LOCAL DERIVATIONS AND AUTOMORPHISMS OF CAYLEY ALGEBRAS

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Abstract. The present paper is devoted to the description of local and 2-local derivations and automorphisms on Cayley algebras over an arbitrary field $\mathbb{F}$. Given a Cayley algebra $\mathcal{C}$ with norm n, let $\mathcal{C}_0$ be its subspace of trace 0 elements. We prove that the space of all local derivations of $\mathcal{C}$ coincides with the Lie algebra $\{d \in \mathfrak{so}(\mathcal{C}, \mathrm{n}) \mid d(1) = 0\}$ which is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(\mathcal{C}_0, \mathrm{n})$. Surprisingly, the behavior of 2-local derivations depends on the Cayley algebra being split or division. Every 2-local derivation on the split Cayley algebra is a derivation, i.e. they form the exceptional Lie algebra $\mathfrak{g}_2(\mathbb{F})$ if $\mathrm{char}\mathbb{F} \neq 2, 3$. On the other hand, on division Cayley algebras over a field $\mathbb{F}$, the sets of 2-local derivations and local derivations coincide, and they are isomorphic to the Lie algebra $\mathfrak{so}(\mathcal{C}_0, \mathrm{n})$. As a corollary we obtain descriptions of local and 2-local derivations of the seven dimensional simple non-Lie Malcev algebras over fields of characteristic $\neq 2, 3$. Further, we prove that the group of all local automorphisms of $\mathcal{C}$ coincides with the group $\{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\}$. As in the case of 2-local derivations the behavior of 2-local automorphisms depends on the Cayley algebra being split or division. Every 2-local automorphism on the split Cayley algebra is an automorphism, i.e. they form the exceptional Lie group $G_2(\mathbb{F})$ if $\mathrm{char}\mathbb{F} \neq 2, 3$. On the other hand, on division Cayley algebras over a field $\mathbb{F}$, the groups of 2-local automorphisms and local automorphisms coincide, and they are isomorphic to the group $\{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\}$.

1. Introduction

Let $\mathcal{A}$ be an algebra (not necessary associative). Recall that a linear mapping (respectively, a linear bijection) $\varphi : \mathcal{A} \to \mathcal{A}$ is said to be a derivation (respectively, automorphism), if $\varphi(xy) = \varphi(x)y + x\varphi(y)$ (respectively, $\varphi(xy) = \varphi(x)\varphi(y)$) for all $x, y \in \mathcal{A}$. A linear mapping Δ is said to be a local derivation (respectively, local automorphism), if for every $x \in \mathcal{A}$ there exists a derivation (respectively, automorphism) φ_x on $\mathcal{A}$ (depending on x) such that $\Delta(x) = \varphi_x(x)$. These notions were introduced and investigated independently by R.V. Kadison [?] and D.R. Larson and A.R. Sourour [?]. The above papers gave rise to a series of works devoted to the description of mappings which are close to automorphisms and derivations of C^*-algebras and operator algebras. In [?] D.R. Larson and A.R. Sourour proved that if $\mathcal{A} = B(X)$, the algebra of all bounded linear operators on a Banach space X, then every invertible local automorphism of $\mathcal{A}$ is an automorphism. Thus automorphisms on $B(X)$ are completely determined by their local actions. In

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[?, Lemma 4] it was shown that the set of all local automorphisms $\mathrm{LocAut}(\mathcal{A})$ of an algebra $\mathcal{A}$ form a multiplicative group.

In 1997, P.Šemrl [?] introduced the concepts of 2-local derivations and 2-local automorphisms. Recall that a mapping $\Delta : \mathcal{A} \to \mathcal{A}$ (not necessary linear) is said to be a 2-local derivation (respectively, 2-local automorphism), if for every pair $x, y \in \mathcal{A}$ there exists a derivation (respectively, automorphism) $\varphi_{x,y} : \mathcal{A} \to \mathcal{A}$ (depending on x, y) such that $\Delta(x) = \varphi_{x,y}(x)$, $\Delta(y) = \varphi_{x,y}(y)$. P. Šemrl [?] described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H, by proving that every 2-local derivation on $B(H)$ is a derivation. A detailed discussion of 2-local derivations on operator algebras can be found in the survey [?].

In [?] a general form of local derivations on the real Cayley algebra $\mathbb{O}$ was obtained. This description implies that the space of all local derivations on $\mathbb{O}$, when equipped with Lie bracket, is isomorphic to the Lie algebra $\mathfrak{so}_7(\mathbb{R})$ of all real skew-symmetric 7×7-matrices. Also it is proved that any 2-local derivation on a Cayley algebra $\mathcal{C}$ over an algebraically closed field $\mathbb{F}$ is a derivation. In [?] it was proved that every local automorphism on the special linear Lie algebra $\mathfrak{sl}_n$ is an automorphism or anti-automorphism. Further M. Constantini [?] extended the above result to an arbitrary simple Lie algebra $\mathfrak{g}$. These results can be reformulate as follows $\mathrm{LocAut}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g}) \ltimes \{\mathrm{id}_{\mathfrak{g}}, -\mathrm{id}_{\mathfrak{g}}\}$, where $\mathrm{id}_{\mathfrak{g}}$ is the identical automorphism of $\mathfrak{g}$.

The present paper is devoted to the description of local and 2-local derivations and automorphisms on Cayley algebras over an arbitrary field, and it is organized as follows.

In Section 2 we give the necessary information concerning Cayley algebras, their derivations and automorphisms.

In Section 3 we find a general form for the local derivations on a Cayley algebra $\mathcal{C}$ over an arbitrary field $\mathbb{F}$ with a norm n. Namely, we prove that the space of local derivations of $\mathcal{C}$ is the Lie algebra $\{d \in \mathfrak{so}(\mathcal{C}, \mathrm{n}) \mid d(1) = 0\}$ (Theorem **??**). Further we consider 2-local derivations on a Cayley algebra $\mathcal{C}$. It turns out that the structure of 2-local derivations on a Cayley algebra $\mathcal{C}$ depends on the norm n. If the norm n is isotropic, then $\mathcal{C}$ is the split Cayley algebra and all 2-local derivations on $\mathcal{C}$ are derivations (Theorem **??**). Further, if the norm n is anisotropic, then $\mathcal{C}$ is a division Cayley algebra, and in this case the spaces of all 2-local derivations and local derivations on $\mathcal{C}$ coincide, and both are isomorphic to $\mathfrak{so}(\mathcal{C}_0, \mathrm{n})$ (Theorem **??**). At the end of this section we apply of the above results to the description of local and 2-local derivations of the seven dimensional simple Malcev algebras over a field of characteristic $\neq 2$.

In Section 4 we prove that the group of all local automorphisms of $\mathcal{C}$ coincides with the group $\{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\}$ (Theorem **??**). Further we prove that again the behavior of 2-local automorphisms depends on the Cayley algebra being split or division. Every 2-local automorphism on the split Cayley algebra is an automorphism, i.e. they form the exceptional Lie group $G_2(\mathbb{F})$ if $\mathrm{char}\mathbb{F} \neq 2, 3$ (Theorem **??**). On the other hand, on division Cayley algebras over a field $\mathbb{F}$, the groups of 2-local automorphisms and local automorphisms coincide, and they are isomorphic to the group $\{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\}$ (Theorem **??**). Again, at the end of the section, we apply these results to the description of local and 2-local automorphisms of the seven dimensional simple Malcev algebras over a field of characteristic $\neq 2$.

In Section 5 we formulate some open problems.

2. Cayley algebras

Let $\mathbb{F}$ be an arbitrary field. Cayley (or octonion) algebras over $\mathbb{F}$ constitute a well-known class of nonassociative algebras. They are unital nonassociative algebras $\mathcal{C}$ of dimension eight over $\mathbb{F}$, endowed with a nonsingular quadratic multiplicative form (the norm) $\mathrm{n} : \mathcal{C} \to \mathbb{F}$. Hence

$$\mathrm{n}(xy) = \mathrm{n}(x)\mathrm{n}(y)$$

for all $x, y \in \mathcal{C}$, and the polar form

$$\mathrm{n}(x, y) := \mathrm{n}(x + y) - \mathrm{n}(x) - \mathrm{n}(y)$$

is a nondegenerate bilinear form.

Any element in a Cayley algebra $\mathcal{C}$ satisfies the degree 2 equation:

$$x^2 - \mathrm{n}(x, 1)x + \mathrm{n}(x)1 = 0. \tag{2.1}$$

The map $x \to \overline{x} = \mathrm{n}(x, 1)1 - x$ is an involution and the trace $t(x) = \mathrm{n}(x, 1)$ and norm $\mathrm{n}(x)$ are given by $t(x)1 = x + \overline{x}$, $\mathrm{n}(x)1 = x\overline{x} = \overline{x}x$ for all $x \in \mathcal{C}$.

From (**??**), we get

$$xy + yx - \mathrm{n}(x, 1)y - \mathrm{n}(y, 1)x + \mathrm{n}(x, y)1 = 0,$$

for all $x, y \in \mathcal{C}$.

Let $\mathcal{C}$ be a Cayley algebra over an arbitrary field $\mathbb{F}$. Note that any derivation on $\mathcal{C}$ is skew-symmetric with respect to the norm n, i.e., $\mathrm{n}\big(x, d(x)\big) = 0$ for all $x \in \mathcal{C}$, and this implies

$$\mathrm{n}(d(x), y) + \mathrm{n}(x, d(y)) = 0 \tag{2.2}$$

for all $x, y \in \mathcal{C}$. In the same vein, any automorphism φ on $\mathcal{C}$ satisfies $\varphi(1) = 1$, it leaves invariant the subspace of traceless octonions $\mathcal{C}_0 = \{x \in \mathcal{C} : \mathrm{n}(x, 1) = 0\}$ and

$$\mathrm{n}(\varphi(x)) = \mathrm{n}(x), \;\; \mathrm{n}(\varphi(x), \varphi(y)) = \mathrm{n}(x, y) \tag{2.3}$$

for all $x, y \in \mathcal{C}$.

Note that two Cayley algebras $\mathcal{C}_1$ and $\mathcal{C}_2$, with respective norms n_1 and n_2, are isomorphic if and only if the norms n_1 and n_2 are isometric (see [**?**, Corollary 4.7]). It must be remarked that Cayley algebras are alternative, that is, the subalgebra generated by any two elements is associative.

