SPECTRAL SETS OF GENERALIZED HAUSDORFF MATRICES ON SPACES OF HOLOMORPHIC FUNCTIONS ON $\mathbb{D}$

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Abstract. Here, we study a family of bounded operators $\mathcal{H}$, acting on Banach spaces of holomorphic functions $X \hookrightarrow \mathcal{O}(\mathbb{D})$, which are subordinated in terms of a C_0-semigroup of weighted composition operators $(v_t C_{\phi_t})$, i.e., $\mathcal{H} = \int_0^\infty v_t C_{\phi_t}\, d\nu(t)$ in the strong sense for some Borel measure ν. This family of operators extends the so-called generalized Hausdorff operators. Here, we obtain the spectrum, point spectrum and essential spectrum of $\mathcal{H}$ under mild assumptions on $(v_t C_{\phi_t}), \nu$ and X. In particular, we obtain these spectral sets for a wide family of generalized Hausdorff operators acting on Hardy spaces, weighted Bergman spaces, weighted Dirichlet spaces and little Korenblum classes. The description for the spectra of the infinitesimal generator of $(v_t C_{\phi_t})$ is the key for our findings.

Introduction

Let Δ be the forward difference operator acting on scalar sequences $a = (a_n)_{n=0}^\infty$, that is, $(\Delta a)_n = a_n - a_{n+1}$. The generalized Hausdorff matrix $H_a^{(\zeta)}$ generated by the sequence a and a real number ζ is the infinite lower triangular matrix given by

$$H_a^{(\zeta)}(n,k) = \begin{cases} 0, & n < k, \\ \binom{n+\zeta}{n-k} (\Delta^{n-k} a)_k, & n \geq k. \end{cases}$$

These matrices were defined independently in [10, 25]. As a countably infinite matrix, each generalized Hausdorff matrix $H_a^{(\zeta)}$ induces an operator on sequence spaces on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, denoted also by $H_a^{(\zeta)}$, determined by

$$(H_a^{(\zeta)} b)_n := \sum_{k=0}^{n} H_a^{(\zeta)}(n,k) b_k, \qquad n \in \mathbb{N}_0, \quad b = (b_n)_{n=0}^\infty.$$

2020 *Mathematics Subject Classification.* Primary: 46E15, 47A25, 47G10. Secondary: 47B33, 47D06, 40G05.

Key words and phrases. Hausdorff operators, Spectrum, Weighted composition semigroups, Holomorphic function Banach spaces.

Authors have been partially supported by Project PID2022-137294NB-I00, DGI-FEDER, of the MCYTS and Project E48-23R, D.G. Aragón, Universidad de Zaragoza, Spain. The second author has also been supported by BES-2017-081552, MINECO, Spain; and by XXXV Grants for Postdoctoral Studies, by Ramón Areces Foundation.

Let $\zeta \geq 0$, let μ be a finite Borel measure on $(0,1]$ and let (μ_n^ζ) be the sequence given by

$$\mu_n^\zeta = \int_0^1 t^{n+\zeta} d\mu(t), \quad n \in \mathbb{N}_0. \tag{0.1}$$

The sequences (μ_n^ζ) generate an interesting family of generalized Hausdorff matrices $H_\mu^{(\zeta)} := H_{(\mu_n^\zeta)}^{(\zeta)}$, for which its non-zero elements are given by

$$H_\mu^{(\zeta)}(n,k) = \binom{n+\zeta}{n-k} \int_0^1 t^{k+\zeta}(1-t)^{n-k}\, d\mu(t), \qquad 0 \leq k \leq n. \tag{0.2}$$

Indeed, the behavior of the operators on sequence spaces induced by the matrices $H_\mu^{(\zeta)}$ has been object of study (or play a central role) in several papers, see for instance [18, 26, 32, 33]. In particular, for $\zeta = 0$, the Hausdorff matrix $H_\mu^{(0)}$ corresponds to the ordinary Hausdorff summability [22]. In this case, it follows from the work of Hardy [21] that if $\int_0^1 t^{-1/p} d\mu(t) < \infty$, then $H_\mu^{(0)}$ induces a bounded operator on $\ell^p(\mathbb{N}_0)$ for $1 < p < \infty$.

Even more, boundedness of ordinary Hausdorff matrices $H_\mu^{(0)}$ has been proved in [13, 14] as operators $\mathcal{H}_\mu^{(0)}$ on spaces of holomorphic functions (Hardy, Bergman, Dirichlet, Bloch and BMOA) on the disc by acting on the sequence of coefficients of the power series of such functions. One of the crucial points in these studies is to represent such operators $\mathcal{H}_\mu^{(0)}$ in terms of averages of weighted composition semigroups.

We note in Section 6 that a similar representation also holds for the generalized Hausdorff matrices of type $H_\mu^{(\zeta)}$ for $\zeta \geq 0$. More precisely, let $\mathcal{H}_\mu^{(\zeta)}$ be the operator on spaces of holomorphic functions induced by $H_\mu^{(\zeta)}$, that is

$$\mathcal{H}_\mu^{(\zeta)} f(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} H_\mu^{(\zeta)}(n,k) a_k \right) z^n, \qquad z \in \mathbb{D}, \tag{0.3}$$

where $f(z) = \sum_{n=0}^\infty a_n z^n$. Let $\psi(t) = \log(1/t)$, $t \in (0,1]$, and set $\nu = \psi(\mu)$, i.e., ν is the image measure (on $[0,\infty)$) of μ. Then

$$\mathcal{H}_\mu^{(\zeta)} f(z) = \int_0^\infty u_t^\zeta(z)\, (C_{\phi_t} f)(z)\, d\nu(t), \qquad z \in \mathbb{D},\ f \in \mathcal{O}(\mathbb{D}),$$

see Proposition 6.1, where $C_{\phi_t} f := f \circ \phi_t$, (ϕ_t) is the semiflow (i.e., composition semigroup) given by

$$\phi_t(z) = \frac{e^{-t} z}{(e^{-t}-1)z + 1}, \qquad z \in \mathbb{D},\ t \geq 0, \tag{0.4}$$

and

$$u_t^\zeta(z) = \left(\frac{\phi_t(z)}{z} \right)^\zeta \frac{1-\phi_t(z)}{1-z} = \frac{e^{-\zeta t}}{((e^{-t}-1)z+1)^{\zeta+1}}, \qquad z \in \mathbb{D},\ t \geq 0, \tag{0.5}$$

which is a semicocycle for (ϕ_t), so $(u_t^\zeta C_{\phi_t})$ is a one-parameter semigroup (see Section 2 for the definition of a semicocycle). Then, it seems reasonable to study operators $\mathcal{H}_\mu^{(\zeta)}$, which we label as generalized Hausdorff operators, from the viewpoint of subordination in terms of weighted composition semigroups related to the semiflow (ϕ_t) and the semicocycle (u_t^ζ) as above.

Motivated by these facts, we consider here operators $\mathcal{H}$ which are averages in terms of weighted composition semigroups $(v_t C_{\phi_t})$, where (v_t) is a semicocycle satisfying some suitable properties. Hence, the operators $\mathcal{H}$ we study here are of the type

$$\mathcal{H}f = \int_0^\infty v_t C_{\phi_t} f \, d\nu(t), \qquad f \in \mathcal{O}(\mathbb{D}), \tag{0.6}$$

where ν is a suitable complex Borel measure on $[0, +\infty)$. As we will see later on, this operator $\mathcal{H}$ can be written as

$$\mathcal{H}f(z) = \frac{1}{\omega(z)} \int_0^z \frac{\omega(\xi)}{\xi(1-\xi)} f(\xi) \, d\nu \left(\log \frac{z(1-\xi)}{\xi(1-z)} \right), \qquad z \in \mathbb{D}, \tag{0.7}$$

where ξ is the integration variable, the integration path is taken through the orbit from z to 0 given by (ϕ_t) (so that $\log \frac{z(1-\xi)}{\xi(1-z)} \in [0, +\infty)$); and where ω is a multivalued non-vanishing function on $\mathbb{D} \setminus \{0\}$ of the type $\omega(z) = z^\alpha g(z)$ for some $\alpha \in \mathbb{C}$ and non-vanishing holomorphic function $g \in \mathcal{O}(\mathbb{D})$. Such ω gives a representation of (v_t) as a coboundary, that is,

$$v_t = \frac{\omega \circ \phi_t}{\omega}, \qquad t \geq 0, \tag{0.8}$$

see Remark 2.6 for more details.

The spectral study of weighted averaging operators has been of interest during last years, see for instance [2, 4, 31, 35]. However, spectral properties of Hausdorff and generalized Hausdorff operators on holomorphic function Banach spaces has not been studied in the literature. Recall that, given a closed operator A on a Banach space X, its spectrum $\sigma(A)$ is given by those complex numbers λ for which $\lambda - A$ has not a bounded inverse on X. Our study focuses on the boundedness and, mainly, the spectrum of operators (0.6) and (0.7) acting on classical Banach spaces of holomorphic functions such as the Hardy spaces, the weighted Bergman spaces, or the little Korenblum classes. To avoid the direct spectral study of such operators $\mathcal{H}$ (which is a rather difficult task), the crucial point is the description of the spectrum of the infinitesimal generator Δ of the semigroup $(v_t C_{\phi_t})$. This spectrum is then transferred to the one of $\mathcal{H}$ via the functional calculus of sectorial operators and the spectral mapping theorems given in [19, 30].

The main reason why the spectral study of generators Δ is considerably easier than the one of operators like $\mathcal{H}$ is that the generators Δ are given by first order linear differential operators of the type

$$\Delta f = \Phi f' + gf = \Phi \left(f' + \frac{\omega'}{\omega} f \right), \qquad f \in \mathrm{Dom}(\Delta),$$

where Φ is the generator of (ϕ_t) and g is the generator of (v_t), that is, $\Phi(z) = \frac{\partial \phi_t(z)}{\partial t}\big|_{t=0} = -z(1-z)$ and $g(z) = \frac{\partial v_t(z)}{\partial t}\big|_{t=0} = \Phi\, \omega'/\omega$ where $\omega : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ is as in (0.8), see [34, Th. 2] and [17, Th. 2.1]. In consequence, the representation of the semicocycle (v_t) in terms of the multivalued holomorphic function ω on $\mathbb{D} \setminus 0$ is convenient to obtain $\sigma(\Delta)$. Indeed, one of our crucial results is that, if v_t is continuous at the repulsive point $z = 1$ of (ϕ_t), then ω has singularities of fractional type at this point, see Proposition 2.8.

This paper is structured as follows. We list in Section 1 the axiomatic properties that we require the Banach spaces X of holomorphic functions on $\mathbb{D}$, as well as we provide several examples of classical Banach spaces satisfying these properties. In Section 2, we study the behavior near the fixed points of (ϕ_t) of the multivalued function ω associated to a semicocycle (v_t) by (0.8). These results of ω are critical to obtain one of our main contributions: the spectrum and (under some extra assumptions) the essential spectrum of the infinitesimal generator Δ in Sections 3 and 4, respectively. These spectral sets are transferred to the ones of $\mathcal{H}$ via spectral mapping theorems in Section 5. Finally, in Section 6 we apply our results to two families of generalized Hausdorff operators: generalized Cesàro operators $\mathcal{C}_\alpha^\zeta$ and generalized Hölder operators $\mathfrak{H}_\alpha^\zeta$. We also show that our proofs can be adapted to the weighted Dirichlet spaces $\mathcal{D}_\sigma^p(\mathbb{D})$ (which do **not** satisfy the axiomatic properties listed in Section 1) in the particular case $\mathcal{H}$ is given by a generalized Hausdorff operator $\mathcal{H}_\mu^{(\zeta)}$.

We note that, in a different direction from the one taken here, spectral properties of weighted composition operators with fixed point in $\mathbb{D}$ have been treated in several settings through different papers, see for instance [3, 5, 15, 16, 28]. In particular, spectral inclusions for weighted composition operators vC_ψ were obtained in [16] under fairly general conditions for a long list of Banach spaces of holomorphic functions with domain the unit ball of a Banach space (for instance, Hardy, Bergman, Korenblum spaces on the polydisc).

Finally, we recall the definitions of some classical objects in spectral theory which are needed through this paper. Let A be a closed operator on a Banach space X. The spectral radius $r(A)$ of a bounded operator A is given by $r(A) := \sup\{|\lambda| \,:\, \lambda \in \sigma(A)\}$. The essential spectrum of A, $\sigma_{ess}(A)$, is given by

$$\sigma_{ess}(A) = \{\lambda \in \mathbb{C} \mid \dim(\ker(\lambda - A)) = \infty \text{ or } \mathrm{codim}(\mathrm{Ran}(\lambda - A)) = \infty\},$$

where $\operatorname{codim}(Y) := \dim(X/Y)$ for any linear subspace $Y \subseteq X$. For our purposes, it is useful to consider the extended essential spectrum $\widetilde{\sigma}_{ess}$, which is given by

$$\widetilde{\sigma}_{ess}(A) := \begin{cases} \sigma_{ess}(A), & \text{if } \operatorname{codim}(\operatorname{Dom}(A)) < \infty, \\ \sigma_{ess}(A) \cup \{\infty\}, & \text{otherwise.} \end{cases}$$

Given a closed operator A with non-empty resolvent set, $\widetilde{\sigma}_{ess}(A)$ is a non-empty compact subset of the Riemann sphere $\mathbb{C}_\infty$, see [30].

Also, given a set Y and two functions $f, g : Y \to [0, \infty]$, we use throughout this work the notation $f \lesssim g$ to denote that there exists a constant $M > 0$ such that $f(y) \leq Mg(y)$ for all $y \in Y$. In addition, by $f \sim g$ we mean that $f \lesssim g \lesssim f$. If Y is a topological space (all topological spaces considered here are first-countable) and $y \in Y$, by $f(y') \lesssim g(y')$ as $y' \to y$ we mean that there exist a neighborhood V of y and a constant $M > 0$ such that $f(y') \leq Mg(y')$ for all $y' \in V$. Similarly, by $f(y') \sim g(y')$ as $y' \to y$ we mean that both $f(y') \lesssim g(y')$ as $y' \to y$ and $g(y') \lesssim f(y')$ as $y' \to y$ are true.

1. Axiomatic spaces

By $Mul(X)$ we denote the space of multipliers of a Banach space $X \subseteq \mathcal{O}(\mathbb{D})$. Let $L(X)$ the Banach algebra of linear bounded operators on X. On the other hand, by B we denote the backshift operator given by

$$(Bf)(z) = \frac{f(z) - f(0)}{z}, \qquad f \in \mathcal{O}(\mathbb{D}).$$

For $\gamma \geq 0$, the Korenblum class $\mathcal{K}^{-\gamma}(\mathbb{D})$ is the Banach space of analytic functions f on $\mathbb{D}$ given by

$$\mathcal{K}^{-\gamma}(\mathbb{D}) := \{f \in \mathcal{O}(\mathbb{D}) : \|f\|_{\mathcal{K}^{-\gamma}} := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\gamma |f(z)| < \infty\},$$

which is a Banach space when endowed with the norm $\|\cdot\|_{\mathcal{K}^{-\gamma}}$. Note that $\gamma = 0$ corresponds to the Banach algebra of bounded and holomorphic functions on $\mathbb{D}$, $H^\infty(\mathbb{D})$.

Fix $\gamma \geq 0$. In this work, we deal with Banach spaces $X \hookrightarrow \mathcal{O}(\mathbb{D})$ which contain the constant functions and satisfy the following conditions

(P1) $Mul(X) = H^\infty(\mathbb{D})$.

(P2) $B(X) \subseteq X$.

(P3) $X \hookrightarrow \mathcal{K}^{-\gamma}(\mathbb{D})$.

(P4) $(\phi_t')^\gamma C_{\phi_t} \in L(X)$ for $t \geq 0$ with

$$\sup_{t \geq 0} \|(\phi_t')^\gamma C_{\phi_t}\|_{L(X)} < \infty.$$

(P5) For every $\varepsilon > 0$, we have

$$\text{if } f \in \mathcal{O}(\mathbb{D}) \text{ with } |f(z)| \lesssim |1-z|^{-\gamma+\varepsilon}, \qquad \text{then} \qquad f \in X.$$

We point out that the property (**P5**) is only needed to give some extra information about $\sigma_{point}(\Delta)$, see Proposition 3.4 and the paragraph preceding it.

Definition 1.1. *Let $\gamma \geq 0$. We say that a Banach space $X \hookrightarrow \mathcal{O}(\mathbb{D})$ is a γ^∞-space if it satisfies properties* (**P1**)-(**P5**).

