An study of cost effective maintenance policies: age replacement versus replacement after N minimal repairs.

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Abstract

In this paper we consider the inspection and maintenance of a system under two types of age-dependent failures, revealed minor failures (R) and unrevealed catastrophic failures (U). Periodic inspections every T units of time are carried out to detect U failures, leading to the system replacement when one is discovered. R failures are followed by a minor repair. In addition the system is preventively replaced at MT or after the N^{th} R failure whichever comes first. The costs of minimal repair and replacement after N minor failures depend on age and history of failures. Non-perfect inspections are assumed, providing false positives when no U failure has happened or false negatives when a U failure is present. The long-run cost per unit of time along with the optimum policy (T^*, M^*, N^*) are obtained. We explore conditions under which both strategies of preventive maintenance are profitable, comparing with suboptimal policies when only one of them is performed. Maintenance of infrastructures illustrates the model conditions.

Keywords: Age replacement; Maintenance; Minor repair; optimum policy

1 Introduction

The design of maintenance policies that extend the useful life of systems is a crucial concern for industry. In so doing companies reduce costs avoiding early replacements of costly equipment. Thus, a number of repairs which are done when needed usually precede the renewal of a system. In practice the maintainer has two choices: either to keep on repairing or else to replace the system. This decision is made based on cost as well as reliability requirements. When repairs hardly reduce system deterioration or do not provide a significant remaining life that compensates maintenance costs, then the system is replaced. This idea often underlies the purchase of a new car. If the age of a car makes it very likely to experience a critical and costly failure in the near future, repairing the one in use can be wasteful. If so, replacing the car seems to be the cost-optimal decision. Kurt and Kharoufeh [14] mention several engineering applications that can only be repaired a number of times because of the risks incurred due to the increasing degradation. They also point out a limited warranty as the reason for a given number of repairs before replacement. Lugtigheid *et al* [15] considered a limited number of repairs in the context of contractual agreement between the OEM (original equipment manufacturer), or contractor, and the equipment owner or user. Qiu *et al* [21] investigate optimal maintenance strategies considering performance-based contracts that try to guarantee higher availability of systems.

In general the successive repairs that precede the system replacement are addressed at minor failures. By a minor one we refer to the type of failure whose repair restores the intended function of the system. After the repair the system is not as perfect as when it was new but the maintainer assumes that the repair cost pays off with an extended lifetime. Sheu *et al* [26] include age-dependent repair costs in the maintenance of a system with minor or catastrophic failures that is replaced at the N minor failure or the first catastrophic failure or at age T, whichever occurs first. Shafiee *et al* [24] present the current application of this idea in the maintenance of offshore wind turbine blades. The works of Do *et al* [11] and Do Van and Bérenguer[12] deal with condition-based maintenance considering both perfect and imperfect maintenance actions for a deteriorating system. In both cases a threshold for the number of imperfect maintenances is assumed. Safaei *et al* [22] describe the maintenance of a system under three types of failures: type I can be fixed $k-1$ times by a minimal repair, type III are catastrophic and the system should be replaced. Type II are minimally repaired with probability $p(t)$ and lead to system replacement with probability $1-p(t)$. Nakagawa and Zhao [17] propose generalized replacement policies under which the unit is replaced at time T, at a working cycle N or at a failure K. Zhao *et al* [35] compare replacement policies that are carried out at some periodic times and a predetermined number of repairs. These authors claim that both are commonly used in total productive maintenance in Japanese industry.

In contrast to revealed failures, detected at the very moment they occur, unrevealed failures require some type of inspection to be discovered.

Thus the maintenance of the latter should include an inspection policy. Vaurio [31] suggests an inspection policy along with a preventive replacement when a failure is detected or after N inspections, whichever comes first. The work of [13] considers a system with periodic imperfect inspections to detect hidden failures and periodic preventive maintenance to correct them with the objective of determining the optimal frequency and quantity of imperfect inspections. Nielsen and Sørensen [18] analyze the optimal planning for inspection and maintenance of offshore wind turbines. Inspections aim at obtaining information about the state of the components avoiding both unnecessary maintenance or lack of repair when needed. Peng *et al* [19] describe the maintenance of a system subject to two types of failures: catastrophic and a two-stage delayed failure. Inspections are carried out to detect defective states of the two-stage delayed failure and age replacement to avoid catastrophic failures. The case study refers to peristaltic pumps used to pump fluids in patients. Wang and Wu [33] deal with the problem of inspections in a production line subject to small stoppages and hard failures.

The works of Badía and Berrade [3], [4] and [5] address the optimization problem under maintenance policies involving two decision variables (T, N) where T is the inspection interval and N the maximum number of failures before the system undergoes a perfect repair. The work in [3] assumes that the N failures are of the type minor and unrevealed. A perfect restoration follows after the N minor unrevealed failures or the first catastrophic failure, whichever comes first. The system in [5] experiences periodic inspections to detect hidden failures and undergoes an imperfect restoration after the $N-1$ first failures while the N^{th} failure is removed by a perfect repair. Sheu *et al* [27] present a maintenance model for a system under two types of unrevealed failures, minor or catastrophic. The system is replaced at the occurrence of the N minor failure, the catastrophic one or a working age T.The authors consider that models of this type present potential applications in oil and gas industries, medicine, and nuclear power plants. A model for inspection and maintenance on a finite time interval of a complex system with soft and hard failures is considered in Taghipour *et al* [29].

A number of authors focus on the extension of current structures lifetime. Zio [36] stands out the interest of the proper use of information about their condition obtained from inspections or measurements. Carretero [9] *et al* apply reliability centered maintenance to railway networks. Podofillini *et al* [20] present an ultrasonic inspection strategy to detect railway cracks. Sheils *et al* [25] develop a two-stage inspection procedure for detection and sizing assessment of defects in infrastructures. Concerning bridges design, Tang [30] analyzes the use of a new concept of orthotropic deck. This author identifies failures of two types: failure of the steel deck and in the pavement. Fatigue and corrosion are within the former group and require inspection or tests to be detected. Cracking and separation in the pavement can be observed when they occur. A recent review on condition based maintenance can be found on Alaswad and Yang [1]. Maintenance of nuclear power plants that should cover all their components [8] provides another example with minor and major failures. Radiation leaks can be catastrophic and require monitoring systems to be detected. Monitoring systems (radiation level, temperature indicators, state of refrigeration pumps. . .) can also fail. The failure of these systems that stop providing information is immediately detected and their replacement can be considered as a minor repair since the reliability of the rest of the system is not affected.

The use of sensors to estimate the state of system and make decisions to avoid catastrophic failures is also crucial for the maintenance of wind turbines

(Byon [7]). Thus, models for inspection and maintenance of systems with both soft revealed and catastrophic unrevealed failures are required.

The current paper presents a policy valid for a system under both revealed and unrevealed failures. We propose a model for inspection and maintenance of a system with two types of age-dependent failures, minor revealed and catastrophic unrevealed. Different approaches to soft and hard failures and hidden failures can be found in Taghipour *et al* [29] and Seyedhosseini *et al* [23]. The system can experience two types of preventive replacements, periodic at MT and random after N minor failures. In order to mimic maintenance contracts, we assume costs depending on both, age and history of failures. We are concerned with the inspection interval, number of inspections and number of minimal repairs, $(T^\star, M^\star, N^\star)$ minimizing the cost rate. Regarding previous literature the first contribution of this model is that it presents two types of maintenance taking into account the two types of failures revealed and unrevealed. Thus, age replacement at MT is based on unrevealed failures and replacement after N minor failures is defined on revealed ones. This dual maintenance can provide a better protection when the probability of minor and catastrophic failures changes. The second contribution is that this model extends that in [4] by introducing age-dependent probabilities of failure and preventive age replacement. To our knowledge, this has not been yet considered in literature.

In addition the comparison with suboptimal policies where only one type of preventive replacement is carried out, $(T^\star, M^\star, \infty)$ and $(\infty, -, N^\star)$ is also of interest as it highlights the way that the parameters involved can determine the superiority of one type of preventive replacement over the other. This model constitutes an extension of those in He *et al* [13] and Badía and Berrade [4]. The former only considers unrevealed failures $(p(t) = 0)$ whereas the latter involves two decision variables $(T^\star, N^\star)$. Thus, age replacement at MT and preventive maintenance after N minor failures can be analyzed and compared.

The structure of this paper is as follows. In the next section we describe the maintenance model and its assumptions, developing the calculations for the long run expected cost per unit time. We determine the expected cost and length of a renewal cycle. Section 3 is devoted to numerical examples that give insight about the dependence of the optimum policy on the parameters of the model as well as the comparison with suboptimal policies. This analysis constitutes a practical guide for maintainers.

2 Maintenance model

In what follows we present the inspection and maintenance of a single-unit system undergoing two types of failures, revealed minor failures (R) and unrevealed catastrophic failures (U). We assume that the occurrence of R and U failures is age-dependent. Thus when a failure happens at time t it is of the type R with probability $p(t)$ $(0 \leq p(t) \leq 1)$ and of the type U with probability $q(t) = 1 - p(t)$. U-failures are detected by periodic inspections carried out at times kT, $k = 1, 2, \ldots, M-1$. Although only the time for the first catastrophic is required for calculations we assume that additional failures of this type can occur before the system is replaced. Concerning R-failures they are followed by a minimal repair that restores the system to the condition just previous to failure ("as-bad-as-old"). We assume a maximum number of allowable minimal repairs, $N-1$. The system is replaced by a new one when a U-failure is detected on inspection, or preventively at MT, or once the N^{th} R-failure occurs whichever comes first. We consider that there is no inspection at MT just replacement but this does not represent a significant restriction and inspection additional to replacement might be assumed. This point has been discussed in Berrade *et al* [6]. Preventive replacement at MT makes the maintainer gain protection against catastrophic failures which usually present high costs. Regarding preventive actions, the perfect repair after N minor failures can be also seen as an opportunity-based maintenance. Thus, T, M and N can be considered decision variables to be optimized.

We also assume that inspections may not be perfect (Badía *et al* [2], [6]). The result can be a false positive (type I error) when the system has not failed but inspection indicates that a U-failure has occurred and a false negative (type II error) when inspection fails to detect an actual U-failure. The corresponding probabilities are denoted by α and β respectively. Following the assumptions of model 1 in [6] and Badía *et al* [2], we assume that a false positive is detected by the OEM in a further inspection so there is no effect in the system reliability and just an additional cost is incurred.

Times of inspections, repairs and preventive maintenance are considered to be negligible. The following notation is used throughout.

