Log concavity preservation by beta operator based on probability tools

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Abstract

A number of papers have dealt with the preservation of log convexity and log concavity based on various operators. For instance, in Badía and Sangüesa [10], the preservation of log convexity and log concavity under Bernstein operators was discussed based on some characteristics of a stochastic process. However, regarding beta-type operators, the preservation of log concavity does not hold based on such a probabilistic method used in Badía and Sangüesa [10]. In this study, we explore the preservation of log concavity for the beta operator using alternative probabilistic tools. Notably, we show results on the preservation of log concavity for monotone log concave functions. Further, some results of application to some specific functions, ageing classes of the deterioration Dirichlet process, related operators and order statistics are provided.

1 Introduction

Beta-type operators are probabilistic approximation operators based on beta or inverted beta distributions. Many studies on beta-type operators and their

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properties have been performed in the literature (see Adell et al. [4, 5, 6], Deo et al. [16], Khan [25], Lupaş[30], and Upreti [41]).

The beta-type operator, which is denoted by $\beta_t f,\ t > 0$, is defined as follows:

$$\beta_t(f,x) = \int_0^1 f(\theta)\frac{1}{B(tx,t(1-x))}\theta^{tx-1}(1-\theta)^{t(1-x)-1}d\theta, \quad x \in (0,1),$$

where $B(\,.,\,.)$ is the beta function and f is a measurable function defined on the interval $(0,1)$ such that $\beta_t(\mid f \mid, x) < \infty$ being $\mid f \mid$ the absolute value of f. If f is defined on 0 or 1, we define $\beta_t(f,i) = f(i)$, $i = 0, 1$. It is well-known that this operator can be written in terms of a random variable which has the beta distribution as follows:

$$\beta_t(f,x) = E\left[f\left(\Omega(tx,t(1-x))\right)\right],$$

where $\Omega(p,q)$ is a random variable that has the beta distribution with parameters p and q, $p,q > 0$ and its probability density function (pdf) denoted by $f_{\Omega(p,q)}$ is given by

$$f_{\Omega(p,q)}(x) = \frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}, \quad x \in (0,1).$$

Based on the inverted beta distribution, other beta-type operator called the inverted beta operator $T_t f$ is defined as

$$T_t(f,x) = \int_0^\infty f(\theta)\frac{1}{B(tx,t)}\frac{\theta^{tx-1}}{(1+\theta)^{tx+t}}d\theta = E[f(\Upsilon(tx,t))], t > 0,$$

where f is any measurable function defined on the interval $(0,\infty)$ such that $T_t(\mid f \mid, x) < \infty$ and $\Upsilon(p,q)$ is a random variable having the inverted beta distribution with parameters p and q, $p,q > 0$. The pdf of $\Upsilon(p,q)$ is given by

$$f_{\Upsilon(p,q)}(x) = \frac{1}{B(p,q)}\frac{x^{p-1}}{(1+x)^{p+q}}, \quad x > 0.$$

Recalling that

$$\frac{\Omega(p,q)}{1-\Omega(p,q)}$$

has the same probability law with $\Upsilon(p,q)$ yields the following relationship between two operators:

$$T_t(f,x) = \beta_{t(1+x)}\left(g, \frac{x}{1+x}\right), \quad t,x > 0,$$

where

$$g(z) = f\left(\frac{z}{1-z}\right), \quad 0 < z < 1.$$

It is also known that beta and inverted beta distributions can be represented in terms of two independent gamma random variables (see Kotz et al. [27]). Let $Z(p)$ and $Z(q)$ be independent gamma random variables with parameters $p > 0$ and $q > 0$, respectively. Then the beta random variable $\Omega(p, q)$ and the inverted beta random variable $\Upsilon(p, q)$ with the parameters p and q can be expressed as follows:

$$\Omega(p, q) \stackrel{d}{=} \frac{Z(p)}{Z(p) + Z(q)}, \tag{1}$$

and

$$\Upsilon(p, q) \stackrel{d}{=} \frac{Z(p)}{Z(q)}, \tag{2}$$

where $\stackrel{d}{=}$ means that both random variables have the same distribution. Recall that the pdf of $Z(p)$ is given by

$$f_{Z(p)}(x) = \frac{1}{\Gamma(p)} x^{p-1} e^{-x}, \quad x > 0,$$

where $\Gamma(\,.\,)$ denotes the standard gamma function.