For $m \in \mathbb{N}_0$, set $Z^m = \{f \in X \, : \, f$ has a zero at 0 of order at least $m\}$ and let P_m be the space of polynomials of degree at most m. Since $P_m \subseteq X$ by (**P1**), then $X = P_m \oplus Z^m$ whence Z^m has finite codimension in X. Moreover, the projection $X \to P_m$ induced by the above decomposition is continuous since $X \hookrightarrow \mathcal{O}(\mathbb{D})$. This implies that the complementary projection $X \to Z^m$ is continuous too. Thus, Z^m is a closed subspace of X.

By (**P1**) and (**P2**), Z^m is the range space of the multiplication operator by the function $z \mapsto z^m$. Also, $\|f\|_X \simeq \|B^m f\|_X$ for all $f \in Z^m$ by the open mapping theorem.

We list below some examples of Banach spaces satisfying the properties (**P1**)-(**P5**).

(1) *Little Korenblum classes.*

If $\gamma > 0$, then the closure of polynomials in $\mathcal{K}^{-\gamma}(\mathbb{D})$ is the Little Korenblum growth class $\mathcal{K}_0^{-\gamma}(\mathbb{D})$ given by

$$\mathcal{K}_0^{-\gamma}(\mathbb{D}) := \{f \in \mathcal{K}^{-\gamma}(\mathbb{D}) \, : \, \lim_{|z|\to 1} (1-|z|^2)^\gamma |f(z)| = 0\},$$

with the norm inherited from $\mathcal{K}^{-\gamma}$.

It is clear that $H^\infty(\mathbb{D})$ satisfies properties (**P1**)-(**P5**) and (**P5**) for $\gamma = 0$. Also, for every $\gamma > 0$, $\mathcal{K}^{-\gamma}(\mathbb{D})$ and $\mathcal{K}_0^{-\gamma}(\mathbb{D})$ satisfies such properties for such γ. Indeed, it is clear that they satisfy (**P1**)-(**P3**). And it is readily seen by Schwarz-Pick lemma that these spaces also fulfill (**P4**).

However, since C_{ϕ_t} is not strongly continuous on $\mathcal{K}^{-\gamma}(\mathbb{D})$ or $H^\infty(\mathbb{D})$, we are only interested in the spaces $\mathcal{K}_0^{-\gamma}(\mathbb{D})$, as we explain later on.

(2) *Hardy spaces.* For $1 \leq p < \infty$, let $H^p(\mathbb{D})$ be the Hardy space on $\mathbb{D}$ formed by all functions $f \in \mathcal{O}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} < \infty,$$

endowed with the norm $\|\cdot\|_{H^p}$. Then $H^p(\mathbb{D})$ satisfies properties (**P1**)-(**P5**) for $\gamma = 1/p$, see for instance [24, Sect. 2.2] and [2, Prop. 3.3].

(3) *Weighted Bergman spaces.* Let $1 \leq p < \infty$ and $\sigma > -1$. $\mathcal{A}^p_\sigma(\mathbb{D})$ denotes the weighted Bergman space formed by all holomorphic functions in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{A}^p_\sigma} := \left(\int_{\mathbb{D}} |f(z))|^p (1-|z|^2)^\sigma dA(z) \right)^{1/p} < \infty,$$

where dA is the Lebesgue measure of $\mathbb{D}$. It is well known that the space $\mathcal{A}^p_\sigma(\mathbb{D})$, with norm $\|\cdot\|_{\mathcal{A}^p_\sigma}$, is a Banach space satisfying (**P1**)-(**P3**) with $\gamma = \frac{\sigma+2}{p}$. Property (**P4**) follows from [2, Prop. 3.3]. It is readily seen that $\mathcal{A}^p_\sigma(\mathbb{D})$ also satisfies (**P5**), see for example [1].

2. Semicocycles

Let $h(z) = z/(1-z)$ which maps bijectively $\mathbb{D}$ onto $\{z \in \mathbb{C} \mid \mathfrak{Re}\,(z) > -1/2\}$. It is well known that

$$h(\phi_t(z)) = e^{-t}h(z), \qquad t \geq 0, z \in \mathbb{D}, \tag{2.1}$$

where (ϕ_t) is the semiflow given in (0.4), see for instance [36]. Such a representation of (ϕ_t) is key for several technical results given here.

A family (v_t) of analytic functions $v_t\colon \mathbb{D} \to \mathbb{C}$ is a (differentiable) semicocycle for (ϕ_t) if

(1) $v_0(z) = 1$ for all $z \in \mathbb{D}$;
(2) $v_{s+t} = v_t \cdot (v_s \circ \phi_t)$ for all s, t;
(3) the mapping $t \mapsto v_t(z)$ is differentiable on $[0, \infty)$ for every $z \in \mathbb{D}$.

In this section, from now on, (v_t) is a differentiable semicocycle for (ϕ_t). This is a necessary assumption for our purposes. Indeed, if $(v_tC_{\phi_t})$ is a C_0-semigroup in any of the spaces considered in Section 1, then (v_t) is a differentiable semicocycle for (ϕ_t), see for instance [17, 27, 29].

We turn to the axiomatic properties of the semicocycles we are concerned with.

(**SCo1**) The limit $v_t(1) := \lim_{z\to 1} v_t(z)$ exists in $\mathbb{C}$ for any $t \geq 0$.

We refer the reader to [1, 7, 9, 24] for the suitability of the condition above when dealing with spectra of invertible weighted composition operators.

If (v_t) is a semicocycle, the function v_t has no zeroes in $\mathbb{D}$ for any $t \geq 0$, see [27, Lemma 2.1b)]. It may happen that $v_t(1) = 0$ for some $t \geq 0$. Following axiom concerns such cases. We exclude the case $\lim_{z\to 1} |v_t(z)| = \infty$ since it would imply that the operator $v_tC_{\phi_t}$ is not a bounded operator on $H^p(\mathbb{D})$, $\mathcal{A}^p_\sigma(\mathbb{D})$, see [16, Cor. 3.7].

(**SCo2**) Let Ω be an open neighborhood in $\mathbb{D}$ of 1 (so $\Omega = V \cap \mathbb{D}$ for some open set $V \subset \mathbb{C}$ containing 1). Then

$$\sup_{z\in\mathbb{D}} |v_t(z)| < \infty, \qquad \inf_{z\in\mathbb{D}\setminus\Omega} |v_t(z)| > 0.$$

Similar conditions as (**SCo2**) arise naturally when studying the strong continuity of the semigroup $(v_tC_{\phi_t})$, see for instance [34]. Note that, if $v_t(1) \neq 0$, then the above implies that $\inf_{z\in\mathbb{D}} |v_t(z)| > 0$ for all $t \geq 0$.

Remark 2.1. Let $\delta \in \mathbb{C}$. Then it is readily seen that $((\phi_t')^\delta)$, which is given by

$$(\phi_t')^\delta(z) = \left(\frac{\phi_t(z)(1-\phi_t(z))}{z(1-z)}\right)^\delta = \frac{e^{-\delta t}}{((e^{-t}-1)z+1)^{2\delta}}, \qquad z \in \mathbb{D},\, t \geq 0,$$

is a semicocycle for (ϕ_t) which satisfies (**SCo1**) and (**SCo2**).

The remainder of this subsection is devoted to give technical results on semicocycles satisfying the properties above.

Remark 2.2. For $R > 0$, set $V_R := \{z \in \mathbb{C} \mid \mathfrak{Re}\,(z) > -1/2,\, |z| \geq R\}$. Note that $V_R = h(\mathbb{D}) \cap \{z \in \mathbb{C} \mid |z| \geq R\}$, where h is the univalent function given in (2.1). Let $\Omega_R := h^{-1}(V_R)$, which is an neighbourhood of 1 in $\mathbb{D}$. Note that the subsets $\mathbb{D}\setminus\Omega_R$ are C_{ϕ_t}-invariant for all $t \geq 0$. That is, $\phi_t(\mathbb{D}\setminus\Omega_R) \subseteq \mathbb{D}\setminus\Omega_R$. It is readily seen that (for each $R > 0$) the functions given by $t \mapsto \sup_{z\in\mathbb{D}} \log|v_t(z)|$, $t \mapsto -\inf_{z\in\mathbb{D}\setminus\Omega_R} \log|v_t(z)|$ are subadditive. Hence,

$$\exists \lim_{t\to\infty} \sup_{z\in\mathbb{D}} |v_t(z)|^{1/t} < \infty, \qquad \exists \lim_{t\to\infty} \inf_{z\in\mathbb{D}\setminus\Omega_R} |v_t(z)|^{1/t} > 0,$$

see for example [23, Th. 7.6.5]. As a consequence, for each $T > 0$, $R > 0$, there exist $M, \rho > 0$ such that

$$\sup_{z\in\mathbb{D}} |v_t(z)| \leq Me^{\rho t}, \qquad \inf_{z\in\mathbb{D}\setminus\Omega_R} |v_t(z)| \geq Me^{-\rho t}, \qquad \text{for all } t \geq T. \tag{2.2}$$

We now study the asymptotic behavior of a semicocycle (v_t), which is crucial in the understanding of the spectrum of the infinitesimal generator of $(v_t C_{\phi_t})$.

Lemma 2.3. *Let (v_t) be a semicocycle for (ϕ_t) satisfying* (**SCo1**) *and* (**SCo2**)*. We have*

$$\lim_{t\to\infty}\left(\sup_{z\in\mathbb{D}}|v_t(z)|\right)^{1/t} = \max\{|v_1(0)|, |v_1(1)|\},$$
$$\lim_{t\to\infty}\left(\inf_{z\in\mathbb{D}}|v_t(z)|\right)^{1/t} = \min\{|v_1(0)|, |v_1(1)|\}.$$

Proof. This proof is inspired by the proof of [24, Lemma 4.4].

Let us prove the identity regarding the supremum. First of all, note that by the subadditivity of $t \mapsto \sup_{z\in\mathbb{D}} \log|v_t(z)|$, it is enough to prove

$$\lim_{n\to\infty}\left(\sup_{z\in\mathbb{D}}|v_n(z)|\right)^{1/n} = \max\{|v_1(0)|, |v_1(1)|\},$$

where $n \in \mathbb{N}_0$, see for instance 2.2.

For each $\varepsilon > 0$, take $R > 1$ big enough, and a neighbourhood $U \ni 0$ such that

$$|v_1(z)| < (1+\varepsilon)\max\{|v_1(0)|, |v_1(1)|\}, \qquad z \in U \cup \Omega_R,$$

where Ω_R is the subset defined in Remark 2.2. Note that such U, R exist by (**SCo1**) and that, by (2.1), there exists $m \in \mathbb{N}_0$ such that, for all $z \in \mathbb{D}$, at most m elements of $\{\phi_n(z) \mid n \in \mathbb{N}_0\}$ belong to $\mathbb{D} \setminus \{U \cup \Omega_R\}$. Since $v_n = \prod_{j=0}^{n-1} v_1 \circ \phi_j$, we have

$$\sup_{z\in\mathbb{D}} |v_n(z)| \leq \|v_1\|_\infty^m \left[(1+\varepsilon)\max\{|v_1(0)|, |v_1(1)|\}\right]^{n-m}.$$

Hence $\lim_{n\to\infty} \left(\sup_{z\in\mathbb{D}} |v_n(z)|\right)^{1/n} \leq \max\{|v_1(0)|, |v_1(1)|\}$. Moreover, since $v_n(1) = v_1(1)^n$ and $v_n(0) = v_1(0)^n$, it follows $\lim_{n\to\infty} \left(\sup_{z\in\mathbb{D}} |v_n(z)|\right)^{1/n} = \max\{|v_1(0)|, |v_1(1)|\}$.

Regarding the infimum, the statement is trivial if $v_1(1) = 0$. Otherwise, the proof is analogous to the one of the supremum. □

Remark 2.4. Let (v_t) be a semicocycle for (ϕ_t) as in the lemma above. Note that, for $0 \leq s \leq t$, one has $\phi_s(\Omega_{Re^t}) \subseteq \Omega_{Re^{t^*}}$ for all $t^* \in [0, t-s]$ (see Remark 2.2 for the definition of $\Omega_{(\cdot)}$). Then, reasoning as in the proof of Lemma 2.3, one gets

$$\lim_{t\to\infty} \left(\sup_{z\in\Omega_{Re^t}} |v_t(z)|\right)^{1/t} = |v_1(1)|, \quad \text{and} \quad \lim_{t\to\infty} \left(\inf_{z\in\Omega_{Re^t}} |v_t(z)|\right)^{1/t} = |v_1(1)|.$$

The elements of the extended real line α_ι found in the following lemma will be called *exponents* of (v_t).

Lemma 2.5. *Let (v_t) be a semicocycle for (ϕ_t) satisfying* (**SCo1**) *and* (**SCo2**). *For $\iota \in \{0,1\}$, there exists $\alpha_\iota \in [-\infty, \infty)$ (with $\alpha_0 \in \mathbb{R}$) such that*

$$|v_t(\iota)| = e^{\alpha_\iota t}, \qquad t > 0,$$

where $e^{-\infty} = 0$.

Proof. Let $\iota \in \{0,1\}$. The mapping $t \mapsto |v_t(\iota)|$ is measurable. This is clear for $\iota = 0$. If $\iota = 1$, note that such a mapping is the limit of a countable family of continuous functions. Indeed, $|v_t(1)| = \lim_{k\to\infty} |v_t(z_k)|$ where $(z_k)_{k=1}^\infty \subset \mathbb{D}$ with $z_k \xrightarrow[k\to\infty]{} 1$. Then, by the semicocycle property, $v_{t+s}(1) = v_t(1)v_s(1)$ for $s, t \geq 0$. Hence, $t \mapsto |v_t(1)|$ is measurable and fulfills the Cauchy's exponential equation.

Thus, for $\iota \in \{0,1\}$, there exists $\alpha_\iota \in \mathbb{R} \cup \{-\infty\}$ such that $|v_t(\iota)| = e^{\alpha_\iota t}$ for all $t > 0$ (see for instance [6, Prop. 8.1.14]), and the proof is done. (Note that $\alpha_0 \in \mathbb{R}$ as $v_t(0) \neq 0$ for $t \geq 0$.) □

Remark 2.6. Given a differentiable semicocycle (v_t) for the semiflow (ϕ_t), there exists a multivalued non-vanishing holomorphic function $\omega : \mathbb{D}\setminus\{0\} \to \mathbb{C}$ (i.e., $\omega(z) = z^\delta r(z)$ for some $\delta \in \mathbb{C}$ and non-vanishing holomorphic $r : \mathbb{D} \to \mathbb{C}$) such that

$$v_t(z) = \frac{\omega \circ \phi_t(z)}{\omega(z)}, \qquad z \in \mathbb{D} \setminus \{0\},\, t \geq 0,$$

see [27, Lemma 2.2b)] and [29, Prop. 4.2.2]. Even more, it holds $\delta = -g(0)$ and $g = \frac{\omega'}{\omega}\Phi$, where $g = \frac{\partial v_t}{\partial t}\big|_{t=0}$ is the generator of (v_t) and $\Phi(z) = \frac{\partial \phi_t(z)}{\partial t}\big|_{t=0} = -z(1-$

$z)$, $z \in \mathbb{D}$, is the generator of (ϕ_t). For such a ω, we say that ω is a multivalued function associated with (v_t).

Lemma 2.7. *Let (v_t) be a differentiable semicocycle for (ϕ_t) satisfying (**SCo1**) and (**SCo2**), and let ω be a multivalued holomorphic function associated with (v_t). Let $\mathcal{K} \subset \mathbb{D}$ be such that $0, 1 \notin \overline{\mathcal{K}}$. Then*

$$\sup_{z\in\mathcal{K}} |\omega(z)| < \infty \quad \textit{and} \quad \inf_{z\in\mathcal{K}} |\omega(z)| > 0.$$

Proof. Set $A := \mathcal{K} \cap \{z \in \mathbb{D} \mid |z| < 3/4\}$. Since $\overline{A}$ is a compact subset of $\mathbb{D} \setminus \{0\}$, then $\sup_{z\in A} |\omega(z)| < \infty$ and $\inf_{z\in A} |\omega(z)| > 0$.

Let $D_{1/2} := \{z \in \mathbb{C} : |z| = 1/2\}$. Since $1 \notin \overline{\mathcal{K}}$, there exists $\tau > 0$ such that, for each $z \in \mathcal{K} \setminus A$, there exists $t_z \in [1/\tau, \tau]$ for which $\phi_{t_z}(z) \in D_{1/2}$, see (2.1). As $\omega(z) = \frac{\omega(\phi_{t_z}(z))}{v_{t_z}(z)}$, one has by (2.2)

$$\sup_{z\in\mathcal{K}\setminus A} |\omega(z)| \lesssim \sup_{z\in D_{1/2}} |\omega(z)| < \infty, \quad \text{and} \quad \inf_{z\in\mathcal{K}\setminus A} |\omega(z)| \gtrsim \inf_{z\in D_{1/2}} |\omega(z)| > 0,$$

and the proof is done. □

As mentioned in Remark 2.6, ω behaves as a fractional power as $z \to 0$. The next theorem gives some information about the behavior of ω near the repulsive point of (ϕ_t), $z = 1$.

