Some insights into the effect of maintenance quality for a protection system

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***Abstract*—This paper considers an inspection and preventive replacement policy for a one-component protection or cold standby system. Inspection is imperfect, and subject to false positives and negatives; preventive replacement may also be of poor quality. We determine conditions relating to the quality of the inspection and preventive replacement under which a maintained system would not benefit from the execution of inspections and preventive maintenance. We present examples with decreasing failure rate component lifetimes in which preventive replacement is cost-optimal, contrary to the classic policy. Such cases arise when inspections do not necessarily detect the failed state.**

***Index Terms*—Inspection, replacement, mixture, cold standby, preparedness, optimum policy**

ABBREVIATIONS

DFR Decreasing failure rate
IFR Increasing failure rate

NOTATION

X component lifetime
$R(x)$ reliability or survival function of X
μ=E[X] expected component lifetime
T inspection interval
α probability of a false positive inspection
β probability of a false negative inspection
M maximum number of inspections until preventive replacement.
K_1 number of inspections previous to failure or a false positive or to preventive replacement whichever occurs first
K_2 number of inspections after failure until its detection or preventive replacement whichever occurs first
U uptime in a cycle
D downtime in a cycle
c_0 cost of inspection
c_m cost of a preventive replacement either at time MT or at a replacement at a false positive inspection
c_r cost of replacement at a true positive inspection
c_d cost-rate of unavailability
c_{ins} expected cost due to all inspections in a cycle
c_{ren} expected cost derived from the replacement of the system
τ length of a cycle
$C(\tau)$ total cost of a cycle
$Q(T, M)$ cost rate (the long run expected cost per unit time)

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I. INTRODUCTION

THIS paper focuses on the quality of maintenance, and in particular on the quality of inspection. Inspection policies are typically used when failures are hidden, where a test or inspection is required to establish the state of the system. This hidden failure condition is the case for systems that are not in continuous operation but alternate between idle periods and periods of use. When a failure can occur during an idle period, if such a failure occurs and there is no regular inspection, the failure will remain undetected until there is an attempted use of the system.

Protection or preparedness systems and cold standby systems are typical examples of systems that operate in this way, and thus experience hidden failures. Apostolakis and Bansal [1] describe safety features of nuclear power plants consisting of redundant systems that are inactive until emergency conditions occur (e.g., pump failure in a cooling system). These systems undergo inspections at regular intervals to ensure their high availability. The consequences of poor quality maintenance, including poorly executed replacements or low quality spare parts [2], are reported by technology users or operators ([3], [4]). In our paper here, we model the quality of inspections, and analyze when low quality has non-negligible consequences on both maintenance procedures and systems reliability.

Inspection policies for systems subject to hidden failures have been studied by many authors. A classic reference is Vaurio [5]. Recent work has considered multi-component systems (e.g [6]), and hard (revealed) and soft (unrevealed) failures of multi-component systems (e.g [7]). Models have been proposed when inspection and replacement downtimes are non-negligible (e.g. [8]).

Inspection policies with false positives or false negatives or both are analysed in [9]- [13]. Okumura *et al.* [9] present an inspection model based on the delay-time model. Human errors are considered in [1], and [10]. Gong [14] investigates a repetitive testing process where the testing equipment may leave the in-control state at random, resulting in different testing errors.

At a false positive event, inspection indicates that the system is failed when it is in fact not failed. At a false negative, inspection does not reveal an existing failure. In Berrade *et al.* [12], the effect of such imperfect inspection is further

analysed by presenting two scenarios, one of which is related to the outsourcing of maintenance, and the other is related to maintenance performed in-house. In our paper, we analyze in detail the second scenario, and in particular we characterise circumstances in which false positives can lead to unnecessary replacements. We also implicitly consider circumstances in which false negatives do the same. This consideration is important because, if false negatives tend to occur with high probability, a typical response might be to do more inspection. Moreover, more inspection will lead to more false positives; and, where we have DFR component lifetimes, more inspection will lead to the possibility of more unnecessary replacements of good components by poor components. We model this low quality replacement by a mixture of two Weibull distributions (see e.g. [16], [20]). Cha and Finkelstein [17] also consider two ordered sub-populations with strong, and weak units respectively for a burn-in procedure.

When the population is heterogeneous, that is, strong and weak spares are mixed, failed components are replaced by new ones, which in turn can be weak or strong. However, in the case of imperfect inspections, another consequence emerges which is that strong, working components can be replaced by weak ones. Thus, an actual decrease of reliability can occur when heterogeneous components may undergo imperfect inspections.

Our paper analyzes this situation just described. Furthermore, whereas Ten Wolde and Ghobbar [15] for example state that increasing the inspection frequency increases the chance that a potential failure will be identified so that the additional cost (of more inspections) leads to improved reliability, this condition is not the case when the consequence of a false positive inspection is the possible replacement of a strong component by a weak one (Berrade *et al.* [13]). Therefore, in our view, it is necessary to consider both cost and system reliability when analyzing the effect of inspection errors and low quality maintenance on inspection policy, and we do so in our paper here.

In the next section, we describe our maintenance policy and its features, including the underlying component reliability model. We then determine conditions under which there exists a finite optimum inspection interval (Section 2.1). In so doing, we also quantify circumstances in which the quality of inspection and replacement is not sufficiently high for the benefit derived from inspections to compensate for their incurred cost. In this way, we provide conditions that relate to the quality of maintenance and spare parts under which a maintained system would not benefit from the execution of preventive maintenance. Proofs of our results are given in the Appendix. In Section 2.2, we further determine the operational reliability of the system. This knowledge may be useful when one wishes not only to consider a maintenance policy that is safety driven rather than cost driven (e.g. Flage *et al.* [18]), but also to consider the possibility that an increasing cost (of inspection) does not imply an increasing reliability. We then consider some numerical examples, the first of which presents a discrete mixture for the failure time distribution (Section 3.1), and the second a continuous mixture (Section 3.2). The examples provide additional insight about how the interaction between inspection errors and low quality maintenance cannot be neglected. We finish with a short discussion of the implications for the management of engineering services.

II. Maintenance policy

Consider a system with failures that are detected only by inspection. The system is inspected periodically every T units up to replacement. The system is replaced at a positive inspection, which is when inspection indicates the occurrence of a failure.

Testing may be imperfect. At an inspection, a false positive can occur. A false positive is the case when the inspection indicates a system failure but the system is actually good (working). Also, an inspection may report that the system is good when in fact it is failed; this is a false negative. The corresponding probabilities of a false positive, and a false negative at an inspection are denoted by α, and β respectively. We assume that, when an inspection is positive, the system is replaced. Thus the system is replaced at a true positive inspection, in error at a false positive inspection, and at MT, whichever occurs first.

One might envisage circumstances in which at a false positive a deeper investigation of the state of the system is carried out to reveal the true system state. Such a secondary inspection would come at an extra cost; but because it would ultimately reveal the system state, in such circumstances false positives would not occur, and a positive inspection would merely imply an additional cost. Thus, we do not consider this scenario. We emphasise that, in the model we develop in this paper, at a positive inspection, the system is replaced. In a real context, a maintainer may not have the capability to carry out the secondary inspection; the maintainer responds to a positive inspection (whether true or false) with an immediate replacement. This response is referred to as in-house maintenance by Berrade *et al.* [12].

