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A Shape Preserving Class of Two-Frequency Trigonometric B-Spline Curves

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Abstract: This paper proposes a new approach to define two frequency trigonometric spline curves with interesting shape preserving properties. This construction requires the normalized B-basis of the space $U_4(I_\alpha) = \text{span}\{1, \cos t, \sin t, \cos 2t, \sin 2t\}$ defined on compact intervals $I_\alpha = [0, \alpha]$, where α is a global shape parameter. It will be shown that the normalized B-basis can be regarded as the equivalent in the trigonometric space $U_4(I_\alpha)$ to the Bernstein polynomial basis and shares its well-known symmetry properties. In fact, the normalized B-basis functions converge to the Bernstein polynomials as $\alpha \to 0$. As a consequence, the convergence of the obtained piecewise trigonometric curves to uniform quartic B-Spline curves will be also shown. The proposed trigonometric spline curves can be used for CAM design, trajectory-generation, data fitting on the sphere and even to define new algebraic-trigonometric Pythagorean-Hodograph curves and their piecewise counterparts allowing the resolution of $C^{(3}$ Hermite interpolation problems.

Keywords: trigonometric curves; B-splines; B-basis; total positivity

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1. Introduction

The definition of new spaces with more flexibility than polynomials, but with the same nice structural properties, such as variation diminishing, containment in the convex-hull, affine invariance, or tangency to the control-polygon at the endpoints, is an interesting trend in Computer-Aided Geometric Design (CAGD).

For a suitable basis $(b_0, \ldots, b_n)$ of a given space of functions defined on $I \subseteq \mathbb{R}$, $\gamma(t) = \sum_{i=0}^{n} P_i b_i(t)$, $t \in I$, provides a parametric representation of curves, where the coefficients are points in a given $\mathbb{R}^d$ determining a polygon $P_0 \cdots P_n$, which is called the control polygon of γ.

A system $(b_0, \ldots, b_n)$ of functions defined on I is normalized if $\sum_{i=0}^{n} b_i(t) = 1$ for all $t \in I$. On the other hand, a totally positive (TP) basis is a basis whose collocation matrices have nonnegative minors. We say that a basis provides a shape preserving representation if the shape of any parametric curve γ imitates the shape of its control polygon. Normalized and totally positive (NTP) bases provide shape preserving representations (cf. [1,2]). In [3], it was proved that a space of functions with NTP bases always has an optimal shape preserving basis that it is called the normalized B-basis and satisfies that the matrix of change of basis of any NTP basis with respect to the normalized B-basis is totally positive and stochastic. Roughly speaking, any parametric curve more faithfully imitates the shape of its control polygon with respect to the normalized B-basis than using other shape preserving representation.

The Bernstein polynomials of a given degree n on a compact interval $[a, b]$ are defined by

$$b_i^n(t) = \binom{n}{i}(t-a)^i(b-t)^{n-i}/(b-a)^n, \quad t \in [a,b], \quad i = 0,\ldots,n.$$

It is well-known that the Bernstein basis $(b_0^n, \ldots, b_n^n)$ is the normalized B-basis of the space of polynomials of degree at most n on the considered interval. The B-spline basis is the normalized B-basis in the case of the space of spline polynomials.

Trigonometric functions are usually considered for the representation of closed curves and periodical functions. The space of n order trigonometric polynomials,

$$U_{2n} := \text{span}\{1, \cos t, \sin t, \cos(2t), \sin(2t), \ldots, \cos(nt), \sin(nt)\},$$

is a classical space of functions with applications in CAGD in order to represent approximately parametric curves arising in civil engineering and robotics, among other fields. Traditionally these curves were approximated by polynomial and rational functions.

Due to the oscillation properties of trigonometric functions, the representation of curves using trigonometric spaces is not shape preserving for very long intervals. However, using a domain of a length $\alpha < \pi$, the space U_{2n} admits a normalized B-basis and solves several shortcomings of the algebraic polynomials. Consequently, it provides an alternative to the rational model for several purposes (see [4]). In the literature one can find many papers introducing shape parameters for a more flexible design of trigonometric curves (see [5–12]). For trigonometric polynomials in U_{2n}, the length α of the parameter domain can also be considered as a shape parameter with a tension-like effect in the parametric curves generated by the corresponding normalized B-bases. Trigonometric spline functions were introduced in [13] (see also [14]) and the recurrence relation for the trigonometric B-splines of arbitrary order was obtained in [15]. In [16,17], interested readers can find a novel technique based on the collocation finite element method for the numerical resolution of the wave equation in which trigonometric cubic B-splines are used as approximate functions.

Spline spaces provide more flexibility for curve design and then common curve and surface representations in CAGD use piecewise polynomials. Integral recurrence formulae for B-splines have been often used in the past. The definition of B-spline functions as a divided difference of a truncated power function and the Hermite–Gennochi formula lead to integral recursions. However, trigonometric polynomial spaces do not admit an integral construction. For this reason, this paper introduces NTP bases of U_{2n}, such that their derivatives at the ends of the interval of $[0, \alpha]$ allow us the definition of regular piecewise trigonometric curves. In [18], one-frequency trigonometric spline curves are defined in the space U_2. This paper proposes and analyzes piecewise trigonometric curves in U_4 providing two different frequencies for the design.

Polynomial Pythagorean-Hodograph (PH) curves have been widely analyzed (see [19–25] and their references). Polynomial PH curves are defined with Bernstein bases and possess a closed-form polynomial representation of their arc lengths, as well as an exact rational parameterization of their offset curves. New Pythagorean-Hodograph (PH) B-spline curves were proposed in [21]. The results in this paper will be considered in the near future to define new algebraic-trigonometric PH curves and their piecewise counterparts for the resolution of $C^{(3}$ Hermite interpolation problems. Moreover, the proposed trigonometric spline curves can also be used for CAM design, trajectory generation or data fitting on the sphere where the use of trigonometric splines provide better results than the conventional polynomial counterpart (see Chapter 12 of [1]).

The paper is organized as follows. Section 2 provides the normalized B-basis of the space $U_4(I_\alpha)$ defined on a given interval with a length less than π. A corner cutting algorithm for the evaluation and subdivision of curves generated in $U_4(I_\alpha)$ is also provided. This algorithm allows us to generate curves through a control polygon in a similar way to the Bézier case [26,27]. In Section 3, we introduce NTP bases of U_4 defined on appropriate intervals $[0, \alpha]$, and such that their derivatives at the ends of the interval allow us the definition of regular spline spaces and their bases called normalized T_4 B-spline basis. The matrices relating the normalized B-bases of U_4 and the spline spaces are also derived and their total positivity analyzed. Piecewise trigonometric curves with nice properties are also defined. Section 4 shows the convergence of the introduced trigonometric curves to polynomial B-spline curves as the parameter $\alpha \to 0$. Finally, Section 5 summarizes the conclusions and future work.

2. The Normalized B-Basis of the Space of Two Frequency Trigonometric Functions

Trigonometric polynomial spaces U_{2n} are invariant under translations and reflexions and, for this reason, they are usually analyzed on compact intervals $I_\alpha := [0, \alpha]$.