- $r(x)$ failure rate of the time to the first failure.
- $H(x)$ cumulative failure rate: $H(x) = \int_0^x r(u)du$.
- $p(t)$: probability of a failure that occurs at t being of the type R.
- $H_R(x) = \int_0^x p(u)r(u)du$.
- $H_U(x) = \int_0^x q(u)r(u)du$.
- Y: time to the first unrevealed catastrophic failure (U failure).

- α: probability of a false positive, that is an inspection outcome indicates that a U-failure has occurred when the system has not failed.
- β: probability of a false negative, thus an inspection outcome fails to detect an actual U-failure.
- T: inspection interval.
- M: parameter associated to the preventive replacement at MT. The maximum number of inspections in a cycle is $M-1$.
- K_1: number of inspections previous to the first U-failure if there is no preventive maintenance: $K_1 = \lfloor \frac{Y}{T} \rfloor$, with $\lfloor . \rfloor$ representing the integer part function.
- L: number of inspections after a U-failure until it is detected when no preventive maintenance is carried out.
- I_0: number of inspections before a U-failure or the N^{th} R failure or preventive replacement at MT whichever occurs first.
- I: number of inspections in a cycle.
- F: number of false positive inspections in a cycle.
- G_i time to the i^{th} revealed minor failure (R failure) $i = 1, 2, \ldots, N$.
- $N[0, t]$: number of R-failures in $[0, t]$.

The following events are also considered:

- R_1: the renewal cycle ends after the preventive replacement at MT.

- R_2: the cycle is completed at the preventive renewal after the N^{th} R failure.

- R_3: the cycle is completed once a U-failure is detected on inspection.

- R_4: the system with an undetected U-failure is preventively replaced at MT.

- R_5: the system, free of a U-failure, is preventively replaced at MT.

The costs assumed are given next :

- c_1: unitary cost of inspection.

- c_{PM}^1: cost of preventive maintenance at MT when the system presents an undetected U-failure.

- c_{PM}^2: cost of preventive maintenance at MT when the system is free of a U-failure.

- c_{r1}: cost of replacement when a U-failure is detected on inspection.

- $c_{r2}(N, t)$: cost of preventive replacement after the N R-failure at time t.

- $c_{mr,j}(t)$: cost of the minimal repair after the j^{th} R-failure at time t, $j = 1, 2, \ldots, N - 1$.

- c_f: cost of a false alarm.

- c_d: cost per unit of time while a U-failure remains undiscovered.

2.1 Preliminary Results

Proposition 1 *Under the model assumptions it follows that*

(i) The density and reliability functions of Y are given as follows

$$\begin{aligned} f_Y(x) &= q(x)r(x)e^{-H_U(x)}, \quad x \geq 0 \\ \overline{F}_Y(x) &= e^{-H_U(x)}, \quad x \geq 0 \end{aligned}$$

(ii) The density and reliability functions of G_i, $i = 1, 2, \ldots, N$ are

$$\begin{aligned} f_{G_i}(x) &= p(x)r(x)\frac{(H_R(x))^{i-1}}{(i-1)!}e^{-H_R(x)}, \quad x \geq 0 \\ \overline{F}_{G_i}(x) &= \sum_{j=0}^{i-1} \frac{(H_R(x))^j}{j!}e^{-H_R(x)}, \quad x \geq 0 \end{aligned}$$

(iii) $N[0,t]$ is a non-homogeneous Poisson process with mean function given by $H_R(t)$.

(iv) Y is independent from both G_N and $N[0,t]$.

The proof can be found in Cha and Finkelstein [10].

Regarding result in (iv), the process of failures is split into two independent ones. At each time t the failure is revealed or unrevealed independently of the previous history of failures with $p(t)$ being the probability of a revealed one. This is similar to the thinning of a Poisson process. Then when the system is new at $t = 0$, two independent times are triggered: that of the N revealed minor failure and the corresponding to a catastrophic unrevealed failure. Nevertheless the unrevealed failure is more likely to occur as time goes by since $p(t)$ tends zero for $t \to \infty$. Hence although both failures are mutually exclusive at each time t, the probabilities change reflecting the proneness of the system to undergo each type of failure. Thus, $Y < G_N$ or $Y \geq G_N$.

Let X_1 and X_2 be non negative absolute continuous and independent random variables and $a > 0$. It follows that

$$E[\min(X_1, X_2, a)] = \int_0^a \bar{F}_{X_1}(x)\bar{F}_{X_2}(x)dx \tag{1}$$

In addition if X_1 and X_2 are non negative discrete and independent random variables and $n = 1, 2, \ldots$, then

$$E[\min(X_1, X_2, n)] = \sum_{x=1}^{n} P(X_1 \geq x)P(X_2 \geq x) \tag{2}$$

The previous result is derived below:

$$\begin{aligned}
E[\min(X, n)] &= \sum_{x=0}^{n-1} xP(X = x) + nP(X \geq n) \\
&= \sum_{x=1}^{n-1}\sum_{j=1}^{x} P(X = x) + nP(X \geq n) \\
&= \sum_{j=1}^{n-1}\sum_{x=j}^{n-1} P(X = x) + nP(X \geq n) \\
&= \sum_{j=1}^{n-1}(P(X \geq j) - P(X \geq n)) + nP(X \geq n) \\
&= \sum_{j=1}^{n} P(X \geq j)
\end{aligned}$$

2.2 The maintenance model

If no preventive replacement is carried out, then K_1 represents the number of inspections previous to the first U failure. It follows that

$$K_1 = \left\lfloor \frac{Y}{T} \right\rfloor$$

In addition, $K_1 = 0, 1, 2, \ldots$ with the probabilities given below

$$P(K_1 = j) = F_Y((j+1)T) - F_Y(jT), \quad j = 0, 1 \ldots$$

If no preventive replacement is carried out, the number of inspections from the first U failure until it is detected, L, follows a geometric distribution with mean value $\frac{1}{1-\beta}$. Thus

$$P(L = j) = \beta^{j-1}(1-\beta), \quad j = 1, 2, \ldots$$

$$P(K_1 + L = j) = \tag{3}$$
$$\sum_{r=0}^{j-1} (F_Y((r+1)T) - F_Y(rT))\beta^{j-r-1}(1-\beta), \quad j = 1, 2, \ldots$$

$$P(K_1 + L \geq j+1) = \tag{4}$$
$$\sum_{r=0}^{j} P(L \geq j+1-r)P(K_1 = r) + P(K_1 \geq j+1) =$$
$$\sum_{r=0}^{j} \beta^{j-r} \left[F_Y((r+1)T) - F_Y(rT))\right] + \bar{F}_Y((j+1)T).$$

The expression of the mean time until the system is renewed is given next. The proof is in the Appendix.

Proposition 2 *The expected length of a renewal cycle, τ, is given by:*

$$E[\tau] = \tag{5}$$
$$\sum_{j=0}^{M-1} \left(\sum_{r=0}^{j} \beta^{j-r} \left[F_Y((r+1)T) - F_Y(rT)\right] + \bar{F}_Y((j+1)T) \right) \int_{jT}^{(j+1)T} \bar{F}_{G_N}(x)dx$$

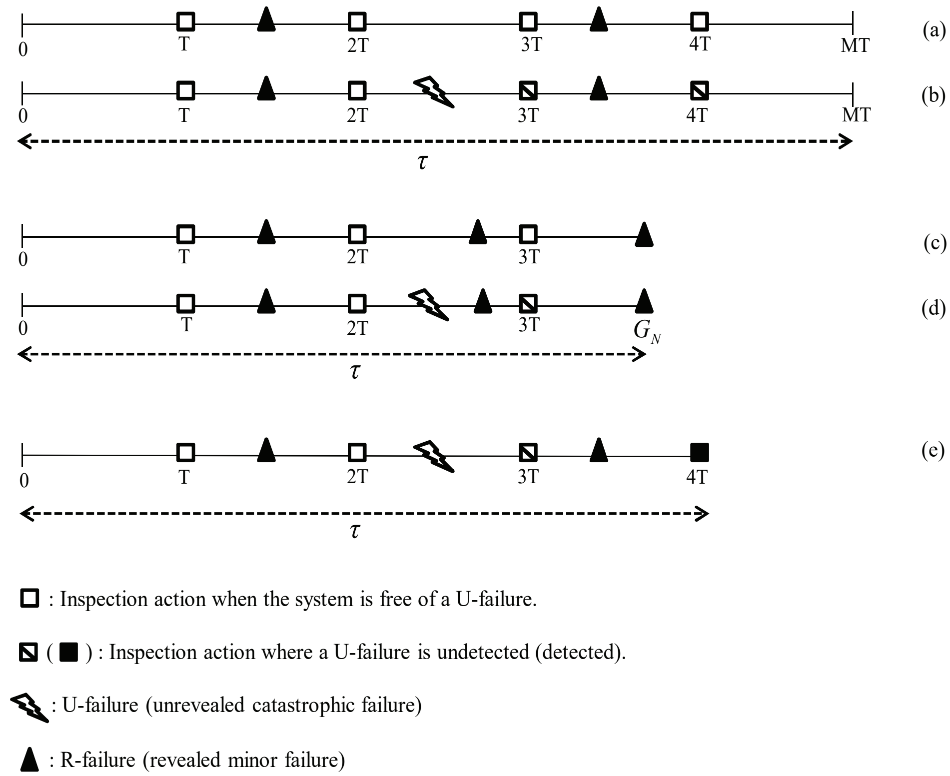

Figure 1: Length of a cycle under the maintenance model.

Figure 1 represents how a cycle is completed under this maintenance model. Replacement at MT for a system that has not experienced a U failure or undergoes a catastrophic but undetected U failure is represented, respectively, in (a) and (b). The system is replaced after N minor failures in (c) and (d). No U failure has occurred before the N minor failure in (c) whereas a U failure not detected by the time that the N minor failure happens is present in (d). The graph in (e) describes a system replaced on inspection after a U failure is detected.

In what follows I_0 denotes the number of inspections previous to a U-failure or the Nth R failure or the preventive maintenance at MT whichever comes first. I and F represent, respectively, the number of inspections and the number of false positive inspections in a cycle. The mean values of I_0, I, and F are given next. The proof can be found in the Appendix.