Recall that the norm n is isotropic if there is a non zero element $x \in \mathcal{C}$ with $\mathrm{n}(x) = 0$, otherwise it is called anisotropic. Note that any Cayley algebra with anisotropic norm is a division algebra.

It is known that, up to isomorphism, there is a unique Cayley algebra whose norm is isotropic. It is called the split Cayley algebra. A split Cayley $\mathcal{C}$ admits a *canonical basis* $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$. The multiplication table in this basis is given in Table **??** (see [**?**, §4.1]).

Table 1.

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

Recall too that $\mathcal{K} = \mathbb{F}e_1 + \mathbb{F}e_2$, which is isomorphic to $\mathbb{F}\times\mathbb{F}$, is the split Hurwitz algebra of dimension 2, and with $\mathcal{U} = \mathbb{F}u_1 + \mathbb{F}u_2 + \mathbb{F}u_3$ and $\mathcal{V} = \mathbb{F}v_1 + \mathbb{F}v_2 + \mathbb{F}v_3$, the decomposition $\mathcal{C} = \mathcal{K}\oplus\mathcal{U}\oplus\mathcal{V}$ is a $\mathbb{Z}_3$-grading: $\mathcal{C}_{\bar 0} = \mathcal{K}$, $\mathcal{C}_{\bar 1} = \mathcal{U}$, $\mathcal{C}_{\bar 2} = \mathcal{V}$. This induces a $\mathbb{Z}_3$-grading on the Lie algebra of derivations $\mathfrak{g} = \mathrm{Der}(\mathcal{C})$, with $\mathfrak{g}_i = \{d\in\mathfrak{g} \mid d(\mathcal{C}_j)\subset\mathcal{C}_{i+j} \text{ for all } j\in\mathbb{Z}_3\}$.

The arguments in [?, §4.4] show that the coordinate matrix in the canonical basis of any derivation of the split Cayley algebra $\mathcal{C}$ is of the form:

$$\begin{pmatrix} 0 & 0 & \alpha' & \beta' & \gamma' & -\alpha & -\beta & -\gamma\\ 0 & 0 & -\alpha' & -\beta' & -\gamma' & \alpha & \beta & \gamma\\ \alpha & -\alpha & a_{11} & a_{12} & a_{13} & 0 & \gamma' & -\beta'\\ \beta & -\beta & a_{21} & a_{22} & a_{23} & -\gamma' & 0 & \alpha'\\ \gamma & -\gamma & a_{31} & a_{32} & a_{33} & \beta' & -\alpha' & 0\\ -\alpha' & \alpha' & 0 & \gamma & -\beta & -a_{11} & -a_{21} & -a_{31}\\ -\beta' & \beta' & -\gamma & 0 & \alpha & -a_{12} & -a_{22} & -a_{32}\\ -\gamma' & \gamma' & \beta & -\alpha & 0 & -a_{13} & -a_{23} & -a_{33} \end{pmatrix}, \tag{2.4}$$

with $a_{11} + a_{22} + a_{33} = 0$.

It is well-known that, over a field $\mathbb{F}$ of characteristic $\neq 2,3$, the Lie algebra of all derivations $\mathrm{Der}(\mathcal{C})$ of a Cayley algebra $\mathcal{C}$ is a central simple Lie algebra of type G_2 and, conversely, any central simple Lie algebra of type G_2 is obtained, up to isomorphism, in this way. (See [?, Theorem IV.4.1] and the references there in.)

Over fields of characteristic 3, $\mathrm{Der}(\mathcal{C})$ is no longer simple, but has a unique minimal ideal, which is simple and isomorphic to the Lie algebra $\mathcal{C}_0$ of trace zero elements with the Lie bracket $[x,y] = xy - yx$, which is a central simple Lie algebra of type A_2. The quotient modulo this unique minimal ideal is again isomorphic to $\mathcal{C}_0$. On the other hand, over fields of characteristic 2, $\mathrm{Der}(\mathcal{C})$ is isomorphic to the split simple Lie algebra $\mathfrak{psl}_4(\mathbb{F})$ of type A_3, and surprisingly this does not depend on whether $\mathcal{C}$ is the split algebra or a division algebra. (See [?, §§3,4] and the references there in.)

3. Local and 2-local derivations on Cayley algebras

3.1. **Local derivations on Cayley algebras.**

The main result of this subsection is the following theorem.

Theorem 3.1. *Let $\mathcal{C}$ be a Cayley algebra over an arbitrary field with norm* n. *Then the space of local derivations of $\mathcal{C}$ is the Lie algebra $\{d\in\mathfrak{so}(\mathcal{C},\mathrm{n}) \mid d(1) = 0\}$.*

Remark 3.2. Let $\mathcal{C}_0$ be the subspace of trace 0 elements. If the characteristic of $\mathbb{F}$ is not 2, then $\mathcal{C} = \mathbb{F}1\oplus\mathcal{C}_0$. Embed $\mathrm{End}_{\mathbb{F}}(\mathcal{C}_0)$ inside $\mathrm{End}_{\mathbb{F}}(\mathcal{C})$ by extending any endomorphism φ of $\mathcal{C}_0$ to an endomorphism of $\mathcal{C}$ by imposing $\varphi(1) = 0$. Then (characteristic not 2) the Lie algebra in Theorem **??** is naturally identified with $\mathfrak{so}(\mathcal{C}_0,\mathrm{n}) = \{d\in\mathrm{End}_{\mathbb{F}}(\mathcal{C}_0) \mid \mathrm{n}\big(v,d(v)\big) = 0\ \forall v\in\mathcal{C}_0\}$. The trace of any element in $\mathfrak{so}(\mathcal{C}_0,\mathrm{n})$ is zero.

However, if the characteristic of $\mathbb{F}$ is 2, then $1\in\mathcal{C}_0$. Here the radical of the restriction of the polarization of n to $\mathcal{C}_0$ is $\mathbb{F}1$. In this case the right definition of $\mathfrak{so}(\mathcal{C}_0,\mathrm{n})$ is $\{d\in\mathrm{End}_{\mathbb{F}}(\mathcal{C}_0) \mid \mathrm{trace}(d) = 0 \text{ and } \mathrm{n}\big(v,d(v)\big) = 0\ \forall v\in\mathcal{C}_0\}$. It follows that any element in $\mathfrak{so}(\mathcal{C}_0,\mathrm{n})$ annihilates 1. Moreover, any element $d\in\mathfrak{so}(\mathcal{C},\mathrm{n})$ with $d(1) = 0$ leaves $\mathcal{C}_0$ invariant, and its restriction to $\mathcal{C}_0$ lies in $\mathfrak{so}(\mathcal{C}_0,\mathrm{n})$. Conversely, pick any element $u\in\mathcal{C}\setminus\mathbb{F}1$, so that $\mathcal{C} = \mathbb{F}u\oplus\mathcal{C}_0$. Any element $d\in\mathfrak{so}(\mathcal{C}_0,\mathrm{n})$ can be extended to an element of $\mathfrak{so}(\mathcal{C},\mathrm{n})$ by defining $d(u)$ as the unique element of $\mathcal{C}$ such that $\mathrm{n}\big(d(u),u\big) = 0$ and $\mathrm{n}\big(d(u),x\big) = \mathrm{n}\big(u,d(x)\big)$ for all $x\in\mathcal{C}_0$. Therefore, the restriction

$$\begin{aligned}\{d \in \mathfrak{so}(\mathcal{C}, \mathrm{n}) \mid d(1) = 0\} &\longrightarrow \mathfrak{so}(\mathcal{C}_0, \mathrm{n})\\ d \quad &\mapsto \quad d|_{\mathcal{C}_0}\end{aligned}$$

is an isomorphism of Lie algebras also in characteristic 2.

Therefore, in any characteristic, we can identify canonically the Lie algebra $\{d \in \mathfrak{so}(\mathcal{C}, \mathrm{n}) \mid d(1) = 0\}$ with $\mathfrak{so}(\mathcal{C}_0, \mathrm{n})$.

The proof of Theorem **??** is based on two lemmas. The first one is surely well-known. For Cayley division algebras over fields of characteristic not two, it follows from the Cayley-Dickson doubling process as in [**?**, Corollary 1.7.2]. We include a proof, valid over arbitrary fields, for completeness.

Lemma 3.3. *Let $\mathcal{C}$ be a Cayley over a field $\mathbb{F}$ with norm n. Any two elements of $\mathcal{C} \setminus \mathbb{F}1$ are conjugate under $\operatorname{Aut}\mathcal{C}$ if and only if they have the same norm and the same trace.*

Proof. Let $a, b \in \mathcal{C} \setminus \mathbb{F}1$ with $t(a) = t(b) = \alpha$, and $\mathrm{n}(a) = \mathrm{n}(b) = \mu$. The minimal polynomial of both a and b is $p(X) = X^2 - \alpha X + \mu$.

If $p(X)$ is separable, then $\mathcal{K}_1 = \mathbb{F}1 + \mathbb{F}a$ and $\mathcal{K}_2 = \mathbb{F}1 + \mathbb{F}b$ are isomorphic two-dimensional composition subalgebras of $\mathcal{C}$, and we use the Cayley-Dickson doubling process to extend any isomorphim between $\mathcal{K}_1$ and $\mathcal{K}_2$ carrying a to b to an automorphism of $\mathcal{C}$.

Otherwise, either the characteristic of $\mathbb{F}$ is not 2 and $p(X) = \big(X - \frac{\alpha}{2}\big)^2$, or the characteristic of $\mathbb{F}$ is 2 and $\alpha = 0$.

Note that if $x \in \mathcal{C} \setminus \mathbb{F}1$ satisfies $x^2 = 0$ and we pick $y \in \mathcal{C}$ with $\mathrm{n}(x, \bar{y}) = 1$, then $1 = \mathrm{n}(x, \bar{y}) = -\bar{x}y - \bar{y}x = xy - \bar{y}x$, and then $e_1 = xy$ is a nonzero idempotent, because $e_1^2 = xyxy = (1 + \bar{y}x)xy = xy + \bar{y}x^2y = e_1$. Also, we have $e_1x = xyx = (1 + \bar{y}x)x = x$ and $xe_1 = x^2y = 0$. Hence x lies in the Peirce component $\{u \in \mathcal{C} \mid e_1x = x,\ xe_1 = 0\} = e_1\mathcal{C}e_2$, and as in [**?**, §4.4] we may take a canonical basis of $\mathcal{C}$ with $x = u_1$. In particular, if $\alpha = 0 = \mu$, then a and b are both conjugate to u_1.

