Oscillation period of the Truncated Simple Pendulum

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Received xxxxxx
Accepted for publication xxxxxx
Published xxxxxx

Abstract

In this manuscript, a non typical kind of simple pendulum called *Truncated Simple Pendulum* is analyzed to obtain its oscillation period, frequency, and angular frequency. It is easily derived that its motion can be viewed as the concatenation of two pendular movements with different lengths, contributing each one to half of the period of the complete oscillation. An analytical formulation is derived and corroborated with an experiment showing high agreement. This experiment could be interesting as a proposed exercise for the students or as a laboratory practical work. Besides, the gravity acceleration can be determined from each singular experiment and averaged to obtain it with lesser experimental error. In addition, it can be used to evaluate energy conservation theorem and small angle approximation (Law of isochronism) for the pendular oscillation. The level of the manuscript makes it appropriate for undergraduate students and introductory physics courses.

Keywords: Pendulum, oscillation, period.

1. Introduction

Oscillatory movement is a fundamental part of every general physics course. Depending on the level of the course, different approaches can be done. The simplest oscillatory element is that formed by a mass hanging on a spring, [1, 2]. The position of the mass in terms of the time can be obtained by solving the following differential equation

$$\frac{d^2\xi}{dt^2}+\frac{k}{m}\xi^2=0, \tag{1}$$

where ξ is the position, k is the elastic constant of the spring and m is the hanging mass, Figure 1a. The general solution to this equation is a harmonic movement such as

$$\xi(t)=A\sin\left(\pm\omega t+\phi\right), \tag{2}$$

where $\omega=\sqrt{k/m}$, A is the amplitude, and ϕ is the initial phase.

Another oscillatory system commonly taught is the simple pendulum, Figure 1b, [3-5]. It consists of a rope without mass and a punctual mass hanging of it. In this case, the differential equation to solve is

$$\frac{d^2\theta}{dt^2}+\frac{g}{L_0}\sin\theta=0, \tag{3}$$

where g is the gravity acceleration, L_0 is the length of the pendulum, and θ is the angle measured from the upright, see Figure 1b.

For small perturbations around the equilibrium state, Eq. (3) can be solved resulting also in a harmonic oscillation

$$\theta(t)=A_\theta\sin\left(\pm\omega_\theta t+\phi\right), \tag{4}$$

where A_0 is the angular amplitude, $\omega_0 = \sqrt{g/L_0}$, and ϕ is the initial phase.

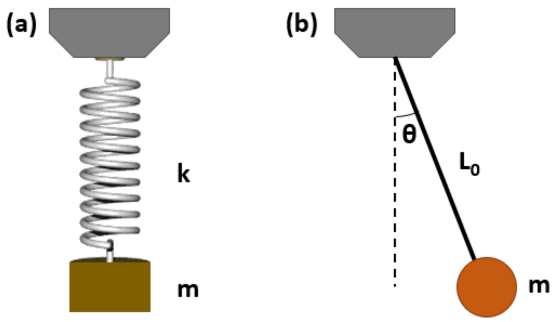

Figure 1. (a) Mass hanging from a spring, (b) simple pendulum.

Other more complex oscillatory examples are the rotary spring or the physical pendulum. Although, both of them can be easily analyzed for small perturbations and the solution is also a harmonic function with $\omega_\delta = \sqrt{k/I}$ for the rotary spring, [6], and $\omega_t = \sqrt{mgd/I}$ for the physical pendulum, [7,8], where k is the rotary spring constant, I the inertia moment in both cases, m the mass of the physical pendulum, and d the distance from the hanging point to the center of mass of the physical pendulum. The oscillation period, $T = 2\pi/\omega$, for all mentioned cases is resumed in Table 1.

Table 1. Period of the four depicted cases of oscillatory movement.