We assume replacement renews the system so that conceptually the system comprises a component located in a socket which together perform an operational function (Ascher and Feingold [19]). In this way, by replacement we mean the replacement of the component in the socket with another functionally new component from a common stockpile of components.

Further, we suppose that, when the component fails, the system fails. The lifetime X of a component has a general distribution F with corresponding reliability function R. In our numerical study in Section 3, we suppose that F may be a mixture, although in the determination of our results the exact nature of F does not need to be specified. In its simplest form, in our first example, the mixture we consider is a mixture of two sub-distributions: one that represents components with relatively short lives, and one that represents components with long lives. A short component life may be the result of using a poor quality spare part, or of poor quality installation. Either way, this is conceptually what we mean by poor quality replacement. This idea has been studied in a reliability context (e.g. [20]), and a maintenance context (e.g. Scarf *et al.* [21]). Furthermore, and more generally in

our second example, we consider a continuous mixture model for the lifetimes of a heterogeneous population of components. Overall, in the numerical study, our purpose is to illustrate the effect of poor quality inspection and component heterogeneity, or poor component quality, upon a preventive maintenance policy.

Regarding the costs, we suppose that the preventive cost c_m is smaller than the corrective cost c_r, even though the component is replaced in both cases. We justify this assumption on the basis that, on replacement, components may be re-conditioned and returned to inventory. In the case of a false positive or a preventive replacement at MT, little re-conditioning may be required. In the case of a true positive, an overhaul of the component may be required. Furthermore, it is worth noting that, as the system is a protection system, c_r is associated with the failure of the protection system, and not the protected system. Thus, failure of the protection system only has an impact on the protected system if the protected system demands protective action from the protection system when the protection system is in the failed state. Such an event is modelled through the cost of unavailability c_d, which is interpreted as the product of notional quantities: the expected cost per unmet demand (for the operational function of the protection system), and the rate of occurrence of unmet demands.

A. Some characterisations of the optimal policy

The random variables K_1 and K_2 take values in $0, 1, 2, \ldots M$.
Now

$$P(K_1 = 0) = 1 - R(T),$$

and for $i = 1, 2, \ldots M-1$

$$P(K_1 = i) = \\ (R(iT) - R((i+1)T))(1-\alpha)^i + R(iT)(1-\alpha)^{i-1}\alpha,$$

and

$$P(K_1 = M) = R(MT)(1-\alpha)^{M-1}.$$

Its expected value is

$$E[K_1] = \sum_{i=1}^{M} R(iT)(1-\alpha)^{i-1}. \tag{1}$$

In the case that there is no inspection at MT, the expectation is

$$E[K_1] = \sum_{i=1}^{M-1} R(iT)(1-\alpha)^{i-1}.$$

For K_2, we have

$$P(K_2 = 0) = \\ \sum_{i=1}^{M-1} R(iT)(1-\alpha)^{i-1}\alpha + R(MT)(1-\alpha)^{M-1};$$

and for $i = 1, 2, \ldots M-1$,

$$P(K_2 = i) = \\ \sum_{k=1}^{M-i} \left(R((k-1)T) - R(kT)\right)(1-\alpha)^{k-1}\beta^{i-1}(1-\beta) + \\ \left(R((M-i)T) - R((M-i+1)T)\right)(1-\alpha)^{M-i}\beta^{i-1},$$

and

$$P(K_2 = M) = (1 - R(T))\beta^{M-1}.$$

$$E[K_2] = \\ \sum_{i=1}^{M} \left(R((i-1)T) - R(iT)\right)(1-\alpha)^{i-1}\frac{1-\beta^{M+1-i}}{1-\beta}. \tag{2}$$

If there is no inspection at MT, the expectation is

$$E[K_2] = \sum_{i=1}^{M-1} \left(R((i-1)T) - R(iT)\right)(1-\alpha)^{i-1}\frac{1-\beta^{M-i}}{1-\beta}.$$

Therefore, the expected length of a cycle is

$$E[\tau] = E[(K_1 + K_2)T] = \\ \sum_{i=1}^{M}(1-\alpha)^{i-1}T\frac{\beta^{M+1-i} - \beta}{1-\beta}R(iT) + \\ \sum_{i=1}^{M}(1-\alpha)^{i-1}T\frac{1-\beta^{M+1-i}}{1-\beta}R((i-1)T). \tag{3}$$

The expected uptime in a cycle, $E[U]$, is

$$E[U] = \sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} t dF(t) + \\ \sum_{i=1}^{M-1} iT\alpha(1-\alpha)^{i-1}R(iT) + MT(1-\alpha)^{M-1}R(MT) = \\ \sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} R(t)dt. \tag{4}$$

In addition, the expected downtime in a cycle is

$$E[D] = E[\tau] - E[U].$$

Regarding costs, the expected cost derived from inspections, c_{ins}, is

$$c_{ins} = c_0\left(E[K_1] + E[K_2]\right).$$

Therefore,

$$c_{ins} = \\ c_0\sum_{i=1}^{M} R(iT)(1-\alpha)^{i-1} + \\ c_0\sum_{i=1}^{M} \left(R((i-1)T) - R(iT)\right)(1-\alpha)^{i-1}\frac{1-\beta^{M+1-i}}{1-\beta}. \tag{5}$$

The system is replaced at a false positive, or when it is detected to be failed (true positive), or preventively at MT, whichever occurs first. Both a false positive and the preventive maintenance at MT incur a cost c_m. The associated cost due

to the replacement of a failed system is c_r. Hence, the cost derived from the replacement of the system is

$$\begin{cases} c_r, & \text{if } X < MT \text{ and there are no false positives,} \\ c_m, & \text{otherwise,} \end{cases}$$

with

$$P(c_r) = \sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} dF(t) = 1-\alpha\sum_{i=1}^{M-1} R(iT)(1-\alpha)^{i-1} - (1-\alpha)^{M-1}R(MT),$$

and

$$P(c_m) = \alpha\sum_{i=1}^{M-1} R(iT)(1-\alpha)^{i-1} + (1-\alpha)^{M-1}R(MT),$$

where $P(c_r)$, and $P(c_m)$ denote the probabilities that the replacement cost is c_r, and c_m respectively. The expected cost derived from the replacement of the system is

$$c_{ren} = (c_m - c_r)\alpha\sum_{i=1}^{M-1} R(iT)(1-\alpha)^{i-1} + (c_m - c_r)(1-\alpha)^{M-1}R(MT) + c_r.$$

The expected total cost incurred in a cycle is then

$$C(\tau) = c_{ins} + c_{ren} + c_d E[D] = c_{ins} + c_{ren} + c_d\left(E[\tau] - \sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} R(t)dt\right).$$

Therefore, the cost-rate (the long-run cost per unit time) turns out to be

$$Q(T,M) = c_d + \frac{c_{ins} + c_{ren} - c_d\left(\sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} R(t)dt\right)}{E[\tau]}, \tag{6}$$

with $E[\tau]$ given in (3). In the case that $M = 1$, the expression in (6) is

$$Q(T,M) = c_d + \frac{c_0 + (c_m - c_r)R(T) + c_r - c_d\int_0^T R(t)dt}{T}.$$

The next lemma gives two limits for $Q(T,M)$ which are the key in the proofs of the results concerning the existence of a finite optimum policy T.