Proposition 3 *The following results apply:*

$$
\begin{aligned}
E[I_0] &= \sum_{j=1}^{M-1} \bar{F}_{G_N}(jT)\bar{F}_Y(jT) \\
E[F] &= \alpha E[I_0] \\
E[I] &= \sum_{j=1}^{M-1} \bar{F}_{G_N}(jT)(\sum_{r=0}^{j-1} \beta^{j-1-r}(F_Y((r+1)T) - F_Y(rT)) + \bar{F}_Y(jT))
\end{aligned}
$$

A cycle is completed whichever of the following events comes first:

- R_1: at MT,
- R_2: after N R-failures at random time G_N,
- R_3: when a U-failure is detected after $K_1 + L$ inspections

R_1 that is renewal at MT, can occur under two mutually exclusive events: the system presents an undetected U failure (denoted by R_4) or the system is free of any U-failure (denoted by R_5). It follows that

$$
\begin{aligned}
R_4 &= \bigcup_{j=0}^{M-1} \{K_1 = j, L \geq M - j, G_N > MT\} \\
R_5 &= \{K_1 \geq M, G_N > MT\} \\
R_1 &= R_4 \bigcup R_5 = \bigcup_{j=0}^{\infty} \{K_1 = j, K_1 + L \geq M, G_N > MT\}
\end{aligned}
$$

The probabilities of the foregoing events are given below:

$$P(R_4) = \bar{F}_{G_N}(MT)\sum_{j=0}^{M-1}(F_Y((j+1)T) - F_Y(jT))\beta^{M-j-1}.$$

$$P(R_5) = \bar{F}_{G_N}(MT)\bar{F}_Y(MT).$$

$$P(R_1) = \bar{F}_{G_N}(MT)\left(\sum_{j=0}^{M-1}(F_Y((j+1)T) - F_Y(jT))\beta^{M-j-1} + \bar{F}_Y(MT)\right)$$

$$P(R_2) = P(G_N \leq MT, G_N \leq (K_1+L)T) =$$
$$\sum_{j=0}^{M-1}\int_{jT}^{(j+1)T} f_{G_N}(x)P(K_1+L \geq j+1)dx =$$
$$\sum_{j=0}^{M-1}\int_{jT}^{(j+1)T}\left(\sum_{r=0}^{j}\beta^{j-r}(F_Y((r+1)T) - F_Y(rT)) + \bar{F}_Y((j+1)T)\right) f_{G_N}(x).$$

$$P(R_3) = P(K_1+L \leq M-1, (K_1+L)T \leq G_N) =$$
$$\sum_{j=1}^{M-1} P(K_1+L=j)\bar{F}_{G_N}(jT).$$

Observe that in the formula of $P(R_2)$ the expression of $P(K_1+L \geq j+1)$ is given in (4).

Next calculations provide the mean cost of replacement when a cycle is completed under any of the events denoted as R_1, R_2, R_3.

In what follows 1_A denotes the indicator function of event A.

Cost of replacement at MT:

$$CPM = c^1_{PM}1_{R_4} + c^2_{PM}1_{R_5}$$

$$E[CPM] = c^1_{PM}P(R_4) + c^2_{PM}P(R_5) = \tag{6}$$
$$c^1_{PM}\bar{F}_{G_N}(MT)\sum_{j=0}^{M-1}(F_Y((j+1)T) - F_Y(jT))\beta^{M-j-1} + c^2_{PM}\bar{F}_{G_N}(MT)\bar{F}_Y(MT)$$

Cost of replacement after the Nth failure of type R:

$$CPR_N = c_{r2}(N, G_N)1_{R_2} = c_{r2}(N, G_N)1_{G_N \leq MT, G_N \leq (K_1+L)T}$$

$$E[CPR_N] = \tag{7}$$
$$\int_0^{MT} f_{G_N}(x) c_{r2}(N,x) P\left(K_1 + L > \frac{x}{T}\right) dx =$$
$$\sum_{j=0}^{M-1} \int_{jT}^{(j+1)T} c_{r2}(N,x) f_{G_N}(x) P(K_1 + L \geq j+1) dx =$$
$$\sum_{j=0}^{M-1} \int_{jT}^{(j+1)T} \left(\sum_{r=0}^{j} \beta^{j-r} (F_Y((r+1)T) - F_Y(rT)) + \bar{F}_Y((j+1)T) \right) c_{r2}(N,x) f_{G_N}(x) dx$$

Cost of replacement when a U-failure is detected:

$$CPR_Y = c_{r1} 1_{R_3}$$

$$E[CPR_Y] = c_{r1} P(R_3) = c_{r1} \sum_{j=1}^{M-1} P(K_1 + L = j) \bar{F}_{G_N}(jT) \tag{8}$$

The following result provides the expected cost derived from minimal repairs in a cycle. The corresponding proof can be found in the Appendix.

Proposition 4 *The expected cost incurred in a cycle due to minimal repairs,* $E[CMR]$, *is given by*

$$E[CMR] = \tag{9}$$
$$\sum_{j=1}^{M-1} P(K_1 + L = j) \sum_{i=1}^{N-1} \int_0^{jT} c_{mr,i}(x) p(x) r(x) \frac{H_R(x)^{i-1}}{(i-1)!} e^{-H_R(x)} dx$$
$$+ P(K_1 + L \geq M) \sum_{i=1}^{N-1} \int_0^{MT} c_{mr,i}(x) p(x) r(x) \frac{H_R(x)^{i-1}}{(i-1)!} e^{-H_R(x)} dx$$

with $P(K_1 + L = j)$ *in (3) and* $P(K_1 + L \geq M)$ *in (4).*

Let D denote the downtime in a cycle. The downtime occurs once a U-failure occurs until it is detected. It follows that

$$D = \tau - \min(Y, G_N, MT)$$

and the expected downtime

$$E[D] = E[\tau] - \int_0^{MT} \bar{F}_Y(x)\bar{F}_{G_N}(x)dx \tag{10}$$

The expected cost of a cycle $E[C(\tau)]$ is obtained by summing the cost due to inspections, false alarms, minimal repairs, preventive maintenance at MT and after N R-failures, corrective maintenance when a U-failure is detected and downtime cost incurred while a U-failure remains undetected. The corresponding formula follows from expressions of $E[I]$ and $E[F]$ in Proposition 3, $E[CMR]$ in Proposition 4 along with $E[CPM]$, $E[CPR_N]$, $E[CPR_Y]$, and $E[D]$ in (6), (7), (8) and (10), respectively. Thus

$$E[C(\tau)] = c_1E[I]+c_fE[F]+E[CPM]+E[CPR_N]+E[CPR_Y]+E[CMR]+c_dE[D]$$

The cost per unit of time is objective function which in the long-run is given by

$$Q(T, M, N) = \frac{E[C(\tau)]}{E[\tau]}$$

2.3 Analysis of the cost function. Optimum policies

Regarding the three decision variables in the model (T, M, N), this section focuses on the existence of an optimum value for any of the three when the other two are given. The following results provide sufficient conditions guaranteing that such an optimum exists. The corresponding proofs are given in the Appendix.

We first analyze the existence $T^\star$ when both M and N are given.

Theorem 1 *Let V_N defined as follows*

$$\begin{aligned} V_N &= \frac{1}{c_d}\left(\int_0^\infty c_{r2}(N,x) f_{G_N}(x)dx + \sum_{i=1}^{N-1}\int_0^\infty c_{mr,i}(x)p(x)r(x)\frac{H_R^{i-1}(x)}{(i-1)!}e^{-H_R(x)}\right) \\ &- \int_0^\infty \overline{F}_Y(x)\overline{F}_{G_N}(x)dx \end{aligned}$$

Assume that M and N are given values. If the following conditions hold: $V_N < 0$, $\lim_{t\to\infty} p(t)r(t) = \infty$, $\overline{F}_{G_N}(x) > 0$, $c_{mr,i}(x) > 0$ and $c_{mr,i}(x)$ increasing in x, then there exists $T^*_{M,N}$, $(0 < T^*_{M,N} < \infty)$ minimizing $Q(T, M, N)$.

The first two terms in V_N represent the ratio of the mean cost incurred due to revealed failures to the unitary downtime cost. In addition, the third term corresponds to the expectation of the minimum of both Y and G_N, that is, the mean time to the first catastrophic failure or the N minor failure whichever comes first. Condition $V_N < 0$ means that if the mean time to replacement is larger than a function of the costs, then it is worth inspecting the system to detect catastrophic failures. The larger c_d, the more profitable inspections are.

Next, we define the auxiliary functions $V_{T,M}$ and $V_{T,N}$. The former depends on T and M, and the latter on T and N.

$$
\begin{aligned}
&V_{T,M} \qquad (11)\\
= \;& \frac{c_1}{c_d}\sum_{j=1}^{M-1}\left(\sum_{r=0}^{j-1}\beta^{j-1-r}[F_Y((r+1)T)-F_Y(rT)]+\overline{F}_Y(jT)\right)+\frac{c_f}{c_d}\alpha\sum_{j=1}^{M-1}\overline{F}_Y(jT)\\
+ \;& \frac{c_{r1}}{c_d}P(K_1+L\le M-1)+\frac{c_{PM}^1}{c_d}\sum_{j=0}^{M-1}\beta^{M-1-j}[F_Y((j+1)T)-F_Y(jT)]\\
+ \;& \frac{c_{PM}^2}{c_d}\overline{F}_Y(MT)+\sum_{j=1}^{M-1}P(K_1+L=j)\sum_{i=1}^{\infty}\int_0^{jT}\frac{c_{mr,i}(x)}{c_d}p(x)r(x)\frac{H_R^{i-1}(x)}{(i-1)!}e^{-H_R(x)}dx\\
+ \;& P(K_1+L\ge M)\sum_{i=1}^{\infty}\int_0^{MT}\frac{c_{mr,i}(x)}{c_d}p(x)r(x)\frac{H_R^{i-1}(x)}{(i-1)!}e^{-H_R(x)}dx-\int_0^{MT}\overline{F}_Y(x)dx
\end{aligned}
$$

and

$$
\begin{aligned}
&V_{T,N} \qquad (12)\\
= \;& \frac{c_1}{c_d}\sum_{j=1}^{\infty}\overline{F}_{G_N}(jT)\left(\sum_{r=0}^{j-1}\beta^{j-1-r}[F_Y((r+1)T)-F_Y(rT)]+\overline{F}_Y(jT)\right)\\
+ \;& \frac{c_f}{c_d}\alpha\sum_{j=1}^{\infty}\overline{F}_Y(jT)\overline{F}_{G_N}(jT)\\
+ \;& \sum_{j=1}^{\infty}\left(\sum_{r=0}^{j-1}\beta^{j-1-r}[F_Y((r+1)T)-F_Y(rT)]+\overline{F}_Y(jT)\right)\int_{jT}^{(j+1)T}c_{r2}(N,x)f_{G_N}(x)dx\\
+ \;& \frac{c_{r1}}{c_d}\sum_{j=0}^{\infty}P(K_1+L=j)\overline{F}_{G_N}(jT)\\
+ \;& \sum_{j=0}^{\infty}P(K_1+L=j)\sum_{i=1}^{N-1}\int_{=}^{jT}\frac{c_{mr,i}(x)}{c_d}p(x)r(x)\frac{H_R^{i-1}(x)}{(i-1)!}e^{-H_R(x)}dx\\
- \;& \int_0^{\infty}\overline{F}_Y(x)\overline{F}_{G_N}(x)dx.
\end{aligned}
$$

In what follows we analyze conditions for the existence of optimum values N^* and M^* when the other two decision variables are given.