Also, if the characteristic of $\mathbb{F}$ is not 2 and $p(X) = \big(X - \frac{\alpha}{2}\big)^2$, then there are automorphisms φ, ψ of $\mathcal{C}$ such that $\varphi(a - \frac{\alpha}{2}1) = u_1 = \psi(b - \frac{\alpha}{2}1)$, and hence $\psi^{-1} \circ \varphi(a) = b$.

Finally, if the characteristic of $\mathbb{F}$ is 2 and $\alpha = 0$ but $\mu \neq 0$, we distinguish two cases:

- If $1 \in \mathbb{F}a + \mathbb{F}b$, then $b = \epsilon 1 + \delta a$ with $\delta \neq 0$, so $\mu = \mathrm{n}(a) = \mathrm{n}(b) = \epsilon^2 + \delta^2\mathrm{n}(a)$. Hence $(1 + \delta)^2\mathrm{n}(a) = \epsilon^2$ and either $\delta = 1$, $\epsilon = 0$ and $a = b$, or $\delta \neq 1$ and $\mu = \mathrm{n}(a) = \mathrm{n}(b) \in \mathbb{F}^2$. Write $\mu = \gamma^2$. In this case, $(a + \gamma 1)^2 = 0 = (b + \gamma 1)^2$ and again there are automorphisms φ, ψ of $\mathcal{C}$ such that $\varphi(a + \gamma 1) = u_1 = \psi(b + \gamma 1)$, so that $\psi^{-1} \circ \varphi(a) = b$.
- If, on the contrary, $1 \notin \mathbb{F}a + \mathbb{F}b$, then $(\mathbb{F}a + \mathbb{F}b)^\perp \not\subseteq (\mathbb{F}1)^\perp$, where $\perp$ denotes the orthogonal subspace relative to n, so we may pick an element $u \in \mathcal{C}$ such that $\mathrm{n}(u, 1) = 1$ and $\mathrm{n}(u, a) = 0 = \mathrm{n}(u, b)$. Then $\mathcal{K} = \mathbb{F}1 + \mathbb{F}u$ is a two-dimensional composition subalgebra of $\mathcal{C}$ and, by the Cayley-Dickson doubling process, the quaternion subalgebra $\mathcal{Q}_1 = \mathcal{K} \oplus \mathcal{K}a$ and $\mathcal{Q}_2 = \mathcal{K} \oplus \mathcal{K}b$ are isomorphic, under an isomorphism that is the identity on $\mathcal{K}$ and takes a to b. Again, by the Cayley-Dickson doubling process, this isomorphism can be extended to an automorphism of $\mathcal{C}$. (See, for instance, [**?**, Corollary 1.7.2].) □

Lemma 3.4. *Let $\mathcal{C}$ be a Cayley algebra over an arbitrary field $\mathbb{F}$, with Lie algebra of derivations* $\mathrm{Der}(\mathcal{C})$*, and let x be any element in $\mathcal{C}\setminus\mathbb{F}1$. Then*

$$\mathrm{Der}(\mathcal{C})x := \{d(x) \mid d\in \mathrm{Der}(\mathcal{C})\} = (\mathbb{F}1+\mathbb{F}x)^{\perp} = \bigl(\mathbb{F}x\bigr)^{\perp}\cap \mathcal{C}_0,$$

where $\perp$ denotes the orthogonal subspace relative to the norm n.

Proof. We may extend scalars and assume that $\mathbb{F}$ is algebraically closed. In particular, $\mathcal{C}$ is split. Take a canonical basis of $\mathcal{C}$ as in Table **??**. If the minimal polynomial of x is separable: $p(X) = (X-\alpha)(X-\beta)$, with $\alpha\neq\beta$, then Lemma **??** shows that x is conjugate to $\alpha e_1+\beta e_2$. But Equation (**??**) shows that

$$\mathrm{Der}(\mathcal{C})(\alpha e_1+\beta e_2) = \mathcal{U}\oplus\mathcal{V} = (\mathbb{F}e_1+\mathbb{F}e_2)^{\perp} = \bigl(\mathbb{F}1+\mathbb{F}(\alpha e_1+\beta e_2)\bigr)^{\perp}$$

and we are done in this case.

Otherwise the minimal polynomial of x is of the form $p(X) = (X-\lambda)^2$ for some $\lambda\in\mathbb{F}$. By Lemma **??**, $x-\lambda 1$ is conjugate to u_1, and hence x is conjugate to $\lambda 1+u_1$. Now Equation (**??**) shows that

$$\mathrm{Der}(\mathcal{C})(\lambda 1+u_1) = \mathrm{Der}(\mathcal{C})u_1 = \mathbb{F}(e_1-e_2)\oplus\mathcal{U}\oplus\mathbb{F}v_2\oplus\mathbb{F}v_3 = (\mathbb{F}1+\mathbb{F}u_1)^{\perp},$$

and we are done. □

Proof of Theorem **??**. If Δ is a local derivation of $\mathcal{C}$, $\Delta(1)=0$ and for any $x\in\mathcal{C}$, we get $\mathrm{n}\bigl(x,\Delta(x)\bigr)=0$, as $\Delta(x)=d(x)$ for a derivation d and $\mathrm{Der}(\mathcal{C})$ is contained in $\{\phi\in\mathfrak{so}(\mathcal{C},\mathrm{n})\mid \phi(1)=0\}$. Hence the space of local derivations of $\mathcal{C}$ is contained in $\{\phi\in\mathfrak{so}(\mathcal{C},\mathrm{n})\mid \phi(1)=0\}$.

Conversely, given any Δ in $\{\phi\in\mathfrak{so}(\mathcal{C},\mathrm{n})\mid \phi(1)=0\}$ and any $x\in\mathcal{C}$, if $x\in\mathbb{F}1$, then $\Delta(x)=0$, while if $x\in\mathcal{C}\setminus\mathbb{F}1$, $\mathrm{n}\bigl(x,\Delta(x)\bigr)=0=\mathrm{n}\bigl(1,\Delta(x)\bigr)$, so $\Delta(x)\in\bigl(\mathbb{F}x\bigr)^{\perp}\cap\mathcal{C}_0$. Hence $\Delta(x)\in\mathrm{Der}(\mathcal{C})x$ by Lemma **??**, and it follows that there is a derivation $d\in\mathrm{Der}(\mathcal{C})$ such that $\Delta(x)=d(x)$. □

Note that Theorem **??** implies the following result (see [**?**, Theorem 1.1]).

Corollary 3.5. *The space* $\mathrm{LocDer}(\mathbb{O})$ *of all local derivations on the real Cayley algebra $\mathbb{O}$ equipped with the Lie bracket is isomorphic to the Lie algebra* $\mathfrak{so}_7(\mathbb{R})$ *of all real skew-symmetric matrices of order* 7.

Remark 3.6. Note that the dimensions of the Lie algebras $\mathrm{LocDer}(\mathcal{C})\cong\mathfrak{so}(\mathcal{C}_0,\mathrm{n})$ and $\mathrm{Der}(\mathcal{C})$ are equal to 21 and 14, respectively. Therefore the Cayley algebra $\mathcal{C}$ admits local derivations which are not derivations.

3.2. **2-Local derivations on split Cayley algebras.**

As we mentioned in Section 2, each Cayley algebra is either the split algebra (if the norm is isotropic) or a division algebra (if the norm is anisotropic) (see [**?**], [**?**]). We shall consider 2-local derivations separately for each case.

Theorem 3.7. *Any* 2*-local derivation of the split Cayley algebra $\mathcal{C}$ over a field $\mathbb{F}$ is a derivation.*

The proof of the following lemma is similar to the proof of [**?**, Lemma 3.4], we include it for the sake of completeness.

Lemma 3.8. *Any* 2*-local derivation on an arbitrary Cayley algebra is linear.*

Proof. Note that any 2-local derivation Δ on $\mathcal{C}$ is 'homogeneous': $\Delta(\lambda x) = \lambda\Delta(x)$ for all $\lambda \in \mathbb{F}$ and $x \in \mathcal{C}$. Indeed, take a derivation $d_{x,\lambda x}$ on $\mathcal{C}$ such that $\Delta(x) = d_{x,\lambda x}(x)$ and $\Delta(\lambda x) = d_{x,\lambda x}(\lambda x)$. Then

$$\Delta(\lambda x) = d_{x,\lambda x}(\lambda x) = \lambda d_{x,\lambda x}(x) = \lambda\Delta(x).$$

Let us show that Δ is additive.

Let $x, z \in \mathcal{C}$. Take a derivation $d_{x,z}$ such that $\Delta(x) = d_{x,z}(x)$ and $\Delta(z) = d_{x,z}(z)$. From Equation (**??**) we obtain

$$\mathrm{n}(\Delta(x), z) = \mathrm{n}(d_{x,z}(x), z) = -\mathrm{n}(x, d_{x,z}(z)) = -\mathrm{n}(x, \Delta(z)).$$

So,

$$\mathrm{n}(\Delta(x), z) = -\mathrm{n}(x, \Delta(z)). \tag{3.1}$$

Now replacing the element x by $x + y$ in (**??**), we obtain that

$$\begin{aligned}\mathrm{n}(\Delta(x+y), z) &= -\mathrm{n}(x+y, \Delta(z)) = -\mathrm{n}(x, \Delta(z)) - \mathrm{n}(y, \Delta(z)) \\ &= \mathrm{n}(\Delta(x), z) + \mathrm{n}(\Delta(y), z) = \mathrm{n}(\Delta(x) + \Delta(y), z).\end{aligned}$$

Thus

$$\mathrm{n}(\Delta(x+y) - \Delta(x) - \Delta(y), z) = 0$$

for all $z \in \mathcal{C}$. Since the bilinear form n is nondegenerate, it follows that $\Delta(x+y) = \Delta(x) + \Delta(y)$. □

So, any 2-local derivation on a Cayley algebra is linear and annihilates 1.

