A simple approach to the suppression of the Gibbs phenomenon in diffractive numerical calculations

Francisco Jose Torcal-Milla

Grupo de Tecnología Óptica Láser, Universidad de Zaragoza, C/ Pedro Cerbuna 12, 50009, Zaragoza (Spain)
Applied Optics Complutense Group, Facultad de Física, Universidad Complutense de Madrid, Plaza de las Ciencias 1, 28040, Madrid (Spain)
fjtorcal@unizar.es

Abstract

The Gibbs phenomenon is a well-known effect that is produced at discontinuities of a function represented by the Fourier expansion when it is truncated to perform numerical calculations. This phenomenon appears because it is not possible to fit a discontinuous function as the summation of continuous functions, such as it is done with the Fourier expansion. Only considering infinite terms of the summation, the Fourier expansion fits the real signal. From a general point of view, it will affect to the final results since the representation of the signal does not include higher frequencies. It is true that the higher is the truncation, the better are the results, but an error is always committed. The Gibbs phenomenon has been studied in electric signal and diffractive optics, where the Fourier expansion is commonly used. In this work, we drop complex mathematics to show the effect of the Gibbs phenomenon on the near field propagation of diffraction gratings (self-imaging phenomenon) and also possible implementations of some corrections which allow diminishing the analytical or numerical errors in comparison with less accurate approaches. Anyway, the conclusions of this work would be applicable to other numerically solved diffractive problems which include sharp edges apertures. Simulations are compared with experiments giving interesting results.

Keywords: Diffraction gratings, Talbot effect, Gibbs phenomenon

1. Introduction

The concept of diffraction was firstly introduced by Francesco Maria Grimaldi in 1660, [1]. It consists of a series of phenomena that occur when light approaches sharp edges, obstacles, and apertures of size similar to the wavelength of the impinging light, [2[3]. Diffraction itself can be explained by using the Huygens-Fresnel principle as the interference between the secondary spherical waves emerging from the edges or the apertures and it has been observed for electromagnetic waves such as light but also happens for acoustic or matter waves, [4[5]. Diffraction has been analyzed by many scientists along the history, from Isaac Newton to Thomas Young, giving explanation based on diffraction theory to very impressive phenomena such as the well-known double slit experiment, [6]. Related to diffraction phenomena, one of the most important optical elements that has been under analysis and used for lots of applications during the years is the diffraction grating. A diffraction grating is an optical element that modulates periodically one or more than one properties of the light transmitted or reflected from it, [7]. Amplitude-based or phase-based diffraction gratings are the most common types but one may find diffraction gratings that modulate the polarization state, [8[9], or spatial coherence state of light, [10[11]. Usually, diffraction gratings present sharp edges and it contributes, join to their feature size, to diffract the light after them. In any case, numerical simulations including periodical objects commonly need to perform some approximations. When one tries to solve an optical problem which involves apertures, they must be usually represented as a truncated Fourier series summation to obtain numerical results. In fact, Fourier series summation is widely used in several branches of science such as physics or mathematics, [12]. Summarizing, being $f(x)$ a continuous periodical and differencing function, one may represent it as an infinite summation as

$$f(x)=\sum_{n=-\infty}^{\infty} c_n \exp\left(i\frac{2\pi}{p}nx\right), \tag{1}$$

where p is the period of the function and c_n are the corresponding Fourier coefficients that are calculated as

$$c_n=\frac{1}{p}\int_{-p/2}^{p/2} f(x)\exp\left(-i\frac{2\pi}{p}nx\right)dx\,. \tag{2}$$

Now, if the infinite Fourier expansion is truncated for numerical simulations, the Gibbs phenomenon appears at discontinuities of the function, [13[15]. This phenomenon appears because it is not possible to fit a discontinuous function as the summation of continuous functions. Only considering infinite terms of the summation, the Fourier expansion fits the real function. From a general point of view, it will affect to the final results since the representation of the function does not include higher frequencies. The Gibbs phenomenon is well-known in electric signal processing and it has been also shown in optics, [16]. Suppression of Gibbs phenomenon has been achieved in computation of diffractive fields, [17[18], but involving very complex methods. In this work, we show a simple visualization of the Gibbs phenomenon effect when we propagate in the Fresnel regime the field after illuminating a binary amplitude-based diffraction grating with a plane wave, showing also possible implementations of some corrections which allow diminishing the numerical errors in comparison with the common Fourier series expansion of the transmittance of the grating. The conclusions of this work are also applicable to other problems which involve diffraction by sharp edges apertures and numerical simulations.