We shall consider the design of parametric trigonometric curves whose components are functions in the space

$$U_4(I_\alpha) = \mathrm{span}\{1, \cos(t), \sin t, \cos(2t), \sin(2t)\}, \quad t \in I_\alpha.$$

In Section 4.2 of [28], the normalized B-basis $(B_0^4, \ldots, B_4^4)$ of $U_4(I_\alpha)$ is given by:

$$\begin{aligned}
B_0^4(t) &:= \frac{(1-\cos(\alpha-t))^2}{(1-\cos\alpha)^2}, \\
B_1^4(t) &:= \frac{2(1-\cos(\alpha-t))(\cos t + \cos(\alpha-t) - \cos\alpha - 1)}{(1-\cos\alpha)^2}, \\
B_2^4(t) &:= \frac{2(1-\cos(\alpha-t))(1-\cos t) + (\cos t + \cos(\alpha-t) - \cos\alpha - 1)^2}{(1-\cos\alpha)^2}, \\
B_3^4(t) &:= \frac{2(1-\cos t)(\cos t + \cos(\alpha-t) - \cos\alpha - 1)}{(1-\cos\alpha)^2}, \\
B_4^4(t) &:= \frac{(1-\cos t)^2}{(1-\cos\alpha)^2},
\end{aligned} \tag{1}$$

for $t \in I_\alpha$ and $0 < \alpha < \pi$. Note that the normalized B-basis can be also defined as follows:

$$\begin{aligned}
B_0^4(t) &= \frac{1}{\sin^4(\alpha/2)} \sin^4((\alpha-t)/2), \quad B_4^4(t) = B_0^4(\alpha - t), \\
B_1^4(t) &= \frac{4\cos(\alpha/2)}{\sin^4(\alpha/2)} \sin^3((\alpha-t)/2)\sin(t/2), \quad B_3^4(t) = B_1^4(\alpha - t), \\
B_2^4(t) &= \frac{2(1+2\cos^2(\alpha/2))}{\sin^4(\alpha/2)} \sin^2((\alpha-t)/2)\sin^2(t/2).
\end{aligned} \tag{2}$$

The basis functions in (1), equivalently in (2), can be regarded as the equivalent to the Bernstein polynomial basis in $U_4(I_\alpha)$ due to the following properties:

1. Symmetry:
$$B_i^4(t) = B_{4-i}^4(\alpha - t), \quad i = 0, \ldots, 4, \quad t \in I_\alpha. \tag{3}$$

2. Positivity:
$$B_i^4(t) \geq 0, \quad i = 0, \ldots, 4, \quad t \in I_\alpha.$$

3. Partition of unity:
$$\sum_{i=0}^{4} B_i^4(t) = 1, \quad t \in I_\alpha.$$

Figure 1 shows the graphs of the normalized B-basis of $U_4(I_\alpha)$ for $\alpha = \pi/3$.

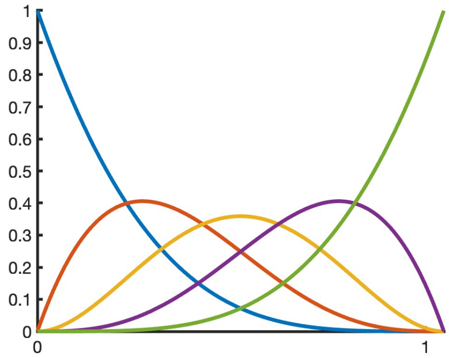


Figure 1. Normalized B-basis of $U_4(I_\alpha)$ for $\alpha = \pi/3$.

Let us note that the parameter α not only controls the length of the domain but also can be considered as a shape parameter with a tension-like effect in the parametric curves generated by the normalized B-basis of U_4. The following result will explain the behavior as $\alpha \to 0$. In order to prevent the normalized B-bases from losing their domain intervals I_α, we shall use a reparametrization so that the new functions are defined on $[0,1]$ allowing the parameter $\alpha \to 0$.

Lemma 1. *The normalized B-basis of $U_4(I_\alpha)$ converges uniformly to the Bernstein basis $(b_0^4,\dots,b_4^4)$ of degree 4 on the interval $[0,1]$ whenever $\alpha \to 0$.*

Proof. Using the change $t = \alpha\tau$, and developing by the Taylor expansion at $\tau = 0$, we have

$$b_4^4(\tau) - B_4^4(\alpha\tau) = \frac{4(1-\cos\alpha)^2 - \alpha^4}{4(1-\cos\alpha)^2}\tau^4 + \frac{2\alpha^6(16\cos(2\alpha\xi) - \cos(\alpha\xi))}{6!(\cos\alpha - 1)^2}\tau^6, \quad \tau \in [0,1],$$

for $\xi \in [0,\tau]$. Then

$$\left|b_4^4(\tau) - B_4^4(\alpha\tau)\right| \le \left|\frac{4(1-\cos\alpha)^2 - \alpha^4}{4(1-\cos\alpha)^2}\right| + \frac{\alpha^6}{(\cos\alpha - 1)^2}, \quad \tau \in [0,1],$$

and, since

$$\lim_{\alpha\to 0}\frac{4(1-\cos\alpha)^2 - \alpha^4}{4(1-\cos\alpha)^2} = 0, \quad \lim_{\alpha\to 0}\frac{\alpha^6}{(\cos\alpha - 1)^2} = 0,$$

we derive

$$\lim_{\alpha\to 0}\max_{0\le\tau\le 1}|b_4^4(\tau) - B_4^4(\alpha\tau)| = 0. \tag{4}$$

By definition, $b_0^4(\tau) = b_4^4(1-\tau)$, $B_0^4(\alpha\tau) = \widetilde{B}_4^4(\alpha(1-\tau))$ and therefore, using (4), we deduce that

$$\lim_{\alpha\to 0}\max_{0\le\tau\le 1}|b_0^4(\tau) - B_0^4(\alpha\tau)| = 0. \tag{5}$$

Using the Taylor expansion and a similar reasoning, we can also write

$$\lim_{\alpha\to 0}\max_{0\le\tau\le 1}|b_3^4(\tau) - B_3^4(\alpha\tau)| = 0, \quad \lim_{\alpha\to 0}\max_{0\le\tau\le 1}|b_1^4(\tau) - B_1^4(\alpha\tau)| = 0 \tag{6}$$

Finally, from the normalization property of the bases, $1 = \sum_{i=0}^4 b_i^4(\tau) = \sum_{i=0}^4 B_i^4(\alpha\tau)$, $\tau \in [0,1]$, and taking into account Formulaes (4)–(6), we conclude

$$\lim_{\alpha\to 0}\max_{0\le\tau\le 1}|b_2^4(t) - \widetilde{B}_2^4(t)| = 0. \tag{7}$$

□

Definition 1. *Let $0 < \alpha < \pi$ and $d \in \mathbb{N}$. Given $p_i \in \mathbb{R}^d$, $i = 0, \ldots, 4$, we say that the parametric trigonometric curve*

$$p_4(t) := \sum_{i=0}^{4} p_i B_i^4(t), \quad t \in I_\alpha, \tag{8}$$

is a T_4-curve.

Let us observe that, using the symmetry property (3), we can write

$$p_4(\alpha - t) = \sum_{i=0}^{4} p_{4-i} B_i^4(t), \quad t \in I_\alpha,$$

and derive that T_4-curves possess a symmetry similar to that of Bézier curves. Furthermore, since T_4-curves are curves expressed in terms of a normalized B-basis, they admit a de Casteljau-type algorithm, which is a corner cutting algorithm providing evaluation and subdivision.