Proposition 2.8. *Let (v_t) be a differentiable semicocycle for (ϕ_t) satisfying (**SCo1**) and (**SCo2**). Let ω be a multivalued function associated with (v_t), and let α_0, α_1 be the exponents of (v_t). Then, for every $\varepsilon > 0$,*

$$|\omega(z)| \lesssim |1 - z|^{\alpha_1 - \varepsilon} \quad \textit{and} \quad |\omega(z)| \gtrsim |1 - z|^{\alpha_1 + \varepsilon}, \qquad \textit{as } z \to 1.$$

Also, there is a non-vanishing holomorphic function $r : \mathbb{D} \to \mathbb{C}$ such that $|\omega(z)| = |z|^{-\alpha_0} |r(z)|$, $z \in \mathbb{D}$, with $\alpha_0 = \mathfrak{Re}\,(g(0))$.

If $\alpha_1 = -\infty$, the above reads as, for each $\beta > 0$, $|\omega(z)| \gtrsim |1 - z|^{-\beta}$ as $z \to 1$.

Proof. For each $z \in \mathbb{D} \setminus \{0\}$, set $t_z = \log(4|h(z)|)$. Note that $t_z \to \infty$ as $\mathbb{D} \ni z \to 1$ and that $z \in \Omega_{e^{t_z}/4}$ by (2.1). Thus, by Remark 2.4, one has

$$\limsup_{\mathbb{D}\ni z\to 1} |v_{t_z}(z)|^{1/t_z} \le \lim_{t_z\to\infty} \left(\sup_{w\in\Omega_{e^{t_z}/4}} |v_{t_z}(w)|^{1/t_z} \right) = |v_1(1)| = e^{\alpha_1},$$

and

$$\liminf_{\mathbb{D}\ni z\to 1} |v_{t_z}(z)|^{1/t_z} \ge \lim_{t_z\to\infty} \left(\inf_{w\in\Omega_{e^{t_z}/4}} |v_{t_z}(w)|^{1/t_z} \right) = |v_1(1)| = e^{\alpha_1}.$$

Fix $\varepsilon > 0$. From the above, one gets

$$|\omega(z)| \ge |\omega(\phi_{t_z}(z))| e^{-t_z(\alpha_1+\varepsilon)}, \qquad |\omega(z)| \le |\omega(\phi_{t_z}(z))| e^{-t_z(\alpha_1-\varepsilon)},$$

for all z near enough the point 1. As $e^{-t_z} = \dfrac{h(\phi_t(z))}{h(z)} = \dfrac{\phi_{t_z}(z)/(1-\phi_{t_z}(z))}{z/(1-z)}$ (see (2.1)), one has

$$|\omega(z)| \geq |\omega(\phi_{t_z}(z))| \left|\frac{\phi_{t_z}(z)/(1-\phi_{t_z}(z))}{z/(1-z)}\right|^{\alpha_1+\varepsilon} \simeq |1-z|^{\alpha_1+\varepsilon},$$
$$|\omega(z)| \leq |\omega(\phi_{t_z}(z))| \left|\frac{\phi_{t_z}(z)/(1-\phi_{t_z}(z))}{z/(1-z)}\right|^{\alpha_1-\varepsilon} \simeq |1-z|^{\alpha_1-\varepsilon},$$

again, for all z near enough the point 1, as we wanted to prove. Note that for both $\simeq$ signs we have used that $|h(\phi_{t_z}(z))| = 1/4$, so $|\omega(\phi_{t_z}(z))|$ and $\left|\frac{\phi_{t_z}(z)}{z(1-\phi_{t_z}(z))}\right|$ are uniformly bounded above and uniformly bounded below by Lemma 2.7 with $\mathcal{K} = h^{-1}(\{z \in \mathbb{C} \mid |z| = 1/4\})$.

Finally, the proof of the identity $|\omega(z)| = |z|^{-\alpha_0}|r(z)|$ is analogous to what we have already proven and the fact that there exist $\delta \in \mathbb{C}$ and non-vanishing holomorphic $r : \mathbb{D} \to \mathbb{C}$ such that $\omega(z) = z^{\delta} r(z)$, $z \in \mathbb{D}$ (see [27, Lemma 2.2b)] and [29, Prop. 4.2.2]). □

3. Spectrum of the infinitesimal generator

Fix $\gamma \geq 0$ and let X be a γ^∞-space through this section. We deal here with the spectral properties of the generator Δ of $(v_t C_{\phi_t})$ on X. For Δ to be well defined, we assume that the semicocycles (v_t) (of a semiflow (ϕ_t)) we are working with fulfill the following condition:

(SCo3) $(v_t C_{\phi_t})$ is a C_0-semigroup of bounded operators on X.

Unfortunately, (**SCo3**) rules out any Banach space X of holomorphic functions for which the inclusions $H^\infty(\mathbb{D}) \subseteq X \subseteq \mathcal{B}_1(\mathbb{D})$ hold (where $\mathcal{B}_1(\mathbb{D})$ denotes the Bloch space) since no weighted composition semigroup is strongly continuous (at 0) in such a space X, see [17, Th. 4.1]. In particular, the results of this work are not applicable in spaces like $H^\infty(\mathbb{D})$ or the Korenblum classes $\mathcal{K}^{-\gamma}(\mathbb{D})$.

If $(v_t C_{\phi_t})$ satisfies (**SCo3**), one has that (v_t) is a differentiable semicocycle and that the infinitesimal generator Δ of the C_0-semigroup $(v_t C_{\phi_t})$ is given by

$$\Delta f = \Phi f' + g f, \qquad f \in \mathrm{Dom}(\Delta), \tag{3.1}$$

with $\mathrm{Dom}(\Delta) = \{f \in X : \Phi f' + g f \in X\}$, $\Phi(z) = -z(1-z)$, and where g is the generator of (v_t), i.e. $g = \frac{\partial v_t}{\partial t}\big|_{t=0}$. See [27, Th. 1], [34, Th. 2] and [17, Th. 2.1] for more details.

The following upper bound for the asymptotic behavior of the norm of $(v_t C_{\phi_t})$ yields the spectral inclusion given in corollary below. Recall that, for $m \in \mathbb{N}_0$, we denote by Z^m the subset of functions f in X which have a zero at 0 of order at least m, and by B we denote the backshift operator.

Proposition 3.1. *Let X be a γ^∞-space for $\gamma \geq 0$, and let (v_t) be a semicocycle for (ϕ_t) satisfying* (**SCo1**) *and* (**SCo2**)*. For $m \in \mathbb{N}_0$, Z^m is an invariant subspace of $(v_t C_{\phi_t})$ and*

$$\lim_{t\to\infty} \|v_t C_{\phi_t}\|_{L(Z^m)}^{1/t} \leq \exp\left(\max\{\alpha_0 + \gamma - m, \alpha_1 - \gamma\}\right),$$

where α_0, α_1 are the exponents of (v_t).

If $\alpha_1 = -\infty$, the above reads as $\lim_{t\to\infty} \|v_t C_{\phi_t}\|_{L(Z^m)}^{1/t} \leq e^{\alpha_0+\gamma-m}$.

Proof. The inclusion $(v_t C_{\phi_t})(Z^m) \subseteq Z^m$ follows from the fact ϕ_t has a zero of order 1 at $z = 0$ for all $t \geq 0$.

Since $t \mapsto \log \|v_t C_{\phi_t}\|_{L(Z^m)}$ is a subadditive function, the limit above exists, see for example [23, Th. 7.6.5]. Hence, by (**P1**) and (**P2**) we have

$$\begin{aligned}\|v_t C_{\phi_t} f\|_X &\simeq \|B^m(v_t C_{\phi_t} f)\|_X = \|(B\phi_t)^m\, v_t C_{\phi_t}\,(B^m f)\|_X \\ &\lesssim \left\|(B\phi_t)^m\, v_t\,(\phi_t')^{-\gamma}\right\|_\infty \|(\phi_t')^\gamma C_{\phi_t}\,(B^m f)\|_X \\ &\lesssim \left\|(B\phi_t)^m\, v_t\,(\phi_t')^{-\gamma}\right\|_\infty \|f\|_X, \quad f \in Z^m,\ t \geq 0,\end{aligned}$$

where we have used $Mul(X) = H^\infty(\mathbb{D})$ by (**P1**), the inequality $\sup_{t\geq 0} \|(\phi_t')^\gamma C_{\phi_t}\|_{L(X)} < \infty$ by (**P4**), and the fact that $\|B^m f\|_X \simeq \|f\|_X$ for all $f \in Z^m$.

In addition, the semicocycle (w_t) given by $w_t := (B\phi_t)^m\, v_t\,(\phi_t')^{-\gamma}$, $t \geq 0$, satisfies properties (**SCo1**) and (**SCo2**) since the semicocycles (v_t), $((\phi_t')^{-\gamma})$, $(B\phi_t)^m$ do so. Hence, Lemma 2.3 yields

$$\begin{aligned}\lim_{t\to\infty} \left\|(B\phi_t)^m\, v_t\,(\phi_t')^{-\gamma}\right\|_\infty^{1/t} &= \max_{w=0,1} \left\{ \left|\frac{\phi_1(w)}{w}\right|^m e^{\alpha_w}\,(\phi_1')^{-\gamma}(w) \right\} \\ &= \exp\left(\max\{\alpha_0 + \gamma - m, \alpha_1 - \gamma\}\right),\end{aligned}$$

and the proof is done. □

Corollary 3.2. *Let X be a γ^∞-space for $\gamma \geq 0$, and let (v_t) be a semicocycle for (ϕ_t) satisfying* (**SCo1**)-(**SCo3**)*. Then*

$$\sigma(\Delta) \subseteq \{\lambda \in \mathbb{C} : \mathfrak{Re}\,(\lambda) \leq \alpha_1 - \gamma\} \cup \sigma_{point}(\Delta),$$

where α_1 is the exponent of (v_t) at $z = 1$.

If $\alpha_1 = -\infty$, the above reads as $\sigma(\Delta) = \sigma_{point}(\Delta)$.

Proof. Set $B := \{\lambda \in \mathbb{C} : \mathfrak{Re}\,(\lambda) > \alpha_1 - \gamma\}$. For $m \in \mathbb{N}$, $\Delta|_{Z^m}$ is the generator of the C_0-semigroup $(v_t C_{\phi_t}|_{Z^m})$ by Proposition 3.1. Also, if $\mathfrak{Re}\,(\lambda) > \max\{\alpha_0 + \gamma - m, \alpha_1 - \gamma\}$, then $\lambda \in \rho(\Delta|_{Z^m})$, see Proposition 3.1 again. Moreover, $(\lambda - \Delta|_{Z^m})^{-1} = \int_0^\infty e^{-\lambda t}(v_t C_{\phi_t})|_{Z^m}\, dt$ by the resolvent formula for semigroup generators, see for example [11, Th. II.1.10].

Since Z^m has finite codimension for all $m \in \mathbb{N}$ ($\mathrm{codim}(Z^m) = m$), we have $\sigma_{ess}(\Delta) = \sigma_{ess}(\Delta|_{Z^m})$. Thus, B lies in the essential resolvent of Δ, i.e., $B \subseteq \mathbb{C} \setminus \sigma_{ess}(\Delta)$. Since

B is a connected open set and $B \cap \rho(\Delta) \neq \emptyset$, the points in $\sigma(\Delta) \cap B$ are isolated eigenvalues, see for instance [8, Sect. I.4], and the claim follows. $\square$

Remark 3.3. Let $h(z) = z/(1-z)$ for $z \in \mathbb{D}$. The point spectrum of Δ is given by

$$\sigma_{point}(\Delta) = \left\{ g(0) - k \, : \, k \in \mathbb{N}_0 \text{ and } \frac{h^{k-g(0)}}{\omega} \in X \right\},$$

where ω is a multivalued function associated with (v_t) and g is the generator of (v_t). Moreover, if $g(0) - k \in \sigma_{point}(\Delta)$, then its eigenspace is one-dimensional and is generated by the function $\frac{h^{k-g(0)}}{\omega}$. In the case ω is holomorphic in $\mathbb{D}$ and for $X = H^p(\mathbb{D})$, the above was proven in [34, Th. 3]. The adaptation of the proof of such a result to our setting is straightforward, hence we omit it (see also [29, Prop. 4.2.4 b)]).

Along the paper, property (**P5**) is only used in the second inclusion of the following result, which gives a little bit more information regarding $\sigma_{point}(\Delta)$.

Proposition 3.4. *Let X be a γ^∞-space with $\gamma \geq 0$, and let (v_t) be a semicocycle for (ϕ_t) satisfying (**SCo1**)-(**SCo3**). Let α_0, α_1 be the exponents of (v_t). Then*

$$\sigma_{point}(\Delta) \subseteq \{ g(0) - k \, : \, k \in \mathbb{N}_0 \text{ and } k \leq \alpha_0 - \alpha_1 + \gamma \},$$

where g is the generator of (v_t). Also

$$\{ g(0) - k \, : \, k \in \mathbb{N}_0 \text{ and } k < \alpha_0 - \alpha_1 + \gamma \} \subseteq \sigma_{point}(\Delta).$$

If $\alpha_1 = -\infty$, the above inclusions read as $\sigma_{point}(\Delta) \subseteq \{g(0) - k \, : \, k \in \mathbb{N}_0\}$ and $\{g(0) - k \, : \, k \in \mathbb{N}_0\} \subseteq \sigma_{point}(\Delta)$ respectively.

Proof. Note first that $\sigma_{point}(\Delta) \subseteq \{g(0) - k \, : \, k \in \mathbb{N}_0\}$ by Remark 3.3. Let $h(z) = z/(1-z)$ for $z \in \mathbb{D}$. Lemma 2.7 and Proposition 2.8 imply, for each $\varepsilon > 0$,

$$|z|^k |1-z|^{\alpha_0 - \alpha_1 - k + \varepsilon} \lesssim \left| \frac{h^{k-g(0)}}{\omega} \right| \lesssim |z|^k |1-z|^{\alpha_0 - \alpha_1 - k - \varepsilon}, \qquad z \in \mathbb{D}.$$

(Recall that $\alpha_0 = \mathfrak{Re}\,(g(0))$ by Proposition 2.8.) If $k > \alpha_0 - \alpha_1 + \gamma$, then $\frac{h^{k-g(0)}}{\omega} \notin \mathcal{K}^{-\gamma}(\mathbb{D})$ by the first inequality above, whence it is not in X since $X \hookrightarrow \mathcal{K}^{-\gamma}(\mathbb{D})$. Thus $g(0) - k \notin \sigma_{point}(\Delta)$ by Remark 3.3.

On the other hand, if $k < \alpha_0 - \alpha_1 + \gamma$, then $\frac{h^{k-g(0)}}{\omega} \in X$ by the second inequality above and property (**P5**). Then $g(0) - k \in \sigma_{point}(\Delta)$ by Remark 3.3, and our claim follows. $\square$

As a consequence of the proposition above, one can improve the asymptotic bound given in Proposition 3.1.

Proposition 3.5. *Let X be a γ^∞-space for $\gamma \geq 0$, and let (v_t) be a semicocycle for (ϕ_t) satisfying (**SCo1**)-(**SCo3**). Then*

$$\lim_{t\to\infty} \|v_t C_{\phi_t}\|_{L(X)}^{1/t} \leq \exp\left(\max\{\alpha_0, \alpha_1 - \gamma\}\right),$$

where α_0, α_1 are the exponents of (v_t).

If $\alpha_1 = -\infty$, the above reads as $\lim_{t\to\infty} \|v_t C_{\phi_t}\|_{L(X)}^{1/t} \leq e^{\alpha_0}$.

