Effective homology of universal covers*

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Abstract

We present an algorithm for constructing the effective homology for the universal cover of simplicial sets with effective homology, provided that a nilpotency condition holds. This can be useful for computing higher homotopy groups, since this process can be seen as the first step in the Whitehead tower. Our algorithm can be applied to some spaces satisfying a particular condition. We also present implementations of this method in SageMath and Kenzo.

1 Introduction

A *cover* of a connected topological space X is a topological space Y with a map $f : Y \to X$ such that for every $x \in X$, there exists an open neighborhood U of x such that $f^{-1}(U)$ is a disjoint union of homeomorphic copies of U. If Y is simply connected, then Y is said to be a *universal cover* of Y, and satisfies $\pi_i(Y) \cong \pi_i(X)$ for all $i \geq 2$. The universal cover of a topological space X provides a convenient way to study X by lifting paths and other geometric objects to the simpler universal cover. This makes some topological and geometric problems on X easier to solve, as they can be reduced to the study of the universal cover. The universal cover can be useful in applying the Whitehead tower method [6], a technique to determine homotopy groups of simply connected spaces. In fact, the universal cover can be seen as a first step in the Whitehead tower.

Effective homology is a technique developed by F. Sergeraert [4] that permits one to carry out some kinds of computations over infinite structures, via homotopy equivalences between chain complexes. Kenzo [2] is a computer algebra system that implements several algorithms to compute homology of infinite structures using effective homology. In addition, it also allows one to compute algorithmically homotopy groups of 1-reduced spaces by combining the Whitehead tower method and the effective homology technique. Topological spaces are represented in Kenzo by means of *simplicial sets*, a combinatorial structure generalizing the notion of simplicial complex.

In this work we present an algorithm to compute a simplicial model of universal covers and combine it with the effective homology and homological perturbation techniques to be able to work with spaces of infinite type with finite fundamental group. Our algorithms have been implemented in the computer algebra systems SageMath [5] and Kenzo, using a Kenzo interface and an optional package for SageMath that we developed in a previous work [1]. In that work, we also integrated an algorithm to compute homotopy groups of simply connected simplicial sets that are not necessary 1-reduced. Combining our new

*The first author has been partially supported by PID2020-114750GB-C31 and E22_20R: Álgebra y Geometría. The second and third authors are supported by grant PID2020-116641GB-I00 funded by MCIN/AEI/ 10.13039/501100011033.

constructions with the Whitehead tower method used in [1], we are able now to determine the homotopy groups of some simplicial sets that are not simply connected.

2 Construction of the universal cover

Given a connected simplicial set X, a presentation of its fundamental group can be found as follows: 1) choose a maximal tree T in the 1-skeleton of X; 2) take a generator g_e for each edge e that is not in T; and 3) for each 2-simplex in X, add a relation given by the product of the generators corresponding to its faces (assuming that the edges in T correspond to the trivial element). The result is a presentation of $G := \pi_1(X)$. Note that we also get a map τ that assigns an element of the group to each edge (assuming that the edges in T correspond to the trivial element). This assignment can be lifted to higher-dimensional simplices by taking the first face. In order to work effectively with this presentation, we need to be able to solve the word problem in this presentation (which cannot be ensured in the general case, but is solved for several families of groups, including the cases of finite, free, abelian, polycyclic or simple groups).

With these pieces of data, we can construct the universal cover of X as the simplicial set $\tilde{X}$:

- The sets of simplices are $\tilde{X}_n = G \times X_n$
- The degeneracy maps are $\tilde{s}_i(h, \sigma) = (h, s_i\sigma)$
- The face maps for an n-dimensional simplex (h, σ) are given by:
  - $\tilde{\partial}_i(h, \sigma) = (h, \partial_i\sigma)$, for $i < n$
  - $\tilde{\partial}_n(h, \sigma) = (h \cdot \tau(\sigma)^{-1}, \partial_n\sigma)$

3 Effective homology for the universal cover

We say that X has effective homology if there exist two chain complexes DX_*, EX_*, where EX_* is a chain complex of finite type (effective) and two reductions (particular cases of homotopy equivalences) $C_*(X) \Lleftarrow DX_* \Rrightarrow EX_*$. In this case, the problem of computing the homology of X can be transfered to EX_*, which allows one to compute different topological properties of X, and of other spaces constructed from X. One of these possible spaces determined by X is the twisted cartesian product $K(G, 0) \times_\tau X$, where $K(G, 0)$ is the Eilenberg–MacLane space of G in dimension 0, and the twisting operator is induced by the map τ defined before. This space is isomorphic to the construction of the universal cover presented in the previous section. In order to obtain a space with effective homology, the following nilpotency condition must be satisfied:

Theorem 1 *Let X be a connected simplicial set. Let us suppose that X has effective homology given by two reductions $\rho_1^X = (f_1^X, g_1^X, h_1^X) : DX_* \Rrightarrow C_*(X)$ and $\rho_2^X = (f_2^X, g_2^X, h_2^X) : DX_* \Rrightarrow EX_*$. Assume that the composition $h_2^X g_1^X \partial_m f_1^X$ satisfies that, for every element $y \in DX_m$, there exists some natural number n with $(h_2^X g_1^X \partial_m f_1^X)^n(y) = 0$, then the universal cover constructed as the twisted cartesian product $K(G, 0) \times_\tau X$ has effective homology.*

This theorem can be proved by means of homological perturbation theory and several lemmas. The complete proof will be included in a full version of this paper.