Given a T_4-curve (8), and a parameter value $t \in [0, \alpha]$, the B-algorithm provides certain values $\lambda_i^k(t)$ that define the intermediate points $p_i^k(t)$ of a de Casteljau-like algorithm:

$$p_i^{k+1}(t) = (1 - \lambda_i^k(t)) p_i^k(t) + \lambda_i^k(t) p_{i+1}^k(t),$$

where $k = 0, 1, 2, 3$ and $i = 0, \ldots, 3 - k$. In fact, beginning with $p_i^0 = p_i$, this algorithm yields the final value

$$p_0^4 = p(t).$$

In addition, the two segments in which the parameter t divides the curve have control points given by $\{p_0^0, p_0^1, p_0^2, p_0^3, p_0^4\}$ and $\{p_0^4, p_1^3, p_2^3, p_3^1, p_4^0\}$, respectively. Here we provide compact expressions of $\lambda_i^k(t)$ for an arbitrary $t \in [0, \alpha]$:

$$\lambda_0^0(t) = \frac{\sin(t/2)}{\sin(\alpha/2)} \frac{\cos(\alpha/2)}{\cos(t/2)}, \quad \lambda_3^0(t) = 1 - \lambda_0^0(\alpha - t),$$

$$\lambda_1^0(t) = \frac{\sin(t/2)}{\sin(\alpha/2)} \frac{1 + 2\cos^2(\alpha/2)}{\cos((\alpha - t)/2) + 2\cos(\alpha/2)\cos(t/2)}, \quad \lambda_2^0(t) = 1 - \lambda_1^0(\alpha - t),$$

$$\lambda_0^1(t) = \frac{\sin(t/2)}{\sin(\alpha/2)} \frac{\cos((\alpha - t)/2) + 2\cos(\alpha/2)\cos(t/2)}{1 + 2\cos^2(t/2)}, \quad \lambda_2^1(t) = 1 - \lambda_0^1(\alpha - t),$$

$$\lambda_1^1(t) = \frac{\sin(t/2)}{\sin(\alpha/2)} \frac{3\cos(t/2) + 2\sin(\alpha/2)\sin((\alpha - t)/2)}{\cos(\alpha/2) + 2\cos(t/2)\cos((\alpha - t)/2)},$$

$$\lambda_0^2(t) = \frac{\sin(t/2)}{\sin(\alpha/2)} \frac{\cos(\alpha/2) + 2\cos(t/2)\cos((\alpha - t)/2)}{3\cos(t/2)}, \quad \lambda_1^2(t) = 1 - \lambda_0^2(\alpha - t),$$

$$\lambda_0^3(t) = \frac{\sin(t/2)}{\sin(\alpha/2)} \cos((\alpha - t)/2). \tag{9}$$

Taking into account Lemma 1, whenever $\alpha \to 0$, the T_4-curve (8) degenerates to an integral Bézier curve with control points p_i, $i = 0, \ldots, 4$, and the coefficients $\lambda_i^k(t)$ of the algorithm degenerate to t/α (see Figure 2). In other words, the B-algorithm reduces to the standard de Casteljau algorithm. This property can be also checked by introducing the Taylor expansion of the functions λ_i^k and taking limits.

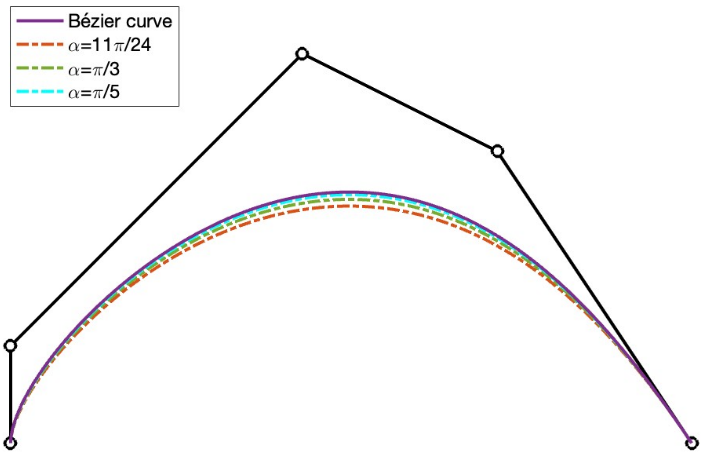


Figure 2. $T4$-curves (8) associated to a control polygon with several parameters α, as well as the Bézier curve corresponding to the mentioned control polygon.

3. Two Frequency Trigonometric Spline Curves

Now, we shall introduce NTP bases of $U_4(I_\alpha)$ such that their derivatives up to the third order at the ends of the interval of $[0, \alpha]$ allow us the definition of regular piecewise curves. We shall use the following notation:

$$C_{p,q} := p + q\cos\alpha, \quad C_{p,q,r} := p + q\cos\alpha + r\cos^2\alpha,$$

for given $p, q, r \in \mathbb{N}$.

Let us define the following systems of functions on I_α:

$$(N_0^4, \ldots, N_4^4) := (B_0^4, \ldots, B_4^4)A_4, \qquad (N_0^{j,4}, \ldots, N_4^{j,4}) := (B_0^4, \ldots, B_4^4)A_{j,4}, \quad j = 0, 1, 2, \quad (10)$$

where $(B_0^4, \ldots, B_4^4)$ is the normalized B-basis of $U_4(I_\alpha)$ (see (1) or (2)),

$$A_4 := \begin{pmatrix} 12C_{1,1} & 0 & 0 & 0 & 0 \\ 0 & 6C_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 2C_{2,1} & 0 & 0 \\ 0 & 0 & 0 & 6C_{1,1} & 0 \\ 0 & 0 & 0 & 0 & 12C_{1,1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & C_{5,6} & C_{5,6} & 1 & 0 \\ 0 & 2C_{1,1} & C_{3,4} & 1 & 0 \\ 0 & 1 & 2C_{1,1} & 1 & 0 \\ 0 & 1 & C_{3,4} & 2C_{1,1} & 0 \\ 0 & 1 & C_{5,6} & C_{5,6} & 1 \end{pmatrix}, \quad (11)$$

and the matrices $A_{j,4}$, for $j = 0, 1, 2$, are defined as follows:

$$A_{0,4} := D_0 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & C_{1,2} & 0 & 0 \\ 0 & 3 & C_{3,4} & 2C_{0,1} & 0 \\ 0 & 3C_{1,2} & C_{7,18,12} & C_{1,10,12} & C_{1,2} \end{pmatrix}, \quad (12)$$

$$D_0 := \operatorname{diag}(1, 1, 2C_{2,1}, 6C_{1,1}, 12C_{1,1}C_{1,2})^{-1},$$

$$A_{1,4} := D_1 \begin{pmatrix} 3C_{1,2} & C_{7,18,12} & C_{1,10,12} & C_{1,2} & 0 \\ 0 & 4C_{1,1}^2 & C_{1,8,8} & C_{1,2} & 0 \\ 0 & 2C_{1,1} & C_{1,6,4} & C_{1,2} & 0 \\ 0 & 1 & C_{1,4} & C_{1,2} & 0 \\ 0 & 2C_{1,1} & 4C_{1,3}C_{1,1} & C_{5,6}C_{1,2} & C_{1,2} \end{pmatrix}, \quad (13)$$

$$D_1 := \operatorname{diag}(12C_{1,1}C_{1,2}, 6C_{1,1}C_{1,2}, 2C_{2,1}C_{1,2}, 3C_{1,2}, 12C_{1,1}C_{1,2})^{-1},$$

$$A_{2,4} := D_2 \begin{pmatrix} 2C_{1,1} & 4C_{1,3}C_{1,1} & C_{5,6}C_{1,2} & C_{1,2} & 0 \\ 0 & 2C_{1,1} & C_{3,4} & 1 & 0 \\ 0 & 1 & 2C_{1,1} & 1 & 0 \\ 0 & 1 & C_{3,4} & 2C_{1,1} & 0 \\ 0 & 1 & C_{5,6} & C_{5,6} & 1 \end{pmatrix}, \quad (14)$$

$$D_2 := \operatorname{diag}(12C_{1,1}C_{1,2}, 6C_{1,1}, 2C_{2,1}, 6C_{1,1}, 12C_{1,1})^{-1}.$$

The following result proves that the systems (10) are all NTP bases of $U_4(I_\alpha)$.