Proof. By the spectral radius formula, $\lim_{t\to\infty} \|v_t C_{\phi_t}\|_{L(X)}^{1/t} = r(v_1 C_{\phi_1})$. Thus, we are done if we prove $r(v_1 C_{\phi_1}) \leq \exp\left(\max\{\alpha_0, \alpha_1 - \gamma\}\right) =: C$. Let $\lambda \in \sigma(v_1 C_{\phi_1})$ with $|\lambda| > e^{\alpha_1 - \gamma}$. Then λ is an eigenvalue of $v_1 C_{\phi_1}$ by Corollary 3.2. One has $\sigma_{point}(v_t C_{\phi_t}) = \exp(t\sigma(\Delta))$, where Δ is the generator of $(v_t C_{\phi_t})$ (see for example [11, Th. IV.3.7]). Since $\mathfrak{Re}\,(g(0)) = \alpha_0$, it follows by Proposition 3.4 that $|\lambda| \leq e^{\alpha_0} \leq C$, and the proof is finished. □

Remark 3.6. Recall that the infinitesimal generator Δ of a C_0-semigroup $(v_t C_{\phi_t})$ is given by the differential operator

$$(\Delta f)(z) = -z(1-z)f' + g(z)f(z) = -z(1-z)\left(f'(z) + \frac{\omega'(z)}{\omega(z)}f(z)\right), \tag{3.2}$$

where ω is a multivalued holomorphic function associated with (v_t), see (3.1). Fix $d \in \mathbb{D}\setminus\{0\}$, $f_0, f_1 \in \mathcal{O}(\mathbb{D})$ and $\lambda \in \mathbb{C}$. By solving the above differential equation, one obtains that $(\lambda - \Delta)f_0 = f_1$ holds if and only if there exist $A \in \mathbb{C}$ for which

$$f_0(z) = (\Lambda_{A,d}^{\lambda,\omega} f_1)(z) := \frac{(1-z)^\lambda}{z^\lambda \omega(z)}\left(A + \int_d^z \frac{\tau^{\lambda-1}}{(1-\tau)^{\lambda+1}}\omega(\tau)f_1(\tau)\,d\tau\right), \quad z \in \mathbb{D}. \tag{3.3}$$

(Note that, in general, $\Lambda_{A,d}^{\lambda,\omega}$ is a multivalued holomorphic function on $\mathbb{D}\setminus\{0\}$.) As a consequence, given $f \in X$ and $\lambda \in \mathbb{C}$, f belongs to $\mathrm{Ran}(\lambda - \Delta)$ if and only if there exists $A \in \mathbb{C}$ such that the multivalued function $\Lambda_{A,d}^{\lambda,\omega} f$ induces a holomorphic function on $\mathbb{D}$ which belongs to X. In that case, $f = (\lambda - \Delta)(\Lambda_{A,d}^{\lambda,\omega} f)$.

The following functionals, which are inspired by the study of the spectra of Cesàro operators in [2, 31], play a central role in the study of the spectrum of Δ. We set

$$L_{\lambda,\omega} f := \int_0^1 \frac{\tau^{\lambda-1}}{(1-\tau)^{\lambda+1}}\omega(\tau)f(\tau)\,d\tau, \quad f \in \mathcal{O}(\mathbb{D}). \tag{3.4}$$

Lemma 3.7. *Let X be a γ^∞-space for $\gamma \geq 0$, and let (v_t) be a semicocycle for (ϕ_t) satisfying (**SCo1**)-(**SCo3**) with exponents α_0, α_1. Let Δ be the infinitesimal generator of $(v_t C_{\phi_t})$, let ω be a multivalued function associated with (v_t), and let $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\,(\lambda) < \alpha_1 - \gamma$. Let $m \in \mathbb{N}_0$ be such that $m > \alpha_0 - \mathfrak{Re}\,(\lambda)$. Then $L_{\lambda,\omega}$ is a continuous functional on Z^m for which $(\lambda - \Delta)(Z^m) \subseteq \ker L_{\lambda,\omega}|_{Z^m}$.*

Proof. As $X \hookrightarrow \mathcal{O}(\mathbb{D})$, given $0 < a < b < 1$, one has

$$\left|\int_a^b \frac{\tau^{\lambda-1}}{(1-\tau)^{\lambda+1}}\omega(\tau)f(\tau)\,d\tau\right| \lesssim \|f\|_X, \qquad f \in X.$$

Recall that, by Proposition 2.8, $|\omega(z)| \simeq |z|^{-\alpha_0}$ as $z \to 0$, and also that, given $m \in \mathbb{N}$, $\|B^m f\|_X \simeq \|f\|_X$ for all $f \in Z^m$. Thus

$$\left|\frac{\tau^{\lambda-1}}{(1-\tau)^{\lambda+1}}\omega(\tau)f(\tau)\right| \lesssim |\tau|^{\mathfrak{Re}\,(\lambda)-\alpha_0+m-1}\|f\|_X, \qquad f \in Z^m, \text{ as } \tau \to 0. \tag{3.5}$$

Therefore, the integral (3.4) is absolutely convergent in a neighborhood of 0 for each $f \in Z^m$ if $m > \alpha_0 - \mathfrak{Re}\,(\lambda)$.

On the other hand, by the inclusion X $\hookrightarrow \mathcal{K}^{-\gamma}(\mathbb{D})$ and Proposition 2.8 we have, for all $\varepsilon > 0$,

$$\left|\frac{\tau^{\lambda-1}}{(1-\tau)^{\lambda+1}}\omega(\tau)f(\tau)\right| \lesssim |1-\tau|^{\alpha_1-\mathfrak{Re}\,(\lambda)-\gamma-1-\varepsilon}\|f\|_X, \tag{3.6}$$

for all $f \in X$ and as $\tau \to 1$ non-tangentially. As a consequence, the integral (3.4) is absolutely convergent in a neighborhood of 1 if $\mathfrak{Re}\,(\lambda) < \alpha_1 - \gamma$. Hence we conclude that $L_{\lambda,\omega}$ is a bounded functional on Z^m, as claimed.

Now fix $f \in Z^m$. By the bounds we have proven above, one obtains that the mapping from $[0,1]$ (including 0 and 1) to $\mathbb{C}$ given by

$$z \mapsto \int_z^1 \frac{\tau^{\lambda-1}}{(1-\tau)^{\lambda-1}}\omega(\tau)f(\tau)\,d\tau,$$

is continuous. Therefore, for $A \in \mathbb{C}$, $d \in \mathbb{D}\setminus\{0\}$ and $f \notin \ker L_{\lambda,\omega}|_{Z^m}$, either

$$\begin{aligned} &|(\Lambda^{\lambda,\omega}_{A,d}f)(z)| \simeq \left|\frac{(1-z)^\lambda}{z^\lambda\omega(z)}\right| \qquad \text{as } z \to 0 \text{ through } [0,1],\\ \text{or}\qquad &|(\Lambda^{\lambda,\omega}_{A,d}f)(z)| \simeq \left|\frac{(1-z)^\lambda}{z^\lambda\omega(z)}\right| \qquad \text{as } z \to 1 \text{ through } [0,1]. \end{aligned} \tag{3.7}$$

So assume $f \in \mathrm{Ran}(\lambda - \Delta|_{Z^m}) \setminus \ker L_{\lambda,\omega}|_{Z^m}$. Then $\Lambda^{\lambda,\omega}_{A,d}f$ induces a holomorphic function on $\mathbb{D}$ which belongs to Z^m for some $A \in \mathbb{C}$, see Remark 3.6. However, in the first case of (3.7), one gets, by by Proposition 2.8,

$$|\Lambda^{\lambda,\omega}_{A,d}f(z)| \simeq |z|^{\alpha_0-\mathfrak{Re}\,(\lambda)} \qquad \text{as } z \to 0 \text{ through } [0,1].$$

So in this case $\Lambda^{\lambda,\omega}_{A,d}f \notin Z^m$, obtaining a contradiction. Hence the second case of (3.7) holds. However, for any $\varepsilon > 0$ one has, by Proposition 2.8 again,

$$|\Lambda^{\lambda,\omega}_{A,d}f(z)| \gtrsim |1-z|^{\mathfrak{Re}\,(\lambda)-\alpha_1+\varepsilon} \qquad \text{as } z \to 1 \text{ through } [0,1].$$

Hence, $\Lambda^{\lambda,\omega}_{A,d}f \notin \mathcal{K}^{-\gamma}(X)$, so $\Lambda^{\lambda,\omega}_{A,d}f \notin X$, reaching a contradiction again.

Therefore, we have $(\lambda - \Delta)(Z^m) \subseteq \ker L_{\lambda,\omega}|_{Z^m}$, and the proof is finished. ☐

Remark 3.8. In the setting of the lemma above, a suitable application of the Stone-Weierstrass theorem shows that $L_{\lambda,\omega}$ is not the zero functional on Z^m. For more details, see [1, Remark 5.5].

The overall discussion carried out in this section leads to the following result.

Theorem 3.9. *Let X be a γ^∞-space for $\gamma \geq 0$, and let (v_t) be a semicocycle for (ϕ_t) satisfying* (**SCo1**)*-*(**SCo3**)*. Let α_0, α_1 be the exponents of (v_t), and let Δ be the infinitesimal generator of $(v_t C_{\phi_t})$. Then*

$$\sigma(\Delta) = \{\lambda \in \mathbb{C} \,|\, \mathfrak{Re}\,(\lambda) \leq \alpha_1 - \gamma\} \cup \sigma_{point}(\Delta).$$

If $\alpha_1 = -\infty$, the above reads as $\sigma(\Delta) = \sigma_{point}(\Delta)$.

Proof. We gave the inclusion $\subseteq$ in Corollary 3.2, so all that we need to prove now is the inclusion $\supseteq$. Let $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\,(\lambda) < \alpha_1 - \gamma$. Lemma 3.7 together with Remark 3.8 yield

$$(\lambda - \Delta)(Z^m) \subseteq \ker L_{\lambda,\omega}|_{Z^m} \subsetneq Z^m,$$

for some $m \in \mathbb{N}_0$ big enough, where ω is a multivalued function associated with (v_t). Therefore $\dim X/((\lambda - \Delta)(Z^m)) > \dim X/Z^m = m + 1$, which implies $\operatorname{codim}((\lambda - \Delta)(X)) \geq 1$. Thus $\lambda - \Delta$ is not surjective, so $\lambda \in \sigma(\Delta)$ and the proof is finished. □

4. Essential spectrum

In this section, we study the essential spectrum of the infinitesimal generator Δ of a C_0-semigroup $(v_t C_{\phi_t})$ as in preceding sections.

The following result is a straightforward consequence of Theorem 3.9 and the next fact: if $d \in \mathbb{C}_\infty$ is an accumulation point of both $\rho(A)$ and $\widetilde{\sigma}(A)$, then $d \in \widetilde{\sigma}_{ess}(A)$, see for example [8, Sect. I.4] and [30, Remark 4.1].

Proposition 4.1. *Let X be a γ^∞-space for $\gamma \geq 0$, and let (v_t) be a semicocycle for (ϕ_t) satisfying* (**SCo1**)*-*(**SCo3**)*. Let α_0, α_1 be the exponents of (v_t), and let Δ be the infinitesimal generator of $(v_t C_{\phi_t})$. Then*

$$\{\lambda \in \mathbb{C} \mid \mathfrak{Re}\,(\lambda) = \alpha_1 - \gamma\} \cup \{\infty\} \subseteq \widetilde{\sigma}_{ess}(\Delta).$$

In order to prove the reverse inclusion, we need to assume that both the Banach space X and the semicocycle (v_t) satisfy some additional properties. Regarding the properties on X, set $\mathbb{D}_0 := \{z \in \mathbb{D} \mid \mathfrak{Re}\, z < 1/2\}$ and $\mathbb{D}_1 := \mathbb{D} \setminus \overline{\mathbb{D}_0}$. It is readily seen that $\phi_t(\mathbb{D}_0) \subseteq \mathbb{D}_0$ for every $t \geq 0$. Then we ask X to satisfy the property

(**P6**) There are two Banach spaces $X_0 \hookrightarrow \mathcal{O}(\mathbb{D}_0)$, $X_1 \hookrightarrow \mathcal{O}(\mathbb{D}_1)$ such that
- $X = \{f \in \mathcal{O}(\mathbb{D}) \mid f|_{\mathbb{D}_\iota} \in X_\iota \text{ for } \iota = 0, 1\}$,
- $Mul(X_\iota) = H^\infty(\mathbb{D}_\iota)$ for each $\iota = 0, 1$.
- $B(X_0) \subseteq X_0$ (where B denotes the backshift operator).

Note that, since $X \hookrightarrow \mathcal{O}(\mathbb{D}) \hookrightarrow \mathcal{O}(\mathbb{D}_\iota)$ (where the second inclusion is given by the restriction $f \mapsto f|_{\mathbb{D}_\iota}$), the closed graph theorem implies $X \hookrightarrow X_\iota$ for $\iota = 0, 1$.

We also ask X to satisfy the following

(**P7**) $\|C_p\|_{L(X)} \lesssim (1 - |p(0)|)^{-\gamma}$ for every polynomial $p : \mathbb{D} \to \mathbb{D}$;

and

(**P8**) X is separable.

The examples of γ^∞-spaces given in Section 1 (namely $\mathcal{K}_0^{-\gamma}(\mathbb{D})$, $H^p(\mathbb{D})$, $\mathcal{A}^p_\sigma(\mathbb{D})$) also satisfy properties (**P6**)-(**P8**). Indeed, it is well known that such spaces satisfy (**P7**) and (**P8**), see for example [2]. To prove that they also satisfy (**P6**) requires technical definitions for X_ι, $\iota = 0, 1$. These definitions are the analogous versions the ones given in [1, Subsect. 2.1], adapted to the subsets $\mathbb{D}_\iota$.

Regarding the cocycle (v_t), we consider the following property

(**SCo4**) $\limsup_{\mathbb{D}\ni z,w\to 1,\, |h(z)|\simeq|h(w)|} \left(\sup_{t\geq 0} \left| \frac{v_t(z)}{v_t(w)} \right| \right) < \infty$, where $h(z) = z/(1-z)$ is the univalent function associated with (ϕ_t).

To be more precise, the above lim sup reads as follows: For each $A \geq 1$, there exists $M > 0$ and a neighborhood Ω of 1 in $\mathbb{D}$ such that

$$\left| \frac{v_t(z)}{v_t(w)} \right| < M, \qquad \text{for all } t \geq 0 \text{ and all } z, w \in \Omega \text{ with } \left| \frac{h(z)}{h(w)} \right| \in [1/A, A].$$

The following lemma characterizes the semicocycles (v_t) satisfying (**SCo4**) in terms of their multivalued function ω. In particular, the item (4) below shows that the semicocycles (0.5) associated to generalized Hausdorff matrices satisfy (**SCo4**).

Lemma 4.2. *Let (v_t) be a semicocycle for (ϕ_t) satisfying (**SCo1**)-(**SCo2**). Let ω be a multivalued function associated with (v_t). Then, the following are equivalent*

(1) *(v_t) satisfies (**SCo4**), i.e.,* $\limsup_{\mathbb{D}\ni z,w\to 1,\, |h(z)|\simeq|h(w)|} \left(\sup_{t\geq 0} \left| \frac{v_t(z)}{v_t(w)} \right| \right) < \infty.$

(2) $\limsup_{\mathbb{D}\ni z,w\to 1,\, |h(z)|\simeq|h(w)|} \left| \frac{\omega(z)}{\omega(w)} \right| < \infty.$

(3) $\sup_{\mathbb{D}\ni z,w,\, |h(z)|\simeq|h(w)|,\, t\geq 0} \left| \frac{v_t(z)}{v_t(w)} \right| < \infty.$

(4) $\sup_{\mathbb{D}\ni z,w,\, |h(z)|\simeq|h(w)|} \left| \frac{\omega(z)}{\omega(w)} \right| < \infty.$

Proof. Let us proof (1) $\implies$ (2) first. Assume that (1) holds and that (2) is false, and let us see that this yields a contradiction. Since (2) is false, there exist $A \geq 1$ and two sequences $z_n, w_n \in \mathbb{D}$ such that both converge to 1 with $|h(w_n)|/A \leq |h(z_n)| \leq A|h(w_n)|$, and such that

$$\lim_{n\to\infty} \left|\frac{\omega(z_n)}{\omega(w_n)}\right| = \infty.$$

Set $\beta_n := \log|h(z_n)|$, so $\beta_n \to \infty$ as $n \to \infty$. Take N for which $\beta_n \geq 0$ for all $n \geq N$. Since $h(\phi_t(z)) = e^{-t}h(z)$ for all $t \geq 0$, $z \in \mathbb{D}$, one has $|h(\phi_{\beta_n}(z_n))| = 1$ for all $n \geq N$. Also, $1/A \leq |h(\phi_{\beta_n}(w_n))| \leq A$ for all $n \geq N$. Thus, $\{\phi_{\beta_n}(z_n), \phi_{\beta_n}(w_n) \mid n \geq N\}$ is a set bounded away from the points 1 and 0. Therefore, by Lemma 2.7, the set

$$\{|\omega(\phi_{\beta_n})(z_n)|, |\omega(\phi_{\beta_n}(w_n))| \,:\, n \geq N\},$$

is bounded away from ∞ and 0. Hence,

$$\left|\frac{v_{\beta_n}(w_n)}{v_{\beta_n}(z_n)}\right| = \left|\frac{\omega(\phi_{\beta_n}(w_n))}{\omega(\phi_{\beta_n}(z_n))}\right|\left|\frac{\omega(z_n)}{\omega(w_n)}\right| \to \infty, \qquad \text{as } n \to \infty,$$

which contradicts (1). Thus (1) $\implies$ (2).