2. Theoretical approach

Firstly, let us consider a periodical diffractive element whose transmittance may be expressed by $f(x)$, being x the axis parallel to the element. Then, the infinite Fourier series expansion of $f(x)$ is given by Eq. (3). For clarifying the meaning of the infinite Fourier series expansion, let us take a diffractive element whose transmittance is square-shaped. For simplicity and without loss of generality, we take a binary amplitude-based periodical element with 0 and 1 transmittance levels. So, the corresponding Fourier coefficients calculated from Eq. (2) result

$$\begin{cases} c_0 = 1/2 \\ c_n = (1/n\pi)\sin(n\pi/2) \end{cases} \quad (3)$$

We show in Table 1 the values of the first Fourier coefficients of the proposed example. Notice that due to the geometry of the problem and the kind of diffractive element, the even orders are strictly zero and the odd orders are symmetric, $c_n = c_{-n}$ (n odd).

In addition, we show in the transmittance given by the Fourier expansion of the considered element for different orders of truncation (values of n). As can be observed, even taking n=101, the real transmittance is not completely restored. It presents values lesser than 0 and higher than 1. This fact has repercussions on the interpretation of numerical results and on the comparison between them and experiments. For example, performing analytically the free propagation of light passing through the considered element, we find very different results depending on the order of truncation of the Fourier series expansion. In Figure 2 we show four examples of free propagation for some cases shown in Figure 1, illuminated with a plane wave of wavelength $\lambda = 500\,\mathrm{nm}$.

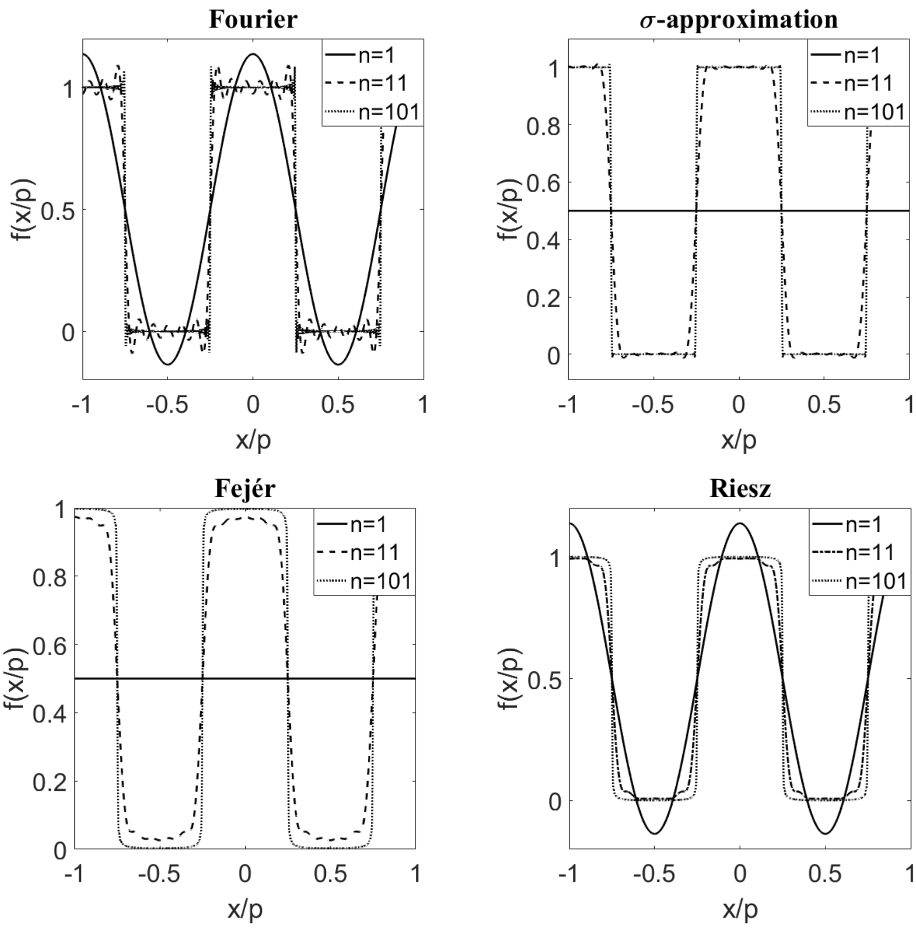

Figure 1.- Series expansion of a binary amplitude diffraction grating transmittance for different orders of truncation and different approaches for summation (Fourier, σ-approximation, Fejér and Riesz). In the Riesz summation δ=0.0001.