Theorem 1. *For $0 < \alpha < \pi/2$, the systems $(N_0^4, \dots, N_4^4)$ and $(N_0^{j,4}, \dots, N_4^{j,4})$, $j = 0, 1, 2$, defined in Formula (10) are NTP basis of $U_4(I_\alpha)$.*

Proof. It can be easily checked that the matrix A_4 in (11), as well as the matrices $A_{j,4}$, $j = 0, 1, 2$, in (12), (13) and (14), respectively, are stochastic for $0 < \alpha < \pi/2$. Moreover, since for $0 < \alpha < \pi/2$

$$\det(A_4) = \frac{1}{5184} \frac{C_{0,1}C_{1,2}^2}{C_{1,1}^4 C_{2,1}} \neq 0,$$

and

$$\det(A_{0,4}) = \frac{1}{72} \frac{C_{0,1}C_{1,2}}{C_{1,1}^2 C_{2,1}} \neq 0, \ \det(A_{1,4}) = \frac{1}{864} \frac{C_{0,1}C_{1,2}}{C_{1,1}^3 C_{2,1}} \neq 0, \ \det(A_{2,4}) = \frac{1}{2592} \frac{C_{0,1}C_{1,2}}{C_{1,1}^3 C_{2,1}} \neq 0,$$

we conclude that they are also nonsingular matrices and the introduced systems are all bases of $U_4(I_\alpha)$.

Since $(B_0^4, \dots, B_4^4)$ is the normalized B-basis of $U_4(I_\alpha)$ (see [4]) , by Corollary 3.9 (iv) of [3], it remains to prove that the matrices are TP. Let us recall that Neville elimination is an alternative procedure to Gaussian elimination and has been used to characterize TP and STP matrices. Applying the Neville elimination to the matrix A_4 in (11), we can factorize A_4 as follows:

$$A_4 = F_{3,4}F_{2,4}F_{1,4}G_{1,4}G_{2,4}G_{3,4}, \quad (15)$$

with

$$F_{3,4} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}, F_{2,4} := \phi_{2,4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & C_{2,1} & C_{1,2} & 0 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}, F_{1,4} := \phi_{1,4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & C_{1,2} & 0 & 0 \\ 0 & 0 & C_{1,1} & C_{0,1} & 0 \\ 0 & 0 & 0 & C_{1,2} & 1 \end{pmatrix},$$

and

$$\phi_{2,4} := \operatorname{diag}(1, 1, 1, 3C_{1,1}, 1)^{-1},$$
$$\phi_{1,4} := \operatorname{diag}(1, 1, 2C_{2,1}, C_{1,2}, 2C_{1,1})^{-1}.$$

Additionally,

$$G_{1,4} := \psi_{1,4} \begin{pmatrix} 1 & C_{11,12} & 0 & 0 & 0 \\ 0 & C_{11,12} & 2C_{2,3} & 0 & 0 \\ 0 & 0 & 2C_{2,3} & C_{5,6} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, G_{2,4} := \psi_{2,4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & C_{5,6} & 6C_{1,1} & 0 & 0 \\ 0 & 0 & 3C_{1,2} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, G_{3,4} := \psi_{3,4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & C_{5,6} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with

$$\psi_{1,4} := \operatorname{diag}(12C_{1,1}, C_{15,18}, C_{5,6}, 1, 1)^{-1},$$
$$\psi_{2,4} := \operatorname{diag}(1, C_{11,12}, 2C_{2,3}, 1, 1)^{-1},$$
$$\psi_{3,4} := \operatorname{diag}(1, 1, 6C_{1,1}, 1, 1)^{-1}.$$

It can be easily checked that, for $0 < \alpha < \pi/2$, the entries of the matrix factors in (15) are nonnegative and consequently, the bidiagonal matrices $F_{k,4}$ and $G_{k,4}$, $k = 1, 2, 3$ are TP. Taking into account that, by Theorem 3.1 of [29], the product of TP matrices is a TP matrix, we conclude that A_4 is TP.

Now, we provide the bidiagonal factorization obtained by applying Neville elimination to the matrices $A_{j,4}$, $j = 0, 1, 2$. It can be checked that

$$A_{0,4} = F_{3,4}F_{2,4}F_{1,4}, \tag{16}$$

where $F_{3,4}$, $F_{2,4}$ and $F_{1,4}$ are the nonsingular, stochastic bidiagonal lower triangular matrices in the factorization (15). Furthermore,

$$A_{1,4} = F_{3,4}F_{2,4}F_{1,4}G^{(1)}_{1,4}G^{(1)}_{2,4}G^{(1)}_{3,4}, \tag{17}$$

where $F_{3,4}$, $F_{2,4}$ and $F_{1,4}$ are the matrices in the factorization (15) and

$$G^{(1)}_{1,4} := \psi^{(1)}_{1,4} \begin{pmatrix} 1 & C_{3,4} & 0 & 0 & 0 \\ 0 & 2C_{1,1}C_{3,4} & (C_{1,2})^2 & 0 & 0 \\ 0 & 0 & 3(C_{1,2})^2 & 2C_{0,1} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad G^{(1)}_{2,4} := \psi^{(1)}_{2,4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & C_{7,18,12} & 2C_{1,6,6} & 0 & 0 \\ 0 & 0 & C_{1,6,6} & C_{7,18,12} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$G^{(1)}_{3,4} := \psi^{(1)}_{3,4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & C_{1,10,12} & C_{1,2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

with

$$\psi_{1,4}^{(1)} := \mathrm{diag}(4C_{1,1}, C_{7,18,12}, C_{3,14,12}, 1, 1)^{-1},$$
$$\psi_{3,4}^{(1)} := \mathrm{diag}(1, 1, 2C_{1,6,6}, 1, 1)^{-1},$$
$$\psi_{2,4}^{(1)} := \mathrm{diag}(1, 3C_{1,2}C_{3,4}, 3C_{1,1}C_{1,2}C_{1,10,12}, 1, 1)^{-1}.$$