We show now that (2) $\implies$ (1). Assume (2) is true. Then, for each $A \geq 1$, there exist a neighborhood Ω of 1 in $\mathbb{D}$ and $M > 0$ such that $|\omega(z)/\omega(w)| < M$ if $w, z \in \Omega$ with $|h(w)|/A \leq |h(z)| \leq A|h(w)|$. Also, it follows by Lemma 2.7 and the fact that $\omega(z) = z^\delta r(z)$ (for some $\delta \in \mathbb{C}$ and non-vanishing holomorphic r) that

$$\sup_{z,w\in\mathbb{D}\setminus\Omega,\, |h(z)|\simeq|h(w)|} \left|\frac{\omega(z)}{\omega(w)}\right| < \infty.$$

Therefore (2) implies

$$\sup_{z,w\in\mathbb{D},\, |h(z)|\simeq|h(w)|} \left|\frac{\omega(z)}{\omega(w)}\right| =: B < \infty. \tag{4.1}$$

Note that if $z, w \in \mathbb{D}$ satisfy $|h(w)|/A \leq |h(z)| \leq A|h(w)|$, then $|h(\phi_t(w))|/A \leq |h(\phi_t(z))| \leq A|h(\phi_t(w))|$ for all $t \geq 0$. As a consequence, (4.1) yields

$$\left|\frac{v_t(z)}{v_t(w)}\right| = \left|\frac{\omega(\phi_t(z))}{\omega(\phi_t(w))}\right|\left|\frac{\omega(w)}{\omega(z)}\right| \leq B^2, \tag{4.2}$$

for all $t \geq 0$ and all $z, w \in \mathbb{D}$ with $|h(w)|/A \leq |h(z)| \leq A|h(w)|$, so (1) holds and (2) $\implies$ (1). Thus (1) and (2) are indeed equivalent.

Finally, the inequalities above (4.1) and (4.2) show that (2) implies both (3) and (4). The implications (3) $\implies$ (1) and (4) $\implies$ (2) are trivial, and the proof is done. $\square$

We require the two lemmas below to prove the main result of this section.

Lemma 4.3. *Let X be a γ^∞-space for $\gamma \geq 0$ satisfying (**P6**) (with spaces X_0, X_1) and (**P7**). Let (v_t) be a semicocycle for (ϕ_t) satisfying (**SCo1**) and (**SCo2**), and let ω be* *a multivalued function associated with (v_t). Set $p_s(z) = sz$ for all $z \in \mathbb{D}$, $s \in [0,1]$, and set $m \in \mathbb{N}_0$. One has*

$$\left\| \frac{\omega \circ p_s}{\omega} C_{p_s} f \right\|_{X_0} \lesssim s^{m-\alpha_0} \|f\|_X, \qquad f \in Z^m \subset X,\, s \in (0,1].$$

where α_0, α_1 are the exponents of (v_t).

Proof. Using the same argument as in the space X, one gets that $\|B^m f\|_{X_0} \simeq \|f\|_{X_0}$ for all $f \in X_0$ with a zero of order m at 0. With a similar reasoning as in the proof of Proposition 3.1, one gets

$$\begin{aligned}\left\| \frac{\omega \circ p_s}{\omega} C_{p_s} f \right\|_{X_0} &\simeq \left\| B^m \left(\frac{\omega \circ p_s}{\omega} C_{p_s} f \right) \right\|_{X_0} = \left\| (Bp_s)^m \frac{\omega \circ p_s}{\omega} C_{p_s} (B^m f) \right\|_{X_0} \\ &\lesssim \left\| (Bp_s)^m \frac{\omega \circ p_s}{\omega} \right\|_{H^\infty(\mathbb{D}_0)} \|C_{p_s}(B^m f)\|_{X_0}, \qquad f \in Z^m \subset X,\, s \in (0,1].\end{aligned}$$

Since $p_s(0) = 0$ for all $s \in (0,1]$, we have $\sup_{s\in[0,1]} \|C_{p_s}\|_{L(X))} < \infty$ by (**P7**). Thus,

$$\|C_{p_s}(B^m f)\|_{X_0} \lesssim \|C_{p_s}(B^m f)\|_X \lesssim \|B^m f\|_X \simeq \|f\|_X, \qquad f \in Z^m,\, s \in (0,1].$$

It follows by Lemma 2.7 and Proposition 2.8 that $|\omega(z)| \simeq |z|^{-\alpha_0}$ for all $z \in \mathbb{D}_0$. Hence,

$$\left\| (Bp_s)^m \frac{\omega \circ p_s}{\omega} \right\|_{H^\infty(\mathbb{D}_0)} = s^m \left\| \frac{\omega \circ p_s}{\omega} \right\|_{H^\infty(\mathbb{D}_0)} \simeq s^{m-\alpha_0}, \qquad s \in (0,1],$$

and the proof is finished. □

Lemma 4.4. *Let X be a γ^∞-space for $\gamma \geq 0$ satisfying (**P6**) (with spaces X_0, X_1) and (**P7**). Let (v_t) be a semicocycle for (ϕ_t) satisfying (**SCo1**), (**SCo2**) and (**SCo4**), and let ω be a multivalued function associated with (v_t). Set $q_s(z) = 1 + s(z-1)$ for all $z \in \mathbb{D}$, $s \in [0,1]$. Let α_0, α_1 be the exponents of (v_t) and assume $\alpha_1 > -\infty$. Take any $\varepsilon > 0$. For all $f \in X$, one has*

$$\left\| \frac{\omega \circ q_s}{\omega} C_{q_s} f \right\|_{X_1} \lesssim s^{\alpha_1-\gamma-\varepsilon} \|f\|_X, \qquad f \in X,\, s \in (0,1].$$

Proof. First note that, for all $f \in X$ and $s \in (0,1]$,

$$\left\| \frac{\omega \circ q_s}{\omega} C_{q_s} f \right\|_{X_1} \lesssim \left\| \frac{\omega \circ q_s}{\omega} \right\|_{H^\infty(\mathbb{D}_1)} \|C_{q_s} f\|_{X_1} \lesssim \left\| \frac{\omega \circ q_s}{\omega} \right\|_{H^\infty(\mathbb{D}_1)} \|C_{q_s} f\|_X.$$

Since $q_s(0) = 1 - s$, one has by (**P7**) that $\|C_{q_s}\|_{L(X)} \lesssim s^{-\gamma}$ for all $s \in (0,1]$. Hence, $\|C_{q_s} f\|_X \lesssim s^{-\gamma}\|f\|_X$ for $f \in X$ and $s \in (0,1]$. Therefore, the claim is proven if we show that, for each $\varepsilon > 0$, we have

$$\left\| \frac{\omega \circ q_s}{\omega} \right\|_{H^\infty(\mathbb{D}_1)} \lesssim s^{\alpha_1-\varepsilon}, \qquad s \in (0,1]. \tag{4.3}$$

Let us prove such an inequality. It is readily seen that $|h(\phi_{-\log s}(q_s(z)))| \simeq |h(z)|$ for all $z \in \mathbb{D}_1$ and $s \in (0,1]$. Indeed,

$$\left|\frac{h(\phi_{-\log s}(q_s(z)))}{h(z)}\right| = \frac{|q_s(z)|}{|z|}, \qquad s \in (0,1],\ z \in \mathbb{D}_1.$$

Take $\varepsilon > 0$. Then, there exists $\delta \in (0,1)$ such that

$$\begin{aligned}\left|\frac{\omega(q_s(z))}{\omega(z)}\right| &= \left|\frac{\omega(q_s(z))}{\omega(\phi_{-\log s}(q_s(z)))}\right|\left|\frac{\omega(\phi_{-\log s}(q_s(z)))}{\omega(z)}\right| \simeq \left|\frac{\omega(q_s(z))}{\omega(\phi_{-\log s}(q_s(z)))}\right| \\ &= \left|\frac{1}{v_{-\log s}(q_s(z))}\right| \lesssim e^{(-\alpha_1+\varepsilon)(-\log s)} = s^{\alpha_1-\varepsilon} \qquad s \in (0,\delta),\ z \in \mathbb{D}_1,\end{aligned}$$

where we have used Lemma 4.2(4) at the $\simeq$ sign, and Remark 2.4 at the $\lesssim$ sign since $q_s(z) \in \Omega_{(2s)^{-1}}$ for all $z \in \mathbb{D}_1$ and $s \in (0,1]$. On the other hand, it is readily seen that $|h(q_s(z))| \simeq |h(z)|$ for all $z \in \mathbb{D}_1$, $s \in [\delta,1]$. Thus, it follows by Lemma 4.2(4) again that $\sup_{s\in[\delta,1], z\in\mathbb{D}_1}\left|\frac{\omega(q_s(z))}{\omega(z)}\right| < \infty$. Putting everything together, one gets that for each $\varepsilon > 0$, the inequality (4.3) holds, so the proof is done. □

Remark 4.5. Let I be a real interval and let $F : I \to \mathcal{O}(\mathbb{D})$ be a Borel measurable function such that $F(I) \subset X$. If X is separable, it is well known that the induced mapping (also denoted by F) $F : I \to X$ is also Borel measurable, hence Bochner measurable (see for instance [37, Cor. 4.5.5]).

Theorem 4.6. *Let X be a γ^∞-space for $\gamma \geq 0$ satisfying (**P6**) (with spaces X_0, X_1), (**P7**) and (**P8**). Let (v_t) be a semicocycle for (ϕ_t) satisfying (**SCo1**)-(**SCo4**), and let Δ be the infinitesimal generator of $(v_t C_{\phi_t})$. Then*

$$\widetilde{\sigma}_{ess}(\Delta) = \{\lambda \in \mathbb{C} \mid \mathfrak{Re}\,(\lambda) = \alpha_1 - \gamma\} \cup \{\infty\},$$

where α_0, α_1 are the exponents of (v_t).

If $\alpha_1 = -\infty$, the above reads as $\widetilde{\sigma}_{ess}(\Delta) = \{\infty\}$.

Proof. We already know the inclusion $\widetilde{\sigma}_{ess}(\Delta) \supseteq \{\lambda \in \mathbb{C} \mid \mathfrak{Re}\,(\lambda) = \alpha_1 - \gamma\} \cup \{\infty\}$ by Proposition 4.1. Also, as we mentioned in the proof of Corollary 3.2, one has

$$\sigma_{ess}(\Delta) \subseteq \{\lambda \in \mathbb{C} \mid \mathfrak{Re}\,(\lambda) \leq \alpha_1 - \gamma\}.$$

Therefore, all we have to proof is that the intersection $\{\lambda \in \mathbb{C} \mid \mathfrak{Re}\,(\lambda) < \alpha_1 - \gamma\} \cap \sigma_{ess}(\Delta)$ is empty.

Thus, take $\lambda \in \mathbb{C}$ such that $\mathfrak{Re}\,(\lambda) < \alpha_1 - \gamma$, and take $m \in \mathbb{N}_0$ such that $m > \alpha_0 - \mathfrak{Re}\,(\lambda)$. Then $L_{\lambda,\omega}$ is a continuous functional on Z^m (where ω is a multivalued function associated with (v_t)) for which $(\lambda - \Delta)(Z^m) \subseteq \ker L_{\lambda,\omega}|_{Z^m}$, see Lemma 3.7. Assume the reverse inclusion also holds, so $(\lambda - \Delta)(Z^m) = \ker L_{\lambda,\omega}|_{Z^m}$. Then $\operatorname{codim}(\operatorname{Ran}(\lambda - \Delta)) \leq \operatorname{codim}(\ker L_{\lambda,\omega}|_{Z^m}) = m + 2 < \infty$. Since $\dim(\ker(\lambda - \Delta)) \leq 1$ by Remark 3.3, one gets $\lambda \notin \sigma_{ess}(\Delta)$, and the proof is finished.

Hence, let us prove the inclusion $(\lambda-\Delta)(Z^m) \supseteq \ker L_{\lambda,\omega}|_{Z^m}$. Take $f \in \ker L_{\lambda,\omega}|_{Z^m}$. By Remark 3.6, f lies in $\mathrm{Ran}(\lambda-\Delta)$ if and only if there exits $A \in \mathbb{C}$, $d \in \mathbb{D}\setminus\{0\}$ such that the multivalued function $\Lambda^{\lambda,\omega}_{A,d} f$ induces a holomorphic function on $\mathbb{D}$ which lies in X. In this case $f = (\lambda-\Delta)\Lambda^{\lambda,\omega}_{A,d} f$ (recall that $\Lambda^{\lambda,\omega}_{A,d} f$ is given by (3.3)). Set

$$\widetilde{f}(z) := \frac{(1-z)^\lambda}{z^\lambda \omega(z)} \int_0^z \frac{\tau^{\lambda-1}}{(1-\tau)^{\lambda+1}} \omega(\tau) f(\tau)\, d\tau, \qquad z \in \mathbb{D}, \tag{4.4}$$

where we take the segment $[0,z]$ as integration path. Reasoning as in the proof of Lemma 3.7, it is readily seen that such integral is absolutely convergent. Since $\omega(z) = z^\delta r(z)$ for some $\delta \in \mathbb{C}$ and non-vanishing $r \in \mathcal{O}(\mathbb{D})$, one has that $\widetilde{f}$ is an holomorphic function in $\mathbb{D}$. Note that $\widetilde{f} = \Lambda^{\lambda,\omega}_{A,d} f$ with

$$A = \int_0^d \frac{\tau^{\lambda-1}}{(1-\tau)^{\lambda+1}} \omega(\tau) f(\tau)\, d\tau.$$

Similarly, since $f \in \ker L_{\lambda,\omega}|_{Z^m}$, then we can choose 1 as starting point for the integral path, i.e.,

$$\widetilde{f}(z) = \frac{(1-z)^\lambda}{z^\lambda \omega(z)} \int_1^z \frac{\tau^{\lambda-1}}{(1-\tau)^{\lambda+1}} \omega(\tau) f(\tau)\, d\tau, \qquad z \in \mathbb{D}, \tag{4.5}$$

where we take the segment $[1,z]$ as integration path.

To see that $\widetilde{f}$ belongs to X, we prove $\widetilde{f}|_{\mathbb{D}_0} \in X_0$ and $\widetilde{f}|_{\mathbb{D}_1} \in X_1$, where X_0, X_1 are the Banach spaces given by (**P6**).

To see that $\widetilde{f}|_{\mathbb{D}_0} \in X_0$, set $p_s(z) = sz$ for all $z \in \mathbb{D}$, $s \in [0,1]$. Then, parameterizing the integration path $[0,z]$ with $s \mapsto p_s(z)$ shows, by (4.4),

$$\widetilde{f}|_{\mathbb{D}_0} = \int_0^1 \frac{(1-(\cdot))^\lambda}{(1-p_s)^{\lambda+1}} s^{\lambda-1} \frac{\omega \circ p_s}{\omega} C_{p_s} f\, ds.$$

Note that the mapping the integrand above is Bochner-measurable on X (hence on X_0 since $X \hookrightarrow X_0$) by Remark 4.5 and (**P8**). Since $\sup_{s\in[0,1]} \left\| \frac{(1-(\cdot))^\lambda}{(1-p_s)^{\lambda+1}} \right\|_{H^\infty(\mathbb{D}_0)} < \infty$, one gets

$$\|\widetilde{f}|_{\mathbb{D}_0}\|_{X_0} \lesssim \int_0^1 \left\| s^{\lambda-1} \frac{\omega \circ p_s}{\omega} C_{p_s} f \right\|_{X_0} ds \lesssim \|f\|_X \int_0^1 s^{m+\mathfrak{Re}\,(\lambda)-\alpha_0-1} < \infty,$$

where we have applied Lemma 4.3. Thus, we conclude $\widetilde{f}|_{\mathbb{D}_0} \in X_0$.