Lemma 1: Given M, $Q(T,M)$ is a continuous function so that

$\lim_{T\to 0} Q(T) = \infty$ and $\lim_{T\to\infty} Q(T) = c_d$.

The next result provides a sufficient condition to ensure the existence of a finite optimum $T^\star$ for a given M when $c_m = c_r$.

Proposition 1: Suppose that $c_m = c_r$, and the number of inspections previous to the preventive maintenance, M, is fixed. If there exists some $T > 0$ satisfying one of two conditions

$$a)\quad TR(MT) > \frac{c_0}{c_d(1-\beta)}\left(1 - \frac{\beta^{M+1} - \beta(1-\alpha)^M}{\beta - (1-\alpha)}\frac{\alpha}{1-(1-\alpha)^M}\right) + \frac{\alpha c_r}{c_d(1-(1-\alpha)^M)} \tag{7}$$

if $\alpha + \beta \neq 1$, or

$$b)\quad TR(MT) > \frac{c_0}{c_d(1-\beta)}\left(1 - \frac{M\beta^M(1-\beta)}{1-\beta^M}\right) + \frac{c_r(1-\beta)}{c_d(1-\beta^M)} \tag{8}$$

if $\alpha + \beta = 1$,

then there exists a finite $T^\star$ minimizing $Q(T,M)$ in (6).

Lemma 2: The following properties apply.

i) $\max(c_r, c_m) \geq c_{ren} \geq \min(c_r, c_m)$.

ii) $E[U] < \mu\frac{1-(1-\alpha)^M}{\alpha}$ for a given M.

The next results give some sufficient conditions under which the optimum inspection interval, $T^\star$, does not exist when the number of inspections M is fixed.

Proposition 2: Given a fixed M, if $\mu = E[X]$ satisfies

$$\mu \leq \frac{\alpha(c_0 + \min(c_r, c_m))}{c_d(1-(1-\alpha)^M)}, \tag{9}$$

then $T^\star = \infty$.

Condition (9) broadly means that, if maintenance costs are large or the false positive probability is large or both, then the best policy is no inspection at all.

Theorem 1: Given a fixed $M \geq 1$, if

i) $c_m \leq c_r$, and $c_0\left(\frac{1-\beta^M}{1-\beta} + 1 - M\right) + c_m \geq c_d\mu\frac{1-(1-\alpha)^M}{\alpha}$; or

ii) $c_m > c_r$, and $c_0\left(\frac{1-\beta^M}{1-\beta} + 1 - M\right) + c_r \geq c_d\mu\frac{1-(1-\alpha)^M}{\alpha}$

then $T^\star = \infty$.

Corollary 1: Given a fixed $M > 1$, let β_{M0} be defined as

$$\beta_{M0} = \min\left\{\beta | \frac{1-\beta^M}{1-\beta} \geq v(\mu, M, \alpha, c_0, c_r, c_m, c_d)\right\}$$

where

$$v(\mu, M, \alpha, c_0, c_r, c_m, c_d) = \frac{c_d\mu\frac{1-(1-\alpha)^M}{\alpha} - \min(c_r, c_m)}{c_0} + M - 1.$$

If $\beta \geq \beta_{M0}$, then $T^\star = \infty$.

Note that Corollary 1 only makes sense when $M > 1$; that is, when there are several inspections, and the inspection may fail to detect a failure, only then does a condition on β exist. Thus, if one insists on conducting an inspection, if β is (sufficiently) large, then such an inspection should be postponed indefinitely.

Next, we aim to obtain a condition for the existence of an optimum number of inspections, $M^\star$, and hence for preventive replacement to be optimal when the time to failure follows

an exponential distribution. In general, $Q(T, M^\star)$ has a local minimum at $M^\star$ if

$$Q(T, M^\star + 1) \geq Q(T, M^\star), \quad Q(T, M^\star - 1) > Q(T, M^\star).$$

Theorem 2: If the time to failure is an exponential distribution, and $\beta = 0$, then, for any given T, $M^\star = \infty$.

The result is not necessarily true in the case of $\beta \neq 0$, as the following counter examples establish. With costs $c_0 = 1$, $c_m = 5$, $c_r = 10$, $c_d = 5000$, $\alpha = 0$, and an exponential time to failure distribution, we obtain

- $\lambda = 100$, $\beta = 0.8$, $M^\star = 1$, $T^\star = 0.0063$, $Q\left(T^\star, M^\star\right) = 2613.751$.
- $\lambda = 1$, $\beta = 0.9$, $M^\star = 1$, $T^\star = 0.0498$, $Q\left(T^\star, M^\star\right) = 247.82$.

It is known (Barlow and Proschan [22]) that, when the failures of a system are revealed as soon as they occur, and the time to failure follows an exponential distribution, then the lack of memory property implies that the best policy is no preventive maintenance ($M^\star = \infty$). Theorem 2 indicates that this classic result still holds when the inspection procedure is only subject to false positives. Thus, there is no need to replace the system before it fails provided that the system in use is as-good-as-new, and the maintainer can be confident that a potential failure will be detected in the next inspection. When false negatives can occur, $M^\star = \infty$ may not be the best policy. The examples show that, although there is nothing to be gained from preventive replacement of a non-ageing system if it is known to be in the good state, if it is not known to be in the good state (because inspections are subject to false negatives) it may be worthwhile to carry out preventive replacement at inspection. This condition is true because replacement guarantees the restoration of system functionality whereas inspection does not. Thus, and this is the crux of our work here, when a protection or cold-standby system has an exponential time to failure, and inspections are subject to false negatives, then contrary to the classic result, preventive replacement may indeed be cost-optimal. Thus, in summary, Theorem 1 describes the consequences of low quality maintenance for an inspection policy when M, the number of inspections until preventive replacement, is given. Theorem 2 shows that, when inspections of a protection or standby system are subject to false positives, the classic result (it is cost-sub-optimal to replace a non-ageing system) still holds; but when inspections are subject to false negatives, our examples show that the classic result is no longer valid.

B. The operational reliability function

In the context of perfect inspection, more frequent inspections will increase the operational reliability of the system while also increasing the cost-rate, so that a reliability or safety requirement will generally dominate policy choice (Scarf *et al.* [23]). With imperfect maintenance, it is possible to simultaneously maximise reliability and minimise cost (Scarf *et al.* [21]). Therefore, for the imperfect inspection policy that we consider in this paper, it is interesting to derive the operational reliability function. We do this now. Note that by the operational reliability function we mean the probability that the system is functioning at time t, and there has been no failure in $(0, t)$ given that the system was new at time $t = 0$ (Lewis [24]). The terminology used here is the same as that in Christer [25] who formulate the reliability function for a delay-time model. The main difference is that, in [25], the inspection procedure is considered to be perfect, and has no effect on the reliability of the system.