Finally,

$$A_{2,4} = F_{3,4}F_{2,4}F_{1,4}G_{1,4}^{(2)}G_{2,4}^{(2)}G_{3,4}^{(2)}, \tag{18}$$

where $F_{3,4}$, $F_{2,4}$ and $F_{1,4}$ are the matrices in the factorization (15) and

$$G_{1,4}^{(2)} := \psi_{1,4}^{(2)} \begin{pmatrix} 1 & C_{5,12} & 0 & 0 & 0 \\ 0 & C_{5,12} & C_{1,6} & 0 & 0 \\ 0 & 0 & C_{1,2}C_{1,6} & 2C_{0,1} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad G_{2,4}^{(2)} := \psi_{2,4}^{(2)} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2C_{1,3} & 3C_{1,2} & 0 & 0 \\ 0 & 0 & 3C_{1,10,12} & 2C_{1,3} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G_{3,4}^{(2)} := \psi_{3,4}^{(2)} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & C_{5,6} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with

$$\psi_{1,4}^{(2)} := \mathrm{diag}(6C_{1,2}, 6C_{1,3}, C_{1,10,12}, 1, 1)^{-1},$$
$$\psi_{2,4}^{(2)} := \mathrm{diag}(1, C_{5,12}, C_{5,36,36}, 1, 1)^{-1},$$
$$\psi_{3,4}^{(2)} := \mathrm{diag}(1, 1, 6C_{1,1}, 1, 1)^{-1}.$$

It can be easily checked that the bidiagonal matrices of the factorization (16), (17) and (18) are TP for $0 < \alpha < \pi/2$ and the result follows. □

We shall see that the NTP bases in (10) allow us to define T_4-B-spline bases and T_4-B-spline curves, that is, piecewise functions and curves, respectively, on $U_4(I_\alpha)$. In the following result, the derivatives up to the third order of the functions of these NTP bases are provided.

Lemma 2. *The functions N_i^4, $N_i^{j,4}$, $j = 0, 1, 2$, $i = 0, \ldots, 4$, of the NTP bases of $U_4(I_\alpha)$ defined in (10) satisfy the following properties (see Figure 3):*

1. $\sum_{i=0}^4 N_i^4(t) = 1$, $\sum_{i=0}^4 N_i^{j,4}(t) = 1$, $t \in I_\alpha$.
2. $(N_0^4)^{(k)}(\alpha) = 0$, $(N_0^{j,4})^{(k)}(\alpha) = 0$, $(N_4^4)^{(k)}(0) = 0$, $(N_4^{j,4})^{(k)}(0) = 0$ *for* $j = 0, 1, 2$.
3. $(N_i^4)^{(k)}(0) = (N_{i+1}^4)^{(k)}(\alpha), \quad k = 0, \ldots, 3.$
4. $(N_i^4)^{(k)}(0) = (N_{i+1}^{2,4})^{(k)}(\alpha), \quad k = 0, \ldots, 3.$
5. $(N_i^{2,4})^{(k)}(0) = (N_{i+1}^{1,4})^{(k)}(\alpha), \quad k = 0, \ldots, 3.$
6. $(N_i^{1,4})^{(k)}(0) = (N_{i+1}^{0,4})^{(k)}(\alpha), \quad k = 0, \ldots, 3.$

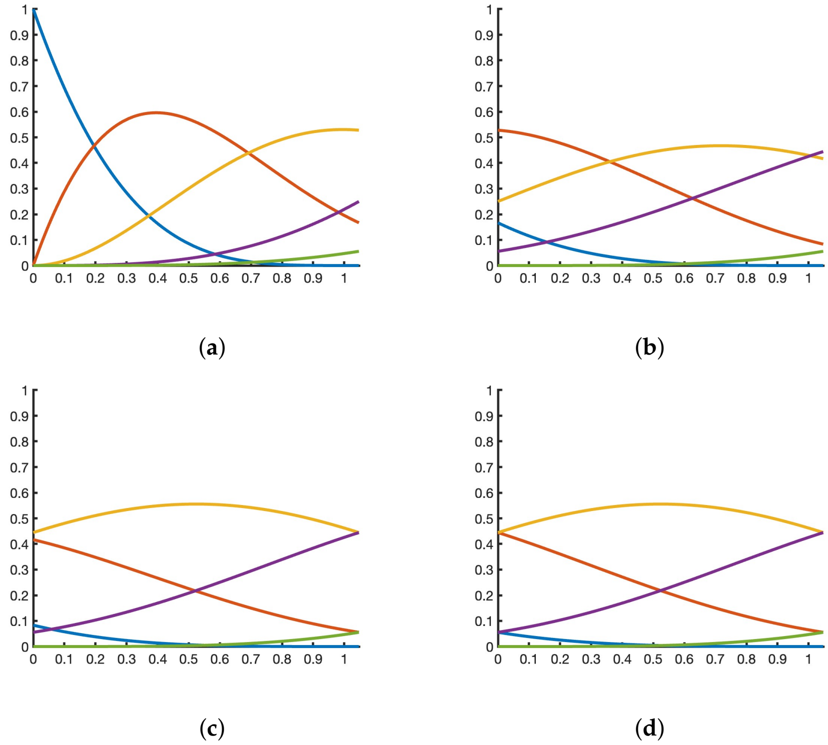


Figure 3. NTP bases defined in (10) for $\alpha = \frac{\pi}{3}$. (**a**) $(N_0^{0,4}, \ldots, N_4^{0,4})$, (**b**) $(N_0^{1,4}, \ldots, N_4^{1,4})$, (**c**) $(N_0^{2,4}, \ldots, N_4^{2,4})$, (**d**) $(N_0^{4}, \ldots, N_4^{4})$.

Given $p \in \mathbb{N}$, $p \geq 4$, we are going to consider the equally spaced partition

$$\boldsymbol{\pi} := \{u_i\}_{i=0}^{p+5} = \{i\,\alpha\}_{i=0}^{p+5}, \tag{19}$$

as well as the partition

$$\boldsymbol{\mu} := \{u_i\}_{i=0}^{p+5}, \tag{20}$$

with

$$0 = u_0 = \cdots = u_4 < u_5 < \cdots < u_p < u_{p+1} = \cdots = u_{p+5},$$

and

$$u_k = (k-4)\alpha, \quad k = 4, \ldots, p+1.$$

Now, we define piecewise functions on either partition $\boldsymbol{\pi}$ or $\boldsymbol{\mu}$. When considering the equally spaced partitions $\boldsymbol{\pi}$, trigonometric B-spline bases will be obtained for the shape-preserving representation of closed curves. On the other hand, using the piecewise functions on partitions $\boldsymbol{\mu}$, clamped trigonometric curves satisfying the tangency to the control-polygon will be defined.

For any $i = 0, \ldots, p$, let

$$N_{i,4}(u) := \begin{cases} N_j^4(u - u_{i+4-j}), & u \in [u_{i+4-j}, u_{i+5-j}), \quad j = 0, \ldots, 4, \\ 0, & \text{else}. \end{cases} \tag{21}$$

Moreover, for the partition $\boldsymbol{\mu}$, we consider the following piecewise functions,

$$\tilde{N}_{i,4}(u) := \begin{cases} N_j^{i-j,4}(u - u_{i+4-j}), & u \in [u_{i+4-j}, u_{i+5-j}), \quad 0 \leq j \leq i, \\ 0, & \text{else}, \end{cases} \tag{22}$$

for $i = 0, \ldots, 3$, with the convention $N_0^{3,4} := N_0^4$, and

$$\tilde{N}_{i,4}(u) := \begin{cases} N_j^{p-(i+4-j),4}(u - u_{i+4-j}), & u \in [u_{i+4-j}, u_{i+5-j}), \quad i+4-p \le j \le 4, \\ 0, & \text{else}, \end{cases} \tag{23}$$

for $i = p-3, \ldots, p$, with the convention $N_4^{3,4} := N_4^4$. In Figure 4 the piecewise functions $N_{i,4}(u)$, $\tilde{N}_{i,4}(u)$, $i = 0, \ldots, 3$ are depicted.