Table 1. First Fourier coefficients of the Fourier expansion of a binary amplitude-based diffraction grating

c_0	c_1	c_3	c_5	c_7	c_9
0.5	-0,.3183	0.1061	-0.0637	0.0455	-0.0354

Then, to obtain more accurate results without nonsense intensity values, we should increase the order of truncation up to infinity. Although, if we look at signal theory on this issue, there are some formulations for adjusting the Fourier summation that eliminate, or at least mitigate, the Gibbs phenomenon, [19[20]. Here we will try the σ-approximation, the Fejér summation and the Riesz summation. All of them consists of using alternative polynomial expansions instead of the Fourier expansion.

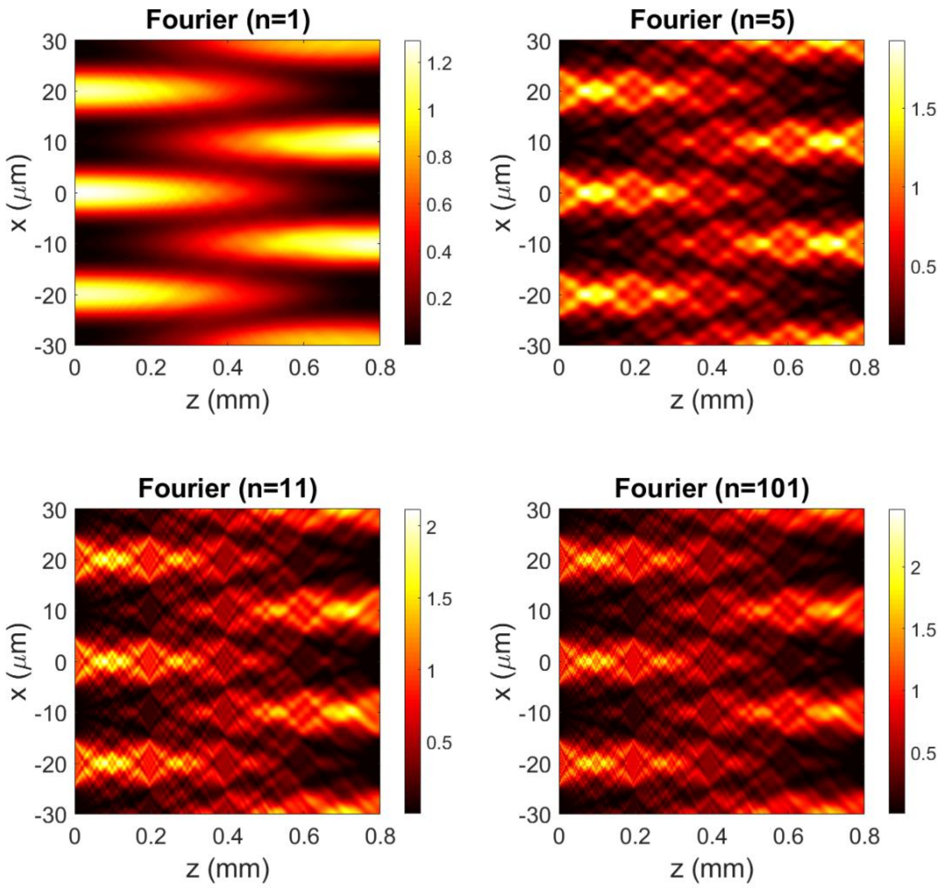

Figure 2.- Free propagation of the diffractive element obtained from the Fourier series expansion for different values of n. Plane wave illumination.

2.1 σ-approximation

A possible formulation of the σ-approximation consists of the following expression,

$$f_{\sigma}(x) = \sum_{k=-(m-1)}^{m-1} \operatorname{sinc}\left(\frac{k}{m}\right) c_k \exp(iqkx) \tag{4}$$

where $\operatorname{sinc}(\alpha) = \sin(\alpha)/\alpha$ is called the Lanczos σ factor, [21].