Until now, there have been many studies on the preservation of convexity and concavity under various operators including beta-type operators in the literature. Adell et al. [4] presented some shape-preserving properties of convexity and monotony for beta-type operators based on a probabilistic representation. The preservation of log convexity and log concavity for Bernstein operators was addressed by Badía [9] and Badía and Sangüesa [10] using the characteristics of stochastic processes. The preservation of log convexity was induced based on a stochastic process with nonnegative, independent and stationary increments whereas an additional condition that the process increases in the likelihood ratio stochastic order was required for that of log concavity. Recently, Xia et al. [42] discussed the preservation of log convexity and concavity using alternative probabilistic tools. In addition, Xia et al. [42], presented results for inverted beta operator and compared them with those in Badía [9] and Badía and Sangüesa [10]; it was shown that such an operator preserves log convexity whereas the property does not hold in general for log concavity.

Goodman [22] showed the preservation for classical Bernstein operator by using analysis tools. Recently, Bienek et al. [12] presented the results related to those derived in Goodman [22] using signature tools in coherent systems. Komisarski [26] provided a concise proof for results on the Bernstein operator.

Now, it should be focused that the probabilistic representations for beta-type operators are associated with stochastic processes with nonnegative and stationary increments except independent increments indicating that the probabilistic method used in Badía and Sangüesa [10] does not allow us to induce the preservation of log concavity under beta-type operators. In this study, we intend to discern results related to the preservation of log concavity for beta-type operators by employing alternative probabilistic methods for those used in Badía and Sangüesa [10]. Specifically, for monotone log concave functions, we show the preservation of log concavity based on the bivariate characterization of the likelihood ratio stochastic order for a probabilistic representation of beta random variables, which is based on suitable bivariate Dirichlet distribution.

The framework of this paper is as follows. In Section 2, some basic concepts and auxiliary results to obtain the main result of this paper are provided including properties of log concave functions, bivariate characterization of likelihood ratio stochastic order and Dirichlet bivariate distribution. In Section 3, we derive the preservation of log concavity for monotone log concave functions by the beta-type operator (see Theorem 1). In addition, it is shown that the key tool for the preservation of log concavity by the beta-type operator does not hold for general nonmonotone log concave functions. In Section 4, some applications of the main result on the preservation of log concavity developed in this study are presented. Finally, the concluding remarks are provided.

2 Preliminary

We now recall the definition of a log concave non negative function as follows: for a non negative function f defined on a convex set I, f is log concave if

$$f(\alpha x + (1-\alpha)y) \geq f(x)^{\alpha} f(y)^{1-\alpha}, \quad x, y \in I, \quad 0 \leq \alpha \leq 1, \tag{3}$$

or equivalently $\ln f$ is concave where $\ln 0 = -\infty$. Log concave functions are crucial in many areas such as reliability, economics, statistics and optimization, and so forth (see e.g., Cowan [15], Fang and Norman [20], Saumard and Wellner [35], and Sengupta ana Nanda [36]). For information on mathematical properties of log concave functions, one can refer to Boyd and Vandenberghe [13, pp 104-108], Dharmadhikari and Joag-Dev [18], and Marshall et al. [31]. Notably, we can find that if a univariate pdf is log concave, it is unimodal; the corresponding hazard rate and reversed hazard rate are increasing and decreasing, respectively; and the left-truncated mean and variance have some nice properties (see An [8] and Bagnoli and Bergstrom [11] and references therein).

Moreover, it is well known that for a measurable function f defined on I, to show that f is log concave, it is sufficient to prove that the inequality in (3) holds when $\alpha = \frac{1}{2}$ (see Sierpiński [39]), that is,

$$f \quad \text{is log concave} \iff f^2\left(\frac{x+y}{2}\right) \geq f(x)f(y), \quad x, y \in I.$$

Accordingly, we can say that for a measurable function f on $[0,1]$ such that $\beta_t(f,x) < \infty$, $\beta_t f$ is log concave if

$$\beta_t^2(f, \frac{x+y}{2}) \geq \beta_t(f,x)\beta_t(f,y), \quad x, y \in [0,1], t > 0. \tag{4}$$

Next we provide some well-known elementary properties of a log concave function in the following lemma which play a crucial role in deriving the main result.