Let $R_{TM}(t)$ denote the reliability function of the system under inspection every T units of time and preventive replacement at MT. $R_{TM}(t)$ represents the probability that a system new at $t = 0$ survives up to t, for all $t > 0$. In addition, let $r_{TM}^{(m)}$ denote the reliability function for the system at time t with $(m-1)T \leq t < mT$. It follows that

$$r_{TM}^{(1)}(t) = R(t), \quad 0 \leq t < T$$

is just the component reliability. When developing $r_{TM}^{(m)}$ for $m > 1$, we have to take into account the possibility of a false positive at each inspection prior to the replacement at MT of a non-failed system.

Therefore, for $T \leq t < 2T$,

$$r_{TM}^{(2)}(t) = (1-\alpha)R(t) + \alpha R(T) r_{TM}^{(1)}(t-T).$$

For $2T \leq t < 3T$, it follows that

$$\begin{aligned} r_{TM}^{(3)}(t) &= (1-\alpha)^2 R(t) + \\ &\alpha R(T) r_{TM}^{(2)}(t-T) + (1-\alpha)\alpha R(2T) r_{TM}^{(1)}(t-2T). \end{aligned}$$

Let us now develop the expression of $r_{TM}^{(3)}(t)$ to explain its meaning. Replacing the corresponding expressions of $r_{TM}^{(2)}(t)$ and $r_{TM}^{(1)}(t)$, we obtain

$$\begin{aligned} &r_{TM}^{(3)}(t) = \\ &(1-\alpha)^2 R(t) + \alpha R(T)(1-\alpha) R(t-T) + \\ &\alpha^2 R^2(T) R(t-2T) + \\ &(1-\alpha)\alpha R(2T) R(t-2T). \end{aligned}$$

The first term represents the probability that the system, new at $t = 0$, survives up to t with no false positive either at T or at $2T$. The second term indicates that the system is renewed at T due to a false positive. There is no false positive at $2T$, and the system survives up to t. The third term corresponds to the case that there are two false positives at T and $2T$, and the component installed at $2T$ survives to t. In the forth term, there is no false positive at T, but one occurs at $2T$, and the system renewed at $2T$ survives up to t.

For $3T \leq t < 4T$,

$$\begin{aligned} &r_{TM}^{(4)}(t) = (1-\alpha)^3 R(t) + \\ &\alpha R(T) r_{TM}^{(3)}(t-T) + (1-\alpha)\alpha R(2T) r_{TM}^{(2)}(t-2T) + \\ &(1-\alpha)^2 \alpha R(3T) r_{TM}^{(1)}(t-3T). \end{aligned}$$

The general formulation for $(m-1)T \leq t < mT$ is

$$\begin{aligned} &r_{TM}^{(m)}(t) = (1-\alpha)^{m-1} R(t) + \\ &\sum_{i=1}^{m-1} \alpha(1-\alpha)^{i-1} R(iT) r_{TM}^{(m-i)}(t-iT). \end{aligned}$$

For $MT \leq t < (M+1)T$,

$$r_{TM}^{(M+1)}(t) = r_{TM}^{(M)}(MT) r_{TM}^{(1)}(t - MT).$$

For $kMT + (j-1)T \leq t < kMT + jT$ with $k = 0, 1, \ldots$, and $j = 1, 2, \ldots, M$, the general formula turns out to be

$$R_{TM}(t) = r_{TM}^{(kM+j)}(t) = (r_{TM}^{(M)}(MT))^k r_{TM}^{(j)}(t - kMT).$$

Observe that there is no effect of false negatives on the operational reliability; these only affect availability.

III. Numerical Examples

In this section, we present some results that aim to illustrate the mathematical dependence on the parameters of both the optimum inspection frequency and optimum number of inspections previous to the preventive maintenance.

A. A discrete mixture failure time distribution

The time to failure is assumed to be a mixture of two Weibull distributions. The failure rate of a Weibull distribution is given by

$$r(t) = s\eta^{-s} t^{s-1}.$$

The reliability function of the Weibull mixture is

$$R(t) = pe^{-(t/\eta_1)^{s_1}} + (1-p)e^{-(t/\eta_2)^{s_2}}.$$

The characteristic lives, and shape parameters of the mixture are $\eta_1 = 500$, $\eta_2 = 7000$, and $s_1 = 2.5$, $s_2 = 4.5$, respectively. The sub-populations in the mixture verify that $r_1(t) \geq r_2(t)$, $t \geq 0$; that is, sub-population 1 is smaller than sub-population 2 in hazard rate order. This mixture is used to model early failures caused by a proportion p of weak items represented by sub-population 1. For additional insight on stochastic orders, see Shaked and Shantikumar [26].

Parameter values in the example relate to a case study considered in Berrade *et al.* [12] regarding a protection device in a soft drink production process. The optimum policy, T and M, for different values of the model parameters is contained in Table I. Table I shows that the inspection interval increases when the probability of false positives, α, increases, whereas T decreases when β increases. Both results are as expected because of the risk of incurring unnecessary replacements in the first case, and to mitigate false negatives in the second.

Whenever η_1, η_2, s_1, s_2 increase, the inspection frequency is relaxed, and the optimum cost-rate decreases. When the proportion of weak items p increases, more intensive inspection is required, and therefore the optimum cost-rate is also higher. The form of the cost-rate is illustrated in Fig. 1.

Fig. 2 illustrates the behaviour of the operational reliability. An inspection of Fig. 2 reveals interesting features. For small values of p, as in Fig. 2(a), $R_{TM}(t) > R(t)$. Thus, maintenance tends to increase the reliability. For large values of time t, the reliability is greater when inspection is carried out at kT than when no inspection is carried out. When the proportion of weak items is high, the reliability function is not monotonic with T as in Fig. 2 (b) and (c). That is, $R_{TM}(t)$ does not always increase when the inspection frequency increases. The high proportion of weak items $p = 0.15$ and $p = 0.3$ explains the non-monotonic behavior. The risk of replacement with a weak item is large; therefore, as T decreases, the risk of a poor quality replacement increases.

Nevertheless, $R_{TM}(t)$ increases as T decreases when the proportion of weak items is low as in Fig. 2(a) ($p = 0.005$). The same reason as before explains this fact: provided that the risk of low quality replacement is low, increasing the inspection frequency is beneficial from the reliability point of view. Moreover, when p increases to very high values as in Fig. 2(c) ($p = 0.3$), $R_{TM}(t)$) drops dramatically, and tends to be even below the reliability when no maintenance is carried out, (baseline reliability $R(t)$).

Fig. 3 illustrates the interaction effect between parameters. In Fig. 3(b), when α increases, the reliability decreases. The same argument as before applies here: the high probability of a low quality replacement ($p = 0.15$) after a false positive makes $R_{TM}(t)$ decrease. In Fig. 3(a), where the value of p is small ($p = 0.005$), the behaviour is just the opposite: when α increases, so does the reliability.

B. Continuous mixture failure time distribution

Following Gupta and Gupta [27], we consider the following mixture.

Let X be a continuous random variable with baseline failure rate $r(x)$, and Z be a nonnegative random variable that models the environmental effect responsible for the heterogeneity in the population. The reliability function conditional upon $Z = z$ is

$$R(t|z) = e^{-z\int_0^t r(x)dx},$$

and the conditional failure rate is

$$r(t|z) = zr(t)$$

In this case, a greater value of z implies harder environmental conditions, and therefore $r(t|z_1) \geq r(t|z_2)$, $z_1 > z_2$.