From a mathematical point of view, the implication of changing the Fourier series expansion by the σ-approximation is merely a multiplicative factor to each ordinary Fourier coefficient, c_k. Obviously, this factor is different for each term of the summation. Then, the Gibbs phenomenon will be quasi-eliminated without many complications. We show in Figure 1b the same cases as for the Fourier summation but using the σ-approximation for fitting the diffractive element. As can be observed, the peaks below zero and over one transmittance are almost cancelled and therefore we almost do not have transmittance values which do not correspond to the real transmittance of the optical element. Besides, considering n=101 the shape of the profile/transmittance is quasi-squared, as it should be.

2.2 Fejer summation

As we have mentioned before, another summation for fitting periodical functions minimizing the Gibbs phenomenon is the so-called Fejér summation, [22]. It is defined as

$$f_{Fejér}\left(x\right)=\frac{1}{n}\sum_{k=0}^{n-1}\left[\sum_{s=-k}^{k}c_S\exp(iqsx)\right]. \qquad (5)$$

This method consists of the summation of arithmetical averages. It is demonstrated that the Fejér summation converges in all cases in which the Fourier summation also converges. We also show in Figure 1c the Fejér summation equivalent to the Fourier summation also shown in Figure 1a. We can observe that for n=101 the Fejér summation almost has the same shape than the original transmittance with only slight curved corners. c_S in Eq. (5) are the conventional Fourier coefficients calculated by using Eq. (2). On the other hand, Fejér summation does not have any sense for n=1 but we plot it for completeness.

2.3 Riesz summation

Finally, another summation method able to eliminate the Gibbs phenomenon is the so-called Riesz summation, [23]. This method is similar to the Fejér summation method but introduces a new parameter δ. With this summation, as it can be demonstrated, the profile results more squared for small values of δ. In this case, the summation is

$$f_{Riesz}(x)=\frac{1}{n}\sum_{k=0}^{n-1}\left[\sum_{s=-k}^{k}\left(\frac{1-s}{n}\right)^{\delta} c_s \exp(iqsx)\right]. \quad (6)$$

We show in Figure 1d the same cases as for the Fourier summation but using the Riesz summation. It is clear that this summation also fits the real transmittance better than the Fourier expansion and does not have values below zero or over one. c_S in Equation (6) are the conventional Fourier coefficients calculated by using Equation (2).

To give a better understanding of all approaches, we show in Figure 3 a comparison between all presented summations for n=101. As can be observed, Fourier summation gives wrong values higher than 1 and very fluctuating. On the other hand, the other three summations are smoother and without fluctuations. In addition, they almost do not give values upper than one, being the σ-approximation the most squared one even though it still has some values upper than one that do not have physical sense.

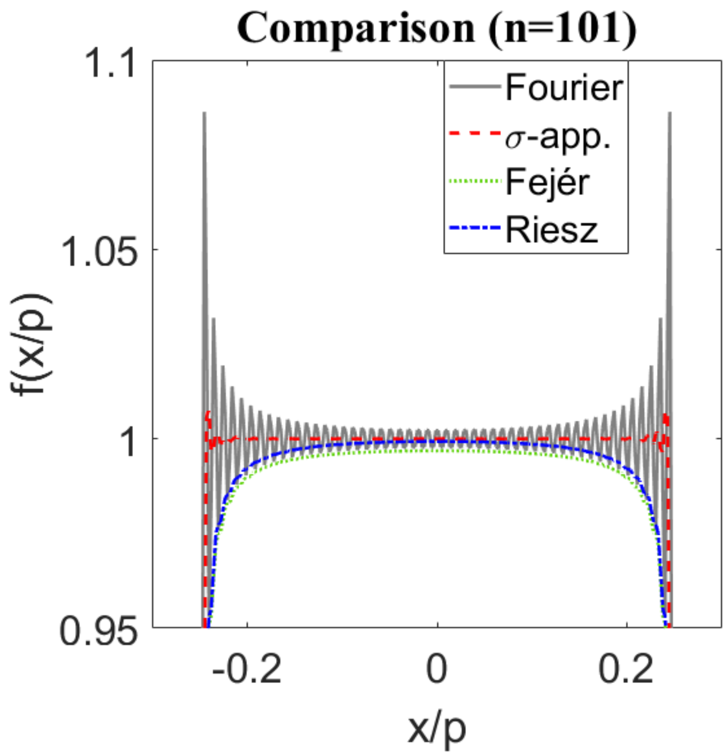

Figure 3.- Numerical comparison between all types of summation considered in this work to fit the transmittance of an amplitude diffraction grating. We only show one of the tops of the transmittance for clearness. δ = 0.0001 for the Riesz summation.