Corollary 3.9. *Any 2-local derivation on an arbitrary Cayley algebra $\mathcal{C}$ is a local derivation, and hence belongs to the orthogonal Lie algebra $\mathfrak{so}(\mathcal{C}_0, \mathrm{n})$.*

Lemma 3.10. *Let Δ be a 2-local derivation on $\mathcal{C}$. If $a, b \in \mathcal{C}$ and $\lambda \in \mathbb{F}$ satisfy $\Delta(a) = 0$, $ab = \lambda b$ (respectively $ba = \lambda b$), then $a\Delta(b) = \lambda\Delta(b)$ (respectively $\Delta(b)a = \lambda\Delta(b)$.*

Proof. Let $d_{a,b}$ be a derivation on $\mathcal{C}$ which coincides with Δ on a and b. Then, if $ab = \lambda b$ (the other case is similar) we get

$$\begin{aligned}\lambda\Delta(b) = \lambda d_{a,b}(b) = d_{a,b}(\lambda b) = d_{a,b}(ab) \\ = d_{a,b}(a)b + a d_{a,b}(b) = \Delta(a)b + a\Delta(b) = a\Delta(b),\end{aligned}$$

as desired. □

Proof of Theorem **??**. Let $\mathcal{C}$ be the split Cayley algebra over the field $\mathbb{F}$. Consider the $\mathbb{Z}_3$-grading $\mathcal{C} = (\mathbb{F}e_1 + \mathbb{F}e_2) \bigoplus \mathcal{U} \bigoplus \mathcal{V}$, with the notations of Section 2. If Δ is a 2-local derivation, let $d \in \mathrm{Der}(\mathcal{C})$ such that $d(e_1) = \Delta(e_1)$. Substituting Δ by $\Delta - d$ we may assume $\Delta(e_1) = 0$, and hence $\Delta(e_2) = \Delta(1 - e_1) = 0$ too.

Since $\mathcal{U} = \{x \in \mathcal{C} \mid e_1 x = x,\ x e_1 = 0\}$, Lemma **??** shows that $\Delta(\mathcal{U}) \subseteq \mathcal{U}$. In the same vein we get $\Delta(\mathcal{V}) \subseteq \mathcal{V}$.

Hence $\Delta \in \{f \in \mathfrak{so}(\mathcal{U} \oplus \mathcal{V}, \mathrm{n}) \mid f(\mathcal{U}) \subseteq \mathcal{U},\ f(\mathcal{V}) \subseteq \mathcal{V}\}$, where $\mathfrak{so}(\mathcal{U} \oplus \mathcal{V}, \mathrm{n})$ is identified with the subspace of the endomorphisms in $\mathfrak{so}(\mathcal{C}, \mathrm{n})$ that annihilate $\mathbb{F}e_1 \oplus \mathbb{F}e_2$.

Since $\mathcal{U}$ and $\mathcal{V}$ are dual relative to n, the elements of $\{f \in \mathfrak{so}(\mathcal{U} \oplus \mathcal{V}, \mathrm{n}) \mid f(\mathcal{U}) \subseteq \mathcal{U},\ f(\mathcal{V}) \subseteq \mathcal{V}\}$ are determined by its restriction to $\mathcal{U}$. Actually, if $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ are dual bases in $\mathcal{U}$ and $\mathcal{V}$, then in these bases, the matrix of the restriction of f to $\mathcal{V}$ is minus the transpose of the matrix of the restriction of f to $\mathcal{U}$. Note that $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ is a canonical basis of $\mathcal{C}$.

However, relative to the $\mathbb{Z}_3$-grading above, $\mathrm{Der}(\mathcal{C})_{\bar 0}$ is the subspace of the derivations that preserve $\mathcal{U}$ and $\mathcal{V}$ and annihilate $\mathbb{F}e_1 \oplus \mathbb{F}e_2$, and this consists of the elements above with the extra condition of the trace of the restriction of f to $\mathcal{U}$ being 0. Thus we have

$$\{f \in \mathfrak{so}(\mathcal{U} \oplus \mathcal{V}, \mathrm{n}) \mid f(\mathcal{U}) \subseteq \mathcal{U},\ f(\mathcal{V}) \subseteq \mathcal{V}\} = \mathrm{Der}(\mathcal{C})_{\bar 0} \oplus \mathbb{F}\varphi,$$

where $\varphi(u_3) = u_3$, $\varphi(v_3) = -v_3$, and $\varphi(e_i) = \varphi(u_i) = \varphi(v_i) = 0$ for $i = 1, 2$.

Therefore, it is enough to prove that φ is not a 2-local derivation. For this, take $a = u_1 - v_1$, so that $\varphi(a) = 0$, and $b = u_2 + v_3$. Then $ab = (u_1 - v_1)(u_2 + v_3) = v_3 + u_2 = b$. Now, according to Lemma **??**, if φ were a 2-local derivation, we would get $a\varphi(b) = \varphi(b)$. But $\varphi(b) = \varphi(u_2 + v_3) = -v_3$, so that $a\varphi(b) = (u_1 - v_1)(-v_3) = -u_2 \neq \varphi(b)$, a contradiction. □

Since any Cayley algebra over an algebraically closed field is split, Theorem **??** implies the following result (see [**?**, Theorem 1.2]).

Corollary 3.11. *Any* 2*-local derivation of a Cayley algebra* $\mathcal{C}$ *over an algebraically closed field is a derivation.*

3.3. 2-Local derivations on division Cayley algebras.

The situation for Cayley division algebras is completely different. In order to deal with it, some preliminaries are needed.

Let $\mathcal{C}$ be the split Cayley algebra over an arbitrary field, and take a canonical basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ as in Table **??**.

Lemma 3.12. *Let* $\mathcal{C}$ *be the split Cayley algebra over an arbitrary field* $\mathbb{F}$*. Consider the following subalgebras of* $\mathrm{Der}(\mathcal{C})$*:*

$$\mathcal{L} = \{d \in \mathrm{Der}(\mathcal{C}) \mid d(e_1) = d(e_2) = 0\}, \quad \mathcal{L}' = \{d \in \mathrm{Der}(\mathcal{C}) \mid d(u_1 + v_1) = 0\}.$$

Then the dimension of each of the following subspaces: $\mathcal{L}(u_1 + v_1)$*,* $\mathcal{L}'(u_1 + \mu v_1)$ *with* $1 \neq \mu \in \mathbb{F}$*, and* $\mathcal{L}'(u_2 + v_2)$*, is always* 5*.*

Proof. These subspaces are easily computed using Equation (**??**), obtaining:

- $\mathcal{L}(u_1 + v_1) = \mathrm{span}\langle u_1 - v_1, u_2, u_3, v_2, v_3\rangle$,
- $\mathcal{L}'(u_1 + \mu v_1) = \mathrm{span}\langle e_1 - e_2, u_2, u_3, v_2, v_3\rangle$ $(\mu \neq 1)$, and
- $\mathcal{L}'(u_2 + v_2) = \mathrm{span}\langle e_1 - e_2, u_1 + v_1, u_2 - v_2, u_3, v_3\rangle$.

The result follows. □

Corollary 3.13. *Let* $\mathcal{C}$ *be a Cayley division algebra over an arbitrary field. Let* $x, y \in \mathcal{C}$ *be elements such that* 1*,* x*, and* y*, are linearly independent. Consider the following subalgebra of* $\mathrm{Der}(\mathcal{C})$*:* $\mathcal{S} = \{d \in \mathrm{Der}(\mathcal{C}) \mid d(x) = 0\}$*. Assume that either:*

- *The restriction of the norm* n *to* $\mathcal{K} = \mathbb{F}1 + \mathbb{F}x$ *is nonsingular (that is,* $\mathcal{K}$ *is a two-dimensional composition subalgebra, and note that this is always the case if the characteristic of* $\mathbb{F}$ *is not* 2*), or*
- *The characteristic of* $\mathbb{F}$ *is* 2 *and* $\mathrm{n}(1, x) = 0 = \mathrm{n}(1, y)$*.*

Then the subspace $\mathcal{S}y$ *equals* $(\mathbb{F}1 + \mathbb{F}x + \mathbb{F}y)^{\perp}$*.*

Proof. Note first that since $\mathrm{Der}(\mathcal{C})$ is contained in the orthogonal Lie algebra $\mathfrak{so}(\mathcal{C}, \mathrm{n})$, we always have $\mathcal{S}y \subseteq (\mathbb{F}1 + \mathbb{F}x + \mathbb{F}y)^{\perp}$, and this is a subspace of dimension 5 because 1, x and y are linearly independent. Therefore, it is enough to check that the dimension of $\mathcal{S}y$ is 5. Also, in order to check this, we may extend scalars to an algebraic closure $\overline{\mathbb{F}}$.

Assume first that $\mathcal{K} = \mathbb{F}1 + \mathbb{F}x$ is a composition subalgebra of $\mathcal{C}$, then $y = a + y'$ with $a \in \mathcal{K}$ and $0 \neq y' \in \mathcal{K}^{\perp}$. Since $\mathcal{C}$ is a division algebra, n is anisotropic, so $\mathrm{n}(y') \neq 0$. Note that $\mathcal{S}y = \mathcal{S}y'$, because $\mathcal{S}$ annihilates both 1 and x. Hence we may assume $y \in \mathcal{K}^\perp$. Extending scalars to $\overline{\mathbb{F}}$, $\mathcal{K} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is a two-dimensional composition subalgebra, and hence it has two orthogonal idempotents: $\overline{\mathcal{K}} := \mathcal{K} \otimes_{\mathbb{F}} \overline{\mathbb{F}} = \overline{\mathbb{F}}e_1 \oplus \overline{\mathbb{F}}e_2$. These two idempotents can be completed to a canonical basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of $\overline{\mathcal{C}} := \mathcal{C} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Denote again by n the norm in $\overline{\mathcal{C}}$. If $\mathrm{n}(y) = \mu$ and we pick a square root $\alpha \in \overline{\mathbb{F}}$ of μ, then $\mathrm{n}(y) = \mathrm{n}\big(\alpha(u_1+v_1)\big)$ and, using the Cayley-Dickson doubling process we can easily define an automorphism φ of $\overline{\mathcal{C}}$ that is the identity on $\overline{\mathcal{K}}$ and satisfies $\varphi(y \otimes 1) = \alpha(u_1 + v_1)$. Then $\varphi(\mathcal{S} \otimes_{\mathbb{F}} \overline{\mathbb{F}})\varphi^{-1} = \{d \in \mathrm{Der}(\overline{\mathcal{C}}) \mid d(e_1) = 0 = d(e_2)\}$, and the result follows from Lemma **??**.