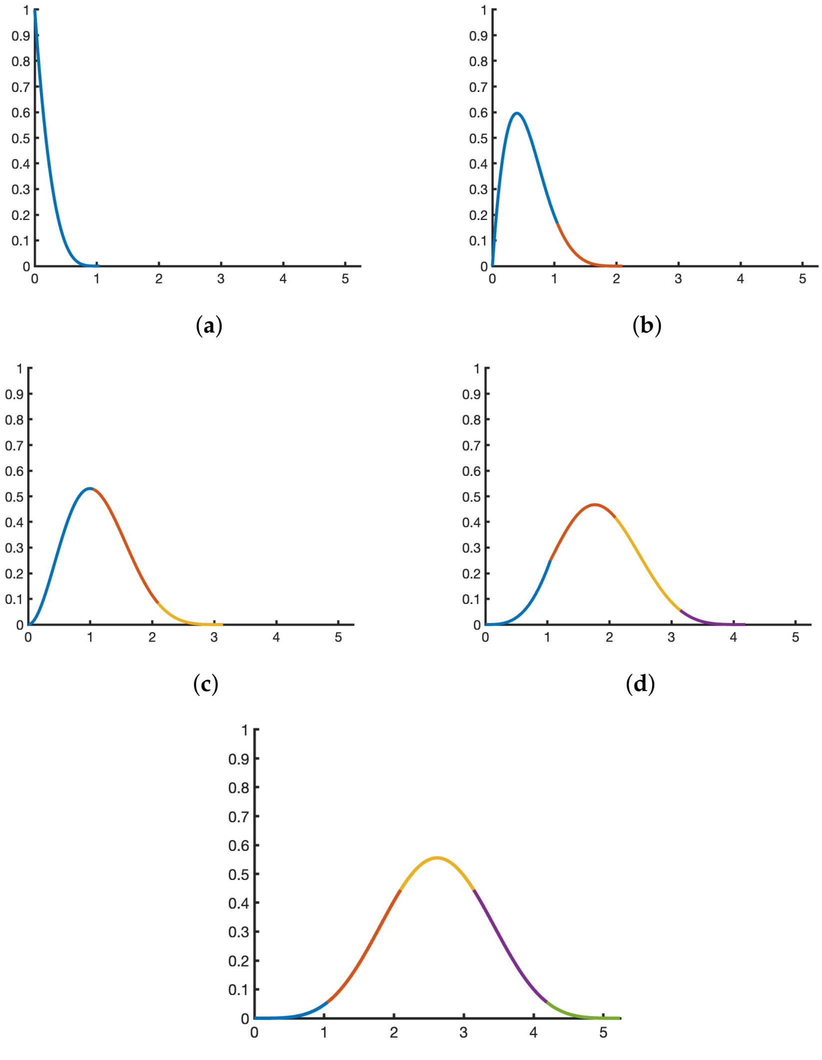


Figure 4. Trigonometric piecewise functions (**a**) $\tilde{N}_{0,4}(u)$, (**b**) $\tilde{N}_{1,4}(u)$, (**c**) $\tilde{N}_{2,4}(u)$, (**d**) $\tilde{N}_{3,4}(u)$ and (**e**) $N_{i,4}(u)$, for $\alpha = \frac{\pi}{3}$.

The following properties of the above introduced piecewise functions can be deduced from their definition and taking into account Theorem 1 and Lemma 2.

Proposition 1. *The functions $N_{i,4}$, $i = 0, \ldots, p$, defined in (21), and the functions $\tilde{N}_{i,4}$, $i = 0, \ldots, 3$ and $i = p-3, \ldots, p$, defined in (22) and (23) satisfy the following properties:*

1. *All the mentioned functions are piecewise trigonometric functions of the space $U_4(I_\alpha)$.*
2. *The functions $N_{i,4}(u)$ are symmetrical with respect to the middle of their supports and they can be obtained by translation, i.e.,*

$$N_{i,4}(u) = N_{0,4}(u - u_i).$$

3. $N_{i,4}(u) > 0$ *and* $\tilde{N}_{i,4}(u) > 0$ *for* $u \in (u_i, u_{i+5})$ *and all applicable indices i. In fact,* $N_{i,4}$ *and* $\tilde{N}_{i,4}$ *have minimal support* $[u_i, u_{i+5}]$.
4. $N_{i,4}(u)|_{[u_l,u_{l+1}]} \neq 0$ *for* $i = l-4,\ldots,l$.
5. *On the partitions* $\boldsymbol{\pi}$ *and* $\boldsymbol{\mu}$,

$$\sum_{i=0}^{p} N_{i,4}(u) = 1, \quad u \in [u_4, u_{p+1}].$$

6. *On the partitions* $\boldsymbol{\pi}$ *and* $\boldsymbol{\mu}$*, the functions* $N_{i,4}(u)$ *for* $i = 0,\ldots,p$ *and* $\tilde{N}_{i,4}(u)$ *for* $i = 0,\ldots,3$ *and* $i = p-3,\ldots,p$ *are* $C^{j-mult(u_k)}$*-continuous, where* $mult(u_k)$ *is the multiplicity of the knot* u_k *in the support of the respective function.*

Due to the analogy to the well-known polynomial B-splines, we will say that the introduced piecewise functions are T_4-B-splines.

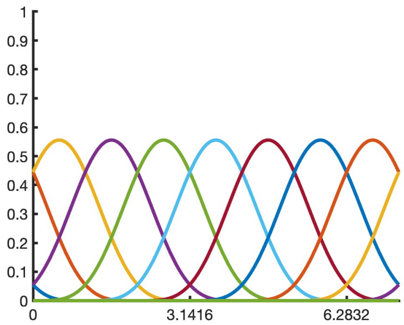
Definition 2. *Given* $p \in \mathbb{N}$, $p \geq 4$*, and partitions* $\boldsymbol{\pi}$ *and* $\boldsymbol{\mu}$ *from* (19) *and* (20)*, respectively. We say that* $(N_{0,4,\boldsymbol{\pi}},\ldots,N_{p,4,\boldsymbol{\pi}})$ *with*

$$N_{i,4,\boldsymbol{\pi}}(u) := N_{i,4}(u), \quad i = 0,\ldots,p,$$

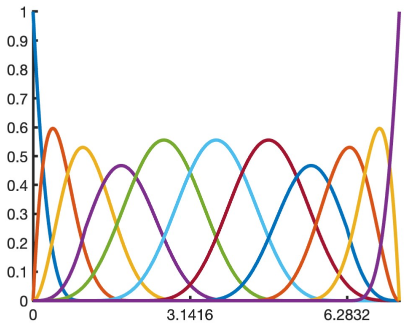
and $N_{i,4}$ *defined in* (21)*, is the normalized* T_4*-B-spline basis over the partition* $\boldsymbol{\pi}$ *(see Figure 5a for an illustration). On the other hand, we say that* $(N_{0,4,\boldsymbol{\mu}},\ldots,N_{p,4,\boldsymbol{\mu}})$ *with*

$$N_{i,4,\boldsymbol{\mu}}(u) := N_{i,4}(u), \quad i = 0,\ldots,p,$$

and $N_{i,4} = \tilde{N}_{i,4}$ *defined in* (22)*, for* $i = 0,\ldots,3$*, and defined in* (23)*, for* $i = p-3,\ldots,p$*, is the normalized* T_4*-B-splines over the partition* $\boldsymbol{\mu}$ *(see Figure 5b for an illustration).*



(**a**)



(**b**)

Figure 5. Normalized T_4-B-splines over a partition $\boldsymbol{\pi}$ (**a**), and $\boldsymbol{\mu}$ (**b**) for $\alpha = \frac{\pi}{3}$.