Similarly, set $q_s(z) = 1 + s(z-1)$ for $z \in \mathbb{D}$, $s \in [0,1]$. Parameterizing the integration path $[1,z]$ with $s \mapsto q_s(z)$ shows, by (4.5),

$$\widetilde{f}|_{\mathbb{D}_1} = -\int_0^1 \frac{(q_s)^{\lambda-1}}{(\cdot)^\lambda} s^{-\lambda-1} \frac{\omega \circ q_s}{\omega} C_{q_s} f\, ds.$$

Again, we have by Remark 4.5 and (**P8**) that the integrand above is a measurable function on X_1. Since $\sup_{s\in[0,1]} \left\| \frac{(q_s)^{\lambda-1}}{(\cdot)^\lambda} \right\|_{H^\infty(\mathbb{D}_1)} < \infty$, we have for small enough $\varepsilon > 0$,

$$\|\widetilde{f}|_{\mathbb{D}_1}\|_{X_1} \lesssim \int_0^1 \left\| s^{-\lambda-1} \frac{\omega \circ q_s}{\omega} C_{q_s} f \right\|_{X_1} ds \lesssim \|f\|_X \int_0^1 s^{\alpha_1-\gamma-\mathfrak{Re}\,(\lambda)-\varepsilon-1} < \infty,$$

where we have applied Lemma 4.4. Thus, we conclude $\widetilde{f}|_{\mathbb{D}_1} \in X_1$, and we have by (**P6**) that indeed $\widetilde{f} \in X$. Finally, it is readily seen from (4.4) that $\widetilde{f}$ has a zero of order at least m, so $\ker L_{\lambda,\omega}|_{Z^m} \subseteq (\lambda - \Delta)(Z^m)$, and the proof is finished. □

Remark 4.7. For a Fredholm operator A, let $\operatorname{ind}(A)$ denote the index of an operator, i.e., $\operatorname{ind}(A) = \dim(\ker A) - \operatorname{codim}(\operatorname{Ran} A) \in \mathbb{Z}$. It is easily deduced from the proof of 4.6 that $\operatorname{ind}(\lambda - \Delta) = -1$ for all $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\,(\lambda) < \alpha_1 - \gamma$. To see this, since the index is a constant in each component of $\mathbb{C} \setminus \sigma_{ess}(\Delta)$, it is enough to prove it for any $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\,(\lambda) < \alpha_1 - \gamma$, see [8, Sect. I.4].

Take any such λ for which $\lambda - \Delta$ is an injective operator. Notice that such λ exists by Proposition 3.4. Since $(\lambda - \Delta)(Z^m) = \ker L_{\lambda,\omega}|_{Z^m}$ for some $m \in \mathbb{N}$ big enough (see the proof of Theorem 4.6), we get $\operatorname{codim}(\lambda - \Delta)(Z^m) = m + 2$. Take an arbitrary $\mu \in \rho(\Delta) \cap \rho(\Delta|_{Z^m})$ and set $\mathfrak{X}_m := (\mu - \Delta)^{-1}(P_m)$ (recall that P_m denotes the polynomials of order less than or equal to m), so $\mathfrak{X}_m \subset \operatorname{Dom}(\Delta)$. Since $\dim \mathfrak{X}_m = m + 1$ and $\mathfrak{X}_m \cap Z^m = \emptyset$, it follows $X = \mathfrak{X}_m \oplus Z^m$. As $\lambda - \Delta$ is injective, one has $(\lambda - \Delta)(X) = (\lambda - \Delta)(Z^m) \oplus (\lambda - \Delta)(\mathfrak{X}_m)$ and $(\lambda - \Delta)(\mathfrak{X}_m) = P_m \simeq \mathbb{C}^{m+1}$. Then

$$\frac{X}{(\lambda - \Delta)(X)} \simeq \frac{X/(\lambda - \Delta)(Z^m)}{(\lambda - \Delta)(X)/(\lambda - \Delta)(Z^m)} \simeq \frac{\mathbb{C}^{m+2}}{\mathbb{C}^{m+1}} \simeq \mathbb{C},$$

so $\operatorname{codim}(\lambda - \Delta)(X) = 1$ and $\operatorname{ind}(\lambda - \Delta) = -1$ as claimed.

5. Spectrum of weighted averaging operators

Here, we apply the results obtained in the preceding sections to study the boundedness and the spectrum, on a γ^∞-space, of an operator subordinated to a weighted composition semigroup $(v_t C_{\phi_t})$.

Along this section, for each semigroup $(v_t C_{\phi_t})$ on a γ^∞-space X, such that (v_t) satisfies properties (**SCo1**)-(**SCo3**), we denote by c the real number $\max\{\alpha_0, \alpha_1 - \gamma\}$, where α_0, α_1 are the exponents of (v_t).

Theorem 5.1. *Let $(v_t C_{\phi_t})$ be a semigroup on a γ^∞-space X, such that (v_t) satisfies properties* (**SCo1**)-(**SCo3**). *Let ν be a complex Borel measure on $[0, +\infty)$, such that $\int_0^\infty e^{(c+\delta)t} |d\nu|(t) < \infty$ for some $\delta > 0$. Let the operator $\mathcal{H}$ be defined by*

$$\mathcal{H} f = \int_0^\infty v_t C_{\phi_t} f \, d\nu(t), \quad f \in X,$$

where the integral above is Bochner-convergent. Then $\mathcal{H}$ is a well-defined bounded operator on X.

Proof. This is a consequence of Proposition 3.5. □

Now we present a technical lemma. Assume that ν is a finite Borel measure on $[0,\infty)$ which is absolutely continuous with respect to the Lebesgue measure, so $d\nu(t) = \rho(t)\,dt$, for some $L^1[0,\infty)$ function ρ. Then its Laplace transform $q(\lambda) := \mathcal{L}(\nu)(\lambda) = \int_0^\infty e^{-\lambda t}\rho(t)\,dt$ is well defined for $\mathfrak{Re}\,\lambda \geq 0$. Also, for $\theta \in (0,\pi]$, we denote by $\sum_\theta$ the complex sector of angle θ, i.e., $\sum_\theta = \{z \in \mathbb{C}\setminus\{0\} : |\arg z| < \theta\}$.

Lemma 5.2. *Suppose that $\rho \in L^1[0,\infty)$ can be extended in an holomorphic way to a sector Σ_θ with $\theta \in (0,\pi/2]$, and that there exist $\eta \in (0,1], \xi \in (0,1)$ satisfying*

$$\sup_{z\in\Sigma_\varepsilon\cap\{|z|\leq 1\}} |z^{1-\eta}\rho(z)| < \infty \quad \textit{and} \quad \sup_{z\in\Sigma_\varepsilon\cap\{|z|\geq 1\}} |z^{1+\xi}\rho(z)| < \infty,$$

for all $\varepsilon \in (0,\theta)$. Then, its Laplace transform $q := \mathcal{L}(\rho)$ can be extended to $\Sigma_{\pi/2+\theta}$, and such extension satisfies

$$\sup_{\lambda\in\Sigma_{\pi/2+\varepsilon}\cap\{|\lambda|\leq 1\}} |\lambda^{-\xi}(q(\lambda)-q(0))| < \infty \quad \textit{and} \quad \sup_{\lambda\in\Sigma_{\pi/2+\varepsilon}\cap\{|\lambda|\geq 1\}} |\lambda^{\eta}q(\lambda)| < \infty,$$

for all $\varepsilon \in (0,\theta)$.

Proof. Let $0 < \varepsilon < \theta$. Then there is $M > 0$ such that $|\rho(z)| \leq M|z|^{\eta-1}$ if $z \in \overline{\Sigma_\varepsilon} \cap \{|z| \leq 1\}\setminus\{0\}$ and $|\rho(z)| \leq \frac{M}{|z|^{\xi+1}}$ if $z \in \overline{\Sigma_\varepsilon} \cap \{|z| \geq 1\}$. Let $\Gamma_\pm$ the paths on the complex plane defined by $\Gamma_\pm := \{se^{\pm i\varepsilon} : 0 \leq s < \infty\}$. Let $\lambda > 0$, by Cauchy's theorem we get

$$q(\lambda) = \int_{\Gamma_\pm} e^{-\lambda z}\rho(z)\,dz = e^{\pm i\varepsilon}\int_0^\infty e^{-\lambda se^{\pm i\varepsilon}}\rho(se^{\pm i\varepsilon})\,ds,$$

since

$$\int_0^{\pm\varepsilon} |e^{-\lambda Re^{i\theta}}\rho(Re^{i\theta})Rie^{i\theta}|\,d\theta \lesssim \frac{e^{-\lambda R\cos\varepsilon}}{R^\xi} \to 0, \quad R \to +\infty.$$

Let now $0 < \tau < \pi/2-\varepsilon$, and $\lambda \in \mathbb{C}$ such that $-\pi/2-\varepsilon+\tau < \arg\lambda < \pi/2-\varepsilon-\tau$. Then $-\pi/2+\tau < \arg(e^{i\varepsilon}\lambda) < \pi/2-\tau$, and therefore $\mathfrak{Re}\,(e^{i\varepsilon}\lambda) \geq |\lambda|\sin\tau$. Then

$$|e^{-\lambda se^{i\varepsilon}}\rho(se^{i\varepsilon})| \leq Me^{-|\lambda|s\sin\tau}s^{\eta-1}, \quad s \in (0,1),$$

and

$$|e^{-\lambda se^{i\varepsilon}}\rho(se^{i\varepsilon})| \leq M\frac{e^{-|\lambda|s\sin\tau}}{s^{\xi+1}}, \quad s > 1.$$

So, the integral

$$q_+(\lambda) := e^{i\varepsilon}\int_0^\infty e^{-\lambda se^{i\varepsilon}}\rho(se^{i\varepsilon})\,ds$$

is absolutely convergent and defines a holomorphic function in the region $-\pi/2-\varepsilon+\tau<\arg\lambda<\pi/2-\varepsilon-\tau$, satisfying

$$|\lambda^\eta q_+(\lambda)|\leq M\Gamma(\eta)/(\sin\tau)^\eta.$$

In a similar way,

$$q_-(\lambda):=e^{-i\varepsilon}\int_0^\infty e^{-\lambda se^{-i\varepsilon}}\rho(se^{-i\varepsilon})\,ds$$

is absolutely convergent and defines a holomorphic function in the region $-\pi/2+\varepsilon+\tau<\arg\lambda<\pi/2+\varepsilon-\tau$, satisfying

$$|\lambda^\eta q_-(\lambda)|\leq M\Gamma(\eta)/(\sin\tau)^\eta.$$

Then q_+ and q_- are holomorphic extensions of q, and they define a holomorphic extension to $\Sigma_{\pi/2+\varepsilon-\tau}$, satisfying $|\lambda^\eta q(\lambda)|\leq M\Gamma(\eta)/(\sin\tau)^\eta$ in the sector. Since $\varepsilon<\theta$ and $0<\tau<\pi/2-\varepsilon$ are arbitrary, we have defined the extension of q in $\Sigma_{\pi/2+\theta}$ such that $\sup_{\lambda\in\Sigma_{\pi/2+\varepsilon}}|\lambda^\eta q(\lambda)|<\infty$ for all $0<\varepsilon<\theta$.

Now observe that by Cauchy's theorem we have $q(0)=e^{\pm i\varepsilon}\int_0^\infty\rho(se^{\pm i\varepsilon})\,ds$, since

$$\int_0^{\pm\varepsilon}|\rho(Re^{i\theta})Rie^{i\theta}|\,d\theta\leq\frac{M\varepsilon}{R^\xi}\to 0,\quad R\to+\infty.$$

So, if $0<\tau<\pi/2-\varepsilon$, and $\lambda\in\mathbb{C}$ is such that $-\pi/2-\varepsilon+\tau<\arg\lambda<\pi/2-\varepsilon-\tau$, since $\mathfrak{Re}\,(e^{i\varepsilon}\lambda)\geq|\lambda|\sin\tau$, one has

$$|(e^{-\lambda se^{i\varepsilon}}-1)\rho(se^{i\varepsilon})|\leq M|\lambda|s^{\eta-1}\int_0^s e^{-|\lambda|u\sin\tau}\,du\leq M|\lambda|,\quad s\in(0,1),$$

and

$$|(e^{-\lambda se^{i\varepsilon}}-1)\rho(se^{i\varepsilon})|\leq M|\lambda|\frac{\int_0^s e^{-|\lambda|u\sin\tau}\,du}{s^{\xi+1}},\quad s>1.$$

Then

$$|q(\lambda)-q(0)|\leq M|\lambda|\Big(\int_0^1 ds+\int_1^\infty\frac{1}{s^{\xi+1}}\int_0^s e^{-|\lambda|u\sin\tau}\,du\,ds\Big).$$

Observe that

$$\begin{aligned}&\int_1^\infty\frac{1}{s^{\xi+1}}\int_0^s e^{-|\lambda|u\sin\tau}\,du\,ds\\ =&\int_0^1 e^{-|\lambda|u\sin\tau}\int_1^\infty\frac{1}{s^{\xi+1}}\,ds\,du+\int_1^\infty e^{-|\lambda|u\sin\tau}\int_u^\infty\frac{1}{s^{\xi+1}}\,ds\,du\\ \lesssim&1+\int_1^\infty\frac{e^{-|\lambda|u\sin\tau}}{u^\xi}\,du=1+(|\lambda|\sin\tau)^{\xi-1}\int_{|\lambda|\sin\tau}^\infty\frac{e^{-v}}{v^\xi}\,dv\\ \lesssim&1+|\lambda|^{\xi-1}.\end{aligned}$$

Therefore $|q(\lambda)-q(0)|\lesssim|\lambda|^\xi$ with $-\pi/2-\varepsilon+\tau<\arg\lambda<\pi/2-\varepsilon-\tau$ and $|\lambda|\leq 1$. Similarly, one gets that $|q(\lambda)-q(0)|\lesssim|\lambda|^\xi$ with $-\pi/2+\varepsilon+\tau<\arg\lambda<$

$\pi/2+\varepsilon-\tau$ and $|\lambda|\le 1$. Since $\varepsilon<\theta$ and $0<\tau<\pi/2-\varepsilon$ are arbitrary, we have $\sup_{\lambda\in\Sigma_{\pi/2+\varepsilon}\cap\{|\lambda|\le 1\}}|\lambda^{-\xi}(q(\lambda)-q(0))|<\infty$ for all $0<\varepsilon<\theta$.

□

A closed operator A is said to be sectorial of angle $\pi/2$ if $\sigma(A)\subseteq\{z\in\mathbb{C}\mid\Re(z)\ge 0\}$ and, for each $\theta\in(\pi/2,\pi)$, one has

$$\sup_{\lambda\notin\sum_\theta}\left\|\frac{\lambda}{\lambda-A}\right\|<\infty.$$

Recall that $\sum_\theta$ denotes the complex sector of angle θ.

Let A be a sectorial operator of angle $\pi/2$. In the context of the functional calculus of sectorial operators, an holomorphic function f on a sector $\sum_\theta$ (for some $\theta\in(\pi/2,\pi)$) belongs to the domain of the functional calculus of A, $\mathcal{E}(A)$, if f is regular at 0 and ∞. We say that a function f is regular if the finite limits $\lim_{z\to 0}f(z)=:d_0$, $\lim_{z\to\infty}f(z)=:d_\infty\in\mathbb{C}$ exist in such a way that, for some $r,R>0$,

$$\int_{\Gamma(\sum_{\theta'}\cap\{|z|<r\})}\left|\frac{f(z)-d_0}{z}\right||dz|<\infty,\qquad\int_{\Gamma(\sum_{\theta'}\cap\{|z|>R\})}\left|\frac{f(z)-d_\infty}{z}\right||dz|<\infty$$

for every $\theta'\in[0,\theta)$. There $\Gamma(\Omega)$ denotes the topological boundary of a subset $\Omega\subset\mathbb{C}$ (so $\Gamma(\sum_{\theta'}\cap\{|z|<r\})=\{z\in\mathbb{C}\setminus\{0\}\mid|\arg z|=\theta'\text{ and }|z|\le r\}\cup\{0\}$). In the case f is regular (so $f\in\mathcal{E}(A)$), the operator $f(A)$ is defined via

$$f(A):=d_\infty+\frac{d_0}{1+A}+\int_{\Gamma(\sum_{\theta'})}\frac{f(z)-d_\infty-\frac{d_0}{z+1}}{z-A}\,dz,$$

where θ' is any number in $(\pi/2,\theta)$, see [19] for more details.

Note that, by Proposition 3.5, for every semigroup $(v_tC_{\phi_t})$ as above, one gets that $e^{-(c+\varepsilon)t}T(t)$ is a uniformly bounded semigroup for each $\varepsilon>0$. Therefore $c+\varepsilon-\Delta$ is sectorial of angle $\pi/2$, where Δ is the infinitesimal generator of $(v_tC_{\phi_t})$, see for example [20, Subsect. 2.1.1]. To avoid cumbersome notation, we write $f\in\mathcal{E}(-\Delta)$ if $f_{c+\varepsilon}\in\mathcal{E}(c+\varepsilon-\Delta)$, where $f_{c+\varepsilon}=f((\cdot)-c-\varepsilon)$. In this case, we set $f(-\Delta):=f_{c+\varepsilon}(c+\varepsilon-\Delta)$.