	Linear spring	*Simple pendulum*	*Rotary spring*	*Physical pendulum*
T	$2\pi\sqrt{m/k}$	$2\pi\sqrt{L_0/g}$	$2\pi\sqrt{I/k}$	$2\pi\sqrt{I/mgd}$

2. The *Truncated Simple Pendulum*. Theoretical approach.

In this manuscript, we analyze a particular case of simple pendulum that we have called *Truncated Simple Pendulum*. It consists of a simple pendulum whose oscillation is suddenly cut at the middle of the oscillation with a stop. Three instants of the oscillation are shown in Figure 2.

Considering that the collision with the stop is totally elastic, it is plausible to think that the oscillation period can be estimated as the summation of half the period of a simple pendulum with length L_0 and half the period of a simple pendulum with length, L, as

$$T_t = \frac{\pi}{\sqrt{g}}\left(\sqrt{L_0} + \sqrt{L}\right). \tag{5}$$

We may express L in terms of L_0 as $L=(1-n)L_0$ with $n \in [0,1]$. Thus, Eq. (5) results

$$T = \pi\sqrt{\frac{L_0}{g}}\left(1 + \sqrt{1-n}\right). \tag{6}$$

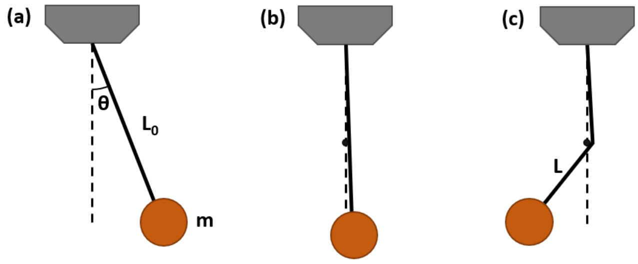

Figure 2. Three instants of the *Truncated Simple Pendulum* oscillation.

We show in Figure 3 (black solid line) the dependence of the relative period of the truncated simple pendulum, T/T_0, on the relative lenghts of both pendular movements, L/L_0.

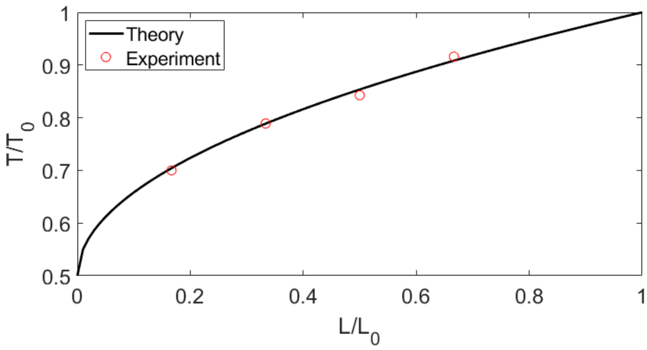

Figure 3. Relative period of the *Truncated Simple Pendulum* oscillation in terms of the stop position, theoretical calculated with Equation (6) (black solid line) and experimental data (red circles).

In addition, from Eq. (6), the frequency, f, and angular frequency, ω, may be derived, resulting

$$f = \frac{1}{\pi\left(1+\sqrt{1-n}\right)}\sqrt{\frac{g}{L_0}},$$
$$\omega = \frac{2}{1+\sqrt{1-n}}\sqrt{\frac{g}{L_0}}. \tag{7}$$

3. The *Truncated Simple Pendulum*. Experimental approach.

To verify the theoretical result, we have performed the corresponding experiment. We show in Figure 4 the used experimental set-up in which we have measured four positions of the stop. Each period has been calculated as the average over ten measurements, Table 2, and the experimental results are also plotted in Figure 4 (red circles). To do it, we have assured that the maximum angles at both sides of the stop remained small enough to be able to verify Eq. (6). As can be observed, the coincidence between theory and experiment is very good, so we can assume that the theoretical prediction was right and we can model the truncated simple pendulum as the concatenation of two simple pendula, one with length L_0 and the other one with length L, each one being half the period of each oscillation.