The reliability function of the mixture, $R^\star(t)$, is expressed as

$$R^\star(t) = \int_0^\infty g(z) e^{-z\int_0^t r(x)dx} dz,$$

with $g(z)$ being the density function of Z.

In what follows, we assume the baseline failure rate corresponding to a Weibull distribution. Hence,

$$R(t|z) = e^{-z(t/\eta)^s},$$

and Z is inverse Gaussian, so

$$g(z) = \frac{1}{(2\pi b z^3)^{\frac{1}{2}}} e^{-\frac{(dz-1)^2}{2bz}}, \quad z, b, d > 0.$$

It can be verified ([27], [28]) that the failure rate of the mixture $r^\star(t)$ is

$i)$ decreasing for $s \leq 1$,
$ii)$ increasing $s \geq 2$, and
$iii)$ non-monotonic of type U for $1 < s < 2$.

The form of $r^\star(t)$ of the type U case implies that there exists a t_0 such that the derivative of $r^\star(t)$, $r^{\star\prime}(t)$, is such that $r^{\star\prime}(t) > 0$ for $t < t_0$, $r^{\star\prime}(t_0) = 0$, $r^{\star\prime}(t) < 0$ for $t > t_0$.

Table II shows some results corresponding to the Weibull-inverse Gaussian mixture.

Note that in all the cases where the mixture has a decreasing failure rate the corresponding $M^\star$ is not finite for moderate values of alpha and beta. This result makes sense provided that in both cases an improvement in the reliability of the system takes place as time goes by. However, we can see that, for β large, or α large and $\beta \neq 0$, then $M^\star \neq \infty$; in fact, pure replacement $M = 1$ is the best policy. In those cases such that $\beta = 0$, then $M^\star = \infty$ no matter what the value of α is, and so we conjecture that the result in Theorem 2 also holds for DFR distributions.

When comparing cases 1 (IFR) and 3 (DFR) in Table II, both mixtures present the same mean time to failure. However, in the DFR case, the cost-rate is higher because in spite of the fact that the failure rate is decreasing it is initially high in comparison to the IFR case so that undetected early failures are likely.

Fig. 4 illustrates the point that if the mixture is DFR then the effect of increasing α is to decrease the reliability, while if the mixture is IFR then the effect of increasing α is to increase the reliability. Thus, if the mixture is DFR, increasing the inspection frequency can produce a negative effect on the reliability. The higher α, the greater this negative effect will be. In addition, increasing α also increases the cost-rate. Note that the reliability function does not depend on the value of β.

IV. Conclusion

This paper focuses on the effect of inspection errors and low quality maintenance when determining the optimal inspection interval and preventive maintenance time for a protection system or cold standby system in which failures are unrevealed. We develop a number of results that characterise the minimum cost-rate policy, and which allow us to quantify circumstances in which inspection is cost-inefficient.

When a safety requirement is prescribed, and it is therefore natural to consider the operational reliability of the system, false negatives affect availability ((3) and (4)), but not reliability. On the other hand, the effect of false positives on operational reliability is complex, and depends on the nature of the component reliability distribution. If this is IFR, then false positives increase operational reliability, so that false positives are a cost issue. For discrete mixtures, false positives provide also a positive effect on the reliability when the proportion of weak items is low. However, if the reliability distribution is not IFR, and replacement is imperfect (in the sense of Scarf *et al.* [2]), then the unnecessary replacements at false positives not only increase the cost-rate but may also decrease operational reliability and decrease availability. The same effect is detected when the proportion of weak items is high. This result is true because a new component introduced at replacement may be worse than the one in use.

When the system under inspection is DFR, and inspections are perfect, no preventive maintenance is recommended. However, we observe that this recommendation is no longer valid when false negatives can occur. If so, preventive maintenance can protect against the occurrence of a failure that remains undetected and causes unavailability and potential catastrophic failure of the protected system.

Thus, when one is dealing with a protection system, it would seem essential to inspect this system because inspection is the only way to know the state of the system. However, we show in this paper that, even if inspection is the only way to know the system state, and hence if the protection system is performing its required function (to protect a critical system), if maintenance is poorly executed (with high probability to commit mistakes at inspection and to introduce poor quality or badly installed components at preventive replacement), then it may be more prudent to do nothing. This kind of conclusion is not according to common engineering sense because in practice one might tend to act more frequently when an action is perceived to be only moderately effective.

Appendix

Proof of Lemma 1

The cost-rate can be alternatively expressed as

$$Q(T, M) = c_d + \frac{c_0}{T} + \frac{c_{ren} - c_d E[U]}{T(E[K_1] + E[K_2])}.$$

The expected uptime is given by

$$E[U] = \int_0^T R(t)dt + \sum_{i=2}^{M} (1-\alpha)^{i-1} \int_{(i-1)T}^{iT} R(t)dt.$$

The second term in the previous expression is bounded as

$$T \sum_{i=2}^{M} (1-\alpha)^{i-1} R(iT) \leq \sum_{i=2}^{M} (1-\alpha)^{i-1} \int_{(i-1)T}^{iT} R(t)dt,$$

and

$$\sum_{i=2}^{M} (1-\alpha)^{i-1} \int_{(i-1)T}^{iT} R(t)dt \leq T \sum_{i=2}^{M} (1-\alpha)^{i-1} R((i-1)T).$$

The upper, and lower bound of the foregoing inequalities tend to zero in both cases when T tends to zero, and when T tends to infinity (see Badía *et al.* [29]); and hence the following conditions hold.

$\lim_{T\to 0} E[U] = 0$ and $\lim_{T\to\infty} E[U] = E[X] = \mu$

In addition,

$\lim_{T\to 0} c_{ren} = c_m, \quad \lim_{T\to\infty} c_{ren} = c_r$

$1 \leq K_1 + K_2 \leq M$. Hence $T \leq E[(K_1 + K_2)T] \leq MT$, and the following two limits hold for a fixed M.

$\lim_{T\to 0} E[(K_1 + K_2)T] = 0$, and

$\lim_{T\to\infty} E[(K_1 + K_2)T] = \infty$.

Thus the two limiting conditions in Lemma 1 are proved.

Proof of Proposition 1

The total number of inspections in a cycle is bounded as follows.

$$E[K_1] + E[K_2] \leq \sum_{i=1}^{M} (1-\alpha)^{i-1} \left(\frac{1-\beta^{M+1-i}}{1-\beta} \right) =$$

$$\frac{1}{1-\beta} \left(\frac{1-(1-\alpha)^M}{\alpha} - \frac{\beta^{M+1} - \beta(1-\alpha)^M}{\beta - (1-\alpha)} \right)$$

provided that $\alpha+\beta\neq 1$.
Next, a lower bound for the expected uptime is

$$E[U]=\sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT}R(t)dt>$$
$$\sum_{i=1}^{M}(1-\alpha)^{i-1}TR(iT)>TR(MT)\frac{1-(1-\alpha)^M}{\alpha}.$$

Then,

$$c_0(E[K_1]+E[K_2])+c_r-cdE[U]<$$
$$\frac{c_0}{1-\beta}\left(\frac{1-(1-\alpha)^M}{\alpha}-\frac{\beta^{M+1}-\beta(1-\alpha)^M}{\beta-(1-\alpha)}\right)+$$
$$c_r-c_dTR(MT)\frac{1-(1-\alpha)^M}{\alpha}.$$

In the case that $\alpha+\beta=1$, it follows that

$$E[K_1]+E[K_2]\leq\sum_{i=1}^{M}(1-\alpha)^{i-1}\left(\frac{1-\beta^{M+1-i}}{1-\beta}\right)=$$
$$\frac{1}{1-\beta}\left(\frac{1-\beta^M}{1-\beta}-M\beta^M\right).$$

Conditions (7) and (8) imply that

$$c_0(E[K_1]+E[K_2])+c_r-c_dE[U]<0$$

for some $T>0$, and Lemma 1 leads to the result.