Assume now that the characteristic is 2 and $\mathrm{n}(1, x) = 0 = \mathrm{n}(1, y)$. In particular $\bar{x} = -x = x$, and we have two different possibilities depending on whether $\mathrm{n}(x, y)$ is 0 or not.

If $\mathrm{n}(x, y) \neq 0$, then $\mathrm{n}(1, yx) = \mathrm{n}(\bar{y}, x) = \mathrm{n}(y, x) \neq 0$, and hence $\mathcal{K} = \mathbb{F}1 + \mathbb{F}yx$ is a composition subalgebra of $\mathcal{C}$. Moreover, $\mathrm{n}(yx, x) = \mathrm{n}(y, x^2) = 0$, as $x^2 = -\mathrm{n}(x)1$, and $(yx)x = \mathrm{n}(x)y$, so $y \in \mathcal{K}x$. Extending scalars as in the previous case, it turns out that there is an automorphism φ of $\overline{\mathcal{C}}$ that is the identity on $\overline{\mathcal{K}}$ and such that $\varphi(x \otimes 1) = \alpha(u_1 + v_1)$ for some $0 \neq \alpha \in \overline{\mathbb{F}}$. Since y is in $\mathcal{K}x$, we get $\varphi(y \otimes 1) \in (\overline{\mathbb{F}}e_1 + \overline{\mathbb{F}}e_2)(u_1 + v_1) = \overline{\mathbb{F}}u_1 + \overline{\mathbb{F}}v_1$. Then we have $\varphi(y \otimes 1) \in \overline{\mathbb{F}}(u_1 + \mu v_1)$, with μ different from 0, as $\mathrm{n}(y) \neq 0$ because n is anisotropic, and different from 1, as x and y are linearly independent. In this case $\varphi(\mathcal{S} \otimes_{\mathbb{F}} \overline{\mathbb{F}})\varphi^{-1} = \{d \in \mathrm{Der}(\overline{\mathcal{C}}) \mid d(u_1 + v_1) = 0\}$, and the result follows from Lemma **??**.

Finally, if $\mathrm{n}(x, y) = 0$, pick an element $u \in (\mathbb{F}x + \mathbb{F}y)^\perp \setminus (\mathbb{F}1)^\perp$ (remember that 1, x, and y are linearly independent). Then $\mathrm{n}(1, u) \neq 0$, so $\mathcal{K} = \mathbb{F}1 + \mathbb{F}u$ is a composition subalgebra of $\mathcal{C}$. Here $\mathrm{n}(x, \mathcal{K}) = 0 = \mathrm{n}(y, \mathcal{K})$. Consider the quaternion subalgebra $\mathcal{Q} = \mathcal{K} \oplus \mathcal{K}x$. As y is orthogonal to $\mathcal{K}$ and to x, it belongs to $(\mathcal{K} + \mathbb{F}x)^\perp = \mathbb{F}x + \mathcal{Q}^\perp$, hence there exists a scalar $\alpha \in \mathbb{F}$ such that $y - \alpha x \in \mathcal{Q}^\perp$. But $\mathcal{S}y = \mathcal{S}(y - \alpha x)$, so we may assume $y \in \mathcal{Q}^\perp$. Extending scalars and using the Cayley-Dickson doubling process, we can easily define an automorphism φ of $\overline{\mathcal{C}}$ such that $\varphi(x) \in \overline{\mathbb{F}}(u_1 + v_1)$ and $\varphi(y) \in \overline{\mathbb{F}}(u_2 + v_2)$. Again we get $\varphi(\mathcal{S} \otimes_{\mathbb{F}} \overline{\mathbb{F}})\varphi^{-1} = \{d \in \mathrm{Der}(\overline{\mathcal{C}}) \mid d(u_1 + v_1) = 0\}$, and the result follows from Lemma **??**. □

Remark 3.14. If $\mathcal{C}$ is a Cayley algebra and $\mathcal{K}$ is a two-dimensional composition subalgebra, the subalgebra $\{d \in \mathrm{Der}(\mathcal{C}) \mid d(\mathcal{K}) = 0\}$ is a special unitary Lie algebra. (See [**?**, §5] and references therein.)

Now we are ready to tackle the case of 2-local derivations on Cayley division algebras.

The following result, combined with Corollary **??**, shows that for Cayley division algebras the notions of local and 2-local derivations coincide.

Theorem 3.15. *Let $\mathcal{C}$ be a Cayley division algebra over an arbitrary field $\mathbb{F}$. Then any local derivation of $\mathcal{C}$ is a 2-local derivation. As a consequence, the space of 2-local derivations coincide with $\{d \in \mathfrak{so}(\mathcal{C}, \mathrm{n}) \mid d(1) = 0\} \simeq \mathfrak{so}(\mathcal{C}_0, \mathrm{n})$.*

Proof. Take an element $\Delta \in \mathfrak{so}(\mathcal{C}, \mathrm{n})$ with $\Delta(1) = 0$, and take two elements $x, y \in \mathcal{C}$. We must show that there is a derivation $d \in \mathrm{Der}(\mathcal{C})$ such that $\Delta(x) = d(x)$ and $\Delta(y) = d(y)$. In other words, Δ coincides with d on $\mathbb{F}1 + \mathbb{F}x + \mathbb{F}y$. If $x \in \mathbb{F}1$ this is trivial, as $\Delta(1) = d(1) = 0$ and Δ is a local derivation. If $y \in \mathbb{F}1 + \mathbb{F}x$ this is trivial too. Hence assume 1, x, and y are linearly independent.

Since Δ is a local derivation, there is a derivation $d' \in \mathrm{Der}(\mathcal{C})$ such that $\Delta(x) = d'(x)$, and hence $\Delta' = \Delta - d'$ annihilates 1 and x. As Δ' is in $\mathfrak{so}(\mathcal{C}, \mathrm{n})$, $\Delta'(y)$ lies in $(\mathbb{F}1 + \mathbb{F}x + \mathbb{F}y)^\perp$, so Corollary **??** shows the existence of a derivation $d \in \mathrm{Der}(\mathcal{C})$ with $d(x) = 0$ such that $\Delta'(y) = d(y)$. Hence Δ and the derivation $d' + d$ coincide on x and y, as desired. □

Remark 3.16. Since $2\mathrm{LocDer}(\mathcal{C}) = \mathrm{LocDer}(\mathcal{C}) \cong \mathfrak{so}(\mathcal{C}_0, \mathrm{n})$, it follows that any division Cayley algebra $\mathcal{C}$ admits 2-local derivations which are not derivations.

3.4. Local and 2-local derivations on 7-dimensional simple Malcev algebras.

An algebra $\mathcal{A}$ over a field $\mathbb{F}$ is called a Malcev algebra if its multiplication is anticommutative and satisfies the following identity:

$$(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y.$$

Let $\mathcal{C}$ be a Cayley algebra over a field $\mathbb{F}$ of characteristic $\neq 2$ and define a bracket $[\cdot,\cdot]$ by

$$[x,y] = \frac{1}{2}(xy - yx),\ x, y \in \mathcal{C}.$$

Then the space of trace 0 elements: $\mathcal{C}_0 = \{x \in \mathcal{C} : t(x) = 0\}$, with the bracket $[\cdot,\cdot]$, is a simple Malcev algebra (see, e.g., [**?**, Theorem 4.10]) that will be denoted $\mathcal{C}_0^-$.

If the characteristic of the ground field $\mathbb{F}$ is $\neq 2, 3$, $\mathcal{C}_0^-$ is a central simple non-Lie Malcev algebra and, conversely, any such algebra is obtained, up to isomorphism, in this way. (See [**?**].)

However, if the characteristic of $\mathbb{F}$ is 3, then any simple Malcev algebra over $\mathbb{F}$ is a Lie algebra ([**?**]). In this case, $\mathcal{C}_0^-$ is a central simple Lie algebra of type A_2 (i.e., a twisted form of $\mathfrak{psl}_3(\mathbb{F})$) and, conversely, any central simple Lie algebra of type A_2 is obtained in this way. (See [**?**] or [**?**, Theorem 4.26].)

It is known [**?**, Chapter IV, Lemma 6.1] that every derivation on a Cayley algebra $\mathcal{C}$ defines a derivation on $\mathcal{C}_0^-$, and conversely. More precisely, if d is a derivation on $\mathcal{C}$, then the restriction $d|_{\mathcal{C}_0^-}$ is a derivation on $\mathcal{C}_0^-$. Conversely, if d is a derivation on $\mathcal{C}_0^-$, then its extension to $\mathcal{C}$ defined as

$$d(\lambda 1 + x) = d(x),\ \lambda \in \mathbb{F},\ x \in \mathcal{C}_0^-,$$

is a derivation on $\mathcal{C}$. Therefore a similar correspondence between local (2-local) derivations of the Cayley algebra $\mathcal{C}$ and the Malcev algebra $\mathcal{C}_0^-$ are also true.

So, from Theorems **??**, **??** and **??** we obtain the following results.

Theorem 3.17. *Let $\mathcal{C}_0^-$ be the central simple Malcev algebra associated with a Cayley algebra $\mathcal{C}$ over a field $\mathbb{F}$ of characteristic $\neq 2$. Then*

(1) *the space of all local derivations of $\mathcal{C}_0^-$ is the Lie algebra $\mathfrak{so}(\mathcal{C}_0, \mathrm{n})$;*
(2) *if $\mathcal{C}$ is a split algebra, then every 2-local derivation on $\mathcal{C}_0^-$ is a derivation;*
(3) *if $\mathcal{C}$ is a division algebra, then*

$$2\mathrm{LocDer}(\mathcal{C}_0^-) = \mathrm{LocDer}(\mathcal{C}_0^-) = \mathfrak{so}(\mathcal{C}_0, \mathrm{n}).$$

Remark 3.18. A Cayley algebra $\mathcal{C}$ over a field $\mathbb{F}$ of characteristic $\neq 2$ equipped with the multiplication

$$x \circ y = \frac{1}{2}(xy + yx)$$

becomes a Jordan algebra $(\mathcal{C}^+, \circ)$. Using the description of derivations of the algebra $\mathcal{C}^+$ [**?**, Chapter IV, Page 175], we obtain the following isomorphisms:

$$\mathrm{LocDer}(\mathcal{C}) \cong \mathrm{LocDer}(\mathcal{C}_0^-) \cong \mathrm{Der}(\mathcal{C}^+) \cong \mathfrak{so}(\mathcal{C}_0, \mathrm{n}).$$

4. Local and 2-local automorphisms on Cayley algebras

4.1. **Local automorphisms.**

Let $\mathrm{LocAut}(\mathcal{C})$ be the set of local automorphisms of a Cayley algebra $\mathcal{C}$ over a field $\mathbb{F}$. Denote by $\mathrm{O}(\mathcal{C}, \mathrm{n})$ the orthogonal group relative to the norm n of $\mathcal{C}$.