Theorem 2 *The following results hold*

a) If $V_{T,M} < 0$ *for given values of* T *and* M*, and*

$$\lim_{N\to\infty} c_{r2}(N,x) = c < \infty \tag{13}$$

then there exists $N^*_{T,M} < \infty$ *such that* $\min_N Q(T,M,N) = Q(T,M,N^*_{T,M})$*.*

b) If $V_{T,N} < 0$ *for given values of* T *and* N*, then there exists* $M^*_{T,N} < \infty$ *verifying that* $\min_M Q(T,M,N) = Q(T,M^*_{T,N},N)$*.*

Mathematical expressions in Theorem 2 have also practical meaning. In case that the only preventive replacement occurs at MT, that is $N = \infty$, condition $V_{T,M} < 0$ in case a) provides a minimum value for the mean time to failure. If the expected system lifetime is greater than that minimum, then including an additional preventive replacement after N minor failures is an advantageous procedure if the corresponding cost is bounded as (13) states. Case b) accounts for the case when replacement after N minor failures constitutes the only preventive action, that is $M = \infty$. In this case replacement is determined by the minimum of both, the time to N minor failures and that of the catastrophic one. $V_{T,N} < 0$ gives a threshold for that minimum time so that a second preventive policy after M inspections is also profitable. Observe that this is so no matter what the costs of replacement at MT are (c^1_{PM} and c^2_{PM}). Conditions regarding V_N in Theorem 1, as well as $V_{T,M}$ and $V_{T,N}$ in Theorem 2 present the downtime cost, c_d, in the denominator of the corresponding expression. Thus, the higher c_d the more likely conditions become. Therefore the full policy consisting of inspections, preventive maintenance either at MT or after N minor failures is more advantageous when compared with a procedure based only on two decision variables. Numerical results in the following Section support this idea.

The complex expression of the cost function $Q(T, M, N)$ prevents from providing conditions that guarantee that there exists a unique global optimum all over the feasible space: $T \in \mathbb{R}^+$, $M, N \in \mathbb{N}$. Most papers under similar conditions use numerical methods to obtain a local optimum. This local optimum $(T^\star, M^\star, N^\star)$ constitutes the global one in the region of interest, that is $M \leq 100$ and $N \leq 100$. Larger values of M and N would lead to maintenance far beyond reasonable life lengths.

- set $i = 1$
- Step 1: set $N = i$
- Step 2: For $j = 1, 2, \ldots 100$, find $T^\star(j, i)$ such that

$$\min_T Q(T, j, i) = Q(T^\star(j, i), j, i)$$

- Step 3: Find $(T^\star(i), M^\star(i))$ such that

$$\min_{j=1\ldots100} Q(T^\star(j, i), j, i) = Q(T^\star(i), M^\star(i), i)$$

- Step 4: Set $i = i + 1$
- Step 5: If $i \leq 100$, go to Step 1, otherwise go to Step 6
- Step 6: Find $(T^\star, M^\star, N^\star)$ such that

$$\min_{i=1\ldots100} Q(T^\star(i), M^\star(i), i) = Q(T^\star, M^\star, N^\star)$$

A restricted version of this algorithm for optimization in the case of a bivariate policy (T, N) is presented in Nakagawa [16] as well as in Zequeira and Bérenguer [34].

3 Numerical examples

Next we develop an study of the dependency of the optimum policy, denoted as $(T^\star, M^\star, N^\star)$, on the parameters of the model. We will assume a failure rate $r(t) = 0.01t^2$ corresponding to a Weibull distribution with shape parameter equal to 3 and scale parameter equal to $300^{\frac{1}{3}}$. The probability that the failure that occurs at t is of the type R is $p(t) = \frac{1}{t+1}$ and the corresponding probability of being of the type U is $q(t) = \frac{t}{t+1}$. Thus, a failure is more likely to be catastrophic rather than minor as time goes by. The probabilities of a false positive and a false negative in the base case are assumed to be $\alpha = 0.05$ and $\beta = 0.1$. Concerning costs, the replacement at MT when the system is free of a U failure is assumed to be the reference value with $c^2_{PM} = 1$. The unitary cost of inspection is $c_1 = 0.001$ and the cost of a false alarm is $c_f = 0.05$. Replacement after N minor failures or after M inspections are illustrated by maintenance contracts including a warranty that reduces the cost if the Nth revealed failure implies an early replacement of the system or if there is an unacceptable number of minor failures in a short period of time. Thus we propose the following functions: $c_{r2}(N, t) = 1.5 + \frac{t}{N+1}$, $c_{mr,j}(t) = 0.5 + \frac{t}{j}$.

Concerning how to arrive at parameters of proposed models it is important to note that due to confidentiality reasons raw data are not usually available. In addition the time until an unrevealed failure occurs is not directly measurable because it happens sometime in-between two inspections. Wang [32] suggests subjective estimation of these parameters by means of expert opinion. The work of Si *et al* [28] presents a review on the remaining useful life of a system.

Table 1 shows the optimum policy $(T^\star, M^\star, N^\star)$ and the two suboptimal ones $(T_0^\star, M_0^\star, \infty)$ and $(\infty, -, N_0^\star)$ when the following parameters change: cost of replacement when a U-failure is detected on inspection, c_{r1}, cost of preventive maintenance at MT when the system presents an undetected U-failure, c_{PM}^1, and cost per unit of time while a U-failure remains undiscovered, c_d. When c_{r1} increases, less inspections $M^\star$ are recommended. In addition $N^\star$ is not decreasing and although $T^\star$ is non monotonic, there is an earlier preventive maintenance at $M^\star T^\star$ to avoid the consequences of a catastrophic failure. When c_{PM}^1 increases, so does $M^\star$ whereas $T^\star$ decreases. Neither $M^\star T^\star$ nor $N^\star$ are monotonic. When c_d decreases, inspection is relaxed and thus the number of inspections is reduced with a larger time interval between them. A similar behaviour is observed in $(T_0^\star, M_0^\star, \infty)$ under changes of c_{r1}, c_{PM}^1 and c_d whereas $N_0^\star$ remains equal to 1. The cost saving when the optimum policy $(T^\star, M^\star, N^\star)$ is compared with the two suboptimal ones $(T_0^\star, M_0^\star, \infty)$ and $(\infty, -, N_0^\star)$ is respectively given in $\Delta_0(\%)$ and $\Delta^0(\%)$. When comparing $\Delta_0(\%)$ and $\Delta^0(\%)$, $(T_0^\star, M_0^\star, \infty)$ is by far a better choice than $(\infty, -, N_0^\star)$ in those cases where $(T^\star, M^\star)$ are both finite. These cases match with medium to large values of c_d and are not appropriate to rely on opportunity-based replacement exclusively. Programmed replacement based on age produces a significant cost reduction. In addition $(\infty, -, N_0^\star)$ can be optimal for small values of c_d, avoiding over maintenance.

The replacement time after $N^\star$ minor failures is denoted by $t_{N^\star}$ in Table 1. In those cases where both preventive maintenances, after N minor failures or at MT are finite, the latter occurs significantly earlier. This is so because replacement after M inspections is specifically designed to prevent a catastrophic failure which is unrevealed. In those cases where c_d is small compared to c_{r1} and c_{PM}^1, the cost incurred while the failure remains undiscovered is minor, then inspections are less important and $T^\star = \infty$. The system will be replaced after $N^\star$ minor failures. Nevertheless it can be observed that $N^\star$ is smaller in those cases where $T^\star = \infty$ than that where both policies are finite. As in this case there is no inspection, the maintainer uses the N minor failure as an opportunity to prevent the catastrophic failure or to reduce the incurred cost if that failure has already occurred. Values of the ratio c_d/c^1_{PM} above a given threshold lead to maintenance at MT to be the preferable choice and $N^\star = \infty$. When the cost derived from an unrevealed failure, c_d, is high enough, then inspection becomes a more valuable tool for maintenance. As c_{r1} increases so does this threshold, reflecting that the economic advantage of replacing when a U-failure is detected, declines.

Table 2 presents the optimum policy under different values of the parameters related to inspection: inspection cost (c_1), cost of a false alarm (c_f), probability of false positive inspection (α), probability of false negative inspection (β). The rest of the parameters correspond to the base case along with $c_{r1} = 2.5$, $c^1_{PM} = 1.5$, and $c_d = 1.5$. When the costs of inspection or false alarm increase, the inspection frequency decreases. If the probability of a false positive inspection, α, increases, so does $T^\star$ whereas its behavior is just the opposite when β increases. The reason for this pattern is that the greater α, the more misleading the inspection and unnecessary costs are incurred. Thus, less inspection reduces this risk.

On the contrary the greater β, the less useful the inspection procedure to detect failures that remain unrevealed otherwise. If so, more frequent inspection increases the probability to detect a current failure. Two additional characteristics can be observed in Table 2: the time for preventive replacement, MT, is robust to changes in the inspection parameters. When there is a huge cost derived from a catastrophic failure, it is worth carrying out a preventive replacement when the system reaches certain age. This is so even when the inspection procedure is not perfect or the associated costs are significant. The second feature is that the number of minor failures before replacement, $N^{\star}$ is not affected by changes as a consequence that minor failures are revealed and therefore they are not concerned with inspection.