The following properties guarantee shape preserving properties of normalized T_4-B-spline systems for knot partitions $\boldsymbol{\mu}$ and $\boldsymbol{\pi}$, with $0 < \alpha < \pi/2$.

Proposition 2. *Given* $0 < \alpha < \pi/2$*, the* T_4*-B-spline functions* $N_{i,4,\boldsymbol{\pi}}$ *and* $N_{i,4,\boldsymbol{\mu}}$, $i = 0,\ldots,p$*, introduced in Definition 2, satisfy the following properties:*

1. $(N_{0,4,\boldsymbol{\pi}},\ldots,N_{p,4,\boldsymbol{\pi}})$ *is an NTP basis of the generated space* $\mathcal{U}$ *of piecewise functions on* $U_4(I_\alpha)$ *defined on* $[u_4, u_{p+1}]$.
2. $(N_{0,4,\boldsymbol{\mu}},\ldots,N_{p,4,\boldsymbol{\mu}})$ *is the normalized B-basis of* $C^{(3}$ *functions on* $[u_0, u_{p+5}]$.

Proof. Let $(N_0^4,\ldots,N_4^4)$ and $(N_0^{j,4},\ldots,N_4^{j,4})$, $j = 0,1,2$, be the NTP bases of $U_4(I_\alpha)$ defined in (10). Given $u \in [u_i, u_{i+1})$,

$$N_{i-4+j,\boldsymbol{\pi}}(u) = N_j^4(u-u_i), \quad j = 0,\ldots,4,$$

for $i = 4, \ldots, p$, and $j = 0, \ldots, 4$, we have

$$N_{i-4+j,\boldsymbol{\mu}}(u) = \begin{cases} N_j^{i-4,4}(u-u_i), & i = 4,5,6, \\ N_j^4(u-u_i), & i = 7,\ldots,p-3, \\ N_j^{p-i,4}(u-u_i), & i = p-2, p-1, p. \end{cases}$$

Consequently, taking into account Theorem 1, we deduce that the restrictions to $[u_i, u_{i+1})$ of $(N_{0,\boldsymbol{\pi}}, \ldots, N_{p,\boldsymbol{\pi}})$ and the restrictions to $[u_i, u_{i+1})$ of $(N_{0,\boldsymbol{\mu}}, \ldots, N_{p,\boldsymbol{\mu}})$ are NTP bases of $U_4(I_\alpha)$ for $0 < \alpha < \pi/2$. Then, it can be deduced that $(N_{0,\boldsymbol{\pi}}, \ldots, N_{p,\boldsymbol{\pi}})$ and $(N_{0,\boldsymbol{\mu}}, \ldots, N_{p,\boldsymbol{\mu}})$ are NTP bases of the corresponding generated space $\mathcal{S}$ of piecewise functions defined on $[u_4, u_{p+1}]$.

In addition, the basis $(N_{0,\boldsymbol{\mu}}, \ldots, N_{p,\boldsymbol{\mu}})$ also satisfies

$$\lim_{t \to u_{j-4}^+} (N_{k,\boldsymbol{\mu}}(t)/N_{j,\boldsymbol{\mu}}(t)) = 0, \qquad \lim_{t \to u_{k+1,\boldsymbol{\mu}}^-} (N_{j,\boldsymbol{\mu}}(t)/N_{k,\boldsymbol{\mu}}(t)) = 0,$$

whenever $0 \le j < k \le p$. Then, by Theorem 3.2 of Chapter 4 of [1], $(N_{0,\boldsymbol{\mu}}, \ldots, N_{p,\boldsymbol{\mu}})$ is the normalized B-basis of $\mathcal{U}$. □

The previous result implies that the T_4-B-spline basis has optimal shape preserving properties (see [3] and Chapter 4 of [1]) and it also has optimal stability properties for the evaluation (cf. Chapter 5 of [1]) Now, we can define the corresponding piecewise trigonometric curves.

Definition 3. *Given $d, p \in \mathbb{N}$, $p \ge 4$, let $s_i \in \mathbb{R}^d$, $i = 0, \ldots, p$, and partitions $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$ from (19) and (20), respectively. The parametric curve defined by*

$$\mathrm{s}(u) := \sum_{i=0}^{p} s_i N_{i,4,\boldsymbol{\pi}}(u), \quad u \in [u_4, u_{p+1}), \tag{24}$$

is called T_4-B-spline curve with respect to the partition $\boldsymbol{\pi}$ and control points $s_0, \ldots, s_p$. In particular, if $p = m$, with $m \in \mathbb{N}$, $m \ge 4$, then we say that $\mathrm{s}(u)$ is an open T_4-B-Spline curve. If $p = m + 4$ and $s_{m+1+i} = s_i$, $i = 0, \ldots, 3$, we say that $\mathrm{s}(u)$ is a closed T_4-B-Spline curve. See Figure 6a for an illustration of a closed T_4-B-Spline curve.

The parametric curve defined by

$$\mathrm{s}(u) := \sum_{i=0}^{p} s_i N_{i,4,\boldsymbol{\mu}}(u), \quad u \in [u_4, u_{p+1}), \tag{25}$$

is called clamped T_4-B-spline curve with respect to the partition $\boldsymbol{\mu}$ and control points $s_0, \ldots, s_p$. See Figure 6b for an illustration of a clamped T_4-B-Spline curve.

T_4-B-Spline curves satisfy the following properties.

Proposition 3. *The T_4-B-Spline curves described in Definition 2 satisfy:*

1. *The relation between a T_4-B-Spline curve and its control points is affinely invariant.*
2. *Any T_4-B-Spline curve $\mathrm{s}(u)$ is locally controlled, i.e., moving a control point s_l only modifies the curve for $u \in [u_l, u_{l+5})$, moreover for $\boldsymbol{\tau} = \boldsymbol{\pi}$ or $\boldsymbol{\tau} = \boldsymbol{\mu}$ we have*

$$\mathrm{s}(u)|_{u \in [u_l, u_{l+1})} = \sum_{i=l-4}^{l} s_i N_{i,4,\boldsymbol{\tau}}(u), \tag{26}$$

   and the curve $\mathrm{s}(u)$ lies in the convex hull of its control points s_i, $i = l-4, \ldots, l$.
3. *The T_4-B-Spline curves are monotonicity preserving: the curve has the same monotonicity as the monotone control points.*

4. *The length of a T_4-B-Spline curve is bounded above by the length of its control polygon.*
5. *If the control polygon of a T_4-B-Spline curve is planar and convex, then the T_4-B-Spline curve is also planar and convex.*
6. *The T_4-B-Spline curve never crosses a hyperplane more often than does the control polygon.*
7. *Clamped T_4-B-Spline curves have end point and end tangent interpolation properties:*

$$\begin{aligned} &\mathrm{s}(0) = s_0, \quad \mathrm{s}(u_{m+1}) = s_m, \\ &s'(0) = \cot(\alpha/2)(s_1 - s_0), \quad s'(u_{m+1}) = \cot(\alpha/2)(s_m - s_{m-1}). \end{aligned}$$

Finally, let us note that Definition 1, relations (10), the corner cutting algorithm for T_4-curves in (9) and the matrix factorizations (11)–(14) result in a corner cutting algorithm for T_4-B-Spline curves analogous to the one for T_2-B-Spline curves detailed in [18].