3. Experimental approach

Following, we present a very common experiment which consists of obtaining the near field diffraction pattern of a binary amplitude-based diffraction grating illuminated by a plane wave. The set-up includes a collimated laser for illumination, a diffraction grating fixed at a plane and a camera which is able to move along the optical axis. As it is well known, Talbot self-images are expected at the near field of the grating, [24[25].

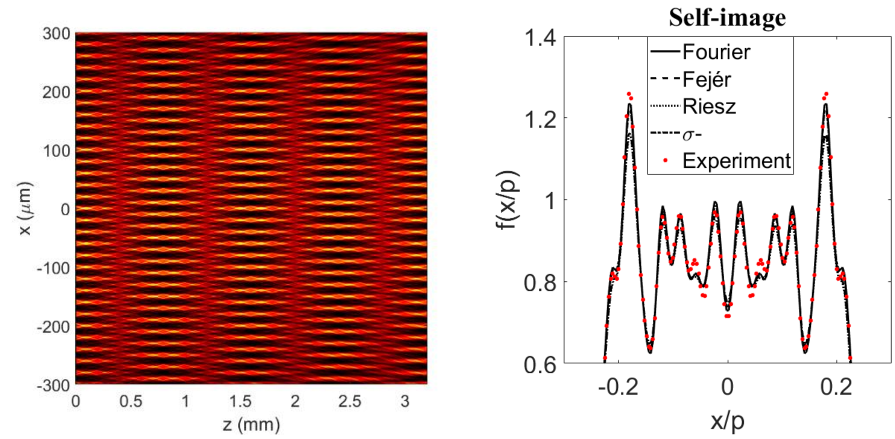

Figure 4.- (left) Experimental propagation of the amplitude-based diffraction grating at the near field, (right) first self-image profile given by experiment and the four summations considered in this work.

Talbot effect consist of the replication of the grating pattern at several distances so-called Talbot distances given by $z_T = 2mp/\lambda$, with m entire. In this experiment, we are going to compare the profile of the first experimental Talbot self-image with those given by the truncated Fourier summation and the three presented alternatives. We show in Figure 4 (left) the 2D carpet obtained from the experiment. The period of the used grating is $p = 20\,\mu\text{m}$. In addition, we show in Figure 4 (right) the profile of the first self-image in the center and all fittings given by summations analyzed along this work up to n=101. The way we have chosen to evaluate which summation fits better the experimental profile of the first self-image is the subtraction of the self-image profile given by the summations from the experimental one and calculation of the standard deviation. The obtained results for different values of n are shown in Table 2. As can be observed, the Riesz summation gives better results for all considered cases. We have used 4096 pixels to define the grating with length equal to 20 times the period so each pixel corresponds to 0.0977 microns.

Table 2. Standard deviations of the subtraction of the experimental first self-image from each numerical summation.

n	5	11	101
$\sigma_{Fourier}$	0.6827	0.6827	0.6827
σ_{Fejer}	0.6124	0.6504	0.6791
σ_{Riesz}	0.5801	0.6351	0.6774
σ_{Sigma}	0.6577	0.6772	0.6826

Conclusions

To conclude, in this work we probe that simple modifications of the finite Fourier series expansion such as the σ-approximation, the Riesz summation or the Fejér summation, fit better the transmittance of optical diffractive elements with sharp edges than the Fourier summation itself. In particular, we show how these special summations widely used in signal processing can be used in optical numerical simulations which included binary amplitude-based diffraction gratings, giving more accurate representation of the transmittance than the conventional Fourier series expansion. These results are also applicable to other diffractive problems which involve numerical calculations on diffraction which involves sharp edges apertures. In addition, the usage of alternative summations instead of Fourier summation could give more exact results with less iterations, which will improve the computing process, by reducing the computation time.