Table 1: The optimal (T^*, M^*, N^*), $Q(T^*, M^*, N^*)$, (T_0^*, M_0^*), $Q_0(T_0^*, M_0^*)$, N_0^*, $Q^0(N_0^*)$ for different c_{r1}, c_{PM}^1 and c_d (α=0.05, β=0.1, c_1=0.001, c_f=0.05 and c_{PM}^2=1 fixed)

c_{r1}	c_{PM}^1	c_d	T^*	M^*	N^*	M^*T^*	t_{N^*}	$Q(T^*,M^*,N^*)$	T_0^*	M_0^*	$Q_0(T_0^*,M_0^*)$	$\triangle_0(\%)$	N_0^*	$Q^0(N_0^*)$	$\triangle^0(\%)$
2.5	1.5	2.0	0.658	6	∞	3.948	∞	0.369	0.658	6	0.369	0	1	1.104	66.58
		1.5	1.326	3	2	3.978	19.66	0.360	1.319	3	0.361	0.28	1	0.828	56.52
		1.0	2.063	2	2	4.126	19.66	0.347	2.050	2	0.348	0.29	1	0.552	37.14
		0.7	2.149	2	2	4.298	19.66	0.337	2.131	2	0.338	0.30	1	0.387	12.92
		0.5	∞	-	1	∞	13.33	0.276	2.197	2	0.331	16.62	1	0.276	0
		0.1	∞	-	1	∞	13.33	0.056	2.379	2	0.314	82.17	1	0.056	0
	2.0	2.0	0.440	9	∞	3.960	∞	0.376	0.440	9	0.376	0	1	1.104	65.94
		1.5	0.570	7	∞	3.990	∞	0.371	0.570	7	0.371	0	1	0.828	55.19
		1.0	1.338	3	2	4.014	19.66	0.363	1.330	3	0.363	0.27	1	0.552	34.24
		0.7	2.048	2	2	4.096	19.66	0.355	2.034	2	0.356	0.28	1	0.387	8.27
		0.5	∞	-	1	∞	13.33	0.276	2.088	2	0.350	21.14	1	0.276	0
		0.1	∞	-	1	∞	13.33	0.056	2.228	2	0.336	83.33	1	0.056	0
	3.0	2.0	0.266	15	∞	3.990	∞	0.386	0.266	15	0.386	0	1	1.104	65.04
		1.5	0.308	13	∞	4.004	∞	0.382	0.308	13	0.382	0	1	0.828	53.86
		1.0	0.366	11	∞	4.026	∞	0.379	0.366	11	0.379	0	1	0.552	31.34
		0.7	0.405	10	∞	4.050	∞	0.376	0.405	10	0.376	0	1	0.387	2.84
		0.5	∞	-	1	∞	13.33	0.276	0.506	8	0.375	26.40	1	0.276	0
		0.1	∞	-	1	∞	13.33	0.056	0.683	6	0.370	84.86	1	0.056	0
3.5	1.5	2.0	1.859	2	2	3.178	19.66	0.376	1.852	2	0.377	0.27	1	1.104	65.94
		1.5	1.937	2	2	3.874	19.66	0.364	1.928	2	0.365	0.27	1	0.828	56.04
		1.0	2.040	2	2	4.080	19.66	0.350	2.027	2	0.352	0.57	1	0.552	36.59
		0.7	2.120	2	2	4.240	19.66	0.341	2.104	2	0.342	0.29	1	0.387	11.89
		0.5	∞	-	1	∞	13.33	0.276	2.165	2	0.335	17.61	1	0.276	0
		0.1	∞	-	1	∞	13.33	0.056	2.331	2	0.319	82.45	1	0.056	0
	2.0	2.0	1.803	2	2	3.606	19.66	0.390	1.214	3	0.390	0	1	1.104	64.67
		1.5	1.872	2	2	3.744	19.66	0.379	1.864	2	0.380	0.26	1	0.828	54.23
		1.0	1.959	2	2	3.918	19.66	0.367	1.948	2	0.368	0.27	1	0.552	33.51
		0.7	2.025	2	2	4.050	19.66	0.359	2.011	2	0.360	0.28	1	0.387	7.24
		0.5	∞	-	1	∞	13.33	0.267	2.061	2	0.354	22.03	1	0.276	0
		0.1	∞	-	1	∞	13.33	0.056	2.191	2	0.340	85.53	1	0.056	0
	3.0	2.0	0.514	7	∞	3.598	∞	0.409	0.514	7	0.409	0	1	1.104	62.95
		1.5	0.724	5	∞	3.620	∞	0.404	0.724	5	0.404	0	1	0.828	51.21
		1.0	1.823	2	2	3.646	19.66	0.396	1.221	3	0.397	0.25	1	0.552	28.26
		0.7	∞	-	1	∞	13.33	0.387	1.860	2	0.391	1.02	1	0.387	0
		0.5	∞	-	1	∞	13.33	0.276	1.894	2	0.386	28.50	1	0.276	0
		0.1	∞	-	1	∞	13.33	0.056	1.976	2	0.376	85.11	1	0.056	0
4.5	1.5	2.0	1.845	2	2	3.690	19.66	0.379	1.838	2	0.380	0.26	1	1.104	65.67
		1.5	1.920	2	2	3.840	19.66	0.367	1.912	2	0.368	0.27	1	0.828	55.68
		1.0	2.006	2	6	4.012	34.83	0.356	2.006	2	0.356	0	1	0.552	35.69
		0.7	2.078	2	7	4.156	37.67	0.346	2.078	2	0.346	0	1	0.387	10.59
		0.5	∞	-	1	∞	13.33	0.276	2.136	2	0.3	18.82	1	0.276	0
		0.1	∞	-	1	∞	13.33	0.056	2.288	2	0.324	82.72	1	0.056	0
	2.0	2.0	1.791	2	2	3.582	19.66	0.393	1.785	2	0.393	0	1	1.104	64.40
		1.5	1.857	2	2	3.714	19.66	0.382	1.849	2	0.383	0.26	1	0.828	53.86
		1.0	1.930	2	6	3.860	34.83	0.371	1.930	2	0.371	0	1	0.552	32.79
		0.7	1.990	2	7	3.980	37.67	0.363	1.990	2	0.363	0	1	0.387	6.20
		0.5	∞	-	1	∞	13.33	0.276	2.036	2	0.357	22.69	1	0.276	0
		0.1	∞	-	1	∞	13.33	0.056	2.157	2	0.344	83.72	1	0.056	0
	3.0	2.0	1.697	2	2	3.394	19.66	0.418	1.692	2	0.418	0	1	1.104	62.14
		1.5	1.747	2	2	3.494	19.66	0.409	1.742	2	0.409	0	1	0.828	50.60
		1.0	1.802	2	6	3.604	34.83	0.400	1.802	2	0.400	0	1	0.552	27.54
		0.7	∞	-	1	∞	13.33	0.387	1.844	2	0.394	1.78	1	0.387	0.0
		0.5	∞	-	1	∞	13.33	0.276	1.877	2	0.389	29.05	1	0.276	0
		0.1	∞	-	1	∞	13.33	0.056	1.954	2	0.379	85.22	1	0.056	0

Table 2: Dependency of the optimum policy (T^*, M^*, N^*) on the inspection parameters c_1, c_f, α, and β

c_1	c_f	α	β	T^*	M^*	N^*	M^*T^*	$Q(T^*, M^*, N^*)$
0.001	0.05	0.05	0.1	1.326	3	2	3.978	0.360
0.005				1.329	3	2	3.987	0.362
0.010				1.960	2	2	3.920	0.364
0.015				1.963	2	2	3.926	0.366
0.020				1.966	2	2	3.932	0.359
0.001	0.01	0.05	0.1	1.324	3	2	3.972	0.359
	0.05			1.326	3	2	3.978	0.360
	0.10			1.328	3	2	3.984	0.361
	0.15			1.958	2	2	3.916	0.362
	0.20			1.959	2	2	3.918	0.363
0.001	0.05	0.01	0.1	1.324	3	2	3.972	0.359
		0.05		1.326	3	2	3.978	0.360
		0.10		1.328	3	2	3.984	0.361
		0.15		1.958	2	2	3.916	0.362
		0.20		1.959	2	2	3.918	0.363
0.001	0.05	0.01	0.01	1.331	3	2	3.994	0.359
			0.05	1.329	3	2	3.987	0.359
			0.10	1.326	3	2	3.978	0.360
			0.15	1.323	3	2	3.969	0.360
			0.20	1.320	3	2	3.960	0.361

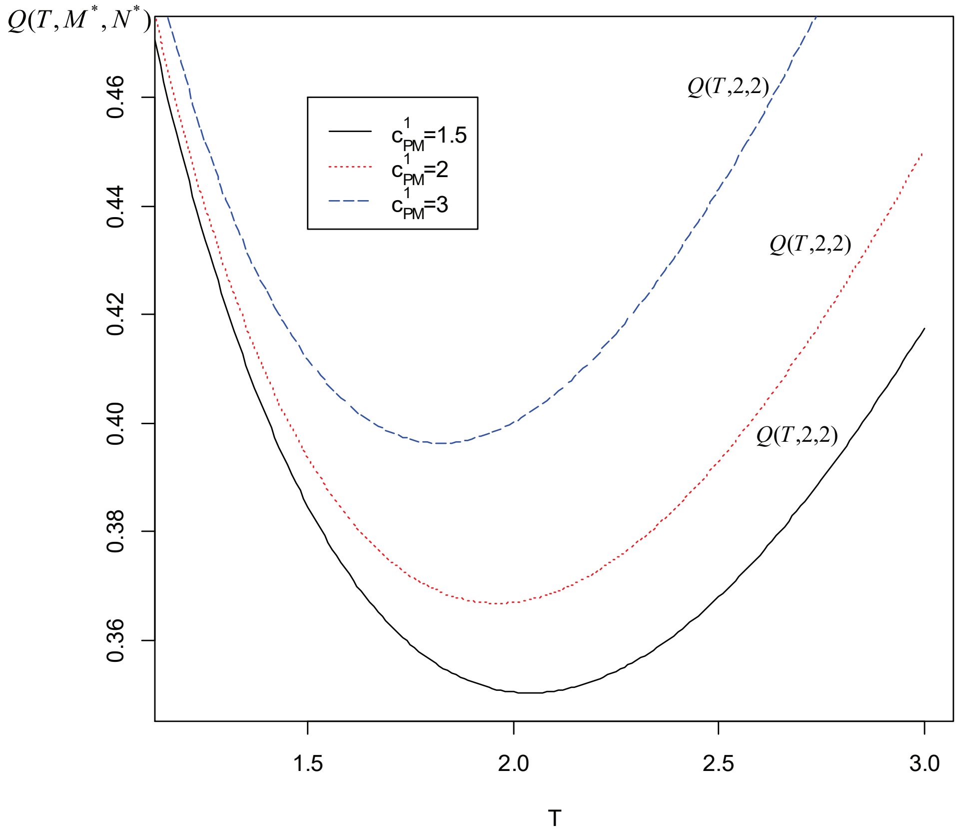

Figure 2: Cost function versus T under different costs of preventive replacement.