4. Convergence of T_4-B-Spline Curves to Quartic Polynomial B-Spline Curves

In this section we prove the convergence, when $\alpha \to 0$, of T_4-B-Spline curves to polynomial B-spline curves.

Theorem 2. *Let $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$ partitions described in (19) and (20), respectively. When $\alpha \to 0$, the T_4-B-Spline curve (24) and the clamped T_4-B-Spline curve (25), with respect to $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$, respectively, and control points $\mathbf{s}_0, \ldots, \mathbf{s}_p$ approaches uniformly to the quartic polynomial B-spline curve with knot vector $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$, respectively and control points $\mathbf{s}_0, \ldots, \mathbf{s}_p$.*

Proof. Let us observe that, by (26), for $\boldsymbol{\tau} = \boldsymbol{\pi}$ or $\boldsymbol{\tau} = \boldsymbol{\mu}$, we can write

$$\mathrm{s}(u)|_{u\in[u_l,u_{l+1})} = \sum_{i=l-4}^{l} s_i N_{i,4,\boldsymbol{\tau}}(u) = \sum_{j=0}^{4} s_{l+j-4} N_j^4(u - u_l).$$

Let $\tau := (u - u_l)/\alpha$ for reparameterizing each segment curve on the interval $0 \le \tau \le 1$. By Lemma 1, as $\alpha \to 0$, the function $B_i^4(\alpha\tau)$ approaches uniformly the Bernstein polynomial $b_i^4(\tau)$, $0 \le \tau \le 1$, for all $i = 0, \ldots, 4$. It can be easily checked that that the matrix A_4 in (11) satisfies

$$\lim_{\alpha\to 0} A_4 = A, \quad A := \begin{pmatrix} 1/24 & 11/24 & 11/24 & 1/24 & 0 \\ 0 & 1/3 & 7/12 & 1/12 & 0 \\ 0 & 1/6 & 2/3 & 1/6 & 0 \\ 0 & 1/12 & 7/12 & 1/3 & 0 \\ 0 & 1/24 & 11/24 & 11/24 & 1/24 \end{pmatrix},$$

and therefore for $\boldsymbol{\tau} = \boldsymbol{\pi}$ we obtain

$$\lim_{\alpha\to 0} \mathrm{s}(u)|_{u\in[u_l,u_{l+1})} = \lim_{\alpha\to 0} \Big(B_0^4(\alpha\tau), \ldots, B_4^4(\alpha\tau)\Big) A_4 \begin{pmatrix} \mathrm{s}_{l-4} \\ \vdots \\ \mathrm{s}_l \end{pmatrix} = \Big(b_0^4(\tau), \ldots, b_4^4(\tau)\Big) A \begin{pmatrix} \mathrm{s}_{l-4} \\ \vdots \\ \mathrm{s}_l \end{pmatrix}, \tag{27}$$

which is the matrix form of a uniform B-spline curve of degree 4.

For $\boldsymbol{\tau} = \boldsymbol{\mu}$, we can follow a similar reasoning taking into account that the matrices of (14) satisfy

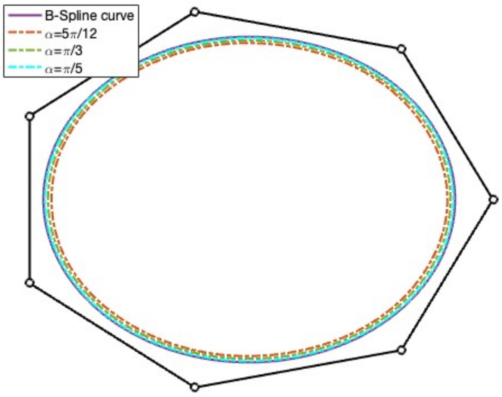
$$\lim_{\alpha\to 0} A_{0,\alpha} = A_0, \quad A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 7/12 & 1/6 & 0 \\ 0 & 1/8 & 37/72 & 23/72 & 1/24 \end{pmatrix},$$

$$\lim_{\alpha \to 0} A_{1,\alpha} = A_1, \quad A_1 = \begin{pmatrix} 1/8 & 37/72 & 23/72 & 1/24 & 0 \\ 0 & 4/9 & 17/36 & 1/12 & 0 \\ 0 & 2/9 & 11/18 & 1/6 & 0 \\ 0 & 1/9 & 5/9 & 1/3 & 0 \\ 0 & 1/18 & 4/9 & 11/24 & 1/24 \end{pmatrix},$$

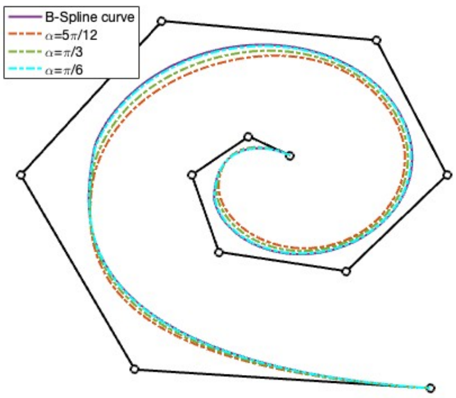
and

$$\lim_{\alpha \to 0} A_{2,\alpha} = A_2, \quad A_2 = \begin{pmatrix} 1/18 & 4/9 & 11/24 & 1/24 & 0 \\ 0 & 1/3 & 7/12 & 1/12 & 0 \\ 0 & 1/6 & 2/3 & 1/6 & 0 \\ 0 & 1/12 & 7/12 & 1/3 & 0 \\ 0 & 1/24 & 11/24 & 11/24 & 1/24 \end{pmatrix}.$$

□



(**a**)



(**b**)

Figure 6. T_4-B-splines curves (closed in (**a**), clamped in (**b**)) associated to a control polygon with several parameters α, as well as the B-spline curve corresponding to the mentioned control polygon.

5. Conclusions and Future Work

We have proposed two frequency trigonometric spline bases with shape preserving properties associated to uniform knot vectors. The corresponding parametric trigonometric spline curves have been also described. It is also shown that these curves share many properties of polynomial spline curves. In fact, they converge to uniform quartic B-spline curves.

There is some worthwhile work to study further. We want to extend the bases to knot vectors with multiple knots and to investigate whether it is possible to construct new Algebraic-Trigonometric Pythagorean-Hodograph B-Splines curves taking into account the results from [21].

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Data Availability Statement: The data and codes used in this work are available under request. The code of the experimentation was developed in Matlab R2022b. All experiments were ran on a Apple M1 Pro chip with 10-Core CPU and 32 GB RAM.

Conflicts of Interest: The authors declare no conflict of interest.

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