Lemma 1 *Let f be a log concave function on an interval $I \subseteq [0,\infty)$ such that $f(x) > 0$, for all $x \in I$. Then the following properties hold:*

(a) If $f(x)$ is decreasing in x, $\frac{f(ax)}{f(x)}$ is an increasing function in x for $0 < a < 1$;

(b) $\frac{f(x+y)}{f(x+z)}$ is increasing in x for $y \leq z$, $x+y, x+z \in I$.

Remark 1 *From Lemma 1, it follows that*

(i) If f is a strictly positive log concave function in (0,1), then Lemma 1 (b) leads to

$$r(u,a) = \frac{f(u+(1-u)a)}{f((1-u)a)} \geq \frac{f(u+(1-u)b)}{f((1-u)b)} = r(u,b), \quad 0 < u < 1, \quad 0 < a \leq b < 1,$$

that is, $r(u,x)$ is decreasing in x;

(ii) From Lemma 1 (a)and f been a decreasing function, we have that

$$\frac{f((1-v)a)}{f((1-u)a)} \leq \frac{f((1-v)b)}{f((1-u)b)}, \quad 0 < u \leq v < 1, \quad 0 < a \leq b < 1,$$

(iii) From Lemma 1 (b),

$$\frac{f(u+(1-u)a)}{f(u+(1-u)b)} \geq \frac{f((1-u)a)}{f((1-u)b)}, \quad 0 < u < 1, \quad 0 < a \leq b < 1.$$

Now we introduce the bivariate definition of the likelihood ratio (lr) order which is useful for the proof of the main result in this study. Recall that for two random variables X and Y with pdfs f_X and f_Y, respectively, X is said to be smaller than Y in the lr order, denoted by $X \leq_{\mathrm{lr}} Y$, iff

$$f_X(x)f_Y(y) \geq f_X(y)f_Y(x), \quad x \leq y.$$

More information on the definitions and properties of stochastic orders can be found in the classical book of Shaked and Shanthikumar [37]. The bivariate characterization of the lr stochastic order is said that for independent random variables X and Y such that $X \leq_{\mathrm{lr}} Y$, if R is a bivariate function satisfying that $L(x,y) = R(x,y) - R(y,x) \geq 0$ for $x \leq y$, then $E[R(X,Y)] \geq E[R(Y,X)]$ (see [38]). In addition, we note that random variables having the gamma and beta distributions are ordered in lr stochastic order under some assumptions on their parameters. Specifically for gamma random variables $Z(p)$ and $Z(q)$ with parameters p and q, respectively, if $p \leq q$, then $Z(p) \leq_{\mathrm{lr}} Z(q)$ and for beta random variables $\Omega(p_i, q_i)$, $i = 1,2$ with parameter sets (p_i, q_i), $i = 1,2$, respectively, if $p_1 \leq p_2$ and $q_1 \geq q_2$, then $\Omega(p_1, q_1) \leq_{\mathrm{lr}} \Omega(p_2, q_2)$. Hereafter, it is assumed that different gamma or beta random variables considered in an expectation are independent.

Now we introduce a definition and related auxiliary result which will be used to prove the the main result on the preservation of log concavity.

Definition 1 *A bivariate random vector is said to follow a bivariate Dirichlet distribution with parameters (c_1, c_2, c_3) with $c_i > 0$, $i = 1,2,3$, iff the bivariate pdf is given by*

$$f_{(X,Y)}(x,y) = \frac{\Gamma(c_1+c_2+c_3)}{\Gamma(c_1)\Gamma(c_2)\Gamma(c_3)} x^{c_1-1} y^{c_2-1} (1-x-y)^{c_3-1}, \quad x, y \geq 0, \quad x+y \leq 1.$$

Lemma 2 *Let (X, Y) be a random bivariate vector that follows the bivariate Dirichlet distribution with parameters (c_1, c_2, c_3) with $c_i > 0$, $i = 1, 2, 3$ and $\Omega(p, q)$ be a random variable that follows the beta distribution with parameters (p, q), then the following properties hold;*

(a) X *and* Y *have the same probability laws as* $\Omega(c_1, c_2 + c_3)$ *and* $\Omega(c_2, c_1 + c_3)$*, respectively;*

(b) $Y|X = u$ *and* $X|Y = u$ *have the same probability laws as* $(1-u)\Omega(c_2, c_3)$ *and* $(1 - u)\Omega(c_1, c_3)$*, respectively;*

(c) $X + Y$ *has the same probability law as* $\Omega(c_1 + c_2, c_3)$.