Figure 2 represents the cost function $Q(T, M^\star, N^\star)$ versus T with $c_{r1} = 3.5$ and $c_d = 1$ for different costs of preventive maintenance at MT when the system presents an undetected U failure. $T^\star$ decreases with c^1_{PM} whereas $(M^\star, N^\star)$ are robust. More frequent inspections increase the probability of replacement on inspection rather than preventively at MT.

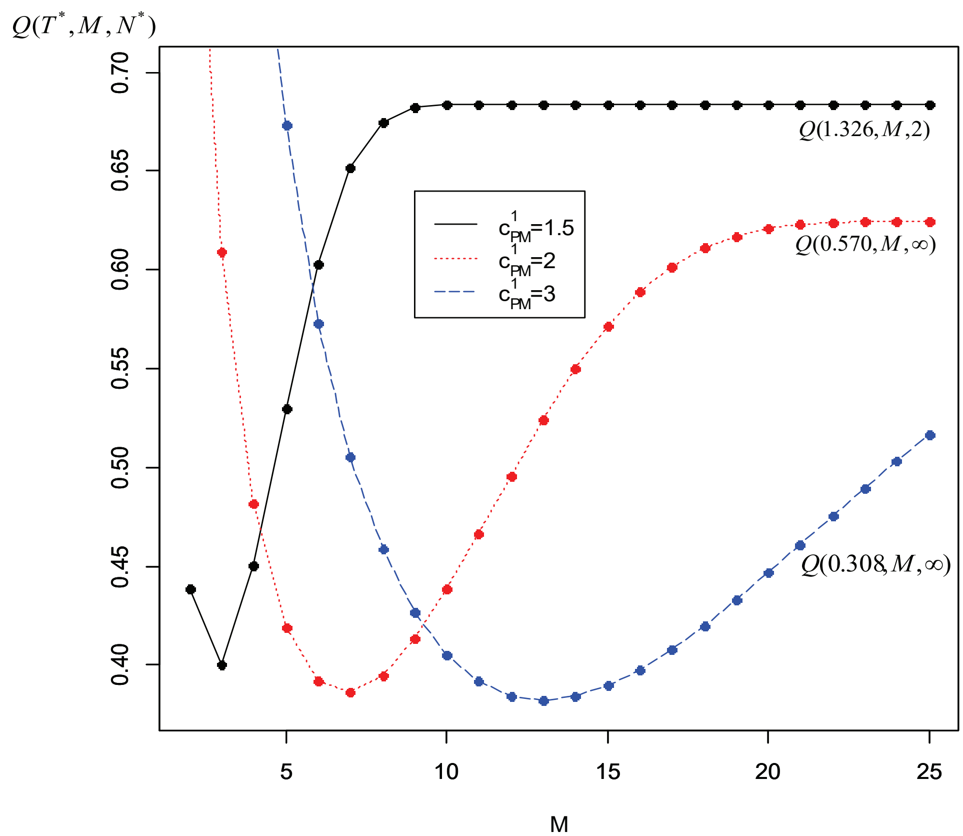

Figure 3: Cost function versus M under different costs of preventive replacement.

Figure 3 contains a similar description for $Q(T, M, N)$ versus M with $c_{r1} = 2.5$, $c_d = 1.5$ for different values of c^1_{PM} and the corresponding optimum $(T^\star, N^\star)$. $M^\star$ increases with c^1_{PM}. According to Table 1, MT is robust although an increasing number of inspections makes more likely to replace on inspection.

The comparison of the optimal policy $(T^\star, M^\star, N^\star)$ with the suboptimal ones $(T^\star_0, M^\star_0, \infty)$ and $(\infty, -, N^\star_0)$ is analyzed in Figure 4. The optimum costs $Q(T^\star, M^\star, N^\star)$, $Q(T^\star, M^\star, \infty) = Q_0(T^\star_0, M^\star_0)$ and $Q(\infty, -, N^\star_0) = Q^0(N^\star_0)$ are represented for different values of the downtime cost incurred while a U failure remains undetected.

The superiority achieved by the optimal policy $(T^\star, M^\star, N^\star)$, is observed when c_d ranges from 0.5 to 0.7. However this cost advantage vanishes out of this interval. Thus, when c_d is low, $Q(T^\star, M^\star, N^\star)$ is similar to $Q^0(N_0^\star)$, indicating that the maintainer can be confident that replacement after the N-th R-failure is enough preventive maintenance. When c_d increases, the situation is reversed and detection of U failures becomes more crucial for not incurring high costs. Thus, inspections are more important and so does replacement after M inspections. The consequence is that $Q_0(T_0^\star, M_0^\star)$ approaches to the optimum, implying that replacement after M inspections can be enough maintenance.

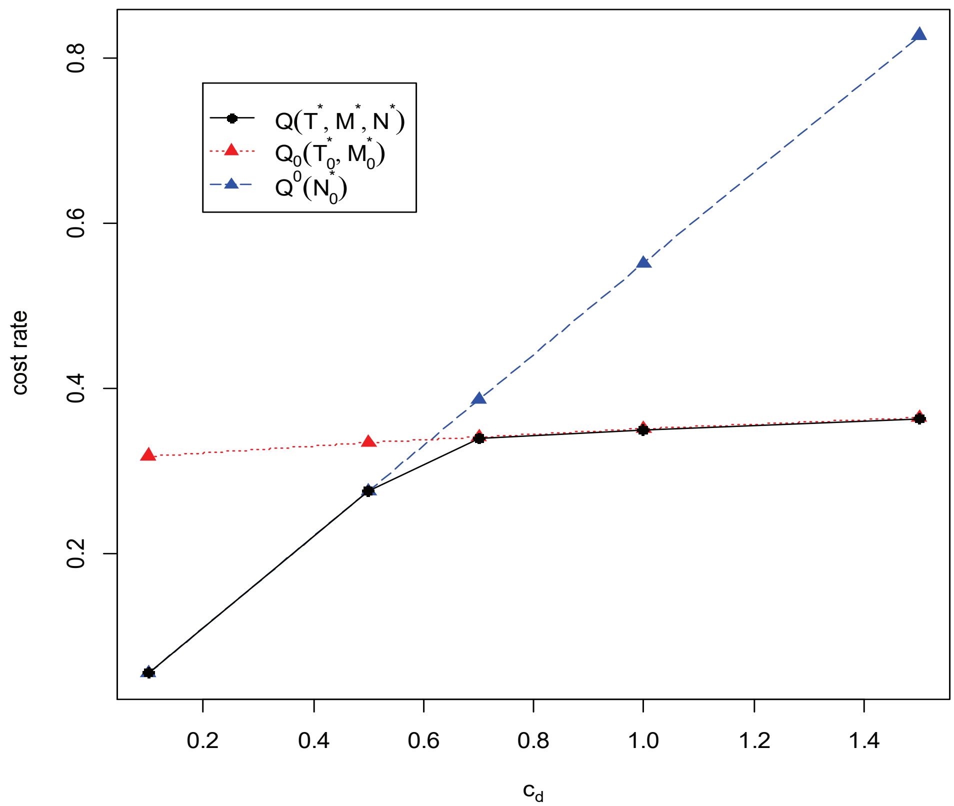

Figure 4: Comparison of the optimal and suboptimal policies for different values of c_d.

4 Conclusion

We consider a one component system that undergoes both minor revealed failures and revealed catastrophic failures. We propose an inspection procedure every T units of time to detect the latter and a preventive maintenance policy to reduce the risk that they occur. The preventive maintenance is carried out after M inspections to account for both age and use that make the system more prone to experience a catastrophic failure. Warranties and maintenance contracts inspire the alternative system replacement after N minor failures. The costs are assumed to depend on both age and history of failures to account for maintenance contracts. The objective is to obtain the cost-minimizing policy $(T^{\star}, M^{\star}, N^{\star})$. The analysis of conditions for the existence of optimum policies provide minimum values for the reliability of the system so that a policy only based on two decision variables can be improved introducing the third one. Both, formulae and numerical results show that increasing values of the downtime cost, c_d, make $(T^{\star}, M^{\star}, N^{\star})$ be a better choice than the policies with only one type of preventive maintenance. Numerical results also indicate that neither the time for preventive replacement at MT or after N failures are affected by changes in the parameters related to inspection. Thus, even the additional costs derived from non-perfect inspections can be compensated, preventing the system from failing. Furthermore the model explores conditions of the parameters that make one of the two alternatives for preventive maintenance preferable to the other, revealing that the value of c_d is crucial to choose between them. For small values of c_d maintenance exclusively based on opportunities can be a suitable option although it implies much higher costs for medium to large values of c_d. All this results can serve as a guide for maintainers. Following the ideas in

[30] it would be interesting to consider that minor and catastrophic failures are dependent and the probability of the latter increases every time that a minor failure occurs. In addition future research could focus on non-constant error probabilities, assuming that a decreasing function of $\beta(t)$ as unrevealed catastrophic failures accumulate. Both issues are worthy of forthcoming research.

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Appendix

Proof of Proposition 2:

The system is replaced after a U-failure is detected, or once the N^{th} R-failure occurs or at MT whichever comes first. Therefore

$$\tau = \min(G_N, (K_1 + L)T, MT)$$

Let's $\bar{F}_{G_N \wedge (K_1+L)T}(x)$ denote the reliability function of the random variable $\min(G_N, (K_1 + L)T)$. Then, for $jT < x < (j + 1)T$ and $j = 1, \ldots, M - 1$ it follows that

$$\bar{F}_{G_N \wedge (K_1+L)T}(x) = \bar{F}_{G_N}(x) P(K_1 + L \geq \left\lfloor \frac{x}{T} \right\rfloor + 1) = \bar{F}_{G_N}(x) P(K_1 + L \geq j + 1),$$

From (11) and (12) we obtain

$$E[\tau] = \int_0^{MT} \bar{F}_{G_N \wedge (K_1+L)T}(x) dx = \sum_{j=0}^{M-1} P(K_1 + L \geq j + 1) \int_{jT}^{(j+1)T} \bar{F}_{G_N}(x) dx$$

and the expression in (5) is obtained after substituting $P(K_1 + L \geq j + 1)$ by (4).