Proof of Lemma 2

Result $i)$ is true because

$$0\leq\left(\alpha\sum_{i=1}^{M-1}R(iT)(1-\alpha)^{i-1}+(1-\alpha)^{M-1}R(MT)\right)<1,$$

and this follows because

$$\alpha\sum_{i=1}^{M-1}R(iT)(1-\alpha)^{i-1}+(1-\alpha)^{M-1}R(MT)<$$
$$\alpha\sum_{i=1}^{M-1}(1-\alpha)^{i-1}+(1-\alpha)^{M-1}=$$
$$\alpha\frac{1-(1-\alpha)^{M-1}}{\alpha}+(1-\alpha)^{M-1}=1.$$

Result $ii)$ follows from the fact that

$$\left(\sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT}R(t)dt\right)\leq$$
$$\left(\sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{0}^{\infty}R(t)dt\right)=$$
$$\sum_{i=1}^{M}(1-\alpha)^{i-1}\mu=\mu\frac{1-(1-\alpha)^M}{\alpha}.$$

Proof of Proposition 2

K_1, and K_2 are s-dependent random variables such that $K_1+K_2\geq 1$, implying that $c_{ins}\geq c_0$. Therefore, for a given M, condition (9) along with Lemma 2 imply that $Q(T,M)\geq c_d$ for all $T>0$, and Lemma 1 leads to the result.

Proof of Theorem 1

Let us analyze the numerator in the expression of the cost-rate given in (6) as follows.

$$c_{ins}+c_{ren}-c_d\left(\sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT}R(t)dt\right)$$

The cost derived from inspections in (5) can be alternatively written as

$$c_{ins}=c_0\left(\frac{1-\beta^M}{1-\beta}+\frac{\beta^M-\beta}{1-\beta}R(T)\right)+$$
$$c_0\sum_{i=2}^{M}R(iT)(1-\alpha)^{i-1}+$$
$$c_0\sum_{i=2}^{M}(R((i-1)T)-R(iT))(1-\alpha)^{i-1}\frac{1-\beta^{M+1-i}}{1-\beta}.$$

Hence,

$$c_{ins}\geq c_0\left(\frac{1-\beta^M}{1-\beta}+\frac{\beta^M-\beta}{1-\beta}R(T)\right).$$

For a given $M\geq 1$, let us denote by $f(\beta)$ the function

$$f(\beta)=\frac{\beta^M-\beta}{1-\beta}.$$

$f(\beta)$ is a decreasing function, hence $f(\beta)\geq\lim_{\beta\to 1}f(\beta)=1-M$, and it follows that

$$c_{ins}\geq c_0\left(\frac{1-\beta^M}{1-\beta}+1-M\right)$$

provided that $f(\beta)$ is a negative function. Hence, if any of the conditions given in $i)$ or $ii)$ verify, then for all $M\geq 1$ and $T>0$, Lemma 2 implies that

$$c_{ins}+c_{ren}\geq c_dE[U].$$

Therefore, for a given $M\geq 1$, $Q(T,M)\geq c_d$ for all $T>0$, and the result follows from Lemma 1.

Proof of Corollary 1

$g(\beta,M)=(1-\beta^M)/(1-\beta)$ is an increasing function with β:

$$\frac{\partial g(\beta,M)}{\partial\beta}=\frac{h(\beta)}{(1-\beta)^2},$$

and $h(\beta)=1-M\beta^{M-1}(1-\beta)-\beta^M$ verifies that $h(0)=1,\quad h(1)=0$, and $h'(\beta)=-M(M-1)\beta^{M-2}(1-\beta)<0$. Therefore, $h(\beta)\geq 0$, and $g(\beta,M)$ is increasing with β. Hence, conditions $i)$ or $ii)$ in Theorem 1 hold for $\beta\geq\beta_{M0}$, and thus the result in Corollary 1 is obtained.

PROOF OF THEOREM 2

If $\beta = 0$, the cost-rate is

$$Q(T,M) = c_d + \frac{c_0}{T} + \frac{a(T,M)}{b(T,M)}$$

with $a(T,M)$, and $b(T,M)$ as

$$\begin{aligned} a(T,M) &= (c_m - c_r)\alpha \sum_{i=1}^{M-1} R(iT)(1-\alpha)^{i-1} + \\ &(c_m - c_r)(1-\alpha)^{M-1}R(MT) + c_r - \\ &c_d \sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} R(t)dt, \\ b(T,M) &= T\sum_{i=1}^{M}(1-\alpha)^{i-1}R((i-1)T). \end{aligned}$$

In addition,

$$Q(T,M+1) = c_d + \frac{c_0}{T} + \frac{a(T,M+1)}{b(T,M+1)},$$

$$\begin{aligned} a(T,M+1) &= a(T,M) + g(T,M), \\ b(T,M+1) &= b(T,M) + h(T,M) \end{aligned}$$

with

$$\begin{aligned} &g(T,M) = \\ &(c_m - c_r)(1-\alpha)^M \left(R((M+1)T) - R(MT)\right) - \\ &c_d(1-\alpha)^M \int_{MT}^{(M+1)T} R(t)dt, \end{aligned}$$

and

$$h(T,M) = T(1-\alpha)^M R(MT).$$

Then $Q(T,M+1) \leq Q(T,M)$ is equivalent to the inequality

$$\frac{a(T,M) + g(T,M)}{b(T,M) + h(T,M)} \leq \frac{a(T,M)}{b(T,M)}$$

which in turn is equivalent to

$$\frac{g(T,M)}{h(T,M)} \leq \frac{a(T,M)}{b(T,M)}$$

provided that $b(T,M) \geq 0$, and $h(T,M) \geq 0$.
The foregoing inequality can be also expressed as

$$\begin{aligned} &\frac{(c_m - c_r)R((M+1)T) - R(MT)}{R(MT)} \\ &- \frac{c_d \int_{MT}^{(M+1)T} R(t)dt}{R(MT)} \\ &\leq \\ &\frac{(c_m - c_r)\alpha \sum_{i=1}^{M-1} R(iT)(1-\alpha)^{i-1}}{\sum_{i=1}^{M}(1-\alpha)^{i-1}R((i-1)T)} + \\ &\frac{(c_m - c_r)(1-\alpha)^{M-1}R(MT) + c_r}{\sum_{i=1}^{M}(1-\alpha)^{i-1}R((i-1)T)} - \\ &\frac{c_d \sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} R(t)dt}{\sum_{i=1}^{M}(1-\alpha)^{i-1}R((i-1)T)}. \end{aligned} \tag{10}$$