Corollary 5.3. *Let $(v_tC_{\phi_t})$ be a semigroup on a γ^∞-space X such that the semicocycle (v_t) satisfies properties (**SCo1**)-(**SCo3**). Let α_0,α_1 be the exponents of (v_t). Let ν be a complex Borel measure on $[0,+\infty)$, such that $\int_0^\infty e^{(c+\delta)t}|d\nu|(t)<\infty$ for some $\delta>0$. Assume $d\nu(t)=\rho(t)\,dt$, and that ρ can be extended in an holomorphic way to a sector Σ_θ with $\theta\in(0,\pi/2]$, and that there exist $\eta\in(0,1],\xi\in(0,1)$ satisfying*

$$\sup_{z\in\Sigma_\varepsilon\cap\{|z|\le 1\}}|z^{1-\eta}\rho(z)|<\infty\ \text{and}\ \sup_{z\in\Sigma_\varepsilon\cap\{|z|\ge 1\}}|z^{1+\xi}e^{(c+\delta)z}\rho(z)|<\infty,$$

for all $0<\varepsilon<\theta$. Then

$$\sigma(\mathcal{H})=\mathcal{L}(\nu)((\gamma-\alpha_1+\Sigma_{\pi/2})\cup-\sigma_{point}(\Delta))\cup\{0\},$$

and

$$\mathcal{L}(\nu)(-\sigma_{point}(\Delta)) \subseteq \sigma_{point}(\mathcal{H}) \subseteq \{0\} \cup \mathcal{L}(\nu)(-\sigma_{point}(\Delta)).$$

If in addition X satisfies (**P6**)-(**P8**)*, and* (v_t) *satisfies* (**SCo4**)*, then*

$$\sigma_{ess}(\mathcal{H}) = \mathcal{L}(\nu)((\gamma - \alpha_1) + i\mathbb{R}) \cup \{0\}.$$

If $\alpha_1 = -\infty$, the above spectral identities read as $\sigma(\mathcal{H}) = \mathcal{L}(\nu)(-\sigma_{point}(\Delta)) \cup \{0\}$ and $\sigma_{ess}(\mathcal{H}) = \{0\}$.

Proof. An application of Lemma 5.2 to the function $t \mapsto e^{(c+\delta)t}\rho(t)$ yields that the function $\mathcal{L}(\nu)$ can be holomorphically extended to the translated sector $-(c+\delta) + \Sigma_{\pi/2+\theta}$, and is regular at $-c-\delta, \infty$ (with $\mathcal{L})(\nu)(\infty) = 0$). Hence, $\mathcal{L}(\nu)$ belongs to $\mathcal{E}(-\Delta)$. Under these conditions, $\mathcal{H} = \mathcal{L}(\nu)(-\Delta)$ (see [20, Prop. 3.3.2]) and the spectral mapping theorem for the usual spectrum σ and for the essential spectrum σ_{ess} holds, that is,

$$\sigma(\mathcal{H}) = \sigma(\mathcal{L}(\nu)(-\Delta)) = \mathcal{L}(\nu)(\widetilde{\sigma}(-\Delta)),$$
$$\sigma_{ess}(\mathcal{H}) = \sigma_{ess}(\mathcal{L}(\nu)(-\Delta)) = \mathcal{L}(\nu)(\widetilde{\sigma}_{ess}(-\Delta)),$$

see [19, Th. 6.4] and [30, Th. 5.4]. Thus, the claims for $\sigma(\mathcal{H})$ and $\sigma_{ess}(\mathcal{H})$ follow by Theorems 3.9 and 4.6.

Regarding the point spectrum, our result is an immediate consequence of the spectral mapping inclusions for the point spectrum given in [19, Cor. 6.6]. ☐

Remark 5.4. More generally, the statement of the corollary above still holds whenever the Laplace transform of $\mathcal{L}(\nu)$ belongs to $\mathcal{E}(-\Delta)$, see [30] for more details. Even more, if there exists $\alpha > 0$ and $\theta \in (\pi/2, \pi)$ such that

$$|\mathcal{L}(\nu)(z)| \gtrsim |z|^{-\alpha}, \qquad \text{as } z \to \infty \text{ through } \Sigma_\theta,$$

then

$$\sigma_{point}(\mathcal{H}) = \mathcal{L}(\nu)(-\sigma_{point}(\Delta)),$$

see [30, Prop. 5.6].

6. Examples

Here we apply our results to some generalized Hausdorff operators $\mathcal{H}_\mu^{(\zeta)}$ on a γ^∞-space X. We also consider these operators on the weighted Dirichlet spaces $\mathcal{D}_\sigma^p(\mathbb{D})$.

Proposition 6.1. *Let μ be a complex Borel bounded measure on $(0,1]$, and set $\nu = \kappa(\mu)$ as the Borel image measure on $[0,\infty)$ by the function $\kappa : (0,1] \to [0,+\infty)$ given by $\kappa(t) = \log(1/t)$. Let $\zeta \geq 0$. Then we have*

$$\mathcal{H}_\mu^{(\zeta)} f(z) = \int_0^\infty u_t^\zeta(z) C_{\phi_t} f(z)\, d\nu(t), \quad z \in \mathbb{D},\ f \in \mathcal{O}(\mathbb{D}),$$

where (ϕ_t) is given by (0.4) and (u_t^ζ) by (0.5).

Proof. Let $f \in \mathcal{O}(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. For each $z \in \mathbb{D}$,

$$\begin{aligned}\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} H_{\mu}^{(\zeta)}(n,k)a_k\right||z|^n &\leq \int_0^1 t^{\zeta}\sum_{k=0}^{\infty}|a_k|\frac{|z|^k t^k}{(1-|z|(1-t))^{k+\zeta+1}}\,d|\mu|(t)\\ &\leq \frac{1}{1-|z|}|\mu|((0,1])\sum_{k=0}^{\infty}|a_k||z|^k,\end{aligned}$$

where we have used the identity for $H_{\mu}^{(\zeta)}$ given in (0.2). Thus, the series defining $\mathcal{H}_{\mu}^{(\zeta)} f(z)$ is absolutely convergent for every $z \in \mathbb{D}$, see (0.3). This absolute convergence implies

$$\begin{aligned}\mathcal{H}_{\mu}^{(\zeta)} f(z) &= \int_0^1 t^{\zeta}\sum_{k=0}^{\infty} a_k \frac{z^k t^k}{(1-z(1-t))^{k+\zeta+1}}\,d\mu(t)\\ &= \int_0^1 \frac{t^{\zeta}}{(1-z(1-t))^{\zeta+1}} f\left(\frac{zt}{1-z(1-t)}\right) d\mu(t)\\ &= \int_0^{\infty} u_t^{\zeta}(z)\,(f\circ\phi_t)(z)\,d\nu(t), \qquad z\in\mathbb{D},\ f\in\mathcal{O}(\mathbb{D}),\end{aligned}$$

and the proof is finished. □

By (3.1), the infinitesimal generator Δ of the C_0-semigroup $(u_t^{\zeta} C_{\phi_t})$ is given by

$$\Delta f(z) = -z(1-z)f'(z) + (\zeta z + z - \zeta)f(z), \qquad z\in\mathbb{D}, f\in \mathrm{Dom}(\Delta).$$

It is readily seen that the exponents of the semicocycle (u_t^{ζ}) are given by $\alpha_0 = -\mathfrak{Re}\,\zeta$ and $\alpha_1 = 1$. Then, by Theorem 3.9,

$$\sigma(\Delta) = \{\lambda\in\mathbb{C} \;:\; \mathfrak{Re}\,\lambda \leq 1-\gamma\}\cup\sigma_{point}(\Delta). \tag{6.1}$$

Remark 6.2. Let X be any of the examples of γ^{∞}-spaces listed in Section 1 (i.e., Hardy spaces, weighted Bergman spaces and little Korenblum classes). Then:

a) The growth bound of the weighted composition semigroup $(u_t^{\zeta} C_{\phi_t})$ described above is $c := \max\{-\mathfrak{Re}\,\zeta, 1-\gamma\}$, that is, there is $M > 0$ such that $\|T(t)\| \leq Me^{ct}$, for $t \geq 0$. So, it is enough to assume $\int_0^{\infty} e^{ct}|d\nu|(t) < \infty$ to get that $\mathcal{H}_{\mu}^{(\zeta)}$ is a bounded operator on X.

b) By Remark 3.3, we obtain

$$\begin{aligned}\sigma_{point}(\Delta) &= \left\{-\zeta-k \;:\; k\in\mathbb{N}_0 \text{ such that } \frac{z^k}{(1-z)^{k+\zeta+1}} \in X\right\}\\ &= \{-\zeta-k \;:\; k\in\mathbb{N}_0 \text{ with } k < \gamma - \mathfrak{Re}\,\zeta - 1\}.\end{aligned}$$

c) By Lemma 4.2, one has that (u_t^{ζ}) also satisfies (**SCo4**). Also, these spaces satisfy properties (**P6**)-(**P8**). Hence, Theorem 4.6 yields

$$\widetilde{\sigma}_{ess}(\Delta) = \{\lambda\in\mathbb{C} \mid \mathfrak{Re}\,\lambda = 1-\alpha_1\}\cup\{\infty\}.$$

6.1. **Generalized Hausdorff operators on weighted Dirichlet spaces.** For $\sigma > -1$ and $1 \leq p < \infty$, let $\mathcal{D}^p_\sigma(\mathbb{D})$ denote the weighted Dirichlet space, consisting of all functions $f \in \mathcal{O}(\mathbb{D})$ such that $f' \in \mathcal{A}^p_\sigma(\mathbb{D})$. Set

$$\|f\|_{\mathcal{D}^p_\sigma} := \left(|f(0)|^p + \|f'\|^p_{\mathcal{A}^p_\sigma} \right)^{1/p} < \infty.$$

Then $\mathcal{D}^p_\sigma(\mathbb{D})$ is a Banach space with norm given by $\|\cdot\|_{\mathcal{D}^p_\sigma}$. When $\sigma > p-1$ one has $\mathcal{D}^p_\sigma(\mathbb{D}) = \mathcal{A}^p_{\sigma-p}(\mathbb{D})$ with equivalent norms, see e.g. [12, Th. 6].

In the case $p-2 < \sigma \leq p-1$, these spaces satisfy all the axioms considered in this work (with $\gamma = \frac{\sigma+2}{p} - 1$) including the axioms presented in Section 4 except for (**P5**) and the ones regarding multipliers, that is, (**P1**) and the second item in (**P6**).

For $\sigma = p-2$, these spaces are the so-called analytic Besov spaces. In this case, they also fail to satisfy (**P1**), (**P5**), and the second item in (**P6**). Even more, they just fulfil weaker versions of properties (**P3**), (**P4**) and (**P7**) with $\gamma = 0$. Namely

- for each $\varepsilon > 0$, $\mathcal{D}^p_{p-2}(\mathbb{D}) \hookrightarrow \mathcal{K}^{-\varepsilon}(\mathbb{D})$;
- $C_{\phi_t} \in L(X)$ for $t \geq 0$ with $\lim_{t\to\infty} \|C_{\phi_t}\|^{1/t}_{L(X)} = 1$;
- fixed $\varepsilon > 0$, $\|C_p\|_{L(\mathcal{D}^p_{p-2})} \lesssim (1 - |p(0)|)^{-\varepsilon}$ for every polynomial $p : \mathbb{D} \to \mathbb{D}$.

We refer the reader to [1, Subsect. 2.1(4)] and [2, Sect. 3] for the proofs of the above statements regarding $\mathcal{D}^p_\sigma(\mathbb{D})$.

As the spaces $\mathcal{D}^p_\sigma(\mathbb{D})$ fail to satisfy (some of) the axioms considered in this paper, our results do not provide the spectral picture neither for C_0-semigroups $(v_t C_{\phi_t})$ (with the semicocycle (v_t) satisfying axioms (**SCo1**)-(**SCo4**)) nor for their infinitesimal generators Δ acting on $\mathcal{D}^p_\sigma(\mathbb{D})$. Nevertheless, all the proofs given here can be adapted to the spaces $\mathcal{D}^p_\sigma(\mathbb{D})$ for the semicocycle (u^ζ_t) given by (0.5) which is associated to the generalized Hausdorff operators (see Proposition 6.1). Thus, we obtain the following result.

Theorem 6.3. *Let $1 \leq p < \infty$, $\sigma > -1$ and $\sigma \geq p-2$. Set $\gamma = \frac{\sigma+2}{p} - 1$ and let $\zeta \in \mathbb{C}$. Then $(u^\zeta_t C_{\phi_t})$ is a C_0-semigroup on $\mathcal{D}^p_\sigma(\mathbb{D})$ with infinitesimal generator Δ, such that*

$$\lim_{t\to\infty} \|u^\zeta_t C_{\phi_t}\|^{1/t}_{L(\mathcal{D}^p_\sigma)} \leq \exp\left(\max\{-\mathfrak{Re}\,\zeta, 1-\gamma\}\right),$$

and

$$\sigma(\Delta) = \{\lambda \in \mathbb{C} \,|\, \mathfrak{Re}\,\lambda \leq 1-\gamma\} \cup \sigma_{point}(\Delta),$$
$$\sigma_{point}(\Delta) = \{-\zeta - k \mid k \in \mathbb{N}_0 \text{ with } k < \gamma - \mathfrak{Re}\,\zeta - 1\},$$
$$\widetilde{\sigma}_{ess}(\Delta) = \{\lambda \in \mathbb{C} \mid \mathfrak{Re}\,\lambda = 1-\gamma\} \cup \{\infty\}.$$

Proof. As stated in the paragraph preceding this theorem, all the proofs presented in the preceding sections regarding the asymptotic behavior of $\|v_t C_{\phi_t}\|$ and the spectrum of Δ can be adapted to the spaces $\mathcal{D}^p_\sigma(\mathbb{D})$ when (v_t) is the semicocycle (u^ζ_t).

Except for the proof of Proposition 3.1, the adaptations of such proofs are natural and straightforward. For instance, to prove the analogous result of Lemma 4.3, one has, for $f \in Z^m$ and $s \in (0,1]$,

$$\begin{aligned}\left\|\frac{\omega\circ p_s}{\omega}C_{p_s}f\right\|_{(\mathcal{D}^p_\sigma)_0} &\simeq \left\|(Bp_s)^m\frac{\omega\circ p_s}{\omega}C_{p_s}(B^m f)\right\|_{(\mathcal{D}^p_\sigma)_0}\\ &\lesssim |u_s(0)\,C_{p_s}(B^m f)(0)| + \|u_s' C_{p_s} f\|_{(\mathcal{A}^p_\sigma)_0} + \|u_s p_s'(f'\circ p_s)\|_{(\mathcal{A}^p_\sigma)_0},\end{aligned}$$

with $u_s(z) = (Bp_s(z))^m \dfrac{\omega\circ p_s(z)}{\omega(z)} = s^{m+\zeta}\dfrac{1-sz}{1-z}$ and $\omega(z) = z^\zeta(1-z)$. Then, for $f \in Z^m$ and $s \in (0,1]$, we have

$$|u_s(0)\,C_{p_s}(B^m f)(0)| = s^{m+\Re\mathfrak{e}\,\zeta}|(B^m f)(0)| \lesssim s^{m+\Re\mathfrak{e}\,\zeta}\|B^m f\|_{\mathcal{D}^p_\sigma} \simeq s^{m+\Re\mathfrak{e}\,\zeta}\|f\|_{\mathcal{D}^p_\sigma};$$

$$\|u_s' C_{p_s} f\|_{(\mathcal{A}^p_\sigma)_0} \le \|u_s'\|_{H^\infty(\mathbb{D}_0)}\,\|C_{p_s} f\|_{(\mathcal{A}^p_\sigma)_0} \lesssim s^{m+\Re\mathfrak{e}\,\zeta}(1-s)\|C_{p_s} f\|_{\mathcal{D}^p_\sigma} \lesssim s^{m+\Re\mathfrak{e}\,\zeta}\|f\|_{\mathcal{D}^p_\sigma},$$

where we used $\|u_s'\|_{H^\infty(\mathbb{D}_0)} \simeq s^{m+\Re\mathfrak{e}\,\zeta}(1-s)$ for $s \in (0,1]$, and $\|g\|_{\mathcal{A}^p_\sigma} \lesssim \|g\|_{\mathcal{D}^p_\sigma}$ for all $g \in \mathcal{D}^p_\sigma(\mathbb{D})$; and

$$\|u_s p_s'(f'\circ p_s)\|_{(\mathcal{A}^p_\sigma)_0} \lesssim s^{m+\Re\mathfrak{e}\,\zeta+1}\|C_{p_s}(f')\|_{\mathcal{A}^p_\sigma} \lesssim s^{m+\Re\mathfrak{e}\,\zeta+1}\|f'\|_{\mathcal{A}^p_\sigma} \lesssim s^{m+\Re\mathfrak{e}\,\zeta}\|f\|_{\mathcal{D}^p_\sigma}.$$

From the above inequalities, the analogous of Lemma 4.3 follows, as stated above.