4 Implementation and examples

The construction of the universal cover has been implemented in SageMath for simplicial sets of finite type (with a finite number of non-degenerate simplices) and finite fundamental group. Our implementation is already available in SageMath10.0.

We illustrate its usage with an example. We start by creating a simplicial set with finite fundamental group. In this case, we take the complex corresponding to the usual presentation of the symmetric group on 3 elements, and take its product with the projective space (note that this simplicial set depends on the specific presentation of the initial group, not on the group itself).

```
sage: G = SymmetricGroup(3).as_finitely_presented_group()
sage: C = simplicial_sets.PresentationComplex(G)
sage: RP3 = simplicial_sets.RealProjectiveSpace(3)
sage: S = C.product(RP3) ; S
Simplicial set with 12 non-degenerate simplices x
RP^3
```

Creating its universal cover takes 22 seconds in an Intel Core i7-10700, using 260MB of RAM:

```
sage: SC = S.universal_cover() ; SC
Simplicial set with 4176 non-degenerate simplices
```

We can check that it is indeed simply connected:

```
sage: SC.fundamental_group()
Finitely presented group <  |  >
```

And now we can compute its usual topological invariants, as any other simplicial set, and compare them with the base space (this computation takes about a minute to complete):

```
sage: [SC.homology(i) for i in range(6)]
[0, 0, Z^11, Z, 0, Z^11]
sage: [S.homology(i) for i in range(6)]
[0, C2 x C2, Z x C2, Z x C2 x C2, C2, Z]
```

The construction of the effective homology for universal covers has been implemented as functions in the Kenzo system (the code can be found at [3]). With these new functions, it is possible to determine a simplicial model of the universal cover of simplicial sets of infinite type (with finite fundamental group) and determine its homology and homotopy groups.

To illustrate our programs and the power of the effective homology theory in our problem, we consider as a didactic example the following simplicial set of infinite type: we build in Kenzo the cartesian product of the projective plane with the "semiline" divided in intervals. The projective plane $\mathbb{R}P$ is given by non-degenerate simplices $\mathbb{R}P_0 = \{v\}$, $\mathbb{R}P_1^{ND} = \{a\}$ and $\mathbb{R}P_2^{ND} = \{t\}$, and faces $\partial_0 a = \partial_1 a = v$, $\partial_0 t = \partial_2 t = a$ and $\partial_1 t = s_0 v$; it is a simplicial set of finite type. The "semiline" is represented as a simplicial set A with non-degenerate simplices given by $A_0 = \{n | n \in \mathbb{N}\}$, $A_1^{ND} = \{[n, n+1] | n \in \mathbb{N}\}$ and $\partial_0([n, n+1]) = n+1$, $\partial_1([n, n+1]) = n$. The simplicial set A has an infinite number of non-degenerate simplices, but we can construct its effective homology in an explicit way as follows. We construct a reduction $\rho_2^A = (f_2^A, g_2^A, h_2^A) : C_*(A) \Rrightarrow C_*(*)$ where $C_*(*)$ is a chain complex with only one generator $*$ in degree 0, f_2^A is defined by $f_2^A(x) = *$ if $x \in A_0$ and $f_2^A(x) = 0$ for all $x \in C_n(A)$ with $n > 0$, g_2^A is given by $g_2^A(*) = 0 \in A_0$ and $h_2^A(n) = [0,1] + [1,2] + \cdots + [n-1, n]$ (and $h_2^A(x) = 0$ for $x \in C_n(A)$ with $n > 0$). It is easy to verify that these maps satisfy the properties of reduction. The left reduction in the effective homology of A_* is the identity reduction $\rho_1^A = \mathrm{Id} : C_*(A) \Lleftarrow C_*(X)$.

Now, the cartesian product $X = \mathbb{R}P \times A$ is also a simplicial set with effective homology (it is built automatically by Kenzo). Moreover, since A is contractible, its fundamental group is equal to the fundamental group of $\mathbb{R}P$, that is, $\pi_1(\mathbb{R}P) = \mathbb{Z}/2\mathbb{Z}$.

It can be checked that the nilpotency condition is satisfied, so we can construct in Kenzo the simplicial model for the universal cover of X with its effective homology as follows. We build the cartesian product of the projective plane and the semiline (we omit the construction of the simplicial sets `proj-plane` and `semiline`) and we store it in the variable `X`. Then, we define the map $\tau : X_1^{ND} \to \mathbb{Z}/2\mathbb{Z}$ as a function that receives an edge and returns an element of the group (in this case, 0 or 1) and we store it in `X-twop-edges`. Finally, we call the function `universal-cover`:

```
> (setf X-univ-cover (universal-cover X (cyclicgroup 2) X-twop-edges))
[K45 Simplicial-Set]
```

As said before, this allows us to determine its homology and homotopy groups. For instance, we compute the homotopy group of dimension 5:

```
> (homotopy X-univ-cover 5)
Homotopy in dimension 5 :
Component Z/2Z
```

which indeed is the correct result, since $\pi_5(\mathbb{S}^2) = \mathbb{Z}/2\mathbb{Z}$.

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