If the time to failure follows an exponential distribution with rate $\lambda = 1/\mu$, then we get

$$\begin{aligned} &\left(\alpha \sum_{i=1}^{M-1}(1-\alpha)^{i-1}R(iT) + (1-\alpha)^{M-1}R(MT)\right) = \\ &\frac{\alpha e^{-\lambda T} + (1-\alpha)^M e^{-\lambda MT}\left(1 - e^{-\lambda T}\right)}{1 - (1-\alpha)e^{-\lambda T}}, \end{aligned}$$

and

$$\sum_{i=1}^{M}(1-\alpha)^{i-1}R((i-1)T) = \frac{1 - (1-\alpha)^M e^{-\lambda MT}}{1 - (1-\alpha)e^{-\lambda T}}.$$

Also,

$$\begin{aligned} &\sum_{i=1}^{M}(1-\alpha)^{i-1}\int_{(i-1)T}^{iT} R(t)dt = \\ &\frac{\left(1 - e^{-\lambda T}\right)\left(1 - (1-\alpha)^M e^{-\lambda MT}\right)}{\lambda\left(1 - (1-\alpha)e^{-\lambda T}\right)}. \end{aligned}$$

Then, the inequality in (10) is given by

$$\begin{aligned} &(c_m - c_r)\left(e^{-\lambda T} - 1\right) \leq \\ &\frac{(c_m - c_r)\left(\alpha e^{-\lambda T} + (1-\alpha)^M e^{-\lambda MT}\left(1 - e^{-\lambda T}\right)\right)}{1 - (1-\alpha)^M e^{-\lambda MT}} + \\ &\frac{c_r\left(1 - (1-\alpha)e^{-\lambda T}\right)}{1 - (1-\alpha)^M e^{-\lambda MT}}. \end{aligned}$$

Given that $1 - (1-\alpha)^M e^{-\lambda MT} \geq 0$ for all T, the previous inequality is in turn equivalent to

$$c_m\left(e^{-\lambda T} - 1\right) \leq c_m \alpha e^{-\lambda T},$$

which is always true for all M and T as the left hand side of the previous expression is less than or equal to zero.

Note that, if $\alpha = 0$, then the last inequality also holds and proves the result for perfect inspections (Barlow and Proschan [22]).
Therefore, under the conditions of Theorem 2, $Q(T,M+1) < Q(T,M)$ for all M, and the result holds. This completes the proof.

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Table I: Discrete two component mixture; minimum cost-rate policy for various parameter values.

Case	Mixed failure distribution parameters					Cost parameters				False -ve and +ve probabilities		Optimum policy		
	s_1	η_1	s_2	η_2	p	c_0	c_m	c_r	c_D	β	α	M^*	T^*	Q^*
1	2.5	500	4.5	7000	0.1	5	55	105	1.35	0.2	0.2	7	606	0.067
2	2.5	500	4.5	7000	0.1	5	55	105	1.35	0.2	0.4	6	733	0.088
3	2.5	500	4.5	7000	0.1	5	55	105	1.35	0.2	0.6	5	847	0.114
4	2.5	500	4.5	7000	0.1	5	55	105	1.35	0	0	10	410	0.042
5	2.5	500	4.5	7000	0.1	5	55	105	1.35	0	0.2	6	687	0.056
6	2.5	500	4.5	7000	0.1	5	55	105	1.35	0.4	0.2	8	527	0.082
7	2.5	500	4.5	7000	0.1	5	55	105	1.35	0.6	0.2	9	449	0.107
8	1.5	500	4.5	7000	0.1	5	55	105	1.35	0.2	0.2	7	593	0.069
9	3.5	500	4.5	7000	0.1	5	55	105	1.35	0.2	0.2	6	656	0.065
10	2.5	400	4.5	7000	0.1	5	55	105	1.35	0.2	0.2	7	588	0.068
11	2.5	600	4.5	7000	0.1	5	55	105	1.35	0.2	0.2	6	670	0.066
12	2.5	500	3	7000	0.1	5	55	105	1.35	0.2	0.2	7	533	0.077
13	2.5	500	6	7000	0.1	5	55	105	1.35	0.2	0.2	7	655	0.062
14	2.5	500	4.5	5000	0.1	5	55	105	1.35	0.2	0.2	6	509	0.079
15	2.5	500	4.5	9000	0.1	5	55	105	1.35	0.2	0.2	8	675	0.060
16	2.5	500	4.5	7000	0.01	5	55	105	1.35	0.2	0.2	1	2557	0.029
17	2.5	500	4.5	7000	0.05	5	55	105	1.35	0.2	0.2	5	740	0.051
18	2.5	500	4.5	7000	0.25	5	55	105	1.35	0.2	0.2	11	469	0.114
19	2.5	500	4.5	7000	0.1	2	55	105	1.35	0.2	0.2	7	588	0.062
20	2.5	500	4.5	7000	0.1	10	55	105	1.35	0.2	0.2	6	686	0.074
21	2.5	500	4.5	7000	0.1	5	55	70	1.35	0.2	0.2	7	608	0.065
22	2.5	500	4.5	7000	0.1	5	55	140	1.35	0.2	0.2	7	605	0.069
23	2.5	500	4.5	7000	0.1	5	55	105	0.8	0.2	0.2	6	724	0.053
24	2.5	500	4.5	7000	0.1	5	55	105	2.2	0.2	0.2	8	510	0.085