Theorem 4.1. *Let $\mathcal{C}$ be a Cayley algebra over a field $\mathbb{F}$ with norm n. Then the set* $\mathrm{LocAut}(\mathcal{C})$ *coincides with* $\{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\}$.

Proof. Any local automorphism fixes the unity 1 of $\mathcal{C}$ and preserves the norm of any element, because so does any automorphism (see (**??**)).

Conversely, given an orthogonal transformation $\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n})$ with $\varphi(1) = 1$, and given any element $x \in \mathcal{C} \setminus \mathbb{F}1$ we have $\mathrm{n}\big(\varphi(x)\big) = \mathrm{n}(x)$, and $\mathrm{n}\big(\varphi(x), 1\big) = \mathrm{n}\big(\varphi(x), \varphi(1)\big) = \mathrm{n}(x, 1)$. Hence $\varphi(x)$ and x have the same norm and trace, and the result follows from Lemma **??**. $\square$

If $\mathrm{char}\,\mathbb{F} \neq 2$, then $\{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\}$ is naturally isomorphic to the orthogonal group $\mathrm{O}(\mathcal{C}_0, \mathrm{n})$, where $\mathcal{C}_0$ is the subspace of trace zero elements, that is, the orthogonal subspace to $\mathbb{F}1$.

However, if $\mathrm{char}\,\mathbb{F} = 2$ there is the natural group homomorphism

$$\Phi : \{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\} \to \mathrm{O}(\mathcal{C}_0, \mathrm{n})$$

obtained by restriction. Take an element $a \in \mathcal{C}$ with $\mathrm{n}(a, 1) = 1$. Then $\mathcal{K} = \mathbb{F}1 + \mathbb{F}a$ is a composition subalgebra of $\mathcal{C}$. Write $W = \mathcal{K}^{\perp}$, so that $\mathcal{C}_0 = \mathbb{F}1 \oplus W$. The kernel of Φ consists of those elements $\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n})$ such that $\varphi(1) = 1$ and $\varphi(w) = w$ for all $w \in W$. Then $\varphi(a)$ is orthogonal to $\varphi(W) = W$, and $\mathrm{n}\big(\varphi(a), 1\big) = \mathrm{n}\big(\varphi(a), \varphi(1)\big) = \mathrm{n}(a, 1) = 1$. Hence $\varphi(a) = a + \mu 1$ for some $\mu \in \mathbb{F}$, and since a and $\varphi(a)$ have the same norm, we must have $\mu + \mu^2 = 0$, so either μ is 0 or 1. In other words, the kernel of Φ is cyclic of order 2. Moreover, Φ is not surjective in general.

Actually, as $\{x \in \mathcal{C}_0 \mid \mathrm{n}(x, \mathcal{C}_0) = 0\} = \mathbb{F}1$, any element ϕ in $\mathrm{O}(\mathcal{C}_0, \mathrm{n})$ fixes 1 and takes any element $w \in W$ to an element of the form $\alpha(w)1 + \sigma(w)$, for a linear map $\alpha : W \to \mathbb{F}$, and an element σ in the symplectic group $\mathrm{Sp}(W, \mathrm{n})$ of W relative to the alternating bilinear form given by the polarization $\mathrm{n}(.,.)$. Moreover, σ and α are related by the condition $\mathrm{n}(w) - \mathrm{n}\big(\sigma(w)\big) = \alpha(w)^2$ for all $w \in W$, so α is determined by σ. The nondegeneracy of n gives a unique element $w_\sigma \in W$ such that $\alpha(w) = \mathrm{n}(w_\sigma, w)1$ for all $w \in W$.

If $\mathbb{F}$ is perfect, then given any $\sigma \in \mathrm{Sp}(W, \mathrm{n})$, the map $\gamma : w \to \mathrm{n}(w) - \mathrm{n}\big(\sigma(w)\big)$ is additive and 'semilinear': $\gamma(\mu w) = \mu^2 \gamma(w)$ for all $\mu \in \mathbb{F}$. Recall that by the definition of a perfect field, $\mu \to \mu^2$ is an automorphism (the Frobenius automorphism) of $\mathbb{F}$. It follows that, for a perfect $\mathbb{F}$, the map γ is always of the form $\alpha(w)^2$ for a linear form α and, therefore, the map $\phi \to \sigma$ gives a well-known group isomorphism $\mathrm{O}(\mathcal{C}_0, \mathrm{n}) \to \mathrm{Sp}(W, \mathrm{n})$. (All this is valid for odd-dimensional regular quadratic forms).

Moreover, for an arbitrary field $\mathbb{F}$ of characteristic 2, take $\phi \in \mathrm{O}(\mathcal{C}_0, \mathrm{n})$, and σ, w_σ as above: $\phi(w) = \sigma(w) + \mathrm{n}(w_\sigma, w)1$ and $\mathrm{n}(w) - \mathrm{n}\big(\sigma(w)\big) = \mathrm{n}(w_\sigma, w)^2$ for all $w \in W$. If ϕ is extended to an element of $\mathrm{O}(\mathcal{C}, \mathrm{n})$, then $\mathrm{n}\big(\phi(a), 1\big) = \mathrm{n}(a, 1) = 1$, so $\phi(a) = \mu 1 + a + w_a$ for a scalar $\mu \in \mathbb{F}$ and an element $w_a \in W$. From $\mathrm{n}\big(\phi(a)\big) = \mathrm{n}(a)$ we get $\mathrm{n}(w_a) = \mu^2 + \mu$, and from

$$\begin{aligned} 0 = \mathrm{n}(a, w) &= \mathrm{n}\big(\phi(a), \phi(w)\big) = \mathrm{n}\big(\mu 1 + a + w_a, \sigma(w) + \mathrm{n}(w_\sigma, w)1\big) \\ &= \mathrm{n}(w_\sigma, w) + \mathrm{n}\big(w_a, \sigma(w)\big) = \mathrm{n}\big(\sigma(w_\sigma), \sigma(w)\big) + \mathrm{n}\big(w_a, \sigma(w)\big) \\ &= \mathrm{n}\big(\sigma(w_\sigma) + w_a, \sigma(w)\big) \end{aligned}$$

for all $w \in W$, we get $w_a = \sigma(w_\sigma)$. Now we have

$$\mu^2 + \mu = \mathrm{n}(w_a) = \mathrm{n}\big(\sigma(w_\sigma)\big) = \mathrm{n}(w_\sigma) + \mathrm{n}(w_\sigma, w_\sigma)^2 = \mathrm{n}(w_\sigma).$$

Therefore, the obstruction for $\phi \in \mathrm{O}(\mathcal{C}_0, \mathrm{n})$ to extend to an element in $\mathrm{O}(\mathcal{C}, \mathrm{n})$ is that $\mathrm{n}(w_\sigma)$ must belong to $\{\mu + \mu^2 \mid \mu \in \mathbb{F}\}$. In particular, Φ is surjective if $\mathbb{F}$ is algebraically closed.

It should be noted that the situation for Lie algebras is easier: $\{d \in \mathfrak{so}(\mathcal{C}, \mathrm{n}) \mid d(1) = 0\}$ is always isomorphic to $\mathfrak{so}(\mathcal{C}_0, \mathrm{n})$ (see Remark **??**).

4.2. **2-Local automorphisms.**
In order to deal with 2-local automorphisms, we need a slight variation of the proof of [**?**, Lemma 3.2].

Lemma 4.2. *Let $\mathcal{C}$ be a Cayley algebra over a field $\mathbb{F}$ with norm n. Then any 2-local automorphism of $\mathcal{C}$ is linear.*

Proof. Consider a 2-local automorphism $\varphi \in \mathrm{2LocAut}(\mathcal{C})$. For any two elements $x, y \in \mathcal{C}$, there is an automorphism $\psi \in \mathrm{Aut}(\mathcal{C})$ such that $\varphi(x) = \psi(x)$, $\varphi(y) = \psi(y)$. As any automorphism preserves the norm, we get

$$\mathrm{n}\big(\varphi(x)\big) = \mathrm{n}(x) \quad \text{and} \quad \mathrm{n}\big(\varphi(x), \varphi(y)\big) = \mathrm{n}(x, y) \tag{4.1}$$

for any $x, y \in \mathcal{C}$.

Pick an arbitrary basis $\{x_i \mid 1 \leq i \leq 8\}$ of $\mathcal{C}$. The matrix $\Big(\mathrm{n}\big(\varphi(x_i), \varphi(x_j)\big)\Big) = \Big(\mathrm{n}(x_i, x_j)\Big)$ is nondegenerate, and hence $\{\varphi(x_i) \mid 1 \leq i \leq 8\}$ is another basis of $\mathcal{C}$. In particular, $\varphi(\mathcal{C})$ spans the whole $\mathcal{C}$.

For $x, y, z \in \mathcal{C}$, using (**??**) we obtain

$$\begin{aligned} \mathrm{n}\big(\varphi(x+y), \varphi(z)\big) &= \mathrm{n}(x+y, z) = \mathrm{n}(x, z) + \mathrm{n}(y, z) \\ &= \mathrm{n}\big(\varphi(x), \varphi(z)\big) + \mathrm{n}\big(\varphi(y), \varphi(z)\big) = \mathrm{n}\big(\varphi(x) + \varphi(y), \varphi(z)\big). \end{aligned}$$

Hence $\varphi(x+y) - \varphi(x) - \varphi(y)$ is orthogonal to all the elements in $\varphi(\mathcal{C})$, and this spans the whole $\mathcal{C}$. The nondegeneracy of n forces $\varphi(x+y) = \varphi(x) + \varphi(y)$.