Proof of Proposition 3:

The random variable I_0 is given by

$$I_0 = \min(S_N, K_1, (M-1))$$

where $S_N = \left\lfloor \frac{G_N}{T} \right\rfloor$.

$$E[I_0] = \sum_{j=1}^{M-1} P(S_N \geq j) P(K_1 \geq j)) = \sum_{j=1}^{M-1} \bar{F}_{G_N}(jT) \bar{F}_Y(jT)$$

The number of false positives in a cycle given I_0, $F|I_0$, follows a binomial distribution $B(I_0, \alpha)$, thus

$$E[F] = E[E[F|I_0]] = \alpha E[I_0]$$

A cycle is completed whichever of the following events comes first:

- R_1: at MT,
- R_2: after N R-failures at random time G_N,
- R_3: when a U-failure is detected after $K_1 + L$ inspections

Hence, the number of inspections in a cycle, I, is

$$I = \min(M-1, S_N, (K_1+L))$$

Thus

$$\begin{aligned} E[I] &= \sum_{j=1}^{M-1} P(S_N \geq j) P(K_1 + L \geq j) \\ &= \sum_{j=1}^{M-1} \bar{F}_{G_N}(jT) (\sum_{r=0}^{j-1} \beta^{j-1-r} (F_Y((r+1)T) - F_Y(rT)) + \bar{F}_Y(jT)) \end{aligned}$$

Proof of Proposition 4:

The maximum number of minimal repairs in a cycle is $N-1$. Hence the cost derived from minimal repairs in a cycle is:

$$CMR = \sum_{i=1}^{\min(N[0,\tau],N-1)} c_{mr,i}(G_i)$$

Moreover

$$\tau = G_N \Longleftrightarrow N[0,\tau] = N$$

$$\tau < G_N \Longleftrightarrow N[0,\tau] = N[0, \min((K_1+L)T, MT)] \leq N-1$$

That is, there will be exactly $N-1$ minimal repairs if event R_2 occurs and $N-1$ at most if any of the events, R_1 or R_3, takes place.

$$\begin{aligned} CMR &= CMR1_{R_2} + CMR1_{\{\tau < G_N\}} = \\ & \sum_{i=1}^{N-1} c_{mr,i}(G_i) 1_{R_2} + \sum_{i=1}^{N[0,\min((K_1+L)T,MT)]} c_{mr,i}(G_i) 1_{\{N[0,\min((K_1+L)T,MT)] \leq N-1\}} \end{aligned}$$

Given $N[0,t] = k$, $G_1, \ldots, G_k$ are the ordered statistics of a random variable with pdf $\frac{p(x)r(x)}{H_R(t)}$, $0 \leq x \leq t$ ([10]). Moreover the density function of

G_i given $N[0,t] = k$ is as follows:

$$f_{G_i|N[0,t]=k}(x) = \frac{k!}{(i-1)!(k-i)!} \frac{p(x)r(x)}{H_R(t)} \left(\frac{H_R(x)}{H_R(t)}\right)^{i-1} \left(1 - \frac{H_R(x)}{H_R(t)}\right)^{k-i}$$

The cost of the minimal repair after the i^{th} R-failure, $c_{mr,i}(x)$, depends on both the number i and the time it occurs, x. Thus, we first obtain the expected cost derived from minimal repair in a cycle conditional to $\min((K_1 + L)T, MT) = y$.

The probability that the number of R-failures in $(0, y)$ is equal to k:

$$\frac{H_R(y)^k}{k!} e^{-H_R(y)}.$$

Therefore,

$$\begin{aligned}
& E(CMR|(K_1+L)T \wedge MT = y) = \\
= \quad & \sum_{k=0}^{N-1} \frac{H_R(y)^k}{k!} e^{-H_R(y)} \\
\times \quad & \sum_{i=1}^{k} \int_0^y c_{mr,i}(x) \frac{k!}{(i-1)!(k-i)!} \frac{p(x)r(x)}{H_R(y)} \left(\frac{H_R(x)}{H_R(y)}\right)^{i-1} \left(1 - \frac{H_R(x)}{H_R(y)}\right)^{k-i} dx \\
+ \quad & \sum_{k=N}^{\infty} \frac{H_R(y)^k}{k!} e^{-H_R(y)} \\
\times \quad & \sum_{i=1}^{N-1} \int_0^y c_{mr,i}(x) \frac{k!}{(i-1)!(k-i)!} \frac{p(x)r(x)}{H_R(y)} \left(\frac{H_R(x)}{H_R(y)}\right)^{i-1} \left(1 - \frac{H_R(x)}{H_R(y)}\right)^{k-i} dx
\end{aligned}$$

The first term of the foregoing sum represents the case where $k \leq N-1$, that is $N-1$ R-failures occur at most in $(0, y)$. The number of minimal repairs will be equal to k. In the second term the number of R-failures is greater than or equal to N and there will be exactly $N-1$ minimal repairs.

Changing the order of summation we obtain

$$
\begin{aligned}
& E(CMR|(K_1+L)T \wedge MT = y) = \\
= \;& \sum_{i=1}^{N-1}\sum_{k=i}^{\infty} \frac{H_R(y)^k}{k!} e^{-H_R(y)} \\
\times \;& \int_0^y c_{mr,i}(x) \frac{k!}{(i-1)!(k-i)!} \frac{p(x)r(x)}{H_R(y)} \left(\frac{H_R(x)}{H_R(y)}\right)^{i-1} \left(1-\frac{H_R(x)}{H_R(y)}\right)^{k-i} dx \\
= \;& \sum_{i=1}^{N-1} \int_0^y c_{mr,i}(x)p(x)r(x)\frac{H_R(x)^{i-1}}{(i-1)!} e^{-H_R(x)} dx
\end{aligned}
$$

Observe that $p(x)r(x)\frac{H_R(x)^{i-1}}{(i-1)!}e^{-H_R(x)}$ represents the density function of the time to failure to the i^{th} R-failure.

The unconditional expectation is obtained considering that $\min((K_1+L)T, MT) = (K_1+L)T$ or $\min((K_1+L)T, MT) = MT$:

$$
\begin{aligned}
& E[CMR] = \\
& \sum_{j=1}^{M-1} P(K_1+L=j) \sum_{i=1}^{N-1} \int_0^{jT} c_{mr,i}(x)p(x)r(x)\frac{H_R(x)^{i-1}}{(i-1)!}e^{-H_R(x)}dx \\
& + P(K_1+L \geq M) \sum_{i=1}^{N-1} \int_0^{MT} c_{mr,i}(x)p(x)r(x)\frac{H_R(x)^{i-1}}{(i-1)!}e^{-H_R(x)}dx
\end{aligned}
$$

And the result in (9) follows.

Proof of Theorem 1:

Straightforward algebra in the formula leads to the following limits

$$
\begin{aligned}
\lim_{T\to 0} E[\tau(T)] &= 0, \\
\lim_{T\to 0} E[C(\tau)] &= (M-1)(c_I + c_f\alpha) + c_{PM}^1 > 0, \\
\lim_{T\to\infty} E[\tau(T)] &= \int_0^{\infty} \overline{F}_{G_N}(x)dx = E[G_N], \qquad (14)
\end{aligned}
$$

$$\lim_{T\to\infty} E[C(\tau)] = c_d(E[G_N] + V_N),$$

$$\lim_{T\to 0} Q(T,M,N) = \infty \tag{15}$$

and

$$\lim_{T\to\infty} Q(T,M,N) = c_d + \frac{V_N c_d}{E[G_N]} \tag{16}$$

Let $A(T,M,N) = \frac{Q(T,M,N)-c_d}{c_d}$

$$Q(T,M,N) = c_d + c_d A(T,M,N), \tag{17}$$

$$\lim_{T\to 0} E[\tau(T)]A(T,M,N) = \frac{1}{c_d}\left((M-1)(c_I + c_f\alpha) + c_{PM}^1\right) > 0,$$

and

$$\lim_{T\to\infty} E[\tau(T)]A(T,M,N) = V_N = \lim_{T\to\infty} B(T) \tag{18}$$

with $B(T)$ given by

$$B(T) =$$
$$\frac{1}{c_d}\left(\int_0^T c_{r2}(N,x)f_{G_N}(x)dx + \sum_{i=1}^{N-1}\int_0^T c_{mr,i}(x)p(x)r(x)\frac{H_R^{i-1}(x)}{(i-1)!}e^{-H_R(x)}\right)$$
$$-\int_0^T \overline{F}_Y(x)\overline{F}_{G_N}(x)dx$$

and its derivative

$$B'(T) = \frac{1}{c_d}\left(c_{r2}(N,T)f_{G_N}(T) + \sum_{i=1}^{N-1} c_{mr,i}(T)p(T)r(T)\frac{H_R^{i-1}(T)}{(i-1)!}e^{-H_R(T)}\right) - \overline{F}_Y(T)\overline{F}_{G_N}(T)$$

which verifies the following inequality

$$B'(T) \geq \overline{F}_{G_N}(T)\left(\frac{\min_i c_{mr,i}(0)}{c_d}p(T)r(T) - \overline{F}_Y(T)\right) > 0, \quad T > T_0$$

The previous inequality is derived from the assumptions $\lim_{t\to\infty} p(t)r(t) = \infty$, $\overline{F}_{G_N}(T) > 0$ and $c_{mr,i}(T) \geq \min_i c_{mr,i}(0)$. Moreover the following limit

$\lim_{T\to\infty} B'(T) = \infty$ holds and thus the existence of $T_0 > 0$ such that $B(T)$ is increasing for $T > T_0$ is concluded.

Then from (18), the assumption $V_N < 0$ and the monotony of $B(T)$ for $T > T_0$, it follows that there exists an increasing sequence $\{T_n : n = 1, 2, \ldots\}$ with $T_1 > T_0$ and $\lim_{n\to\infty} T_n = \infty$ such that

$$E[\tau(T_n)]A(T_n, M, N) < V_N - \frac{1}{n} \leq 0, \quad n = 1, 2, \ldots$$

and by (14)

$$E[\tau(T_n)] \leq E[G_N] + \frac{1}{n}, \quad n = 1, 2, \ldots$$

Thus, applying last two inequalities on equation (17)

$$Q(T_n, M, N) < c_d + \frac{(V_N - \frac{1}{n})c_d}{E[G_N] + \frac{1}{n}} = c_d + c_d R(V_N, E[G_N], n), \quad n = 1, 2, \ldots \tag{19}$$

with

$$R(V_N, E[G_N], n) = \frac{V_N - \frac{1}{n}}{E[G_N] + \frac{1}{n}}$$

In addition

$$\begin{aligned}
&E[G_N] + V_N = \\
&\frac{1}{c_d}\left(\int_0^\infty c_{r2}(N, x) f_{G_N}(x)dx + \sum_{i=1}^{N-1}\int_0^\infty c_{mr,i}(x)p(x)r(x)\frac{H_R^{i-1}(x)}{(i-1)!}e^{-H_R(x)}\right) + \\
&\int_0^\infty \overline{F}_{G_N}(x)\left(1 - \overline{F}_Y(x)\right)dx
\end{aligned}$$

$E[G_N] + V_N \geq 0$, implying that

$$R(V_N, E[G_N], n) \leq R(V_N, E[G_N], n+1), \quad n = 1, 2, \ldots$$

Then, from (16) and (19) the condition below holds

$$Q(T_1, M, N) < c_d + c_d \lim_{n\to\infty} R(V_N, E[G_N], n+1) = c_d + \frac{c_d V_N}{E[G_N]} = Q(\infty, M, N)$$

The previous inequality along with (15) lead to

$$Q(T_1, M, N) < Q(\infty, M, N) < \infty = \lim_{T\to 0} Q(T, M, N)$$

$Q(T, M, N)$ is continuous in $(0, \infty)$ and hence there exists $T^*_{M,N}$ $(0 < T^*_{M,N} < \infty)$ minimizing $Q(T, M, N)$.