Acknowledgements

This work has been partially supported by Gobierno de Aragón-Fondo Social Europeo, Grupo de Tecnología Óptica Láser (project E44_20R) and project JIUZ-2020-CIE-06 of the Universidad de Zaragoza y Fundación Ibercaja.

References

[1] A. R. Hall, Beyond the Fringe: Diffraction as seen by Grimaldi, Fabri, Hooke and Newton, Notes and Records of the Royal Society of London, 44 (1990) 13-23.

[2] C. J. Bouwkamp, Diffraction theory, Reports on progress in physics, 17 (1954) 35-100.

[3] M. Born M and E. Wolf, *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light* (Elsevier, 2013).

[4] R. Dixon, Acoustic diffraction of light in anisotropic media, IEEE J. of Quantum Electronics, **3**(2) (1967) 85-93.

[5] Č. Brukner and A. Zeilinger, Diffraction of matter waves in space and in time, Phys. Review A, 56(5) (1997) 3804.

[6] T. Young, The Bakerian Lecture. Experiments and calculations relative to physical optics, Phil. Trans. R. Soc. Lond., 94(1) (1804) 1-16.

[7] C. A. Palmer and E. G. Loewen, *Diffraction grating handbook Vol. 5* (New York: Thermo RGL, 2002).

[8] C. Oh and M. J. Escuti, Achromatic diffraction from polarization gratings with high efficiency, Opt. Lett., 33(20) (2008) 2287-2289.

[9] G. Cincotti, Polarization gratings: design and applications, IEEE Journal of Quantum Electronics, 39(12) (2003) 1645-1652.

[10] L. M. Sanchez-Brea, F. J. Torcal-Milla and E. Bernabeu, Far field of gratings with rough strips, J. Opt. Soc. Am. A, 25(4) (2008) 828-833.

[11] F. J. Torcal-Milla, L. M. Sanchez-Brea and E. Bernabeu, Self-imaging of gratings with rough strips, J. Opt. Soc. Am. A, 25(10) (2008) 2390-2394.

[12] G. P. Tolstov, *Fourier series* (Courier Corporation, 2012).

[13] A. J. Jerri, *The Gibbs phenomenon in Fourier analysis, splines and wavelet approximations (Vol. 446)* (Springer Science & Business Media, 2013).

[14] D. Gottlieb and C. W. Shu, On the Gibbs phenomenon and its resolution, *SIAM review,* 39(4) (1997) 644-668.

[15] W. J. Thomson, Fourier series and the Gibbs phenomenon, American Journal of Physics, 60 (1992) 425-429.

[16] G. L. Rogers, The abbe theory of Microscopic Vision and the Gibbs phenomenon, American Journal of Physics, 22 (1954) 384-389.

[17] K. Falaggis, T. Kozacki, and M. Kujawinska, Computantion of highly off-axis diffracted fields using band-limited angular spectrum method with suppressed Gibbs phenomenon, App. Opt., 52 (2013) 3288-3297.

[18] J. G. Wangüemert-Pérez, R. Godoy-Rubio, A. Ortega-Moñux, and I. Molina-Fernández, Removal of Gibbs phenomenon and its application to fast-Fourier-transform-based mode solvers, J. Opt. Soc. Am. A, 24 (2007) 3772-3780.

[19] C. Pan, Gibbs phenomenon removal and digital filtering directly through the fast Fourier transform, IEEE Transactions on Signal Processing, 49(2) (2001) 444-498.

[20] E. Tadmor, Filters, mollifiers and the computation of the Gibbs phenomenon, Acta Numerica, 16 (2007) 305-378.

[21] J.F. Kiviet and G. D. Phillips, Alternative bias approximations in regressions with a lagged-dependent variable, Econometric theory, 1 (1993) 62-80.

[22] G. A. Vanasse, Fejer Series Approximation to Spectral Functions Obtained by Interferometric Techniques, J Opt. Soc. Am., 52(4) (1962) 472-472.

[23] C. D. Sogge, On the convergence of Riesz means on compact manifolds, Annals of Mathematics, 126(3) (1987) 439-447.

[24] K. Patorski, The self-imaging phenomenon and its applications, Progress in optics, 27 (1989) 1-108.

[25] H. F. Talbot, LXXVI. Facts relating to optical science. No. IV, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 9(56) (1836) 401-407.