On the other hand, for any $x \in \mathcal{C}$ and $\lambda \in \mathbb{F}$, there is an automorphism $\psi \in \mathrm{Aut}(\mathcal{C})$ such that $\varphi(x) = \psi(x)$ and $\varphi(\lambda x) = \psi(\lambda x)$, so that $\varphi(\lambda x) = \psi(\lambda x) = \lambda\psi(x) = \lambda\varphi(x)$, and we conclude that φ is linear. □

In particular, any 2-local automorphism of a Cayley algebra is a local automorphism, and hence it belongs to $\{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\}$.

The proof of the next lemma is straightforward. The similar result for local automorphisms is valid too.

Lemma 4.3. *Let φ be a 2-local automorphism and ψ an automorphism of a nonassociative algebra $\mathcal{A}$. Then $\psi\varphi$ and $\varphi\psi$ are 2-local automorphisms too.*

The situation in the split case is quite simple, and the proof uses arguments close to those for Theorem **??**.

Theorem 4.4. *Let $\mathcal{C}$ be the split Cayley algebra over a field $\mathbb{F}$ with norm n. Then any 2-local automorphism of $\mathcal{C}$ is an automorphism:* $\mathrm{2LocAut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{C})$.

Proof. Consider a canonical basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ as in Table **??**. Let φ be a 2-local automorphism of $\mathcal{C}$, and pick an automorphism ψ such that $\varphi(e_1) = \psi(e_1)$ and $\varphi(e_2) = \psi(e_2)$. Then the 2-local automorphism $\varphi' = \psi^{-1}\varphi$ fixes both e_1 and e_2.

The subspace $\mathcal{U} = \mathbb{F}u_1 + \mathbb{F}u_2 + \mathbb{F}u_3$ is the 'Peirce component' $\{u \in \mathcal{C} \mid e_1 u = u = u e_2\}$. For $u \in \mathcal{U}$, let ϕ be an automorphism such that $e_1 = \varphi'(e_1) = \phi(e_1)$ and $\varphi'(u) = \phi(u)$. Then $\phi(e_2) = \phi(1 - e_1) = 1 - e_1 = e_2$, and hence ϕ preserves the Peirce component $\mathcal{U}$. In particular $\varphi'(\mathcal{U}) = \mathcal{U}$. In the same vein, the Peirce component $\mathcal{V} = \mathbb{F}v_1 + \mathbb{F}v_2 + \mathbb{F}v_3 = \{v \in \mathcal{V} \mid e_2 v = v = v e_1\}$ is fixed too by φ'.

Since $\mathcal{U}$ and $\mathcal{V}$ are isotropic subspaces, $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ are dual bases relative to n, if $A = \big(\alpha_{ij}\big)_{1\leq i,j\leq 3}$ is the coordinate matrix of $\varphi'|_{\mathcal{U}}$ in the basis $\{u_1, u_2, u_3\}$, then the coordinate matrix of $\varphi'|_{\mathcal{V}}$ in the basis $\{v_1, v_2, v_3\}$ is $(A^t)^{-1}$. (A^t denotes the transpose of A).

If $\det(A) = 1$, then φ' is an automorphism and we are done.

Otherwise, if $\det(A) = \lambda \neq 1$, we can factor A as

$$A = \bar{A}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

for a matrix $\bar{A}$ of determinant 1. Consider the linear map ψ that fixes e_1, e_2 and the subspaces $\mathcal{U}$ and $\mathcal{V}$, and such that the coordinate matrix of $\psi|_{\mathcal{U}}$ in the above basis is $\bar{A}^{-1}$ and the coordinate matrix of $\psi|_{\mathcal{V}}$ is $\bar{A}^t$. Then ψ is an automorphism. Therefore the 2-local automorphism $\varphi'' = \psi\varphi'$ fixes the elements $e_1, e_2, u_1, u_2, v_1, v_2$ and sends $u_3 \to \lambda u_3$, $v_3 \to \lambda^{-1} v_3$. Pick now the elements $x = v_1 - u_1$ and $y = u_3 + v_2$. Then $xy = y$ and x is fixed by φ''. There is an automorphism $\tau \in \mathrm{Aut}(\mathcal{C})$ such that $\tau(x) = \varphi''(x) = x$ and $\tau(y) = \varphi''(y) = \lambda u_3 + v_2$. But a simple computation gives

$$\lambda u_3 + v_2 = \tau(y) = \tau(xy) = \tau(x)\tau(y) = x\tau(y) = (v_1 - u_1)(\lambda u_3 + v_2) = u_3 + \lambda v_2,$$

and this is a contradiction as we have $\lambda \neq 1$. □

We turn our attention now to the case of division Cayley algebras. Here, as was the case for 2-local derivations, the situation is different.

Lemma 4.5. *Let $\mathcal{C}$ be a division Cayley algebra over a field $\mathbb{F}$ with norm* n. *Let $\mathcal{K}$ be a two-dimensional composition subalgebra of $\mathcal{C}$ (that is, $\mathrm{n}|_{\mathcal{K}}$ is regular), let x be an arbitrary element of $\mathcal{C}$, and let φ be a local automorphism of $\mathcal{C}$. Then there is an automorphism ψ of $\mathcal{C}$ such that $\psi|_{\mathcal{K}} = \varphi|_{\mathcal{K}}$ and $\psi(x) = \varphi(x)$.*

Proof. Recall that Theorem **??** gives $\mathrm{LocAut}(\mathcal{C}) = \{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\}$.

Pick an element $a \in \mathcal{K} \setminus \mathbb{F}1$. There is an automorphism $\phi \in \mathrm{Aut}(\mathcal{C})$ such that $\phi(a) = \varphi(a)$, and hence the local automorphism $\varphi' = \phi^{-1}\varphi$ fixes $\mathcal{K}$ elementwise.

Write $x = b+y$ with $b \in \mathcal{K}$ and $y \in \mathcal{K}^\perp$. If $y = 0$, we are done as φ and ϕ coincide on $\mathcal{K}$ and hence also on x. Otherwise, as $\mathcal{C}$ is a division algebra $\mathrm{n}(y) \neq 0$. Since φ' is an orthogonal transformation, $\varphi'(y)$ lies too in $\mathcal{K}^\perp$, and it has the same norm as y. As in the proof of [**?**, Theorem 1.7.1], the Cayley-Dickson doubling process, together with Witt's Cancellation Theorem, shows that there is an automorphism $\tau \in \mathrm{Aut}(\mathcal{C})$ which is the identity on $\mathcal{K}$ and such that $\tau(y) = \varphi'(y)$. Then $\tau(x) = \tau(b + y) = b + \tau(y) = b + \varphi'(y) = \varphi'(x) = \phi^{-1}\varphi(x)$. The automorphism $\psi = \phi\tau$ coincides with φ both on $\mathcal{K}$ and on x. □

This result settles the situation easily if the characteristic of the ground field is not 2. Over fields of characteristic 2, the problem is more difficult. Note that the only Cayley (or quaternion) algebra over a perfect field of characteristic 2 is the split one, so we are forced to deal with fields $\mathbb{F}$ of characteristic 2 with $\mathbb{F} \neq \mathbb{F}^2$.

Theorem 4.6. *Let $\mathcal{C}$ be a division Cayley algebra over a field $\mathbb{F}$. Then any local automorphism is a 2-local automorphism:* $\mathrm{LocAut}(\mathcal{C}) = 2\mathrm{LocAut}(\mathcal{C})$.

Proof. Let φ be a local automorphism of $\mathcal{C}$, and let $x, y \in \mathcal{C}$. If $x \in \mathbb{F}1$, then any automorphism ψ such that $\varphi(y) = \psi(y)$ satisfies $\varphi(x) = \psi(x)$ too. Therefore we assume from now on that x, y are not in $\mathbb{F}1$.

If the characteristic of $\mathbb{F}$ is not 2, $\mathcal{K} = \mathbb{F}1 + \mathbb{F}x$ is a composition subalgebra of $\mathcal{C}$, because n is anisotropic. Lemma **??** shows that there is an automorphism $\psi \in \mathrm{Aut}(\mathcal{C})$ such that φ and ψ coincide on both x and y. The same argument works if the characteristic of $\mathbb{F}$ is 2 and $\mathrm{n}(x,1)\neq 0$ (or $\mathrm{n}(y,1)\neq 0$), as this forces $\mathcal{K}=\mathbb{F}1+\mathbb{F}x$ to be a composition subalgebra.

Hence assume that $\operatorname{char}\mathbb{F}=2$ and $\mathrm{n}(x,1)=0=\mathrm{n}(y,1)$. In particular $\bar{x}=-x=x$ and $\bar{y}=-y=y$. We may also assume that 1, x and y are linearly independent. We are left with two different cases, depending on $\mathrm{n}(x,y)$ being 0 or not. Write $\alpha=\mathrm{n}(x)$, $\beta=\mathrm{n}(y)$, and $\gamma=\mathrm{n}(x,y)$, and note that $\alpha\neq 0\neq\beta$, since n is anisotropic, and $\mathrm{n}(xy)=\alpha\beta$.

If $\mathrm{n}(x,y)\neq 0$, then $\mathrm{n}(xy,1)=\mathrm{n}(x,\bar{y})=\mathrm{n}(x,y)\neq 0$, so that $\mathcal{K}=\mathbb{F}1+\mathbb{F}xy$ is a composition subalgebra of $\mathcal{C}$. The multiplicative property of n gives $\mathrm{n}(x,xy)=\mathrm{n}(x)\mathrm{n}(1,y)=0$, so x is orthogonal to $\mathcal{K}$, and hence $\mathcal{Q}=\mathcal{K}\oplus\mathcal{K}x$ is a quaternion subalgebra of $\mathcal{C}$. Besides, using that $xy+yx=x\bar{y}+y\bar{x}=\mathrm{n}(x,y)1$, we obtain

$$
\begin{aligned}
x(xy)&=x^2y=\mathrm{n}(x)y=\alpha y\\
(xy)x&=\bigl(\mathrm{n}(x,y)1-yx\bigr)x=\gamma x-yx^2=\gamma x+\alpha y,
\end{aligned}
$$

so that $\{1,xy,x,y\}$ is a basis of $\mathcal{Q}$, and the multiplication on $\mathcal{Q}$ is completely determined in the following Table **??**.

TABLE 2.