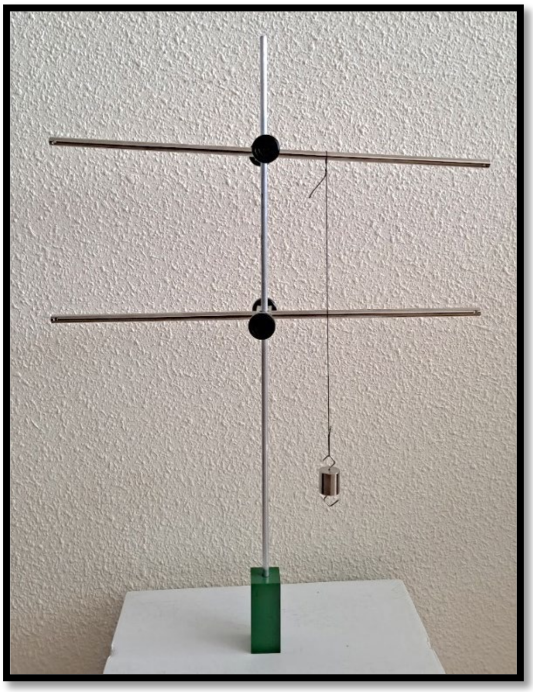

Figure 4. Experimental set-up used to measure the period of the *Truncated Simple Pendulum*. m=50 g and L_0=30 cm.

4. Determination of *g*.

As it is well known, the simple pendulum can be used to determine the gravity acceleration by measuring the period of the oscillation and its length. The *Truncated Simple Pendulum* can be also used to obtain it by measuring the same magnitudes. Besides, by displacing the stop up and down, several experiments can be performed easily. We show in Table 2 the experimental period for different L/L_0 values and the corresponding g, determined by using Eq. (5). From the four values, the mean gravity acceleration results g= 9.7845 m/s^2, which is close to the tabulated value.

Table 2. Experimental data used to determine the gravity acceleration, g, by using Eq. (5).

L/L_0	2/3	1/2	1/3	1/6
T_{exp} (s)	1.007	0.939	0.867	0.769
g (m/s)	9.6345	9.7945	9.7924	9.9167

5. Energy conservation and Law of Isochronism.

Any oscillation in air or other medium suffers from friction forces that eventually produce the object to stop. The decay of the velocity depends on the velocity itself and the shape of the object, among some properties of the medium. For relatively small velocities, the friction force or drag force is proportional to them, $F \propto v$, [9]. Although, for higher velocities, the friction force is proportional to the velocity to a higher power, n, $F \propto v^n$, [9]. The maximum velocity of a pendular motion happens at the lower point of the trajectory and depends on the initial angle, θ_0, and the length of the pendulum. For the first oscillation in air, we may consider that the energy remains constant, as an approximation. Then, by applying energy conservation, the maximum velocity is obtained as, [9],

$$v_{\max} = \sqrt{2gL_0(1-\cos\theta_0)}. \tag{8}$$

We show in Figure 5 the maximum velocity in terms of the initial angle for several pendulum lengths. The maximum velocity is proportional to the root square of the pendulum length for a fixed initial angle, as usual.

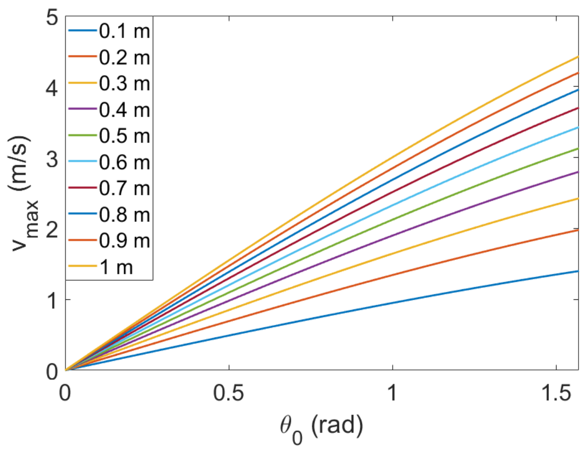

Figure 5. Maximum velocity of the pendulum in terms of the initial angle for several lengths, L, Eq. (8).

