedded into $\mathbb{R}^3$ in the following way:

[7] These two conditions were omitted in Remark 3(2) of Candeal (2022).

$v(x_1) = (1, 2, 1)$, $v(x_2) = (2, 3, 5)$, $v(x_3) = (4, 6, 6)$, $v(x_4) = (5, 4, 2)$, $v(x_5) = (6, 5, 4)$ and $v(x_6) = (3, 1, 3)$.

Note, also, that $\dim(X, \precsim) = 3$. Indeed, consider the following total orders, denoted by $\precsim_1$, $\precsim_2$ and $\precsim_3$, on X:

(a) $x_1 \precsim_1 x_2 \precsim_1 x_6 \precsim_1 x_3 \precsim_1 x_4 \precsim_1 x_5$,
(b) $x_6 \precsim_2 x_1 \precsim_2 x_2 \precsim_2 x_4 \precsim_2 x_5 \precsim_2 x_3$,
(c) $x_1 \precsim_3 x_4 \precsim_3 x_6 \precsim_3 x_5 \precsim_3 x_2 \precsim_3 x_3$.

Then, $\precsim = \bigcap_{i=1}^{3} \precsim_i$.

4. Conclusions and future research

In this paper, we have presented a novel approach to deal with what may be considered a classical problem in mathematics and economics; to wit, the characterization of those binary relations defined on a nonempty set X that can be embedded into the n-dimensional Euclidean space $\mathbb{R}^n$ endowed with its usual partial order $\leq$. In particular, we have focused on the two-dimensional case (which is referred to in the literature as the two-agent case) providing a very simple and natural characterization of such orderings. Our result does not impose cardinality constraints on X and, apparently, paves the way to determine all two-agent Pareto representable partial orders, at least, when X is finite. However, we have not developed this aspect thoroughly. In addition, we have also discussed to some extent the connection of our results with the concept of the dimension of a poset as established by Dushnik and Miller (1941). Moreover, hopefully, our approach might be extended to cover the general case even considering the situation of continuous preferences. We expect to pursue all these items in a next future paper.

Acknowledgments

The paper has greatly benefited from the comments and remarks of two anonymous referees. The comments and support of Professors F. Roberts (guest editor) and A. Diederich (editor-in-chief) are acknowledged too. This work has been partially supported by the Research Project PID2019-108348RA-I00 (Spain).

References

Baker, K. A., Fishburn, P. C., & Roberts, F. S. (1972). Partial orders of dimension 2. *Networks*, *2*, 11–28.
Bogart, K. (1973). Maximal dimensional partially ordered sets I. Hiraguchi's theorem. *Discrete Mathematics*, *5*, 21–31.
Bosi, G., & Herden, G. (2012). Continuous multi-utility representations of preorders. *Journal of Mathematical Economics*, *48*, 212–218.
Bridges, D. S., & Mehta, G. B. (1995). *Representations of preference orderings*. Berlin: Springer Verlag.
Candeal, J. C. (2022). Bi-utility representable orderings on a countable set. *Economics Letters*, *217*, Article 110658.
Debreu, G. (1954). Representation of a preference ordering by a numerical function. In R. Thrall, C. Coombs, & R. Davies (Eds.), *Decision Processes* (pp. 159–165). New York: John Wiley.
Debreu, G. (1964). Continuity properties of Paretian utility. *International Economic Review*, *5*, 285–293.
Donaldson, D., & Weymark, J. (1998). A quasiordering is the intersection of orderings. *Journal of Economic Theory*, *78*, 382–387.
Dushnik, B., & Miller, E. (1941). Partially ordered sets. *American Journal of Mathematics*, *63*, 600–610.
Evren, O., & Ok, E. (2011). On the multi-utility representation of preference relations. *Journal of Mathematical Economics*, *47*, 554–563.
Fishburn, P. C. (1997). Generalizations of semiorders: A review note. *Journal of Mathematical Psychology*, *41*, 357–366.
Fishburn, P. C., & Trotter, W. (1999). Split semiorders. *Discrete Mathematics*, *195*, 111–126.
Hack, P., Braun, D. A., & Gottwald, S. (2022). Representing preorders with injective monotones. *Theory and Decision*, *93*, 663–690.
Hack, P., Braun, D. A., & Gottwald, S. (2023). The classification of preordered spaces in terms of monotones: Complexity and optimization. *Theory and Decision*, *94*, 693–720.
Hiraguchi, T. (1951). On the dimension of partially ordered sets. *Science Reports of Kanazawa University*, *1*, 77–94.
Jaffray, J.-Y. (1975). Existence of a continuous utility function: An elementary proof. *Econometrica*, *43*, 981–983.
Kelly, D. (1977). The 3-irreducible partially ordered sets. *Canadian Journal of Mathematics*, *XXIX*, 367–383.
Ok, E. (2002). Utility representation of an incomplete preference relation. *Journal of Economic Theory*, *104*, 429–449.
Qi, S. (2015). Paretian partial orders: The two-agent case. *Journal of Mathematical Economics*, *57*, 38–48.
Qi, S. (2016). A characterization of the n-agent Pareto dominance relation. *Social Choice and Welfare (2016)*, *46*, 695–706.
Rébillé, Y. (2023). A representation of interval orders through a bi-utility function. *Journal of Mathematical Psychology*, *115*, Article 102778.
Sprumont, Y. (2001). Paretian quasi-orders: The regular two-agent case. *Journal of Economic Theory*, *101*, 437–456.
Trotter, W. (1975). Inequalities in dimension theory for posets. *Proceedings of the American Mathematical Society*, *47*(2), 311–316.
Trotter, W. (1992). *Combinatorics and partially ordered sets*. Baltimore, MD: Johns Hopkins University Press.
Trotter, W., & Moore, J. (1976). Characterization problems for graphs, partially ordered sets, lattices, and families of sets. *Discrete Mathematics*, *16*, 361–381.