Thus, we finish the proof this theorem by giving a detailed proof of the analogous result of Proposition 3.1. For $m \in \mathbb{N}_0$, one has

$$\left\|u_t^\zeta C_{\phi_t} f\right\|_{\mathcal{D}^p_\sigma} \simeq \left\|B^m(u_t^\zeta C_{\phi_t} f)\right\|_{\mathcal{D}^p_\sigma} = \left\|\widetilde{u}_t^\zeta C_{\phi_t}(B^m f)\right\|_{\mathcal{D}^p_\sigma}, \qquad f \in Z^m,\ t \ge 0,$$

where $\widetilde{u}_t^\zeta(z) := ((B\phi_t)^m u_t^\zeta)(z) = \dfrac{e^{-(\zeta+m)t}}{((e^{-t}-1)z+1)^{\zeta+m+1}}$, $z \in \mathbb{D}, t \ge 0$. In consequence,

$$\left\|u_t^\zeta C_{\phi_t} f\right\|_{\mathcal{D}^p_\sigma} \lesssim \left|\widetilde{u}_t^\zeta(0)(B^m f)(0)\right| + \left\|(\widetilde{u}_t^\zeta)' C_{\phi_t}(B^m f)\right\|_{\mathcal{A}^p_\sigma} + \left\|\widetilde{u}_t^\zeta \phi_t' C_{\phi_t}((B^m f)')\right\|_{\mathcal{A}^p_\sigma},$$

for all $f \in Z^m$, $t \ge 0$. Then,

$$\left|\widetilde{u}_t^\zeta(0)(B^m f)(0)\right| \lesssim e^{-(\Re\mathfrak{e}\,\zeta+m)t}\|B^m f\|_{\mathcal{D}^p_\sigma} \simeq e^{-(\Re\mathfrak{e}\,\zeta+m)t}\|f\|_{\mathcal{D}^p_\sigma}, \qquad f \in Z^m,\ t \ge 0.$$

Also,

$$\begin{aligned}\left\|(\widetilde{u}_t^\zeta)' C_{\phi_t}(B^m f)\right\|_{\mathcal{A}^p_\sigma} &\le \left\|\frac{(\widetilde{u}_t^\zeta)'}{(\phi_t')^{\gamma+1}}\right\|_\infty \|(\phi_t')^{\gamma+1} C_{\phi_t}(B^m f)\|_{\mathcal{A}^p_\sigma} \lesssim \left\|\frac{(\widetilde{u}_t^\zeta)'}{(\phi_t')^{\gamma+1}}\right\|_\infty \|B^m f\|_{\mathcal{A}^p_\sigma}\\ &\simeq \left\|\frac{(\widetilde{u}_t^\zeta)'}{(\phi_t')^{\gamma+1}}\right\|_\infty \|f\|_{\mathcal{A}^p_\sigma} \lesssim \left\|\frac{(\widetilde{u}_t^\zeta)'}{(\phi_t')^{\gamma+1}}\right\|_\infty \|f\|_{\mathcal{D}^p_\sigma}, \quad f \in Z^m,\ t \ge 0,\end{aligned}$$

where we have used that the weighted Bergman spaces $\mathcal{A}^p_\sigma(\mathbb{D})$ satisfy (**P4**) with parameter $\gamma+1 = \frac{\sigma+2}{p}$. It is readily seen that $(\widetilde{u}_t^\zeta)'(z) = (1-e^{-t})(\zeta+m+1)\widetilde{u}_t^\zeta(z)\frac{1-\phi_t(z)}{1-z}$. If $\zeta = -m-1$, then $(\widetilde{u}_t^\zeta)' = 0$. Otherwise, by Lemma 2.3 we get

$$\begin{aligned}\limsup_{t\to\infty}\left\|\frac{(\widetilde{u}_t^\zeta)'}{(\phi_t')^{\gamma+1}}\right\|_\infty^{1/t} &= \limsup_{t\to\infty}\left\|\frac{\widetilde{u}_t^\zeta}{(\phi_t')^{\gamma+1}}\frac{1-\phi_t(z)}{1-z}\right\|_\infty^{1/t}\\ &= \exp\left(\max\{-\mathfrak{Re}\,\zeta+\gamma-m+1, 1-\gamma\}\right).\end{aligned}$$

In consequence, $\limsup_{t\to\infty}\left\|(\widetilde{u}_t^\zeta)'C_{\phi_t}(B^m f)\right\|_{\mathcal{A}^p_\sigma}^{1/t} \leq \exp\left(\max\{-\mathfrak{Re}\,\zeta+\gamma-m+1, 1-\gamma\}\right)$. On the other hand, one has

$$\begin{aligned}\left\|\widetilde{u}_t^\zeta\phi_t'C_{\phi_t}((B^m f)')\right\|_{\mathcal{A}^p_\sigma} &\lesssim \left\|\frac{\widetilde{u}_t^\zeta}{(\phi_t')^\gamma}\right\|_\infty \left\|(\phi_t')^{\gamma+1}C_{\phi_t}((B^m f)')\right\|_{\mathcal{A}^p_\sigma} \lesssim \left\|\frac{\widetilde{u}_t^\zeta}{(\phi_t')^\gamma}\right\|_\infty \|(B^m f)'\|_{\mathcal{A}^p_\sigma}\\ &\lesssim \left\|\frac{\widetilde{u}_t^\zeta}{(\phi_t')^\gamma}\right\|_\infty \|B^m f\|_{\mathcal{D}^p_\sigma} \simeq \left\|\frac{\widetilde{u}_t^\zeta}{(\phi_t')^\gamma}\right\|_\infty \|f\|_{\mathcal{D}^p_\sigma}, \qquad f\in Z^m,\ t\geq 0,\end{aligned}$$

where we have used again that $\mathcal{A}^p_\sigma(\mathbb{D})$ satisfies (**P4**) with parameter $\gamma+1 = \frac{\sigma+2}{p}$. Thus, another application of Lemma 2.3 to the semicocycle $\widetilde{u}_t^\zeta/(\phi_t')^\gamma$ yields

$$\limsup_{t\to\infty}\left\|\widetilde{u}_t^\zeta\phi_t'C_{\phi_t}((B^m f)')\right\|_{\mathcal{A}^p_\sigma}^{1/t} \leq \exp\left(\max\{-\mathfrak{Re}\,\zeta+\gamma-m, 1-\gamma\}\right).$$

Putting everything together, we conclude

$$\limsup_{t\to\infty}\left\|u_t^\zeta C_{\phi_t}\right\|_{L(Z^m)}^{1/t} \leq \exp\left(\max\{-\mathfrak{Re}\,\zeta+\gamma-m+1, 1-\gamma\}\right), \qquad m\in\mathbb{N}_0. \tag{6.2}$$

Note that such a bound is weaker than the bound given in Proposition 3.1 (recall that $\alpha_0 = -\mathfrak{Re}\,\zeta$, $\alpha_1 = 1$ for the semicocycle (u_t^ζ)). Nevertheless, (6.2) is good enough for our purposes. Indeed, when adapting the results of Sections 3 and 4 to the spaces $\mathcal{D}^p_\sigma(\mathbb{D})$, we can take $m\in\mathbb{N}_0$ big enough on (6.2) to obtain the required asymptotic bound of $\left\|u_t^\zeta C_{\phi_t}\right\|_{L(Z^m)}$ as $t\to\infty$. Hence, the proof is finished. □

6.2. **Generalized Cesàro operators.** Let $\alpha\in\mathbb{C}$ with $\mathfrak{Re}\,\alpha > 0$. Let μ_α the Borel measure on $(0,1]$ such that $d\mu_\alpha(t) = \alpha(1-t)^{\alpha-1}\,dt$, so $(\mu_\alpha)_n = \frac{\Gamma(\alpha+1)\Gamma(n+\zeta+1)}{\Gamma(n+\zeta+\alpha+1)}$ for $n\in\mathbb{N}_0$. For $\mathfrak{Re}\,\zeta > -1$, the generalized Cesàro operator $\mathcal{C}_\alpha^\zeta$ is defined as the associated Hausdorff operator to μ_α and ζ, that is, $\mathcal{C}_\alpha^\zeta = \mathcal{H}_{\mu_\alpha}^{(\zeta)}$. It is readily seen that

$$(\mathcal{C}_\alpha^\zeta f)(z) = \frac{\alpha}{z^{\zeta+\alpha}}\int_0^z \frac{w^\zeta(z-w)^{\alpha-1}}{(1-w)^\alpha} f(w)\,dw, \quad z\in\mathbb{D},\ f\in\mathcal{O}(\mathbb{D}).$$

Now, let X either be a γ^∞-space for some $\gamma > 0$, or let $X = \mathcal{D}^p_\sigma(\mathbb{D})$ for $1 \leq p < \infty$, $\sigma > -1$ and $\sigma > p-2$, with $\gamma = \frac{\sigma+2}{p} - 1 > 0$ in this case. Then, for each $\delta \in (0, \min\{\mathfrak{Re}\,\zeta + 1, \gamma\})$, one has

$$\int_0^\infty e^{(c+\delta)t} |d\nu_\alpha|(t) = |\alpha| \int_0^\infty e^{\delta t} e^{\max\{-\mathfrak{Re}\,(\zeta+1), -\gamma\} t} (1-e^{-t})^{\mathfrak{Re}\,\alpha - 1}\, dt < \infty,$$

where, following the notation of this section, $d\nu_\alpha(t) := d\kappa(\mu_\alpha)(t) = (1-e^{-t})^{\alpha-1} e^{-t}\, dt$, see Proposition 6.1. As a consequence, $\mathcal{C}^\zeta_\alpha$ is a well-defined bounded operator on X, see Theorem 5.1. In addition, it is readily seen that

$$\mathcal{L}(\nu_\alpha)(z) = \int_0^\infty e^{-zt}\, d\nu_\alpha(t) = \alpha \mathbb{B}(z+1, \alpha), \quad \mathfrak{Re}\, z > -1,$$

where $\mathbb{B}$ denotes the Beta function. Hence, $\mathcal{L}(\nu_\alpha)$ is regular (one can see this by using Lemma 5.2 or directly in the equality above), belongs to $\mathcal{E}(-\Delta)$ and satisfies the conditions of Remark 5.4. As a consequence, we obtain the following

Theorem 6.4. *Let $X = H^p(\mathbb{D}), \mathcal{A}^p_\sigma(\mathbb{D}), \mathcal{K}_0^{-\widetilde{\gamma}}(\mathbb{D}), \mathcal{D}^p_\sigma(\mathbb{D})$ for $p \geq 1, \sigma > -1, \widetilde{\gamma} > 0$ and set $\gamma = 1/p, \frac{\sigma+2}{p}, \widetilde{\gamma}, \frac{\sigma+2}{p} - 1$ respectively. Assume also $\sigma > p-2$ if $X = \mathcal{D}^p_\sigma(\mathbb{D})$. Let $\mathfrak{Re}\,\alpha > 0$ and $\mathfrak{Re}\,\zeta > -1$. Then $\mathcal{C}^\zeta_\alpha$ is a bounded operator on X such that*

$$\sigma(\mathcal{C}^\zeta_\alpha) = \{0\} \cup \{\alpha \mathbb{B}(z, \alpha) \mid \mathfrak{Re}\, z \geq \gamma\} \cup \sigma_{point}(\mathcal{C}^\zeta_\alpha),$$
$$\sigma_{point}(\mathcal{C}^\zeta_\alpha) = \{\alpha \mathbb{B}(\zeta + k, \alpha) \mid k \in \mathbb{N} \text{ and } k < \gamma - \mathfrak{Re}\,\zeta\},$$
$$\sigma_{ess}(\mathcal{C}^\zeta_\alpha) = \{0\} \cup \{\alpha \mathbb{B}(z, \alpha) \mid \mathfrak{Re}\, z = \gamma\}.$$

Proof. The statement follows from the comments above together with Remark 5.4, (6.1), Remark 6.2 and Theorem 6.3. □

6.3. **Hölder operators.** Let $\alpha \in \mathbb{C}$ with $\mathfrak{Re}\,\alpha > 0$. Let μ_α the Borel measure on $(0,1]$ such that $d\mu_\alpha(t) = \frac{1}{\Gamma(\alpha)} \Big(\log(1/t)\Big)^{\alpha-1} dt$, so $\mu_n = \frac{1}{(n+\zeta+1)^\alpha}$ for $n \in \mathbb{N}_0$. For $\mathfrak{Re}\,\zeta > -1$, the generalized Hölder operator $\mathfrak{H}^\zeta_\alpha$ is defined as the associated Hausdorff operator to μ_α and ζ, that is, $\mathfrak{H}^\zeta_\alpha = \mathcal{H}^{(\zeta)}_{\mu_\alpha}$. It is readily seen that

$$(\mathfrak{H}^\zeta_\alpha f)(z) = \frac{1}{\Gamma(\alpha)} \frac{1}{z^{\zeta+1}} \int_0^z \frac{w^\zeta}{1-w} \left(\log \frac{z(1-w)}{w(1-z)}\right)^{\alpha-1} f(w)\, dw, \quad z \in \mathbb{D},\ f \in \mathcal{O}(\mathbb{D}).$$

Now, let X either be a γ^∞-space for some $\gamma > 0$, or let $X = \mathcal{D}^p_\sigma(\mathbb{D})$ for $1 \leq p < \infty$, $\sigma > -1$ and $\sigma > p-2$, with $\gamma = \frac{\sigma+2}{p} - 1 > 0$ in this case. Then, for each $\delta \in (0, \min\{\mathfrak{Re}\,\zeta + 1, \gamma\})$, one has

$$\int_0^\infty e^{(c+\delta)t} |d\nu_\alpha|(t) = \frac{1}{|\Gamma(\alpha)|} \int_0^\infty e^{\delta t} e^{-\min\{\mathfrak{Re}\,\zeta+1, \gamma\} t} t^{\mathfrak{Re}\,\alpha - 1}\, dt < \infty,$$

where $d\nu_\alpha(t) := d\kappa(\mu_\alpha)(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t}\, dt$, see Proposition 6.1. As a consequence, $\mathfrak{H}_\alpha^\zeta$ is a well-defined bounded operator on X, see Theorem 5.1. In addition, it is readily seen that

$$\mathcal{L}(\nu_\alpha)(z) = \int_0^\infty e^{-zt}\, d\nu_\alpha(t) = \frac{1}{(z+1)^\alpha}, \quad \mathfrak{Re}\, z > -1.$$

Hence, $\mathcal{L}(\nu_\alpha)$ is regular (one can see this by using Lemma 5.2 or directly in the equality above), belongs to $\mathcal{E}(-\Delta)$ and satisfies the conditions of Remark 5.4. As a consequence, we obtain the following

Theorem 6.5. *Let $X = H^p(\mathbb{D}), \mathcal{A}_\sigma^p(\mathbb{D}), \mathcal{K}_0^{-\widetilde{\gamma}}(\mathbb{D}), \mathcal{D}_\sigma^p(\mathbb{D})$ for $p \geq 1, \sigma > -1, \widetilde{\gamma} > 0$ and set $\gamma = 1/p, \frac{\sigma+2}{p}, \widetilde{\gamma}, \frac{\sigma+2}{p} - 1$ respectively. Assume also $\sigma > p-2$ if $X = \mathcal{D}_\sigma^p(\mathbb{D})$. Let $\mathfrak{Re}\, \alpha > 0$ and $\mathfrak{Re}\, \zeta > -1$. Then $\mathfrak{H}_\alpha^\zeta$ is a bounded operator on X such that*

$$\sigma(\mathfrak{H}_\alpha^\zeta) = \{0\} \cup \{z^{-\alpha} \mid \mathfrak{Re}\, z \geq \gamma\} \cup \sigma_{point}(\mathfrak{H}_\alpha^\zeta),$$
$$\sigma_{point}(\mathfrak{H}_\alpha^\zeta) = \{(\zeta + k)^{-\alpha} \mid k \in \mathbb{N} \text{ and } k < \gamma - \mathfrak{Re}\, \zeta\},$$
$$\sigma_{ess}(\mathfrak{H}_\alpha^\zeta) = \{0\} \cup \{z^{-\alpha} \mid \mathfrak{Re}\, z = \gamma\}$$

Proof. The statement follows from the comments above together with Remark 5.4, (6.1), Remark 6.2 and Theorem 6.3. □

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