For details of the properties of the bivariate Dirichlet distribution presented in Lemma 2, we can refer to Kotz et al. [28]. Next, we provide a result on the preservation of log concavity under the beta operator as follows.

3 Log concavity for monotone function preservation by the beta operator

In this section, we study the preservation of log concavity for monotone functions by beta operator. The claim is in the following theorem.

Theorem 1 *If a measurable function f is log concave and monotone on $[0, 1]$ with $\beta_t(f, x) < \infty$, $x \in (0, 1)$, then $\beta_t f$ is log concave.*

Proof: To prove that $\beta_t f$ is log concave, we need to show that the inequality in (4) holds. To this end, we first consider, without loss of generality the case $0 \le x \le y \le 1$ (the roles of x and y can be interchanged otherwise). Let (X_1, Y_1) and (X_2, Y_2) be two random bivariate vectors having the bivariate Dirichlet distribution with parameters (p_0, p_1, q_1) and (p_2, p_1, q_2), respectively, where $p_0 = tx, p_1 = t\frac{(y-x)}{2}, q_1 = t\left(1 - \frac{(y+x)}{2}\right)$, $p_2 = t\frac{y+x}{2}$ and $q_2 = t(1-y)$. Let us note that $p_0+p_1 = p_2 = t-q_1$ and $(p_1+p_2) = t-q_2 = ty$. Then by Lemma 2 (a),(b), and (c), we have that

$$\begin{aligned}
\beta_t(f, \frac{x+y}{2}) &= E[f(\Omega(t\frac{x+y}{2}, t(1 - \frac{x+y}{2})))] \\
&= E[f(\Omega_{X_1+Y_1}(p_0 + p_1, q_1))] = E[f(X_1 + Y_1)] \\
&= E[f(\Omega(p_1, q_0)) + (1 - \Omega(p_1, q_0))\Omega(p_0, q_1))],
\end{aligned}$$

where $q_0 = t - p_1 = p_0 + q_1 = p_2 + q_2$ and

$$\begin{aligned}\beta_t(f,x) = E[f(\Omega(tx, t(1-x)))] &= E[f(\Omega_{X_1}(p_0, p_1+q_1))] = E[f(X_1)] \\ &= E[f((1-\Omega(p_1,q_0))\Omega(p_0,q_1))].\end{aligned}$$

Likewise, we obtain

$$\begin{aligned}\beta_t(f,y) = E[f(\Omega(ty, t(1-y)))] &= E[f(\Omega_{X_2+Y_2}(p_2+p_1, q_2))] = E[f(X_2+Y_2)] \\ &= E[f(\Omega(p_1,q_0) + (1-\Omega(p_1,q_0))\Omega(p_2,q_2))]\end{aligned}$$

and

$$\beta_t(f, \frac{x+y}{2}) = E[f(\Omega_{X_2}(t\frac{x+y}{2}, t(1-\frac{x+y}{2})))] = E[f(X_2)] = E[f((1-\Omega(p_1,q_0))\Omega(p_2,q_2))].$$

The expression $\beta_t^2(f, \frac{x+y}{2}) - \beta_t(f,x)\beta_t(f,y)$ can be represented by

$$\begin{aligned} & E[f(\Omega(p_1,q_0) + (1-\Omega(p_1,q_0))\Omega(p_0,q_1))]E[f((1-\Omega(p_1,q_0))\Omega(p_2,q_2))] \\ - \; & E[f(\Omega(p_1,q_0) + (1-\Omega(p_1,q_0))\Omega(p_2,q_2))]E[f((1-\Omega(p_1,q_0))\Omega(p_0,q_1))] \\ = \; & \int_0^1\int_0^1\int_0^1\int_0^1 (f((1-u)b)f(v+(1-v)a) - f((1-v)a)f(u+(1-u)b)) \\ \times \; & f_{\Omega(p_1,q_0)}(u) f_{\Omega(p_1,q_0)}(v) f_{\Omega(p_2,q_2)}(b) f_{\Omega(p_0,q_1)}(a)\,dudvdadb. \end{aligned} \tag{5}$$