	1	xy	x	y
1	1	xy	x	y
xy	xy	$\gamma xy+\alpha\beta 1$	$\gamma x+\alpha y$	βx
x	x	αy	$\alpha 1$	xy
y	y	$\beta x+\gamma y$	$\gamma 1+xy$	$\beta 1$

Since φ is an orthogonal transformation that fixes 1, the elements $x'=\varphi(x)$ and $y'=\varphi(y)$ also satisfy the conditions $\mathrm{n}(x',1)=0=\mathrm{n}(y',1)$, $\mathrm{n}(x')=\alpha$, $\mathrm{n}(y')=\beta$, $\mathrm{n}(x',y')=\gamma$, and hence the quaternion subalgebra $\mathcal{Q}'=\mathbb{F}1+\mathbb{F}x'y'+\mathbb{F}x'+\mathbb{F}y'$ has the same multiplication table as in Table **??**. Therefore there is an isomorphism $\psi:\mathcal{Q}\to\mathcal{Q}'$ with $\psi(x)=x'=\varphi(x)$ and $\psi(y)=y'=\varphi(y)$. Besides, using the Cayley-Dickson doubling process (as in [**?**, Corollary 1.7.3]), ψ extends to an automorphism of $\mathcal{C}$, and we are done in this case.

Finally, assume $\mathrm{n}(x,y)=0$. In this case we have $xy=yx$ and $\mathrm{n}(xy,1)=\mathrm{n}(x,y)=0$, $\mathrm{n}(xy,x)=\mathrm{n}(x)\mathrm{n}(y,1)=0$, and $\mathrm{n}(xy,y)=\mathrm{n}(x,1)\mathrm{n}(y)=0$, so the subspace $\mathcal{S}=\mathbb{F}1+\mathbb{F}x+\mathbb{F}y+\mathbb{F}xy$ satisfies $\mathrm{n}(\mathcal{S},\mathcal{S})=0$, that is, $\mathcal{S}$ is totally isotropic for the polar bilinear form. The dimension of $\mathcal{S}$ is 4, because otherwise we would have $xy=\delta 1+\mu x+\nu y$ for some scalars $\delta,\mu,\nu\in\mathbb{F}$, and this would give $y=(x-\nu 1)^{-1}(\delta 1+\mu x)\in\mathbb{F}1+\mathbb{F}x$, a contradiction with $1,x,y$ being linearly independent. Since φ is an orthogonal transformation that fixes 1, the same conditions hold for $x'=\varphi(x)$ and $y'=\varphi(y)$. So that $\mathcal{S}'=\mathbb{F}1+\mathbb{F}x'+\mathbb{F}y'+\mathbb{F}x'y'$ is again totally isotropic for the polar form. Since $\mathrm{n}(x)=\mathrm{n}(x')$, $\mathrm{n}(y)=\mathrm{n}(y')$, and $\mathrm{n}(xy)=\mathrm{n}(x)\mathrm{n}(y)=\mathrm{n}(x')\mathrm{n}(y')=\mathrm{n}(x'y')$, the subspaces $\mathcal{S}$ and $\mathcal{S}'$ are isometric, relative to the quadratic form n, with an isometry sending $1\to 1$, $x\to x'$, $y\to y'$, and $xy\to x'y'$. Witt's Extension Theorem (see, for instance, [**?**, Theorem 8.3]) shows that there is an orthogonal transformation $\sigma\in\mathrm{O}(\mathcal{C},\mathrm{n})$ such that $\sigma(1)=1$, $\sigma(x)=x'=\varphi(x)$, $\sigma(y)=y'=\varphi(y)$, and $\sigma(xy)=x'y'=\varphi(x)\varphi(y)$. Pick an element u such that $\mathrm{n}(u,x)=\mathrm{n}(u,y)=\mathrm{n}(u,xy)=0$ and $\mathrm{n}(u,1)=1$. This is always possible because $1,x,y,xy$ are linearly independent and n is nondegenerate. Write $u'=\sigma(u)$ and consider the composition subalgebras $\mathcal{K}=\mathbb{F}1+\mathbb{F}u$ and $\mathcal{K}'=\mathbb{F}1+\mathbb{F}u'$. The restriction of σ to $\mathcal{K}$ gives an algebra isomorphism $\psi:\mathcal{K}\to\mathcal{K}'$. Note that x is orthogonal to $\mathcal{K}$, so that $\mathcal{Q}=\mathcal{K}\oplus\mathcal{K}x$ is a quaternion subalgebra and, in the same vein, $\mathcal{Q}'=\mathcal{K}'\oplus\mathcal{K}'x'$ is another quaternion subalgebra. The Cayley-Dickson doubling process shows that ψ can be extended to an isomorphism $\mathcal{Q}\to\mathcal{Q}'$, also denoted by ψ, that satisfies $\psi(x) = x' = \varphi(x)$. Finally, y is orthogonal to $\mathcal{K}$ and $\mathrm{n}(\mathcal{K}x, y) = \mathrm{n}(\mathcal{K}, y\bar{x}) = \mathrm{n}(\mathcal{K}, xy) = 0$, so that y is orthogonal to $\mathcal{Q}$, and hence we get $\mathcal{C} = \mathcal{Q} \oplus \mathcal{Q}y$. In a similar way $\mathcal{C} = \mathcal{Q}' \oplus \mathcal{Q}'y'$, and the isomorphism $\psi : \mathcal{Q} \to \mathcal{Q}'$ can be extended to an automorphism of $\mathcal{C}$ with $\psi(y) = y' = \varphi(y)$. □

4.3. **Local and 2-local automorphisms on 7-dimensional simple Malcev algebras.**

As for the seven dimensional central simple Malcev algebras, the situation is analogous to Theorem **??**, thanks to Theorems **??**, **??**, and **??**.

Theorem 4.7. *Let $\mathcal{C}_0^-$ be the central simple Malcev algebra associated with a Cayley algebra $\mathcal{C}$ over a field $\mathbb{F}$ of characteristic $\neq 2$. Then*

(1) *the set of all local automorphisms of $\mathcal{C}_0^-$ is the orthogonal group $\mathrm{O}(\mathcal{C}_0, \mathrm{n})$,*
(2) *if $\mathcal{C}$ is a split algebra, then every 2-local automorphism on $\mathcal{C}_0^-$ is an automorphism;*
(3) *if $\mathcal{C}$ is a division algebra, then*

$$2\mathrm{LocAut}(\mathcal{C}_0^-) = \mathrm{LocAut}(\mathcal{C}_0^-) = \mathrm{O}(\mathcal{C}_0, \mathrm{n}).$$

For the proof, it suffices to note that any automorphism φ of $\mathcal{C}_0^-$ extends to an automorphism of $\mathcal{C}$ by imposing $\varphi(1) = 1$, and the same happens for local and 2-local automorphisms.

5. Open problems

In this section we will restrict ourselves to the real or complex field, but similar arguments can be discussed over arbitrary fields, if one replaces Lie group by affine algebraic group.

Let $\mathcal{A}$ be a finite dimensional (not necessary associative) algebra over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and denote by $\mathrm{Aut}(\mathcal{A})$ the group of its automorphisms. Let G be a Lie group and let $\mathrm{Lie}(G)$ be the its tangent Lie algebra (see for details [**?**, **?**]).

It is well-known that $\mathrm{Lie}(\mathrm{Aut}(\mathcal{A})) \cong \mathrm{Der}(\mathcal{A})$ (see [**?**, Page 316]).

Note that in the case $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, by Theorem **??**, it follows that $\mathrm{LocAut}(\mathcal{C}) = \{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\}$ is a Lie group, because $\{\varphi \in \mathrm{O}(\mathcal{C}, \mathrm{n}) \mid \varphi(1) = 1\}$ is a closed subgroup in the Lie group $\mathrm{O}(\mathcal{C}, \mathrm{n})$. By Theorem **??** the space of all local derivations of a Cayley algebra $\mathcal{C}$ is the Lie algebra $\{d \in \mathfrak{so}(\mathcal{C}, \mathrm{n}) \mid d(1) = 0\}$. Thus we obtain that $\mathrm{Lie}(\mathrm{LocAut}(\mathcal{C})) \cong \mathrm{LocDer}(\mathcal{C})$. These observations lead us formulate the following problems:

Problems 5.1.

(1) Is $\mathrm{LocAut}\,(\mathcal{A})$ a Lie group?
(2) Is $(\mathrm{LocDer}\,(\mathcal{A})\,, [\cdot, \cdot])$ a Lie algebra?
(3) If the above two assertions are true, are the Lie algebras $\mathrm{Lie}\,(\mathrm{LocAut}(\mathcal{A}))$ and $\mathrm{LocDer}\,(\mathcal{A})$ isomorphic?

It is well known that the space of all derivations $\mathrm{Der}(\mathcal{A})$ is a Lie algebra with respect to the Lie bracket. At the same time, it is not clear whether the space of all local derivations $\mathrm{LocDer}(\mathcal{A})$ forms a Lie algebra.

As we have already noted for Cayley algebras, Problems **??** have a positive solution. Below we list some other classes of algebras for which Problems **??** also have a positive solution:

- finite-dimensional complex simple Lie algebras [**?**, Theorem 3.1], [**?**, Theorem 3.14]. In fact, since $\mathrm{LocAut}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g}) \ltimes \{\mathrm{id}_{\mathfrak{g}}, -\mathrm{id}_{\mathfrak{g}}\}$ for a simple Lie algebra $\mathfrak{g}$, we have that

$$\mathrm{Lie}\,(\mathrm{LocAut}(\mathfrak{g})) = \mathrm{Lie}\,(\mathrm{Aut}(\mathfrak{g})) \cong \mathrm{Der}\,(\mathfrak{g}) = \mathrm{LocDer}\,(\mathfrak{g})\,.$$

- complex Leibniz algebras of the form $\mathfrak{g} = \mathfrak{sl}_\mathfrak{n} \ltimes \mathcal{I}$, where $\mathfrak{sl}_\mathfrak{n}$ is the special linear Lie algebra and $\mathcal{J}$ is the subspace generated by squares of $\mathfrak{g}$ (see [**?**, Theorem 20], [**?**, Theorem 4]).
- the algebra $NT(3, \mathbb{F})$ of lower niltriangular matrices of order 3 over $\mathbb{F}$ (see [**?**, Theorem 2]).

In the last two cases, as in the first one, the isomorphism $\mathrm{Lie}\,(\mathrm{LocAut}(\mathcal{A})) \cong \mathrm{LocDer}\,(\mathcal{A})$ follows directly from the descriptions of $\mathrm{LocAut}(\mathcal{A})$ and $\mathrm{LocDer}\,(\mathcal{A})$.

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