Once the pendulum reaches the stop and applying energy conservation again, the mass oscillates up to a final angle measured from the upright, given by

$$\theta = \text{acos}\left[1 - \frac{1}{L/L_0}(1-\cos\theta_0)\right]. \tag{9}$$

Figure 6 shows this dependence. When θ reaches π, it means that the mass has enough energy to pass over the stop and complete a lap. As an example, it is observed that for $L/L_0 = 0.5$ (green line in Figure 6) and $\theta_0 = \pi/2$, the mass reaches the upright after hitting the stop. In this case, if the initial angle is slightly bigger, the mass would go over the stop and would complete the lap.

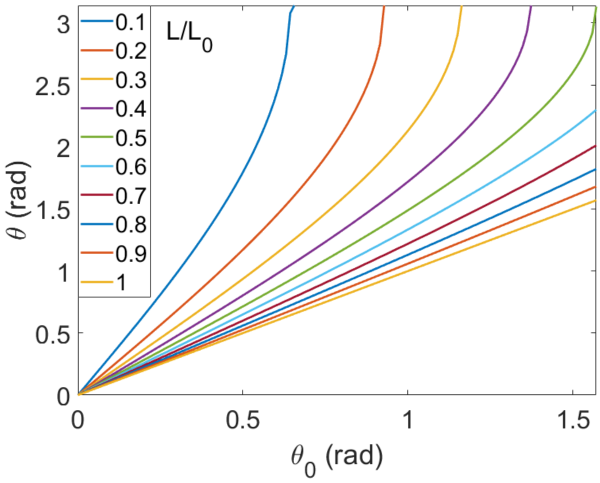

Figure 6. Final angle, θ, for the first oscillation in terms of the initial angle, θ_0, for several values of L/L_0, Eq. (9).

Secondly, Law of Isochronism says that the simple pendulum period does not depend on the oscillation amplitude. This is true only for small perturbations, that is to say, small angles with respect to the upright. By looking at Figure 6 and Eq. (9), we may extract the maximum initial angle that would be possible to do not break the Law of Isochronism, taking into account that the final angle will be higher than the initial one in all cases except for $L/L_0 = 1$, obviously. Usually, small angles are considered for $\theta_{\max} < 0.1\,\mathrm{rad}$. From Eq. (9), the maximum initial angle is given by

$$\theta_{0,\max} = \mathrm{acos}\left[1 - L/L_0\,(1-\cos\theta_{\max})\right], \qquad (10)$$

which results in $\theta_{0,\max} = 0.03\,\mathrm{rad}$. Despite this angle is quite small, the equations derived along this manuscript can be applied for bigger angles just assuming a certain error in the obtained predictions.

Conclusions

In this manuscript, a non usual kind of pendulum has been analyzed to determine its dynamics and its period. It has been called *Truncated Simple Pendulum* and consists of a simple pendulum which is stopped when it reaches the upright, becoming a pendulum of shorter length. The dynamics of the pendulum can be examined as that of two concatenated pendular movements with different length. In a similar fashion as the conventional simple pendulum, the gravity acceleration can be determined from each singular exeriment and averaged to obtain it with smaller error. In addition, we have obtained the initial angle for that the mass is able to reach the upright and pass over the stop, completing a lap. The obtention of the period or the gravity acceleration of the *Truncated Simple Pendulum* can be used as a proposed exercise for the students or even made as a practical one in the laboratory. The contents of this manuscript and the conclusions are adequate for undergraduate students or even first course of university degrees.

Acknowledgements

To my father. Thank you for teaching me to become how I am. Rest in peace.

A mi padre. Gracias por enseñarme a ser como soy. Descansa en paz.

Funding

This work has been partially supported by Gobierno de Aragón – Fondo Social Europeo (Grupo de Tecnologías Ópticas Láser E44_20.

Ethical statement

No conflict of interest.

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