Then, the condition in (4) follows provided that (5) is positive which is proven next. Consider the following function

$$R(a,b,u,v) = f((1-v)b)f(u+(1-u)a) + f((1-u)b)f(v+(1-v)a).$$

Divide the integral in u into the parts $[0,v]$ and $[v,1]$, then apply Fubini‘s Theorem to the second term and interchange the roles of u and v afterwards. Thus, Equation (5) can be expressed as follows

$$\begin{aligned} & \int_0^1\int_0^v (E[R(\Omega(p_0,q_1), \Omega(p_2,q_2), u, v)] - E[R(\Omega(p_2,q_2), \Omega(p_0,q_1), u, v)]) \\ \times \quad & f_{\Omega(p_1,q_0)}(u) f_{\Omega(p_1,q_0)}(v)\,dudv, \end{aligned} \tag{6}$$

for $0 \le u \le v \le 1$, $a, b \in [0,1]$, where $\Omega(p_0,q_1)$ and $\Omega(p_2,q_2)$ are independent. Observe that $p_0 \le p_2$ and $q_1 \ge q_2$, which yields that $\Omega(p_0,q_1) \le_{\text{lr}} \Omega(p_2,q_2)$. Therefore, considering the bivariate characterization of the lr stochastic order, to complete the proof we have to show that

$$L_f(a,b,u,v) = R(a,b,u,v) - R(b,a,u,v) \ge 0, \quad 0 < u \le v < 1, \quad 0 < a \le b < 1, \tag{7}$$

leading to

$$E[R(\Omega(p_2,q_2),\Omega(p_0,q_1),u,v)] \leq E[R(\Omega(p_0,q_1),\Omega(p_2,q_2),u,v)],$$

and, thus, Equation (5) is proven to be positive.
■

3.1 Proof of claim (7)

In this subsection, we show that claim (7) holds when a log concave function f is monotone. To this end, we assume that f is decreasing such that $f(v+(1-v)b) > 0$, for $0 < u \leq v < 1$, $0 < a \leq b < 1$. Obviously, it follows that $f(r) \geq f(v+(1-v)b) > 0$ when $r = (1-u)b, (1-v)b$, and $u+(1-u)b$ since f is nonincreasing. Note that

$$\begin{aligned}
& L_f(a,b,u,v) = R(a,b,u,v) - R(b,a,u,v) \\
= \;& f((1-v)b)f(u+(1-u)b)\left(\frac{f(u+(1-u)a)}{f(u+(1-u)b)} - \frac{f((1-v)a)}{f((1-v)b)}\right) \\
+ \;& f((1-u)b)f(v+(1-v)b)\left(\frac{f(v+(1-v)a)}{f(v+(1-v)b)} - \frac{f((1-u)a)}{f((1-u)b)}\right) \\
\geq \;& [f((1-v)b)f(u+(1-u)b) - f((1-u)b)f(v+(1-v)b)] \\
\times \;& \left(\frac{f((1-u)a)}{f((1-u)b)} - \frac{f((1-v)a)}{f((1-v)b)}\right) \\
\geq \;& 0
\end{aligned}$$

where the first inequality follows by applying Lemma 1 (b) (Remark 1 (iii)) and the second one holds from the facts that $(1-v)b < (1-u)b$, $u+(1-u)b < v+(1-v)b$, and f is decreasing and Lemma 1 (a) (Remark 1 (ii)). Thus, the case $f(v+(1-v)b) > 0$ holds. To prove the statement in the case $f(v+(1-v)b) = 0$, we consider the following two exhaustive subcases (i) $f(u+(1-u)b) = 0$ and (ii) $f(u+(1-u)b) > 0$. Obviously in subcase (i) $L_f(a,b,u,v) \geq 0$ as f is nonnegative. Finally, in subcase (ii), since $(1-v)b \leq (1-u)b \leq u+(1-u)b$ and f is non increasing, we have that

$$\begin{aligned}
& L_f(a,b,u,v) = f((1-u)b)f(v+(1-v)a) - f((1-u)a)f(v+(1-v)b) \\
+ \;& f((1-v)b)f(u+(1-u)b)\left(\frac{f(u+(1-u)a)}{f(u+(1-u)b)} - \frac{f((1-v)a)}{f((1-v)b)}\right) \\
\geq \;& f((1-v)b)f(u+(1-u)b)\left(\frac{f((1-u)a)}{f((1-u)b)} - \frac{f((1-v)a)}{f((1-v)b)}\right) \geq 0
\end{aligned}$$

where both inequalities apply by using the same arguments than those in previous inequalities. Since we have proven the statement in the two exhaustive cases, then the result in (7) holds for a decreasing function f and, thus $\beta_t f$ is log concave.