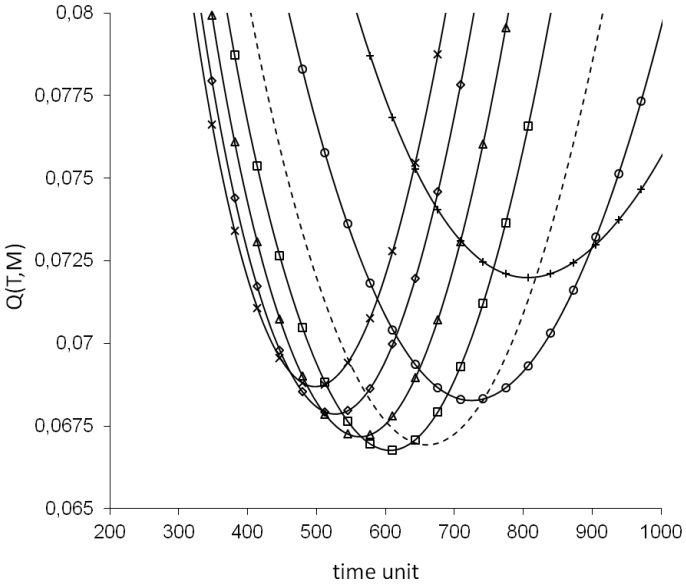
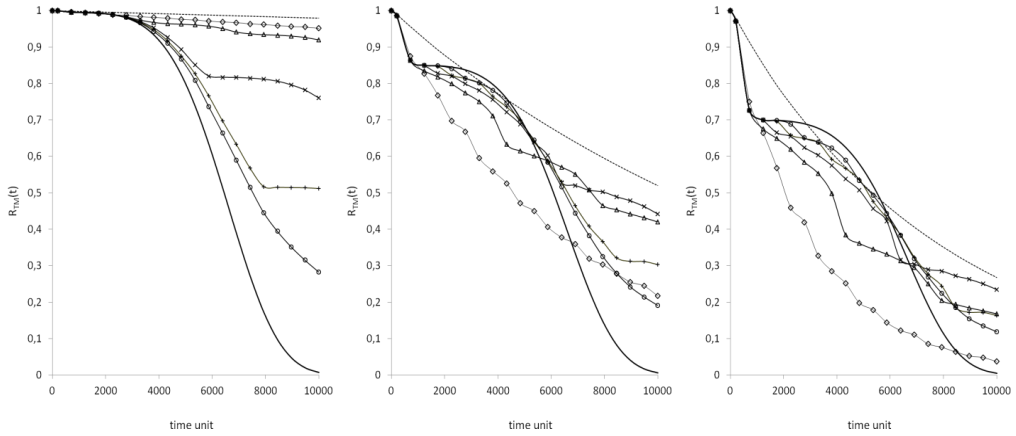
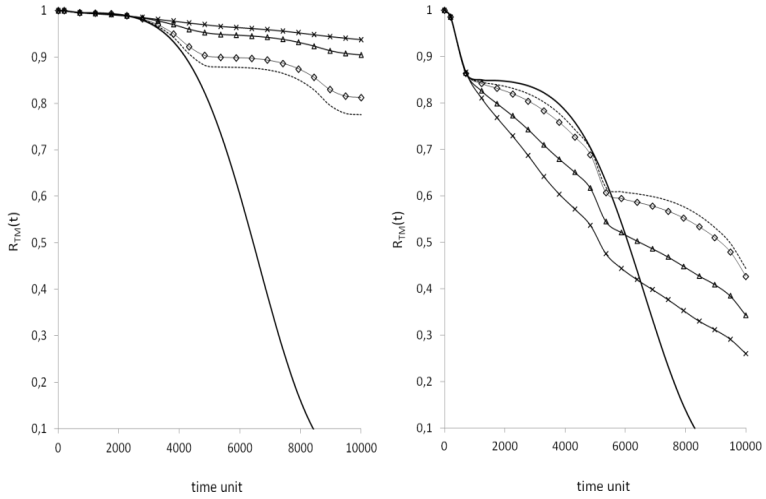


Figure 1. Cost-rate, $Q(T,M)$, as a function of T for M=4 (—+—); M =5 (—o—); M =6 (- - -); M =7(—□—); M =8 (—Δ—); M =9(—◊—); M=10(— X—).
Parameter values: s_1 =2.5, η_1 = 500, s_2 =4.5 , η_2 =7000, p = 0.10, α = 0.2, β = 0.2, c_0 = 5, c_m = 55 c_r =105, c_D = 1.35.



(a) (b) (c)

Figure 2. Reliability function $R_{TM}(t)$ as a function of t for M=5 and α =0.2, for various values of T: T = 50 (- - - -) ; T = 266 (—◊—); T = 700 (—Δ—) ; T = 1133 (—X—); T = 1566 (— + —); T = 2000 (—○—) ; baseline reliability $R(t)$ (——). Discrete mixture (a) p = 0.005; (b) p = 0.15; (c) p = 0.3; other parameter values: s_1 =2.5, η_1 =500, s_2 =4.5, η_2 =7000; costs as base case.

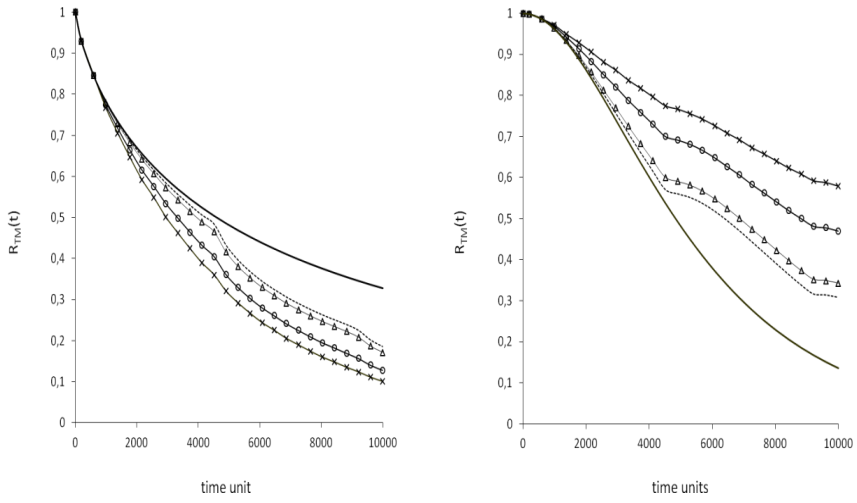


(a) (b)

Figure 3. Reliability function $R_{TM}(t)$ as a function of t for M=7 and T= 655, for various values of α: α =0.05 (- - - -); α =0.1 (—◊—); α =0.3 (—Δ—); α =0.5 (—X—); baseline reliability $R(t)$ (——). Discrete mixture: (a) p=0.005; (b) p=0.15. Other parameter values: s_1 =2.5, η_1 =500, s_2 =4.5 , η_2 =7000; costs as base case.

Table II: Continuous mixture: Weibull-inverse Gaussian. Minimum cost-rate policy for various parameter values.

Case	Mixed failure distribution parameters		Cost parameters						False -ve and +ve probabilities		Optimum policy		
	s	η	b	d	c_0	c_m	c_r	c_D	β	α	M^*	T^*	Q^*
1	2	5008	1	1	5	55	105	1.35	0.2	0.2	2	731	0.076
2	1.5	5008	1	1	5	55	105	1.35	0.2	0.2	∞	340	0.094
3	1	2897	1	1	5	55	105	1.35	0.2	0.2	∞	272	0.157
4	1	2897	1	1	5	55	105	1.35	0.4	0.2	∞	217	0.189
5	1	2897	1	1	5	55	105	1.35	0.6	0.2	2	348	0.236
6	1	2897	1	1	5	55	105	1.35	0.8	0.2	1	578	0.238
7	1	2897	1	1	5	55	105	1.35	0.2	0.4	∞	339	0.198
8	1	2897	1	1	5	55	105	1.35	0.2	0.6	∞	395	0.230
9	1	2897	1	1	5	55	105	1.35	0.2	0.8	1	578	0.238
10	1	2897	1	1	5	55	105	1.35	0.0	0.8	∞	529	0.218
11	1	2897	1	1	5	55	105	1.35	0.0	0.6	∞	475	0.195
12	1	2897	1	1	5	55	105	1.35	0.0	0.5	∞	444	0.182
13	1	2897	1	1	5	55	105	1.35	0.0	0.4	∞	412	0.168
14	2	5008	1	1	5	55	105	1.35	0.1	0.1	6	392	0.069



(a) (b)

Figure 4. Reliability function $R_{TM}(t)$ as a function of t for M=7 and T = 655, for various values of α: α=0.05 (- - -); α =0.1 (—Δ—); α =0.3 (—o—); and α =0.5 (—×—) ; baseline reliability $R(t)$ (——). Continuous mixture: (a) s = 0.8 (DFR), (b) s = 2 (IFR); other parameter values: b =1; d =1, η =5008; costs as base case.