Proof of Theorem 2a:

Let us consider the following auxiliary function

$$O(T, M, N) = E[\tau]\frac{Q(T, M, N) - c_d}{c_d}.$$

Then $Q(T, M, N)$ can be **alternatively** written

$$Q(T, M, N) = c_d + c_d\frac{O(T, M, N)}{E[\tau]}.$$

Next equations provide the asymptotic behavior of $E[\tau]$ for N and M.

$$\begin{aligned}
\lim_{N\to\infty} E[\tau] = C_{T,M} & & (20)\\
&= T\sum_{j=0}^{M-1}\left(\sum_{r=0}^{j}\beta^{j-r}[F_Y((r+1)T) - F_Y(rT)] + \overline{F}_Y((j+1)T)\right)\\
&= E[\min((K_1 + L)T, MT)] \geq E[\min(Y, MT)] & (21)\\
\lim_{M\to\infty} E[\tau] = C_{T,N} & & (22)\\
&= T\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\beta^{j-r}[F_Y((r+1)T) - F_Y(rT)] + \overline{F}_Y((j+1)T)\right)\int_{jT}^{(j+1)T}\overline{F}_{G_N}(x)dx\\
&= E[\min((K_1 + L)T, G_N)] \geq E[\min(Y, G_N)] & (23)
\end{aligned}$$

K_1 and L in the previous expressions denote, respectively, the number of inspections before the catastrophic failure and after it occurs until its detection. Both inequalities are derived since $T\left(\left[\frac{Y}{T}\right] + 1\right) \geq Y$ leads to $(K_1+L)T \geq_{\text{st}} Y$ where st denotes the usual stochastic order.

Moreover

$$\begin{aligned}
&\sum_{j=0}^{M-1}\sum_{r=0}^{j}\beta^{j-r}[F_Y((r+1)T)-F_Y(rT)]+\overline{F}_Y((j+1)T) \\
\leq &\sum_{j=0}^{M-1}\sum_{r=0}^{j}\beta^{j-r}+\overline{F}_Y(T) \\
= &\sum_{j=0}^{M-1}\frac{1-\beta^{j+1}}{1-\beta}+\overline{F}_Y(T) \\
\leq &\sum_{j=0}^{M-1}\frac{1}{1-\beta}+\overline{F}_Y(T)
\end{aligned}$$

Therefore, $E[CPR_N]$ given in (7) verifies

$$\limsup_{N\to\infty} E[CPR_N] \leq \limsup_{N\to\infty}\left(\frac{M}{1-\beta}+\overline{F}_Y(T)\right)\int_0^{MT} c_{r2}(N,x)f_{G_N}(x)dx.$$

From $\lim_{N\to\infty} F_{G_N}(x) = 0$ and (13), it follows that the limit on the right hand side in the foregoing inequality is equal to zero. Thus

$$\limsup_{N\to\infty} E[CPR_N] = 0 \tag{24}$$

Next, we define the auxiliary function $S_{T,M}(N)$. It depends on N for fixed values of T and M and is increasing with N.

$$
\begin{aligned}
&S_{T,M}(N) = \\
&\frac{c_1}{c_d}\sum_{j=1}^{M-1}\left(\sum_{r=0}^{j-1}\beta^{j-1-r}[F_Y((r+1)T)-F_Y(rT)]+\overline{F}_Y(jT)\right)+\frac{c_f}{c_d}\alpha\sum_{j=1}^{M-1}\overline{F}_Y(jT)\\
+\ &\frac{c_{r1}}{c_d}P(K_1+L\leq M-1)+\frac{c_{PM}^1}{c_d}\sum_{j=0}^{M-1}\beta^{M-1-j}[F_Y((j+1)T)-F_Y(jT)]\\
+\ &\frac{c_{PM}^2}{c_d}\overline{F}_Y(MT)+\sum_{j=1}^{M-1}P(K_1+L=j)\sum_{i=1}^{N-1}\int_0^{jT}\frac{c_{mr,i}(x)}{c_d}p(x)r(x)\frac{H_R^{i-1}(x)}{(i-1)!}e^{-H_R(x)}dx\\
+\ &P(K_1+L\geq M)\sum_{i=1}^{N-1}\int_0^{MT}\frac{c_{mr,i}(x)}{c_d}p(x)r(x)\frac{H_R^{i-1}(x)}{(i-1)!}e^{-H_R(x)}dx-\int_0^{MT}\overline{F}_Y(x)dx
\end{aligned}
$$

In addition $\lim_{N\to\infty}\overline{F}_{G_N}(x)=1$ and (24) lead to

$$
\lim_{N\to\infty}O(T,M,N)=\lim_{N\to\infty}S_{T,M}(N)=S_{T,M}(\infty)=V_{T,M}. \tag{25}
$$

with $V_{T,M}$ given in (11)

To complete the proof for Theorem $2a$, it is sufficient to prove that $Q(T,M,N)<Q(T,M,\infty)$.

For $n=1,2,\ldots$, there exists an increasing sequence in n, N_n, of natural numbers with $\lim_{n\to\infty}N_n=\infty$ such that for all n the following two inequalities hold from (20), (25), and the monotony of $S_{T,M}$

$$
O(T,M,N_n)<S_{T,M}(\infty)-\frac{1}{n}=V_{T,M}-\frac{1}{n}\leq 0
$$

and

$$
E[\tau(N_n)]\leq C_{T,M}+\frac{1}{n}
$$

with $E[\tau(N_n)]$ the expected length of a cycle in (5) when the renewal occurs after N_n failures. The expression of $C_{T,M}$ is given in (20).

Then

$$Q(T,M,N_n) = c_d + c_d \frac{O(T,M,N_n)}{E[\tau]} < c_d + c_d \frac{(V_{T,M} - \frac{1}{n})}{(C_{T,M} + \frac{1}{n})} = R(n, V_{T,M}, C_{T,M}), \quad n = 1, \ldots$$

In addition

$$C_{T,M} + S_{T,M}(\infty) \geq C_{T,M} - \int_0^{MT} \overline{F}_Y(x)dx \geq 0.$$

The first inequality is derived from the expression of $S_{T,M}$ and the second from (21) since $\overline{F}_Y(x)dx = E[\min(Y, MT)]$.

We also have

$$\frac{(S_{T,M}(\infty) - \frac{1}{n})}{(C_{T,M} + \frac{1}{n})} \leq \frac{(S_{T,M}(\infty) - \frac{1}{n+1})}{(C_{T,M} + \frac{1}{n+1})} \Leftrightarrow C_{T,M} + S_{T,M}(\infty) \geq 0$$

which, in turn, leads to $R(n, V_{T,M}, C_{T,M})$ to be an increasing sequence in n. Thus

$$Q(T,M,N_1) < c_d + c_d \lim_{n\to\infty} R(n, V_{T,M}, C_{T,M}) = c_d + c_d \frac{V_{T,M}}{C_{T,M}} Q(T,M,\infty)$$

and the result in Theorem 2a follows.

Proof of Theorem 2b:

The existence of the optimum $M^*_{T,N}$ minimizing $Q(T,M,N)$ can be proved in a similar way to Theorem 2a . From (22), it follows that

$$\lim_{M\to\infty} Q(T,M,N) = c_d + c_d \frac{V_{T,N}}{C_{T,N}} = c_d + c_d \lim_{n\to\infty} R(n, V_{T,N}, C_{T,N})$$

with $V_{T,N}$ in (12) and $C_{T,N}$ in (22).

A new auxiliary function $Z_{T,N}(M)$ is defined. It depends on M for fixed values of T and N and it is increasing with M.

$$
\begin{aligned}
&Z_{T,N}(M) = \\
&\frac{c_1}{c_d}\sum_{j=1}^{\infty}\overline{F}_{G_N}(jT)\left(\sum_{r=0}^{j-1}\beta^{j-1-r}[F_Y((r+1)T)-F_Y(rT)]+\overline{F}_Y(jT)\right) \\
&+\ \frac{c_f}{c_d}\alpha\sum_{j=1}^{\infty}\overline{F}_Y(jT)\overline{F}_{G_N}(jT) \\
&+\ \sum_{j=1}^{\infty}\left(\sum_{r=0}^{j-1}\beta^{j-1-r}[F_Y((r+1)T)-F_Y(rT)]+\overline{F}_Y(jT)\right)\int_{jT}^{(j+1)T}c_{r2}(N,x)f_{G_N}(x)dx \\
&+\ \frac{c_{r1}}{c_d}\sum_{j=0}^{M-1}P(K_1+L=j)\overline{F}_{G_N}(jT) \\
&+\ \sum_{j=0}^{\infty}P(K_1+L=j)\sum_{i=1}^{N-1}\int_0^{jT}\frac{c_{mr,i}(x)}{c_d}p(x)r(x)\frac{H_R^{i-1}(x)}{(i-1)!}e^{-H_R(x)}dx \\
&-\ \int_0^{\infty}\overline{F}_Y(x)\overline{F}_{G_N}(x)dx
\end{aligned}
$$

It follows that

$$\lim_{M\to\infty}O(T,M,N)=\lim_{M\to\infty}Z_{T,N}(M)=Z_{T,N}(\infty)=V_{T,N}.$$

and from (23) we obtain

$$Z_{T,N}(\infty)+C_{T,N}=V_{T,N}+C_{T,N}\geq 0$$

and, thus, $R(n,V_{T,N},C_{T,N})$ is increasing in n. The rest of the details, similar to case $2a$, are omitted.

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