Consider the case in which f is log concave and increasing letting

$$L_f(a,b,u,v) = L_{\overline{f}}(1-b,1-a,u,v)$$

where $\overline{f}(x) = f(1-x)$. Observe that f is log concave and increasing on $[0,1]$ iff $\overline{f}(\theta) = f(1-\theta)$ is log concave and decreasing, and $\Omega(r_1,r_2) \leq_{\text{lr}} \Omega(s_1,s_2)$ entails $1-\Omega(r_1,r_2) \geq_{\text{lr}} 1-\Omega(s_1,s_2)$. Accordingly, following the proof for the decreasing case, we can show directly that $L_f(a,b,u,v)$ is positive, which is the completed proof for the increasing case. ■

Remark 2 *Observe that*

$$\Omega(q,p) \stackrel{d}{=} 1-\Omega(p,q), \quad p,q>0,$$

which yields the following identity:

$$\beta_t(f(\theta),x) = \beta_t(f(1-\theta),1-x).$$

Therefore, f is log concave and decreasing (increasing) on $[0,1]$ iff $f(1-\theta)$ is log concave and increasing (decreasing). This yields that the establishment of Theorem 1 in the case when f is increasing (decreasing) entails that of when f is decreasing (increasing).

Remark 3 *Theorem 1 is equivalent to the following statement. If f is log concave on $[0,1]$, monotone and $E\left[f\left(\Omega(p,q-p)\right)\right] < \infty$, $0<p<q$, then*

$$E\left[f\left(\Omega(p,q-p)\right)\right], \quad 0<p<q$$

is log concave in p.

For general log concave f, its shape preservation by the beta operator is an open problem that can not be solved with the tools employed in the proof of Theorem 1 since inequality (7) is not true for general log-concave function f. Actually, setting $u=0$ and $v=1$, we can check if $L_f(a,b,0,1) = (f(a)-f(b))(f(0)-f(1))$ is positive or not. When we take a log-concave function with $f(0) > f(1)$ and $a<b<m$ such that $f(a)<f(b)$ assuming an unimodal function with mode $0<m<1$, we have that $L_f(a,b,0,1)<0$. We cannot overcome this difficulty because 0 and 1 are extreme points. Indeed, for this function f we can take u_n converging to 0 and v_n converging to 1 and then, realize that $L_f(a,b,u_n,v_n)<0$.

4 Applications

In this section we address three applications of Theorem 1.

4.1 Log concave functions

Applying Theorem 1 and Remark 3 to some specific functions, we can derive some log concavity properties of the relevant functions. For instance, for the log concave function f on $[0,1]$, defined by $f(x) = x^{\alpha}$ for $\alpha \geq 0$, we can obtain the fact that

$$\frac{\Gamma(p+\alpha)}{\Gamma(p)}, \quad 0 < p,$$

is a log concave function in p and accordingly, the digamma function (Ψ) is concave (see Said et al. [33]), where the digamma function is the logarithm derivative of the gamma function. This results holds since the following function is decreasing with x being positive and h non negative

$$\Psi(x+h) - \Psi(x) = \frac{d \ln\left(\frac{\Gamma(x+h)}{\Gamma(x)}\right)}{dx}.$$

If we consider the function $f_a(x) = 1_{[0,a]}(x)$ being 1_A the indicator function of a set A and $0 \leq a, x \leq 1$ we obtain the normalized incomplete beta function given by

$$\beta_t(f_a, x) = \frac{1}{B(tx, t(1-x))} \int_0^a u^{tx-1}(1-u)^{t(1-x)-1} du.$$

As f_a is log concave and decreasing we derive by Theorem 1 that the previous incomplete beta function is log concave in x. In Karp [24] other types of log concavity on parameters of the incomplete beta function are explored.

In what follows, we consider another example.

Example 1 *Recently, Karp and Sitnik [23] derived the log concavity properties of some particular cases of the generalized hypergeometric functions. Here, we should note that the properties can be proven by Theorem 1 and the Euler's formula as well. The generalized hypergeometric function with parameters p, q, which are natural numbers, such that $p \leq q+1$, $a_1, \ldots, a_p$,*

$b_1, \ldots, b_q$, and $b_i \neq 0, -1, -2, \ldots$, $i = 1, \ldots, q$, denoted as ${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}\right)$ *are defined by*

$$ {}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!} x^n, $$

where

$$ (a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} $$

is Pochhammer's symbol. Observe that hypergeometric Kummer confluent and Gauss hypergeometric function coincide with the generalized hypergeometric functions when $(p,q) = (1,1)$ and $(p,q) = (2,1)$, respectively. Using Euler's formula, we have that

$$ {}_1F_1\left(\begin{array}{c} a_1 \\ b_1 \end{array} \middle| x \right) = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1 - a_1)} \int_0^1 e^{xy} y^{a_1} (1-y)^{b_1 - a_1} dy, \quad 0 < a_1 < b_1 $$

and

$$ {}_2F_1\left(\begin{array}{c} a_1, a_2 \\ b_1 \end{array} \middle| x \right) = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1 - a_1)} \int_0^1 (1-xy)^{-a_2} y^{a_1} (1-y)^{b_1 - a_1} dy, \quad 0 < a_1 < b_1. $$

Observe that in the above integral formulas, the functions e^{yx}, $x \in (-\infty, \infty)$, and $(1-xy)^{-a_2}$, $x \in [0,1]$, are log concave and monotone functions for $y \in (0,1)$ and $a_2 \leq 0$. Therefore, from Theorem 1 (see also Remark 3), we conclude that for $0 < a_1 < b_1$, ${}_1F_1\left(\begin{array}{c} a_1 \\ b_1 \end{array} \middle| x \right)$ is log concave in a_1 for $x \in (-\infty, \infty)$ and ${}_2F_1\left(\begin{array}{c} a_1, a_2 \\ b_1 \end{array} \middle| x \right)$ is log concave in a_1 for $x \in [0,1]$ and $a_2 \leq 0$.

4.2 Ageing classes in deterioration models

Deterioration of a system is an important topic in the reliability area and it is modeled with a stochastic process $\{X(t) : t \geq 0\}$ depending on the age (time). In most cases a system under stochastic deterioration breaks down when the deterioration reaches a threshold, which can be randomly determined. Let us denote Y as a threshold and its cumulative distribution function (cdf) and survival function (sf) as F and $\overline{F}$, respectively. We are interested in a random time τ in which the first breakdown occurs due to deterioration in the system, i.e.,

$$\tau = \inf_{t\geq 0}\{X(t) \geq Y\}. \tag{8}$$

In the case when the stochastic process has right-continuous increasing paths, its sf and cdf (see [2]), denoted by $\overline{G}$ and G, respectively, are given by

$$\overline{G}(t) = E[\overline{F}(X(t))] \tag{9}$$

and

$$G(t) = E[F(X(t))]. \tag{10}$$

In addition, G is increasing and $\overline{G}$ is decreasing.

Now we intend to investigate some ageing properties of τ. For this, it should be noted that increasing failure rate (IFR) or decreasing reversed hazard rate (DRHR) property holds when the sf (cdf) of τ is log concave. The ageing properties of τ for gamma and Levy processes were studied in Abdel-Hameed [1, 3] and similar results were derived in Sangüesa et al. [34] for deterioration processes with independent increments.

For our application, we consider that the deterioration process is a Dirichlet process. Dirichlet process was introduced in Fergurson [21]. Rigorous definition of such a process and its applications can also be found in Fergurson [21]. Let us now consider that the probability measure associated with the Dirichlet process is a Stieltjes-Lebesgue on $[0, r]$ for some $r > 0$ with associated function:

$$\frac{\Lambda(y)}{\Lambda(r)}, 0 \leq y \leq r,$$

where Λ is an increasing function on $[0, r]$. From Equation (1), it is not difficult to prove that the corresponding process $\{X(t) : 0 \leq t \leq r\}$ has a version defined by $\{\frac{S(\Lambda(r))}{S(\Lambda(t))} : 0 \leq t \leq r\}$ for a homogeneous gamma process $\{S(t) : t \geq 0\}$ with unit scale parameter that has increasing right-continuous paths. The proof of the claim is simple as multivariate Dirichlet random vector of dimension n with parameters $p_i > 0$, $i = 1, 2, \ldots, n+1$ has the same probability distribution as the random vector

$$\left(\frac{Z(p_1)}{\sum_{i=1}^{n+1} Z(p_i)}, \ldots, \frac{Z(p_r)}{\sum_{i=1}^{n+1} Z(p_i)}, \ldots, \frac{Z(p_n)}{\sum_{i=1}^{n+1} Z(p_i)}\right),$$

where $Z(p_i)$, $i = 1, \ldots, n+1$ are independent gamma random variables.

Theorem 2 *Let $\{X(t) : 0 \leq t \leq r\}$ be a Dirichlet process with Stieltjes-Lebesgue on $[0, r]$ with associated function $\frac{\Lambda(y)}{\Lambda(r)}$, $0 \leq y \leq r$ for an increasing function Λ. Let τ be the hitting time of random threshold Y defined as (8), then the following hold:*

(a) if Λ is convex and Y is IFR, τ is IFR;

(b) if Λ is concave and Y is DRHR, τ is DRHR.

Proof: (a) To prove that τ is IFR, note that $\overline{F}$ is log concave and decreasing, and $X(t) \overset{d}{=} \Omega(\Lambda(t), \Lambda(r) - \Lambda(t))$. Then, by Theorem 1 (see Remark 3), we have that

$$E[\overline{F}(\Omega(p, \Lambda(r) - p))]$$

is log concave in p and it can be seen that it is decreasing in p (see Adell et al. [4]). In addition, considering the facts that Λ is convex and the composition of a log concave decreasing function with a convex function is a log concave function, the conclusion follows by (9).

(b) Following similar arguments as the proof of (a), by (10), we conclude that τ is DRHR. ■

4.3 Log concavity preservation by Stancu-Mühlbach and Bernstein-Durrmeyer operator

We apply our results to show the preservation of log concavity of two Bernstein operators which are generalizations of the classical Bernstein polynomial. A Bernstein polynomial of degree n acting on a function f on $[0, 1]$ is defined as the function

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1].$$

It is well known that the Bernstein operator preserves monotony and log concavity. One of the extensions is called the Stancu-Mühlbach operator (see, e.g., Stancu [40], Mühlbach [32], De La Cal [14] and references therein) denoted as $B_n^\alpha f$, which is defined by

$$B_n^\alpha(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} \frac{\left(\frac{x}{\alpha}\right)_k \left(\frac{1-x}{\alpha}\right)_{n-k}}{\left(\frac{1}{\alpha}\right)_n}, \quad x \in [0, 1], \alpha > 0, n = 1, 2, \ldots.$$

Another one is known as the Bernstein-Durrmeyer operator; it is denoted by D_n (see Durrmeyer [19], Adel et al. [7], and Derriennic [17]). D_n can be expressed as follows:

$$D_n(f,x) = (n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_0^1\binom{n}{k}t^k(1-t)^{n-k}f(t)dt, \quad x\in[0,1], n=1,2,\ldots.$$

Notably, the following formula indicates that the Stancu-Mühlbach operator can be represented by the Bernstein and beta operators (De La Cal [14] and Mühlbach [32]).

$$B_n^{\alpha}(f,x) = \beta_{\frac{1}{\alpha}}(B_n f, x) \tag{11}$$

Moreover, the Bernstein-Durrmeyer operator can be presented as follows.

$$D_n(f,x) = B_n(\beta_{n+2} f_n, x), \tag{12}$$

where $f_n(x) = f\left(\frac{n}{n+2}x + \frac{1}{n+2}\right)$. Therefore by formulas (11) and (12), Theorem 1, the preservation of monotony by the beta operator (Adell et al. [4]) and the preservation of log concavity and monotony by Bernstein operators (Goodman [22] and Lorentz [29]), we have the following result.

Corollary 1 *Stancu-Mühlbach and Bernstein-Durrmeyer operators preserve log concavity for monotone functions.*

4.4 Order statistics

Order statistics play an important role in many areas of applied probability. Consider a sequence of n independent and identically distributed random variables with F being the common marginal distribution function. Let $X_{k:n}$ denote the k-th order statistic. The pdf of $X_{k:n}$ is given as follows ($k = 1,\ldots,n$)

$$\frac{1}{B(k,n+1-k)}F(x)^{k-1}(1-F(x))^{n-k}f(x),$$

for x in the domain of the random variable X, where F and f are the cdf and pdf of X, respectively. It is not hard to prove that for a function h

$$E[h(X_{k:n}) = E[g(\Omega(k,n+1-k))] \tag{13}$$

with $g(x) = h(F^{-1}(x))$, where F^{-1} is the quantile function of X. Based on Remark 3 we show that (13) is a log concave sequence in k if g is monotone and log concave. Sufficient conditions under which g is log concave in terms of h and F^{-1} are the following:

- h is log concave and decreasing; and F^{-1} is convex.
- h is log concave and increasing; and F^{-1} is concave.

It is straightforward to check that F^{-1} is convex (concave) if and only if F is concave (convex).

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